V3F1 Stochastic Processes – Problem Sheet 8

Distributed May 24th, 2019. At most in groups of 2. Solutions have to be handed in before noon on Friday May 31st into the marked post boxes opposite to the maths library. Please clearly specify your names and your tutorial group on top of your homework.

Exercise 1. [Pts 8] Let $(X_n)_{n\geqslant 0}$ be a martigale such that $X_n \in L^2$ for all $n \geqslant 0$. Recall that $(\langle X\rangle_n)_{n\geqslant 0}$ is the increasing previsible process in Doob's decomposition of the sub-martingale $(X_n^2)_{n\geqslant 0}$, and is called *compensator*. Prove that $(X_n)_n$ converges almost surely on the event $\{\langle X\rangle_\infty < +\infty\}$. Hint: consider the process $(X_{n\wedge T_N})_{n\geqslant 0}$ where $T_N = \inf\{n\geqslant 0 \colon \langle X\rangle_n \geqslant N\}$ for some $N<\infty$.

Exercise 2. [Pts 2+2+2+2] Let X, $(\xi_n)_{n\geqslant 1}$ random variables such that $X\sim\mathcal{N}(0,1)$ and $\xi_n\sim\mathcal{N}(0,\varepsilon_n^2)$ with $\varepsilon_n>0$ for all $n\geqslant 1$. Let $Y_n = X+\xi_n$ and

$$X_n = \mathbb{E}[X|\mathscr{F}_n]$$

where $\mathscr{F}_n = \sigma(Y_1,\ldots,Y_n)$. We can interpret X as a quantity of interest and X_n as our best prediction (in the L^2 sense) given a sequence of n noisy observations $Y_1,\ldots,Y_n$ of X affected by additive independent noise terms $(\xi_n)_n$. The question is to see if $X_n\to X$ as $n\to\infty$.

a) Show that $(X_n)_n$ is a martingale bounded in L^2, i.e. $\sup_n \mathbb{E}[X_n^2]<\infty$;

b) Show that the sequence $(X_n)_n$ converge almost surely towards $X_\infty\in L^2$.

c) Show that for all $n\geqslant 1$ and all $1\leqslant i\leqslant n$ we have $\mathbb{E}[Z_nY_i]=0$ where Z_n is defined by

$$Z_n = X - \frac{1}{1+\sum_{k=1}^n \varepsilon_k^{-2}} \sum_{k=1}^n \varepsilon_k^{-2} Y_k$$

and that $X_n = X - Z_n$.

d) Compute $\mathbb{E}[(X-X_n)^2]$ and deduce that $X_n\to X$ in L^2 iff $\sum_{k=1}^\infty \varepsilon_k^{-2} = +\infty$.

As a consequence, if $\sum_{k=1}^\infty \varepsilon_k^{-2} < +\infty$ we cannot “filter” X even observing an infinite number of $(Y_n)_n$.

Exercise 3. [Pts 2+2+4] Take $(X_k)_{k\geqslant 1}$ i.i.d random variables such that $\mathbb{E}[e^{uX_i}]<\infty$ for all $u\in\mathbb{R}$. We want to show that for any $\bar\rho>0$ there exists a constant $C(\bar\rho)>0$ such that, for all n and all functions $f\colon\mathbb{R}^n\to\mathbb{R}$ differentiable and such that

$$\sup_{x\in\mathbb{R}^n, k=1,\ldots,n} \left|\frac{\partial f}{\partial x_k}(x_1,\ldots,x_n)\right| \leqslant 1,$$

we have

$$\mathbb{P}(|F-\mathbb{E}(F)|>\rho n) \leqslant 2e^{-n\rho^2/2C(\bar\rho)}, \qquad 0\leqslant\rho\leqslant\bar\rho, \tag{1}$$

where $F = f(X_1,\ldots,X_n)$.

a) Introduce the martingale $F_k = \mathbb{E}[F|\mathscr{F}_k]$ where $\mathscr{F}_k = \sigma(X_1,\ldots,X_k)$ for $k=1,\ldots,n$ and $\mathscr{F}_0 = \{\emptyset,\Omega\}$ and write $F-\mathbb{E}[F] = F_n - F_0$. Use ideas from the proof of the martingale CLT to prove that there exists a positive increasing function $\lambda\mapsto Q(\lambda)$ such that if $\Delta F_k = F_k - F_{k-1}$ then

$$\mathbb{E}[e^{\lambda\Delta F_k}|\mathscr{F}_{k-1}] \leqslant 1+\lambda^2 Q(\lambda), \qquad k=1,\ldots,n.$$

b) Deduce from this a bound on $\mathbb{E}[e^{t(F-\mathbb{E}(F))}]$ for all $t\in\mathbb{R}$;

c) Conclude that the bound (1) holds.