Lecture 13 · 1.6.2021 · 14:15–16:00 via Zoom

Closed martingales

We have already seen that (sub-,super-)martingales converge a.s. quite easily. A positive supermartingale converge a.s. and also a submartingale bounded in L^1.

However the convergence in not in general in L^1 (we have seen a counterexample of a martingale bounded in L^1 with average always 1 but which converges to 0).

We are interested in understanding conditions under which we have stronger types of convergences e.g. in L^p for some $p \geqslant 1$ beside the a.s. converge.

Let's look at some particular situation.

Martingales bounded in L^2

Theorem 1. *Let $(M_n)_{n \geqslant 0}$ be a martingale which is bounded in $L^2(\mathbb{P})$ (i.e. $\alpha = \sup_{n \geqslant 0} \mathbb{E}[|M_n|^2] < \infty$), Then it converges a.s. and in $L^2(\mathbb{P})$ to a r.v. $M_\infty \in L^2(\mathbb{P})$ and moreover*

$$M_n = \mathbb{E}[M_\infty | \mathscr{F}_n], \qquad n \geqslant 0.$$

Remark. The interesting fact is that the limit M_∞ retain the full information about the martingale in these that we can reconstruct the martingale as a orthogonal projection of $M_\infty \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ onto $L^2(\Omega, \mathscr{F}_n, \mathbb{P})$.

Definition. *A martingale $(M_n)_{n \geqslant 0}$ is **closed** (by Z) if there exists a r.v. in $Z \in L^1$ such that*

$$M_n = \mathbb{E}[Z | \mathscr{F}_n]$$

for all $n \geqslant 0$. We say that the martingale is closed in L^p for $p \geqslant 1$ if $Z \in L^p$.

Remark. If a martingale $(M_n)_{n \geqslant 0}$ is closed by Z in L^p then it is easy to see that also $(M_n)_{n \geqslant 0}$ is bounded in L^p and in particular we have that $M_n \to M_\infty$ a.s. as $n \to \infty$. And it is an interesting exercise for you to show that $M_\infty = \mathbb{E}[Z | \mathscr{F}_\infty]$ where recall that $\mathscr{F}_\infty = \vee_{n \geqslant 0} \mathscr{F}_n = \sigma((\mathscr{F}_n)_{n \geqslant 0})$. In general $M_\infty \neq Z$.

Remark. Note that a martingale bounded in L^1 is not necessarily closed (see the above counterexample). This will become clear later on.

Corollary. *A martingale is bounded in L^2 iff it is closed in L^2.*

Proof. This follows from the above theorem of convergence of L^2 martingales. □

Proof of Th 1. The proof is an application of Pythagoras theorem. We write the martingale as the sum of its increments:

$$M_n = M_0 + \sum_{k=1}^{n} \underbrace{\Delta M_k}_{M_k - M_{k-1}}$$

and observe that if $k > k' > 0$ then $\Delta M_{k'} \hat{\in} \mathscr{F}_k$ and $\Delta M_k \Delta M_{k'} \in L^1$

$$\mathbb{E}[\Delta M_k \Delta M_{k'}] = \mathbb{E}\Big[\underbrace{\mathbb{E}[\Delta M_k | \mathscr{F}_k]}_{=0} \Delta M_{k'}\Big] = 0$$

which means that all the increments are orthogonal in L^2. Then (Pythagoras!)

$$\mathbb{E}[M_n^2] = \mathbb{E}[M_0^2] + \sum_{k=1}^{n} \mathbb{E}[(\Delta M_k)^2]$$

which implies that the sequence $(\mathbb{E}[M_n^2])_{n \geqslant 0}$ is increasing and that

$$\mathbb{E}[M_0^2] + \sum_{k=1}^{\infty} \mathbb{E}[(\Delta M_k)^2] = \sup_{n \geqslant 0} \mathbb{E}[M_n^2] =: \alpha < \infty.$$

By a similar argument one deduces that for $k > k' \geqslant n$ we have

$$\mathbb{E}[(M_k - M_{k'})^2] = \sum_{\ell = k'+1}^{k} \mathbb{E}[(\Delta M_\ell)^2] \leqslant \sum_{\ell = n+1}^{\infty} \mathbb{E}[(\Delta M_\ell)^2] \to 0$$

as $n \to \infty$ since the series converges. This means that $(M_k)_{k \geqslant 0}$ is a Cauchy sequence in L^2 so it converges in L^2 to a limit $X \in L^2$, i.e.

$$X = L^2(\mathbb{P}) - \lim_{n \to \infty} M_n \in L^2.$$

Since the martingale is bounded in L^2 it is also bounded in L^1 and therefore it converges a.s. by Doob's submartingale convergence theorem (seen last week) we call the corresponding almost sure limit $M_\infty = (a.s.-)\lim_{n \to \infty} M_n$. Apriori $X \neq M_\infty$, however since $(M_n)_{n \geqslant 0}$ converges in L^2 to M_∞ we can always extract a subsequence $(M_{n_k})_{k \geqslant 0}$ such that $M_{n_k} \to X$ almost surely (we did it in detail when discussing the conditional expectation, try to recall that argument).

Therefore for this subsequence we have both

$$M_{n_k} \to X, \qquad M_{n_k} \to M_\infty$$

since a subsequence of a sequence which converges a.s. to a limit converges a.s. to the same limit. We concludet that $X = M_\infty$ a.s. This proves that $M_n \to M_\infty$ a.s. and in L^2.

We have still to prove that M_∞ closes $(M_n)_{n \geqslant 0}$, that is that $M_n = \mathbb{E}[M_\infty | \mathscr{F}_n]$.

We know that for all $m > n$ we have

$$M_n = \mathbb{E}[M_m | \mathscr{F}_n]$$

(by the martingale property) so we have to pass to the limit as $m \to \infty$ in this equation. We proceed as follows: for every $m > n$ we have

$$\begin{aligned}
\|M_n - \mathbb{E}[M_\infty|\mathscr{F}_n]\|_{L^2(\mathbb{P})} &= \|\mathbb{E}[M_m|\mathscr{F}_n] - \mathbb{E}[M_\infty|\mathscr{F}_n]\|_{L^2(\mathbb{P})} && \text{(by mart. prop.)} \\
&= \|\mathbb{E}[M_m - M_\infty|\mathscr{F}_n]\|_{L^2(\mathbb{P})} && \text{(by linearity)} \\
&\leqslant \|M_m - M_\infty\|_{L^2(\mathbb{P})} && \text{(by contractivity of cond. exp.)} \\
&\to 0 && \text{as } m \to \infty \\
& && (\text{since}\, M_m \to M_\infty \,\text{in}\, L^2)
\end{aligned}$$

which shows that $M_n - \mathbb{E}[M_\infty | \mathscr{F}_n] = 0$ in L^2 and therefore a.s.. □

Exercise. Try to prove that M_∞ can be chosen to be $\mathscr{F}_\infty$-measurable. In the above proof we only have that $M_\infty \hat{\in} \mathscr{F}$.

In order to extend these considerations to large classes of martingales we need to develop tools to deal with martingales which are bounded in other L^p spaces with $p \neq 2$ where we do not have at our disposal the geometry to the Hilbert space.

Doob's maximal inequalities

Theorem 2. *(Doob's maximal inequality) Let $(X_n)_{n \geqslant 0}$ be a positive submartingale and let*

$$X_n^* = \sup_{0 \leqslant k \leqslant n} X_k$$

be the running supremum of X. Then for all $\lambda > 0$ and $n \geqslant 0$ we have

$$\lambda \mathbb{P}(X_n^* \geqslant \lambda) \leqslant \mathbb{E}[X_n \mathbb{1}_{X_n^* \geqslant \lambda}] \leqslant \mathbb{E}[X_n].$$

Proof. Let $T := \inf\{n \geqslant 0 : X_n \geqslant \lambda\}$. Then $\{T \leqslant n\} = \{X_n^* \geqslant \lambda\}$ and $T \wedge n$ is a bounded stopping time. By the positivity of $(X_n)_{n \geqslant 0}$ we have

$$\mathbb{E}[X_n] = \mathbb{E}[X_n \mathbb{1}_{T \leqslant n} + X_n \mathbb{1}_{T > n}] \geqslant \mathbb{E}[X_n \mathbb{1}_{T \leqslant n}] = \mathbb{E}[X_n \mathbb{1}_{X_n^* \geqslant \lambda}]$$

which proves the second inequality. By the optional stopping theorem applied to the two bounded stopping times n and $T \wedge n$ we have $\mathbb{E}[X_n | \mathscr{F}_{T \wedge n}] \geqslant X_{T \wedge n}$ then using the fact that $\{T \leqslant n\} \in \mathscr{F}_{T \wedge n}$ (check it!) and that $X_T \geqslant \lambda$ we have

$$\mathbb{E}[X_n \mathbb{1}_{T \leqslant n}] = \mathbb{E}[\mathbb{E}[X_n | \mathscr{F}_{T \wedge n}] \mathbb{1}_{T \leqslant n}] = \mathbb{E}[X_{T \wedge n} \mathbb{1}_{T \leqslant n}] = \mathbb{E}[X_T \mathbb{1}_{T \leqslant n}]$$

$$\geqslant \lambda \mathbb{E}[\mathbb{1}_{T \leqslant n}] = \lambda \mathbb{P}(T \leqslant n) = \lambda \mathbb{P}(X_n^* \geqslant \lambda)$$

so we prove the theorem. □

We want not to extend this to all (super-/sub-)martingales without positivity assumption.

Corollary 3. *Let $(X_n)_{n \geqslant 0}$ be a (super-/sub-)martingale and let $|X|_n^* := \sup_{0 \leqslant k \leqslant n} |X_k|$. Then for all $\lambda > 0$ and all $n \geqslant 0$ we have*

$$\lambda \mathbb{P}(|X|_n^* \geqslant 3\lambda) \leqslant 3\mathbb{E}[|X_0|] + 4\mathbb{E}[|X_n|].$$

Proof. Is left as exercise. Hint: use *Doob's decomposition* for X_n to have a martingale M and a previsible process I (either decreasing or increasing). Bound the running maximum of the previsible process is easy, and bound the running maximum of the martingale can be done with Doob's inequality for positive submartingales. □

Theorem 4. *(Doob's L^p inequalities) Let $(X_n)_{n \geqslant 0}$ a martingale or a positive submartingale, then for all $p > 1$ we have for the running maximum:*

$$\|X_n^*\|_{L^p} \leqslant \frac{p}{p-1} \|X_n\|_{L^p}, \qquad n \geqslant 0.$$

Proof. If X is a martingale then $(|X_n|)_{n\geqslant 0}$ is a positive submartingale, then we can just discuss this last case. For $L>0$ we have

$$
\begin{aligned}
\mathbb{E}[(X_n^*\wedge L)^p] &= \mathbb{E}\left[\int_0^{X_n^*\wedge L} p x^{p-1}\mathrm{d}x\right] \\
&= \mathbb{E}\left[\int_0^L p x^{p-1}\mathbb{1}_{X_n^*\geqslant x}\mathrm{d}x\right] \\
&= \int_0^L p x^{p-1}\mathbb{E}[\mathbb{1}_{X_n^*\geqslant x}]\mathrm{d}x && \text{(Fubini)} \\
&\leqslant \int_0^L p x^{p-1}\frac{\mathbb{E}[X_n\mathbb{1}_{X_n^*\geqslant x}]}{x}\mathrm{d}x && \text{(Doobs maximal ineq.)} \\
&\leqslant \int_0^L p x^{p-2}\mathbb{E}[X_n\mathbb{1}_{X_n^*\geqslant x}]\mathrm{d}x \\
&\leqslant \frac{p}{p-1}\mathbb{E}\left[X_n\int_0^L (p-1)\,x^{p-2}\mathbb{1}_{X_n^*\geqslant x}\mathrm{d}x\right] && \text{(Fubini again)} \\
&\leqslant \frac{p}{p-1}\mathbb{E}[X_n(X_n^*\wedge L)^{p-1}] \\
&\leqslant \frac{p}{p-1}\|X_n\|_{L^p}\|X_n^*\wedge L\|_{L^p}^{p-1} && \text{(Hölder inequality)}
\end{aligned}
$$

we applied Hölder inequality with exponents $p, \left(1-\frac{1}{p}\right)^{-1}=\frac{p}{p-1}$. At this point we divide both sides by $\|X_n^*\wedge L\|_{L^p}^{p-1}$ to get

$$\|X_n^*\wedge L\|_{L^p}\leqslant \frac{p}{p-1}\|X_n\|_{L^p}$$

since $\mathbb{E}[(X_n^*\wedge L)^p]=\|X_n^*\wedge L\|_{L^p}^p$. Now by monotone convergence as $L\to\infty$ we have

$$\|X_n^*\|_{L^p}=\lim_{L\to\infty}\|X_n^*\wedge L\|_{L^p}\leqslant \frac{p}{p-1}\|X_n\|_{L^p}$$

which proves the theorem. □

Remark. The L^p inequality is false when $p=1$, i.e. there is no constant C such that

$$\|X_n^*\|_{L^1}\leqslant C\|X_n\|_{L^1}$$

This is hinted also by the fact that on the r.h.s. the constant go to infinity. When $p=1$ we only have the weaker inequality given by the Doob's maximal inequality:

$$\lambda\mathbb{P}(X_n^*>\lambda)\leqslant \mathbb{E}[X_n]=\|X_n\|_{L^1}.$$

Note that by Markov's inequality

$$\lambda\mathbb{P}(X_n^*>\lambda)\leqslant \|X_n^*\|_{L^1}$$

this is why Doob's maximal inequality is “weaker”.