Lecture 1 · 13.4.2021 · 14:15–16:00 via Zoom

Topics covered in the course: tools to study and solve probabilistic problems involving *stochastic processes* (i.e. random variables taking values in space of sequences (maybe infinite) or functions)

Example:

- Number of individuals in a given populations (e.g. bacteria) depending on generations: $(N_n)_{n\geqslant 1}$: $\Omega\to\mathbb{Z}_{\geqslant 0}^{\mathbb{N}}$
- Temperatures every day $X(\omega)=(X_n(\omega))_{n\geqslant 1}$ $X:\Omega\to\mathbb{R}^{\mathbb{N}}$
- Brownian motion in $d$-dimensions $B:\Omega\to C([0,1],\mathbb{R}^d)$, $(B_t:\Omega\to\mathbb{R})_{t\in[0,1]}$

Mainly we will concentrate on discrete time processes.

Tools:

- **Conditional expectation**: a way to approximate a r.v. using only partial knowledge about it (i.e. give n a particular $\sigma$-algebra $\mathcal{G}$ obtain a "projection" $\mathbb{E}[X|\mathcal{G}]$ of a r.v. $X$ which better represent $X$ with only the information contained in $\mathcal{G}$. You know a definition valid in the elementary setting and we will give a more general one which is valid *always* (ok…) but is more abstract, and therefore it has to be handled with care.
- Two main classes of stochastic processes (in discrete time):
  - **Martingales**. (Doob in '50) Martingales are a very natural class of stochastic processes which behaves "unpredictably" (in a precise sense). This unpredictability give rise to very nice mathematical theory for them, in particular we can prove limit theorems (for lont time) for martingales and also very strong probabilistic estimates. A very simple example of a martigale $(M_n)_{n\geqslant 1}$ is the sequence

    $$M_n=X_1+\cdots+X_n, \tag{1}$$

    where $(X_n)_{n\geqslant 1}$ is a family of i.i.d. integrable (i.e. $L^1(\mathbb{P})$) random variables with mean zero. The interest is that in general martingales behave like $(M_n)_n$ even if they are not made by independent increments. They are fundamental in continous time, i.e. in stochastic analysis, next semester to study processes like Brownian motions and similar.
  - **Markov processes.** A (discrete) Markov process $(X_n)_{n\geqslant 1}$ is a family of r.v. such that what will happen in the "future" depends only on the "present" and not on the "past". We need of course to define good enough concepts to give a precise meaning to this notion of "passing of time". The random walk (1) is also an example of a Markov process, indeed it satisfy the discrete recursion

    $$M_{n+1}=M_n+X_{n+1}$$

    where $X_{n+1}$ do no depends on what happened to $(X_1,\ldots,X_n)$ and therefore to $(M_1,\ldots,M_n)$ but only on the "present" value $M_n$ and on a random perturbation $X_{n+1}$ of it. So Markov processes corresponds to *random dynamical systems*:

    $$M_{n+1}=F_{n+1}(M_n)$$

    where the family of functions $(F_n:\Omega\times\mathbb{R}\to\mathbb{R})_{n\geqslant 1}$ is random and i.i.d. and in the particular case above is given by

    $$F_n(\omega):x\in\mathbb{R}\mapsto F_n(\omega)(x)=x+X_n(\omega).$$

- **Gaussian processes.** These are families $(X_j)_{j\in I}$ for a general set $I$ such that for any finite dimensional subset $J\subseteq I$ and numbers $(\alpha_j)_{j\in J}\in\mathbb{R}^J$ the r.v.

  $$Z\colon \omega\in\Omega\mapsto Z(\omega):=\sum_{j\in J}\alpha_j X_j(\omega)$$

  is a Gaussian random variable.

  Example: one dimensional white noise $(X_f)_{f\in I}$ is the Gaussian process with $I=L^2(\mathbb{R})$ (the Hilbert space of measurable and square integrable functions $\mathbb{R}\to\mathbb{R}$ wrt. Lebesgue measure) such that for any $f\in L^2(\mathbb{R})$ the r.v. $X_f$ is Gaussian with mean zero and variance given by $\|f\|^2_{L^2(\mathbb{R})}$, i.e.

  $$\mathbb{E}[X_f]=0,\qquad \mathbb{E}[X_f^2]=\|f\|^2_{L^2(\mathbb{R})}.$$

  This is a good model of the white noise of electrical engineers, i.e. of a signal which contains the same power in every frequency range (more later). Problems: does such a process exists? How to construct it?

  This problem was solved by Wiener '40 using the newly introduced integration theory of Lebsegue, when he introduced the rigorous construction of the Brownian motion. Brownian motion was suggested by Einstein (~1900) and by Bachelier (~1900), the first (Einstein) looked at it as a model of the pollen particles observed by Brown (when??) while the second (Bachelier) used the same model for describing the evolution of individual stocks in a market.

  > Brownian motion $(B_t)_{t\geqslant 0}$ is a *continuous random function with independent increments*.

  The magic here is that this definition implies that Brownian motion is a Gaussian process with $I=\mathbb{R}$ with mean zero and covariance

  $$\operatorname{Cov}(B_t,B_s)=\mathbb{E}[B_tB_s]=(t\wedge s),\qquad t,s\in\mathbb{R}.$$

  The link between Brownian motion and white noise is that we can take $B_t=X_{\varphi_t}$ with

  $$t\in\mathbb{R}_{\geqslant 0}\mapsto\varphi_t:=\mathbb{1}_{[0,t]}\in L^2(\mathbb{R})$$

  so that

  $$\mathbb{E}[B_t^2]=\mathbb{E}[X_{\varphi_t}^2]=\|\varphi_t\|^2_{L^2(\mathbb{R})}=\int_0^\infty\mathbb{1}_{[0,t]}(s)^2\mathrm{d}s=t.$$

For all these class of processes one can ask questions about:

- modelisation : which kind of problems give rise to the three classes of processes?
- optimisation : e.g. how to “stop” optimally one of these processes (e.g. in gambling)?
- what happens when I know some partial informations about the process? E.g. how does it looks like a population of bacteria $(N_n)_{n\geqslant 1}$ for which I know that the population survived 1000 generations, i.e. I'm on the event $\{N_{1000}>0\}\in\mathscr{F}$. This is maybe an unlikely event so the kind of histories which give rise to this even should look very different from the typical history of the population.
- asymptotic behaviour :
  - ergodic theorems, generalisation of the Law of Large Numbers for independent sequences
  - fluctuations : generalisations of the Central Limit Theorem
  - large deviations : asking about probabilities of deviations form LLN which are much larger than typical CLT deviations

Many of these questions are relevant to construct statistical methods for stochastic processes with applications in the sciences and in the applied world.

### Proposed slot for tutorials

- Mo 8-10 (von Basis)
- ~~Mo 10-12 (von Basis)~~
- Mi 12-14 (von Basis)
- Mo 16-18 (zusätzlich)
- Di 12-14 (zusätzlich)

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