Lecture 11 · 18.5.2021 · 14:15–16:00 via Zoom

Martingales (2)

Definition. *A real, adapted and integrable stochastic process* $(X_n)_{n\geqslant 0}$ *is called*

a) A ***martingale*** *iff* $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$ *for all* $n \geqslant 0$*;*

b) A ***submartingale*** *iff* $\mathbb{E}[X_{n+1}|\mathscr{F}_n] \geqslant X_n$ *for all* $n \geqslant 0$*;*

c) A ***supermartingale*** *iff* $\mathbb{E}[X_{n+1}|\mathscr{F}_n] \leqslant X_n$ *for all* $n \geqslant 0$*;*

Image we play a game of head and tails and at every time we gain if "head" come out and you loose otherwise. In this case we can consider the i.i.d. sequence $(Y_n)_{n\geqslant 1}$ of r.v. which are ± 1 according to whether we win or loose with probability $1/2$ each (we play a fair game).

The natural filtration in this problem is given by $\mathscr{F}_n = \sigma(Y_1, \ldots, Y_n)$ with $\mathscr{F}_0 = \{\emptyset, \Omega\}$.

Provided we play n rounds then we gain

$$X_n = Y_1 + \cdots + Y_n.$$

Two lectures ago we looked at stopping strategies given by a stopping time T and we proved that provided T is integrable then

$$\mathbb{E}[X_T] = 0.$$

We know that $(X_n)_{n\geqslant 0}$ is a martingale wrt. $\mathscr{F}_\bullet := (\mathscr{F}_n)_{n\geqslant 1}$

But we can play different games, in particular we could bet a different amount $(C_n)_{n\geqslant 1}$ at every step and then the total gain Z_n at time n would be given by

$$\Delta Z_n = Z_n - Z_{n-1} = C_n Y_n = C_n \Delta X_n,$$

for every $n \geqslant 1$ with $Z_0 = 0$ (for example, where negative amount stands for borrowed money). If C_n is negative it means I bet on "tail". Therefore

$$Z_n = \sum_{k=1}^{n} C_k \Delta X_k =: (C \bullet X)_n,$$

An important property of the process $(C_n)_{n\geqslant 0}$ is that it has to be previsible wrt. $\mathscr{F}_\bullet$, i.e.

$$C_n \hat{\in} \mathscr{F}_{n-1} = \sigma(Y_1, \ldots, Y_{n-1})$$

for all $n \geqslant 1$. Note that in this way $(Z_n)_n$ is an adapted process.

For example we can choose

$$C_n = C_n^T := \mathbb{1}_{n\leqslant T},$$

with T a stopping time. Indeed note that $\{n \leqslant T\} \in \mathscr{F}_{n-1}$ so $(C_n^T)_{n\geqslant 1}$ is previsible. With this choice we have

$$Z_n = \sum_{k=1}^{n} C_k^T \Delta X_k = X_{T\wedge n} - X_0 = X_{T\wedge n}.$$

In particular if T is bounded by N (deterministic) then $Z_N = X_T$. So these new strategies include the kind of strategies which are implement via stopping times.

Since $(X_n)_n$ is a martingale, i.e. a fair game, we don't expect to be able to gain even using more general strategies. In particular we expect that

$$\mathbb{E}[Z_n] = \mathbb{E}[(C \bullet X)_n] = 0,$$

for all $n \geqslant 1$ and all previsible processes $(C_n)_n$. Of course for this to be true (or even meaningful) we need to precise some conditions.

Definition. *Let $(X_n)_{n\geqslant 0}$ an adapted stochastic process and $(C_n)_{n\geqslant 1}$ a previsible stochastic process, then we call the new process $((C \bullet X)_n)_{n\geqslant 0}$ defined as*

$$(C \bullet X)_n := \sum_{k=1}^{n} C_k \Delta X_k, \qquad n \geqslant 1,$$

with $(C \bullet X)_0 = 0$, the ***martingale transform*** *of X by C.*

Lemma. *Let $(C_n)_{n\geqslant 1}$ a previsible and uniformly bounded process (i.e. there exists a constant $L < \infty$ such that $|C_n| \leqslant L$ for all $n \geqslant 1$). Then*

a) If $(X_n)_{n\geqslant 0}$ is a martingale then $((C \bullet X)_n)_{n\geqslant 0}$ is also a martingale;

b) If $(X_n)_{n\geqslant 0}$ is a super-martingale (resp. sub-martingale) and $C_n \geqslant 0$ for all $n \geqslant 1$ then $((C \bullet X)_n)_{n\geqslant 0}$ is also a super-martingale (resp. sub-martingale);

If both $(C_n)_{n\geqslant 1}$ and $(X_n)_{n\geqslant 0}$ are square integrable (i.e. $C_n, X_n \in L^2(\mathbb{P})$ for all n) then the same results are true without the uniform boundedness condition.

Proof. I leave you to check that the martingale transform $((C \bullet X)_n)_{n\geqslant 0}$ is an adapted and integrable process. Then it is enough to note that

$$\mathbb{E}[\Delta(C \bullet X)_n | \mathscr{F}_{n-1}] \underset{\text{def.}}{=\!=} \mathbb{E}[C_n \Delta X_n | \mathscr{F}_{n-1}] \underset{\text{prev.}}{=\!=} C_n \mathbb{E}[\Delta X_n | \mathscr{F}_{n-1}]$$

and conclude either with $\mathbb{E}[\Delta X_n | \mathscr{F}_{n-1}] = 0$ for case a) or $\mathbb{E}[\Delta X_n | \mathscr{F}_{n-1}] \gtreqless 0$ in case b).

In the square integrable case one just note that $C_n \Delta X_n \in L^1$ provided C_n, $\Delta X_n \in L^2$ by Cauchy–Schwarz or Hölder inequalities. □

Definition. *If T is a stopping time and $(X_n)_{n\geqslant 0}$ a stochastic process then we define the stopped process $(X_n^T)_{n\geqslant 0}$ by*

$$X_n^T := X_{T \wedge n}, \qquad n \geqslant 0.$$

Note that if $X_\bullet$ is adapted then also $X_\bullet^T$ is adapted. If $T = +\infty$ then $X_\bullet^T = X_\bullet$.

Lemma. *If T is a stopping time and $(X_n)_n$ is a (super-)martingale then the stopped process $(X_n^T)_{n\geqslant 0}$ is again a (super-)martingale and*

$$\mathbb{E}[X_{n\wedge T}] \leqslant \mathbb{E}[X_0], \qquad n \geqslant 1,$$

in the supermartingale case with equalitiy for martingales.

Proof. It is enough to note that

$$Z_n := X_0 + (C^T \bullet X)_n = X_n^T$$

with $C_n^T = \mathbb{1}_{n \leqslant T}$ as above. In the supermartingale case $(C^T \bullet X)_\bullet$ will also be a supermartingale by the previous lemma and therefore by the supermartingale property of $Z_\bullet$ we have

$$\mathbb{E}[X_n^T] = \mathbb{E}[Z_n] \leqslant \mathbb{E}[Z_0] = \mathbb{E}[X_0^T] = \mathbb{E}[X_0]. \qquad \square$$

This theorem is interesting because there are no conditions on the stopping time.

Theorem. *(Optional stopping theorem) Let T be a stopping time and $(X_n)_{n \geqslant 0}$ a (super-)martingale. Then X_T is integrable and*

$$\mathbb{E}[X_T] \leqslant \mathbb{E}[X_0]$$

in the following cases:

a) T is bounded (i.e. $\exists$ constant $N < \infty$ such that $T \leqslant N$);

b) There exist $Y \in L^1(\mathbb{P})$ and $Y \geqslant 0$ such that $|X_n| \leqslant Y$ for all $n \geqslant 1$, (i.e. $\sup_{n \geqslant 1} |X_n| \in L^1(\mathbb{P})$) and $T < \infty$ a.s.;

c) $\mathbb{E}[T] < \infty$ and there exists a constant $K < \infty$ such that $|\Delta X_n| \leqslant K$ for all $n \geqslant 1$;

d) $X_n \geqslant 0$ for all $n \geqslant 0$ and $T < \infty$ a.s.

Remark. In all the cases a),b),c),d) we have that $T < \infty$ a.s. therefore the natural definition of X_T is $X_T = X_{T(\omega)}(\omega)$ on $\{\omega \in \Omega : T(\omega) < \infty\}$ and we can take it arbitrarily on $\{\omega \in \Omega : T(\omega) = +\infty\}$, e.g. $X_T = 4397493274932$.

Proof. *Case a).* Let N be such that $T \leqslant N$. Then we have

$$|X_T| \leqslant \sum_{n=1}^{N} |X_n| \mathbb{1}_{T=n} \leqslant \sum_{n=1}^{N} |X_n| \in L^1(\mathbb{P})$$

so X_T is integrable. Moreover we know that

$$\mathbb{E}[X_{n \wedge T}] = \mathbb{E}[X_0]$$

so it is enough to take $n = N$ to have

$$\mathbb{E}[E_T] = \mathbb{E}[X_{N \wedge T}] = \mathbb{E}[X_0].$$

Case b). We note that

$$|X_{n \wedge T} - X_0| \leqslant Y + |X_0| \in L^1(\mathbb{P})$$

for all $n \geqslant 1$ and that

$$\lim_{n \to \infty} (X_{n \wedge T} - X_0) = X_T - X_0$$

a.s. therefore by dominated convergence we obtain that

$$\mathbb{E}[X_T - X_0] = \mathbb{E}\Big[\lim_{n \to \infty} (X_{n \wedge T} - X_0)\Big] \underset{\text{dom}}{=\!=} \lim_{n \to \infty} \mathbb{E}[X_{n \wedge T} - X_0] = 0.$$

Case c). We observe that

$$X_{T\wedge n}=X_0+\sum_{k=1}^{\infty}\mathbb{1}_{k\leqslant T\wedge n}\Delta X_k$$

therefore

$$|X_{T\wedge n}|\leqslant|X_0|+\sum_{k=1}^{\infty}\mathbb{1}_{k\leqslant T\wedge n}|\Delta X_k|\leqslant|X_0|+K\sum_{k=1}^{\infty}\mathbb{1}_{k\leqslant T\wedge n}\leqslant|X_0|+KT\in L^1(\mathbb{P})$$

and again we can use dominated convergence to conclude that $\mathbb{E}[X_T]=\mathbb{E}[X_0]$.

Case d). In this case we have $X_n\geqslant 0$ and no conditions on T (apart a.s. finiteness). The positivity of the process imply that we can use Fatou's lemma to conclude that

$$0\leqslant\mathbb{E}[X_T]=\mathbb{E}\Big[\lim_n X_{T\wedge n}\Big]\leqslant\liminf_n\mathbb{E}[X_{T\wedge n}]=\mathbb{E}[X_0]<\infty.$$

(In this last case we have only inequality). □

Lemma. *Let $(X_n)_{n\geqslant 0}$ be a martingale (resp. a sub- or super- martingale) and $T\geqslant S$ two bounded stopping times, then*

$$\mathbb{E}[X_T|\mathscr{F}_S]=X_S,$$

(resp. $\geqslant$, $\leqslant$).

Proof. Let us prove it in the martingale case. Let N be a deterministic bound for S,T, i.e. $S\leqslant T\leqslant N$ and $N\in\mathbb{N}$. We want to prove that $\mathbb{E}[X_T-X_S|\mathscr{F}_S]=0$. It will be enough to prove that

$$\mathbb{E}[(X_T-X_S)\mathbb{1}_B]=0$$

for all $B\in\mathscr{F}_S$. We split now according to values of S and T:

$$\mathbb{E}[(X_T-X_S)\mathbb{1}_B]=\sum_{n=0}^{N}\sum_{m=n}^{N}\mathbb{E}[(X_T-X_S)\mathbb{1}_{B\cap\{S=n\}}\mathbb{1}_{T=m}]$$

$$=\sum_{n=0}^{N}\sum_{m=n}^{N}\mathbb{E}[(X_m-X_n)\mathbb{1}_{B\cap\{S=n\}}\mathbb{1}_{T=m}]$$

and using the martingale property of X one see that $X_m=\mathbb{E}[X_N|\mathscr{F}_m]$ and $X_n=\mathbb{E}[X_N|\mathscr{F}_n]$ so

$$=\sum_{n=0}^{N}\sum_{m=n}^{N}\mathbb{E}[(\mathbb{E}[X_N|\mathscr{F}_m]-X_n)\mathbb{1}_{B\cap\{S=n\}}\mathbb{1}_{T=m}]$$

because $B\cap\{S=n\}\cap\{T=m\}\in\mathscr{F}_m$,

$$=\sum_{n=0}^{N}\sum_{m=n}^{N}\mathbb{E}[(\mathbb{E}[X_N\mathbb{1}_{T=m}|\mathscr{F}_m]-X_n\mathbb{1}_{T=m})\mathbb{1}_{B\cap\{S=n\}}]$$

$$=\sum_{n=0}^{N}\sum_{m=n}^{N}\mathbb{E}[(\mathbb{E}[X_N\mathbb{1}_{T=m}|\mathscr{F}_n]-X_n\mathbb{1}_{T=m})\mathbb{1}_{B\cap\{S=n\}}]$$

because $B \cap \{S = n\} \in \mathscr{F}_n$,

$$= \sum_{n=0}^{N} \mathbb{E}\left[\left(\mathbb{E}\left[X_N \underbrace{\sum_{m=n}^{N} \mathbb{1}_{T=m}}_{=1} | \mathscr{F}_n\right] - X_n \underbrace{\sum_{m=n}^{N} \mathbb{1}_{T=m}}_{= \mathbb{1}_{T \geqslant S} = 1}\right) \mathbb{1}_{B \cap \{S=n\}}\right]$$

$$= \sum_{n=0}^{N} \mathbb{E}[(\mathbb{E}[X_N | \mathscr{F}_n] - X_n) \mathbb{1}_{B \cap \{S=n\}}] = 0$$

because $\mathbb{E}[X_N | \mathscr{F}_n] - X_n = 0$ by the martingale property.

To prove it when X is a submartingale (the supermartingale case is analogous) one use Doob's decomposition and writes $X_n = X_0 + M_n + I_n$ where I_n is increasing previsible process and M a martingale. Then

$$\mathbb{E}[X_T | \mathscr{F}_S] = \mathbb{E}[X_0 + M_T + I_T | \mathscr{F}_S] = X_0 + \underbrace{\mathbb{E}[M_T | \mathscr{F}_S]}_{= M_S} + \underbrace{\mathbb{E}[I_T | \mathscr{F}_S]}_{\geqslant \mathbb{E}[I_S | \mathscr{F}_S] = I_S}$$

$$= X_0 + M_S + I_S = X_S.$$

because $I_T \geqslant I_S$. □

Remark that in the last lecture Doob's decomposition was wrongly stated: the conditions on the previsible process are sufficient and necessary:

Proposition. *(Doob's decomposition) Let $(X_n)_{n \geqslant 0}$ be an adapted and integrable stochastic process, then there exists a unique decomposition*

$$X_n = X_0 + M_n + I_n, \qquad n \geqslant 0,$$

where $(M_n)_{n \geqslant 0}$ is a martingale and $(I_n)_{n \geqslant 0}$ a previsible process with $I_0 = 0$. Moreover:

1. *$I_n = 0$ for all $n \geqslant 0$ **iff** X is a martingale,*
2. *I is increasing **iff** X is a submartingale,*
3. *I is decreasing **iff** X is a supermartingale.*

Indeed recall that $I_\bullet$ in Doob's decomposition is defined by

$$\Delta I_n := \mathbb{E}[\Delta X_n | \mathscr{F}_{n-1}] - X_{n-1},$$

and therefore the necessity and sufficiency are obvious.