Institute for Applied Mathematics – SS19
Massimiliano Gubinelli | Immanuel Zachhuber | Nikolay Barashkov

V3F1 Stochastic Processes – Problem Sheet 7

Distributed May 17th, 2019. At most in groups of 2. Solutions have to be handed in before 4pm on Thursday May 23th into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [Pts 1+2+2+1] Let $(Y_n)_n$ an i.i.d. sequence with $Y_n \geqslant 0$ and $\mathbb{E}[Y_n] = 1$. Let $X_n = \prod_{k=1}^n Y_k$ for all $n \geqslant 1$ and $X_0 = 1$.

a) Show that $(X_n)_n$ is a martingale wrt. the natural filtration of $(Y_n)_n$.

b) Assume that $Y_n \geqslant \delta$ for some $\delta > 0$. Show that $\mathbb{E}[\log Y_1] < \infty$ and use the law of large numbers to show that if $\mathbb{P}(Y_1 = 1) < 1$ then $X_n \to 0$ almost surely.

c) Let $Z_n = \max(\delta, Y_n)$. Show that there exists $\delta > 0$ such that $\mathbb{E}[\log Z_n] < \infty$ and conclude that, if $\mathbb{P}(Y_1 = 1) < 1$ then $X_n \to 0$ almost surely, without additional hypothesis on Y.

d) Conclude that, in general, the convergence $X_n \to X_\infty$ in Doob's martingale convergence theorem is not in L^1 but only almost surely.

Exercise 2. [Pts 4] Let $(X_n)_{n \geqslant 0}$ be a super- or sub-martingale and let $|X|_n^* = \sup_{0 \leqslant k \leqslant n} |X_k|$, prove that there exists a constant $C > 0$ such that for all $\lambda > 0$ and all $n \geqslant 0$,

$$\lambda \mathbb{P}(|X|_n^* \geqslant \lambda) \leqslant C \mathbb{E}[|X_0| + |X_n|].$$

(Hint: use the maximal inequality and Doob's decomposition)

Exercise 3. [Pts 2+2+3+3] (Polya's urn) At time 0 we have a urn which contains one red and one green ball. At every instant $n = 1, 2, \ldots$ we draw at random a ball from the urn and we put it back adding another ball of the same color. Let S_n the number and $X_n = S_n / (n+2)$ the proportion of red balls in the urn at time n.

a) Show that $(X_n)_n$ is a martingale with respect to its natural filtration and compute $\mathbb{E}[X_n]$.

b) Show that $X_n \to X_\infty$ almost surely and in L^1.

c) For all $k \geqslant 1$ let

$$Z_n^{(k)} = \frac{S_n(S_n+1)\cdots(S_n+k-1)}{(n+2)\ldots(n+k+1)}.$$

Show that $(Z_n^{(k)})_{n \geqslant 0}$ is a martingale for all $k \geqslant 1$ and compute $\mathbb{E}[Z_n^{(k)}]$.

d) Show that

$$\mathbb{E}[X_\infty^k] = \mathbb{E}[Z_0^{(k)}] = \frac{1}{k+1}$$

and deduce that X_∞ is distributed uniformly on $[0,1]$.