Lecture 12 · 21.5.2021 · 10:15–12:00 via Zoom

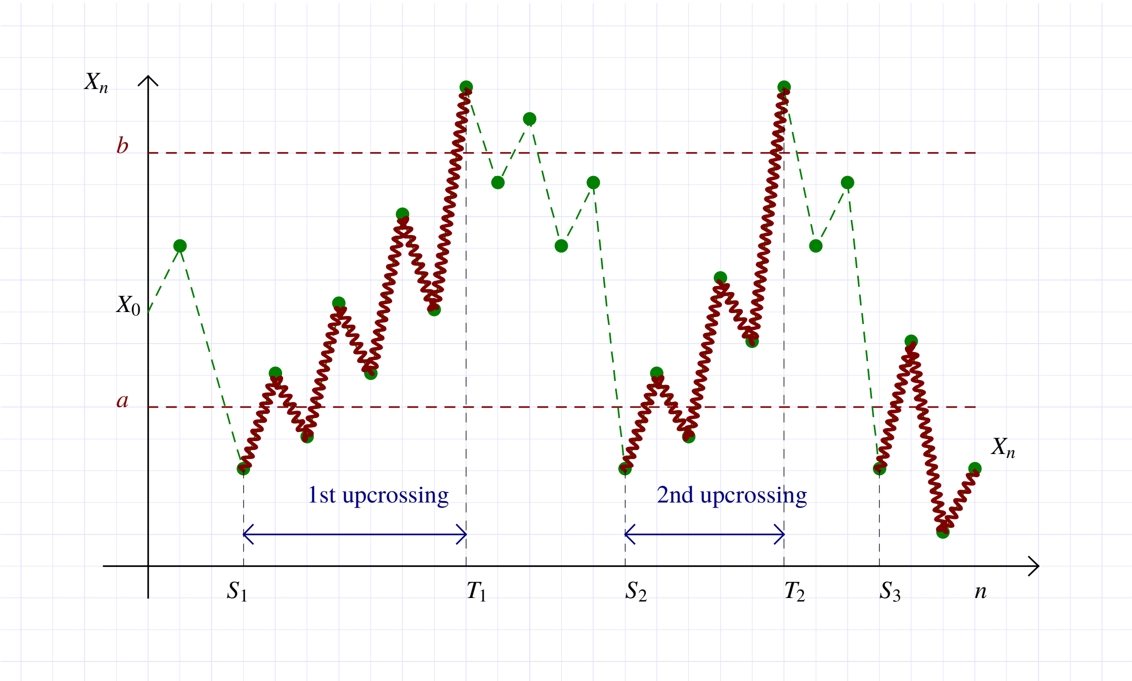
Asymptotic behaviour of martingales

We start the study of the long time behaviour of (super-,sub-)martingales. Part of the interest in martingale theory is that they possess a nice asymptotic behaviour.

Let's start by considering a supermartingale $(X_n)_{n \geqslant 0}$. As usual we can interpret it as the total gain in an unfavorable game. Indeed in average we expect to loose money : $\mathbb{E}[\Delta X_{n+1} | \mathscr{F}_n] \leqslant 0$.

We can also imagine that X is the value of a stock in a stock market (it is the stock for a company whose future is not bright…)

We want to trade this stock and we use the following strategy: we fix two values $a < b$ and every time X falls below a we buy one unit of this stock and then wait that its value goes above b, at which point we sell it.



We call S_1 the first time $X_k \leqslant a$ and T_1 the first time that $X_k \geqslant b$ for $k \geqslant S_1$ and we proceed in the same way to define $S_2, T_2, S_3, T_3, \ldots$ (write a recursive definition for these times). These are random times which depends on our supermartingale, they are indeed stopping times for the filtration generated by $(X_n)_{n \geqslant 0}$. (exercise prove it).

Let W_n our total gain at time n by using this strategy. For example in the drawing above we will have

$$W_n = \underbrace{(X_{T_1} - X_{S_1})}_{\geqslant b-a} + \underbrace{(X_{T_2} - X_{S_2})}_{\geqslant b-a} + \underbrace{(X_n - X_{S_3})}_{\geqslant X_n - a}$$

and in general it is clear that we have an inequality of the form

$$W_n \geqslant (b-a) U_n(a,b) - (X_n - a)_- \tag{1}$$

where $U_n(a,b)$ is a (integer valued) random variable which counts the number of **upcrossing** of the interval $[a,b]$ that is the number of times the process is first $\leqslant a$ and then $\geqslant b$ (in the sense above). In the picture above we have $U_n(a,b) = 2$.

Let's give a look at the process $(W_n)_{n\geqslant 0}$. We can write it as a martingale transform of $(X_n)_{n\geqslant 0}$, that is

$$W_n = W_0 + \sum_{k=1}^{n} H_n \Delta X_n = W_0 + (H \bullet X)_n$$

where $H_n = 0,1$ according to whether at time n we have the stock or not in the pocket. For example we can define it with the following recurrence relation

$$H_{n+1} = \mathbb{1}_{H_n=0, X_n \leqslant a} + \mathbb{1}_{H_n=1, X_n < b}, \qquad H_0 = 0,$$

which defines rigorously my strategy and show that this strategy is indeed previsible (which can be proven by induction on n). Using the process we can also express the number of upcrossings:

$$U_n(a,b) = \sum_{k=1}^{n} \mathbb{1}_{H_{k-1}=1, H_k=0}.$$

Since $H_n \geqslant 0$, H is previsible, bounded, and X is supermartingale then we know that $(W_n)_{n\geqslant 0}$ is a supermartingale, i.e. it will decrease in average at each step, more precisely we will have

$$\mathbb{E}[W_{n+1} | \mathscr{F}_n] \leqslant W_n.$$

The supermartingale propery implies that the process decrease in average, that is:

$$0 \geqslant \mathbb{E}[W_n - W_0]$$

and using (1) we have

$$0 \geqslant \mathbb{E}[W_n - W_0] \geqslant (b-a)\mathbb{E}[U_n(a,b)] - \mathbb{E}[(X_n - a)_-] \geqslant (b-a)\mathbb{E}[U_n(a,b)] - \mathbb{E}[|X_n - a|]$$

which means that we have proven a basic result:

Lemma. *(Doob's upcrossing inequality) For all $a < b$ and $n \geqslant 1$ we have that the average number of upcrossings $U_n(a,b)$ of the interval (a,b) by a supermartingale $(X_n)_{n\geqslant 0}$ is bounded by*

$$\mathbb{E}[U_n(a,b)] \leqslant \frac{\mathbb{E}[|X_n - a|]}{b-a}.$$

Proof. Essentially already given above. □

The interest of this inequality is that it says something about the full history of the process $(X_k)_{k\geqslant 0}$ using only informations for its final value X_n at n.,

Remark. Most of the argument holds if we replace a, b with deterministic functions of n, i.e. $a_n < b_n$ or even with stochastic processes which are previsible wrt. the filtration generated by $(X_n)_{n\geqslant 0}$. But then it is not clear what one can say about the upcrossing number.

Corollary. *Let X be a supermartingale which is bounded in* L^1 *(i.e.* $\sup_{n\geqslant 0}\mathbb{E}[|X_n|]<\infty$*). Then if we denote*

$$U_\infty(a,b) := \lim_{n\to\infty} U_n(a,b) = \sup_{n\geqslant 0} U_n(a,b)$$

the number of upcrossings of (a,b) *by the whole process* $(X_n)_{n\geqslant 0}$*. Then for any* $a<b$ *we have*

$$\mathbb{P}(U_\infty(a,b)<\infty)=1,$$

that is, a.s. the process X do only finitely many upcrossings of any interval (a,b)*.*

Proof. First note that the process $(U_n(a,b))_{n\geqslant 0}$ is increasing so the limit $n\to\infty$ exists almost surely (but it could be $+\infty$). By monotone convergence and by Doob's upcrossing inequality we have

$$\mathbb{E}[U_\infty(a,b)] \underset{\text{mon.}}{=\!=\!=} \lim_{n\to\infty}\mathbb{E}[U_n(a,b)] = \sup_n \mathbb{E}[U_n(a,b)] \underset{\text{Doob}}{\leqslant} \sup_n \frac{\mathbb{E}[|X_n-a|]}{b-a}$$

$$\leqslant \frac{\sup_n \mathbb{E}[|X_n|]+|a|}{b-a}<\infty$$

by assumption of boundedness in L^1. Therefore we conclude that $U_\infty(a,b)$ is integrable which implies in particular that $\{U_\infty(a,b)=+\infty\}$ has probability 0. □

We discovered that if we fix an interval (a,b) then a supermartingale will cross it only finitely many times, therefore eventually it will always be above, below or in between (a,b).

An easy consequence of this behaviour is the following convergence result.

Theorem. *(Doob's submartingale convergence theorem) A submartingale* $(X_n)_{n\geqslant 0}$ *bounded in* L^1*, i.e.*

$$\sup_n \mathbb{E}[|X_n|]<\infty,$$

converges a.s. towards a limit $X_\infty\in L^1$*, i.e.*

$$\lim_{n\to\infty} X_n = X_\infty, \qquad a.s.$$

and $X_\infty\in L^1$*.*

Proof. Take $Y_n=-X_n$ so that Y is a supermartingale bounded in L^1, i.e. $\sup_n\mathbb{E}[|Y_n|]<\infty$. Let

$$L_+ = \limsup_{n\to\infty} Y_n, \qquad L_- = \liminf_{n\to\infty} Y_n,$$

which always exists, they are r.v. and are such that $L_+\geqslant L_-$. The existence of the a.s. limit for X is equivalent to say that $\mathbb{P}(L_+=L_-)=1$. We proceed by contradiction: assume that $\mathbb{P}(L_+>L_-)>0$. In this case we can always find $a<b$ such that $\mathbb{P}(L_+>b>a>L_-)>0$ (exercise, do it!!). Now one can show that

$$\{L_-<a<b<L_+\}\subseteq\{U_\infty(a,b)=+\infty\}$$

since the process has to oscillate infinitely many times between a neighborhood of L_- and a neighborhood of L_+, in particular it has to have ∞-many upcrossings of (a,b). This implies that

$$0<\mathbb{P}(L_-<a<b<L_+)\leqslant\mathbb{P}(U_\infty(a,b)=+\infty)$$

in contradiction with the fact that $U_\infty(a,b) = +\infty$ has probability zero by Doobs' upcrossing inequality since Y is bounded in L^1. We conclude that $\mathbb{P}(L_- < L_+) = 0$ and therefore the almost sure limit of Y exists and then also that X. Let $X_\infty = \lim_n X_n$, by Fatou's lemma we have

$$\mathbb{E}[|X_\infty|] \leqslant \liminf_n \mathbb{E}[|X_n|] \leqslant \sup_n \mathbb{E}[|X_n|] < \infty$$

again by boundedness in L^1 of $(X_n)_{n\geqslant 0}$. □

Remark. This theorem obviously holds also for supermatingale but for supermatingales we have a more interesting result which do not require boundedness in L^1.

Theorem. *(Supermartingale convergence theorem) Let $(X_n)_{n\geqslant 0}$ a positive supermartingale ($X_n \geqslant 0$) then it converges a.s. towards a limit $X_\infty \in L^1$. (Note that $X_\infty \geqslant 0$)*

Proof. Easy. Note that by positivity and by the supermartingale property:

$$\mathbb{E}[|X_n|] = \mathbb{E}[X_n] \leqslant \mathbb{E}[X_0]$$

therefore a positive supermartingale is automatically bounded in L^1 and we conclude by the previous theorem applied to the submartingales $Z_n = -X_n$. □

Warning. Even if a submartingale which is bounded in L^1 converges a.s. to an L^1 random variable (as we have just seen), this does not implies that the convergence takes place in L^1 (i.e. that $\|X_n - X_\infty\|_{L^1} \to 0$). Let's us give a counterexample.

Let $(Z_n)_{n\geqslant 1}$ and i.i.d. sequence with values in ± 1 with $\mathbb{P}(Z_n = +1) = p \in (0,1)$ and let $u > 1$, $X_0 = x > 0$ and let

$$X_{n+1} = u^{Z_{n+1}} X_n$$

which defines a new process $(X_n)_{n\geqslant 0}$. We can choose $p = p(u)$ so that $(X_n)_{n\geqslant 1}$ is a positive martingale (exercise) and therefore $\mathbb{E}[X_n] = \mathbb{E}[X_0] = x > 0$. By the strong law of large numbers

$$\frac{1}{n}\sum_{k=1}^{n} Z_k \to \mathbb{E}[Z_1] = 2p - 1, \qquad a.s.$$

and a consequence

$$\left(\frac{X_n}{x}\right)^{1/n} = u^{\frac{1}{n}\sum_{k=1}^{n} Z_k} \to u^{2p-1}, \qquad a.s.$$

However one verifies that in order for X to be a martingale one must have $\mathbb{E}[u^{Z_1}] = 1$ which implies that $2p - 1 < 0$ since $u > 1$ (check it).

Therefore for any small $\varepsilon > 0$ there exists $N_0(\varepsilon)$ (depending on ω) such that for any $n > N_0(\varepsilon)$ we have almost surely

$$\left(\frac{X_n}{x}\right)^{1/n} \leqslant u^{2p-1}(1+\varepsilon)$$

which implies

$$0 \leqslant X_n \leqslant x[(u^{2p-1})(1+\varepsilon)]^n$$

but we have $u^{2p-1} < 1$ and choosing ε small enough we can also have that $(u^{2p-1})(1+\varepsilon) < 1$ from which we conclude that

$$0 \leqslant X_n \leqslant x[(u^{2p-1})(1+\varepsilon)]^n \to 0$$

so $X_n \to 0$. However by the martingale property we have $\mathbb{E}[X_n] = 1$ for all n. This shows that X_n cannot converge to $X_\infty = 0$ in L^1 since otherwise this would imply convergence of the average:

$$1 = |\mathbb{E}[X_\infty] - \mathbb{E}[X_n]| \leqslant \mathbb{E}[|X_\infty - X_n|]$$

so $\|X_n - X_\infty\|_{L^1} = 1$ for all n while $X_n \to X_\infty$ almost surely.