Stochastic processes SS21. Example questions for the oral exam. (Updated 16.7.2021)

The exam lasts ~30min.

1. What is your preferred theorem from the course? Why? Proof?
2. State the monotone class theorem and give an example of application.
3. Prove completeness of L^p spaces.
4. Define uniform integrability of a family of random variables. Give an example of a UI family and of a non-UI family.
5. Explain in which sense UI is the best condition for convergence of integrals. (you might want to give also some ideas of the proof)
6. Definition of conditional expectation. Proof of uniqueness and of some basic properties.
7. Proof of existence of conditional expectation (first in L^2 then in general).
8. How to compute the conditional expectation $\mathbb{E}[g(Y)|X]$ knowing that (X, Y) have a joint density $f_{X,Y}$?
9. Example that $\mathbb{E}[\mathbb{E}[X|\mathscr{F}]|\mathscr{G}] \neq \mathbb{E}[\mathbb{E}[X|\mathscr{G}]|\mathscr{F}]$ in general?
10. Relation of conditional expectations with independence.
11. Explain the concept of regular conditional probability.
12. Define a stopping time and its σ-algebra. Some properties of stopping times.
13. Which among $T \wedge S, T \vee S$ and $T + S$ are stopping times? And why?
14. Define (sub-,super-)martingales and give some of their basic properties. Explain why martingales represents games which do not allow "winning" stopping strategies.
15. Sketch a method to compute the ruin probability $\mathbb{P}(X_{T_{a,b}} = a|T_{a,b} < \infty)$ for the simple random walk X on $X = \{a, \dots, b\}$ starting at $x \in X$.
16. What is Doob's decomposition? Why is unique? Give an situation where we used it.
17. What happens to a non-negative martingale which touches 0?
18. What can we say about a positive supermartingale X such that $\mathbb{E}[X_n] = 1$ for all $n \geqslant 0$.
19. Explain what is a martingale transform and how to use it to prove the optimal stopping theorem (with statement).
20. What are sufficient conditions for almost sure convergence of super-/sub- martingales? Give counterexample that in general the convergence is not stronger than that, e.g. does not take place in L^1.
21. State Doobs' upcrossing inequality and the ideas behind its proof. How this is used to prove convergence of martingales?
22. What can you say about a martingale that it is bounded in L^2? How the proofs go?
23. Explain the Robbins–Monroe method and give some ideas of its proof of convergence.
24. What is a closed martingale? Examples? Under which conditions a martingale can be closed?
25. State Doobs' maximal and L^p inequalities and gives some ideas of their proofs.
26. What can you say about UI martingales?

27. What happens if you try to stop a closed martingale at a stopping time?
28. Define the tail σ-algebra $\mathscr{T}$ of a stochastic process and give examples of events in $\mathscr{T}$ and of random variables which are measurable wrt. $\mathscr{T}$ with motivation.
29. What is Kolmogorov 0/1 law? Proof?
30. State and give ideas of the proof of Kolmogorov's strong law of large numbers using backwards martingales.
31. State and give some ideas of the proof of Kakutani's theorem.
32. How can you prove Radon-Nykodim theorem using martingales?
33. What is the Snell envelope of a stochastic process? What is its relation with optimal stopping problems. Can you give an example of an optimal stopping problem?
34. Define a Markov process and give some properties. What is the canonical space for a Markov process. State the strong Markov property and give an example of its use.
35. Give an example of a process which is Markov and of a process which is not Markov.
36. State the relation between a Markov process and a martingale problem and give some details of the proof.
37. Define transience and recurrence for a Markov chain in discrete space and give an idea of the proof that an irreducible chain is either transient or recurrent. Give an example of a recurrent chain and of a transient one.
38. Explain the relation between a Markov chains and certain linear equations for its generator.
39. What is Doob's h-transform. How can be used to compute conditional probabilities of Markov chains?
40. What is an invariant measure for a Markov chain. Why is also called stationary measure?
41. Explain how to construct invariant measures for (irreducible) recurrent discrete chains.
42. Conditions for existence and uniqueness of invariant measures? Ideas of proofs?
43. Explain the relation between invariant probability and expected return time to a state in the context of recurrent irreducible discrete chains.