Martingales

We assume we are given a prob. space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $(\mathscr{F}_n)_{n\geqslant 0}$ which will be fixed all along this lecture (unless specified otherwise). The given of $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\geqslant 0}, \mathbb{P})$ it is called a filtered probability space.

We want to characterise the class $\mathscr{M}$ of real-valued stochastic processes $(X_n)_{n\geqslant 0}$ which are adapted, integrable (i.e. $X_n \in L^1(\mathbb{P})$ for all $n \geqslant 0$) and such that for all bounded stopping times T

$$\mathbb{E}[X_T] = \mathbb{E}[X_0], \tag{1}$$

This class models the total gain (or loss) in a "fair" games of chance.

Lemma. *An adapted and integrable process $(X_n)_{n\geqslant 0}$ satisfies (1) iff for all $n \geqslant 0$ we have*

$$\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n. \tag{2}$$

Proof. Let's start to show that (1) implies (2). The idea is to find a appropriate stopping time. For any $n \geqslant 0$ and any $A \in \mathscr{F}_n$ we can define the stopping time (check)

$$T_{n,A}(\omega) = \begin{cases} n+1 & \text{if } \omega \in A \\ n & \text{otherwise} \end{cases}$$

This is a stopping time bounded by $n+1$ and therefore by (1) we have

$$0 = \mathbb{E}[X_{T_{n,A}}] - \mathbb{E}[X_0] = \mathbb{E}\big[X_{n+1} \underbrace{\mathbb{1}_A}_{\in \mathscr{F}_n} + X_n \mathbb{1}_{A^c}\big] - \mathbb{E}[X_0]$$

$$= \mathbb{E}[\mathbb{E}[X_{n+1}|\mathscr{F}_n]\mathbb{1}_A] - \mathbb{E}[X_n \mathbb{1}_A] + \underbrace{\mathbb{E}[X_n] - \mathbb{E}[X_0]}_{=0, \text{ by (1), since } n \text{ is a bounded stopping time}}$$

So we have that for all $n \geqslant 0$ and $A \in \mathscr{F}_n$ we have

$$\mathbb{E}[\mathbb{E}[X_{n+1}|\mathscr{F}_n]\mathbb{1}_A] = \mathbb{E}[X_n \mathbb{1}_A]$$

which implies that $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$ ($\mathbb{P}$-a.s.).

Let's now prove that (2)⇒(1). Consider an arbitrary stopping time T bounded by some $N \in \mathbb{N}$ (i.e. $T(\omega) \leqslant N$ for all $\omega \in \Omega$). By decomposing the prob. space according to the values of T we have

$$\mathbb{E}[X_T] = \sum_{n=0}^{N} \mathbb{E}[X_T \mathbb{1}_{T=n}] = \sum_{n=0}^{N} \mathbb{E}[X_n \mathbb{1}_{T=n}]$$

Now we observe that if (2) holds for any $n \geqslant 0$ then for any $k \geqslant n+1$

$$\mathbb{E}[X_k|\mathscr{F}_n] = \mathbb{E}\big[\underbrace{\mathbb{E}[X_k|\mathscr{F}_{k-1}]}_{=X_{k-1}}|\mathscr{F}_n\big] = \mathbb{E}[X_{k-1}|\mathscr{F}_n] = \underbrace{\cdots}_{\text{induction}} = X_n$$

and therefore we have

$$\mathbb{E}[X_N|\mathscr{F}_n] = X_n \tag{3}$$

for any $n \in \{0, \ldots, N\}$. So now

$$\mathbb{E}[X_T] = \sum_{n=0}^{N} \mathbb{E}\big[\mathbb{E}[X_N|\mathscr{F}_n]\underbrace{\mathbb{1}_{T=n}}_{\in \mathscr{F}_n}\big] = \sum_{n=0}^{N} \mathbb{E}[X_N \mathbb{1}_{T=n}] = \mathbb{E}[X_N].$$

(Stopping at the random time T is in average equivalent to stopping at the final time N). But now using again (3) with $n=0$ we have

$$\mathbb{E}[X_T] = \mathbb{E}[X_N] = \mathbb{E}[\mathbb{E}[X_N|\mathscr{F}_0]] = \mathbb{E}[X_0]$$

which is want we wanted to prove. □

Definition. *A real, adapted and integrable stochastic process $(X_n)_{n\geqslant 0}$ is called*

*a) A **martingale** iff $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$ for all $n \geqslant 0$;*

*b) A **submartingale** iff $\mathbb{E}[X_{n+1}|\mathscr{F}_n] \geqslant X_n$ for all $n \geqslant 0$;*

*c) A **supermartingale** iff $\mathbb{E}[X_{n+1}|\mathscr{F}_n] \leqslant X_n$ for all $n \geqslant 0$;*

In the game intepreation a martingale is a “fair game”, a submartingale is a “favorable game”, a supermartingale is an “unfavorable game”.

Note that a (super-,sub-)martingale satisfies

$$\mathbb{E}[\Delta X_{n+1}|\mathscr{F}_n] \overset{\geqslant}{\underset{\leqslant}{=}} 0$$

with $\Delta X_{n+1} := X_{n+1} - X_n$.

The name of these objects is related to a corresponding naming of object in theory of harmonic functions (e.g. superharmonic, subharmonic). There is a precise relation between the theory of martingales and theory of harmonic functions, we will see it later on.

Example.

1. Let $X \in L^1(\mathscr{F})$ and let $X_n := \mathbb{E}[X|\mathscr{F}_n]$, then the process $(X_n)_{n\geqslant 0}$ is a martingale. Check the three properties: adaptedness, integrability and the martingale relation (i.e. $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$). That's the fundamental example of martingales.

2. A process $(A_n)_{n\geqslant 0}$ which is integrable, adapted, increasing (resp. decreasing) then it is a submartingale (resp. supermartingale).

3. Note that a martingale is both a supermart. and a submart.

4. Let $(X_n)_{n\geqslant 1}$ a sequence of i.i.d. r.v.s which are integrable and with $\mathbb{E}[X_1] = 0$. Let

$$Y_n = X_1 + \cdots + X_n$$

for all $n \geqslant 1$ with $Y_0 = 0$. Take moreover $(\mathscr{F}_n^Y = \sigma(Y_0, ..., Y_n))_{n\geqslant 0}$ to be the filtration generated by $(Y_n)_{n\geqslant 0}$, then $Y_\bullet = (Y_n)_{n\geqslant 0}$ is a martingale wrt. the filtration $(\mathscr{F}_n^Y)_{n\geqslant 0}$. If $\mathbb{E}[X_1] \geqslant 0$ the $Y_\bullet$ is a submartingale while if $\mathbb{E}[X_1] \leqslant 0$ it is a supermartingale. The process $Y_\bullet$ is called also the random walk with increments $(X_n)_{n\geqslant 0}$.

5. If $(X_n)_{n\geqslant 0}$ is a (super-,sub-)mart. wrt. a filtration $(\mathscr{G}_n)_{n\geqslant 0}$, then it is also a (super,sub)-mart. with respect to its own filtration, i.e. $(\mathscr{F}_n^X = \sigma(X_0, \ldots, X_n))_{n\geqslant 0}$.

Stopping times. A typical example of stopping time is something like

$$T = \inf\{n \geqslant 0 : X_n \in A\}$$

the first time the process $(X_n)_n$ enters the set A. The idea is that the event $\{T = n\}$ correspond to the choice of stopping at time n and this depends on $\mathscr{F}_n$, that is on the information available at time n.

In the particular case when $(\mathscr{F}_n)_n = (\mathscr{F}_n^Z)_n$ is generated by a real-valued stochastic process $(Z_n)_{n\geqslant 0}$ then for any $n \geqslant 0$ there exists a measurable function $h_n : \mathbb{R}^{n+1} \to \{0, 1\}$ such that

$$\mathbb{1}_{T=n} = h_n(Z_0, Z_1, \ldots, Z_n).$$

The given of T is equivalent to the family of functions $(h_n)_{n\geqslant 0}$.

Recall that a previsible process $(Y_n)_{n\geqslant 0}$ is a process such that $Y_{n+1} \hat{\in} \mathscr{F}_n$ for all $n \geqslant 0$.

Proposition. *(Doob's decomposition) Let $(X_n)_{n\geqslant 0}$ be an adapted and integrable stochastic process, then there exists a unique decomposition*

$$X_n = X_0 + M_n + I_n, \qquad n \geqslant 0,$$

where $(M_n)_{n\geqslant 0}$ is a martingale and $(I_n)_{n\geqslant 0}$ a previsible process with $I_0 = 0$. Moreover:

1. *$I_n = 0$ for all $n \geqslant 0$ **iff** X is a martingale,*
2. *I is increasing **iff** X is a submartingale,*
3. *I is decreasing **iff** X is a supermartingale.*

Proof. For existence one observe

$$\Delta X_{n+1} = X_{n+1} - X_n = X_{n+1} - \mathbb{E}[X_{n+1} | \mathscr{F}_n] + \mathbb{E}[X_{n+1} | \mathscr{F}_n] - X_n$$

and let $\Delta M_{n+1} := X_{n+1} - \mathbb{E}[X_{n+1} | \mathscr{F}_n]$ and $\Delta I_{n+1} := \mathbb{E}[X_{n+1} | \mathscr{F}_n] - X_n$ with $M_0 = 0, I_0 = 0$. This defines two processes $M_\bullet$ and $I_\bullet$ and I leave you to check that they satisfy the properties stated in the proposition. For example, note that

$$\mathbb{E}[\Delta M_{n+1} | \mathscr{F}_n] = \mathbb{E}[X_{n+1} - \mathbb{E}[X_{n+1} | \mathscr{F}_n] | \mathscr{F}_n] = \mathbb{E}[X_{n+1} | \mathscr{F}_n] - \mathbb{E}[X_{n+1} | \mathscr{F}_n] = 0.$$

For the uniqueness assume that there exists another pair $M'_\bullet, I'_\bullet$ which satisfy the same assumptions as $M_\bullet, I_\bullet$, then we have that

$$M_n - M'_n + I_n - I'_n = X_n - X_0 - X_n + X_0 = 0, \qquad n \geqslant 0$$

which implies that

$$M_n - M'_n = I'_n - I_n, \qquad n \geqslant 0.$$

In particular the process $N_n := M_n - M_n'$ is both a martingale (as difference of two martingales, you can check that martingales indeed for a vector space over $\mathbb{R}$) and it is also a previsible process since $N_n = I_n' - I_n \hat{\in} \mathscr{F}_{n-1}$ for all $n \geqslant 1$. Now the point is that a previsible martingale is constant process:

$$N_{n+1} \underset{\text{prev.}}{=} \mathbb{E}[N_{n+1} | \mathscr{F}_n] \underset{\text{mart.}}{=} N_n = \cdots = N_0 = 0$$

which proves that $N_n = 0$ for all $n \geqslant 0$ and therefore that $M = M'$ and $I = I'$, uniqueness of the decomposition.

The char. of the decomp. for (super-,sub-)mart. is left as exercise. □

Proposition. *Let $(X_n)_{n \geqslant 0}$ be a martingale (resp. sub-martingale) and $\Phi: \mathbb{R} \to \mathbb{R}$ a convex function (resp. convex and increasing) such that $(\Phi(X_n))_{n \geqslant 0}$ is an integrable process, then $(\Phi(X_n))_{n \geqslant 0}$ is a submartingale.*

Proof. (exercise) □

Example 1. If $(X_n)_{n \geqslant 0}$ is a martingale then $(|X_n|^p)_{n \geqslant 0}$ is a sub-martingale for all $p \geqslant 1$ provided $X_n \in L^p$ for all $n \geqslant 1$.

Proposition. *Let $(X_n)_{n \geqslant 0}$ be a square-integrable martingale (i.e. a martingale such that $X_n \in L^2$ for all $n \geqslant 0$) then the sub-martingale $(X_n^2)_{n \geqslant 0}$ has the decomposition*

$$X_n^2 = X_0^2 + N_n + [X]_n$$

where

$$N_n := 2 \sum_{k=1}^{n} X_{k-1} \Delta X_k, \qquad [X]_n := \sum_{k=1}^{n} (\Delta X_k)^2.$$

The process $(N_n)_{n \geqslant 0}$ is a martingale and the process $([X]_n)_{n \geqslant 0}$ is an increasing process called the quadratic variation of X.

Proof. (exercise) □

Remark. The process $([X]_n)_{n \geqslant 0}$ is not previsible therefore this is not Doob's decomposition, but it is still a useful decomposition and a natural one for L^2 martingales.
