Lecture 14 · 4.6.2021 · 14:15–16:00 via Zoom

Closed martingales (II)

Definition. *A martingale $(M_n)_{n\geqslant 0}$ is **closed** (by Z) if there exists a r.v. in $Z\in L^1$ such that*

$$M_n = \mathbb{E}[Z|\mathscr{F}_n]$$

for all $n\geqslant 0$. We say that the martingale is closed in L^p for $p\geqslant 1$ if $Z\in L^p$.

We saw already that a martingale bounded in L^2 is closed (and this is an iff).

Then on Tuesday we proved Doob's maximal inequality and in particular Doob's L^p inequality which controls the size of the running maximum $X_n^* = \sup_{\leqslant k\leqslant n} X_n$: if $(X_n)_{n\geqslant 0}$ is a positive sub-martingale then for all $p>1$

$$\|X_n^*\|_{L^p} \leqslant \frac{p}{p-1}\|X_n\|_{L^p}.$$

Martingales in L^p

We look not at martingales which are bounded in L^p for $p>1$ and use Doob's inequality.

Theorem. *Let $X_\bullet = (X_n)_{n\geqslant 0}$ be a martingale and $p>1$. Then the following statements are equivalent:*

a) $X_\bullet$ is bounded in L^p (i.e. $\sup_{n\geqslant 0}\|X_n\|_{L^p}<\infty$);

b) $X_\bullet$ converges a.s. and in L^p;

c) There exists a random variable $X_\infty\in L^p$ such that $X_n = \mathbb{E}[X_\infty|\mathscr{F}_n]$ for all $n\geqslant 0$, i.e. X_∞ closes the martingale $X_\bullet$.

Proof. a) ⇒ b). Being bounded in L^p implies being bounded in L^1 and therefore by Doob's sub-martingale convergences theorem we have that $X_n \to X_\infty \in L^1$ a.s.

By Doob's L^p inequality $(|X|_n^*)_{n\geqslant 0}$ satisfies (recall that $|X|_n^* = \sup_{\leqslant k\leqslant n}|X_n|$)

$$\||X|_n^*\|_{L^p} \lesssim \|X_n\|_{L^p} \lesssim \sup_n \|X_n\|_{L^p} < \infty.$$

Note that $|X|_{n+1}^* \geqslant |X|_n^*$ for all $n\geqslant 0$. By monotone convergence we have therefore that

$$\||X|_\infty^*\|_{L^p} = \left\|\lim_{n\to\infty}|X|_n^*\right\|_{L^p} \leqslant \lim_{n\to\infty}\||X|_n^*\|_{L^p} \leqslant \frac{p}{p-1}\sup_n\|X_n\|_{L^p}<\infty$$

and in particular

$$|X_n - X_\infty| \leqslant |X_n| + |X_\infty| = |X_n| + \lim_{m\to\infty}|X_m| \leqslant 2|X|_\infty^* \in L^p$$

for all $n\geqslant 0$. By dominated convergence we conclude that

$$\lim_{n\to\infty}\|X_n - X_\infty\|_{L^p} = \left\{\lim_{n\to\infty}\mathbb{E}[|X_n - X_\infty|^p]\right\}^{1/p} = 0,$$

which means that $X_n \to X_\infty$ in L^p. (note indeed that $|X_n - X_\infty|^p \leqslant (2|X|_\infty^*)^p \in L^1$ and $X_n \to X_\infty$ a.s.).

b) ⇒ c). Let $X_\infty := \lim_n X_n$ when X_n coverges and let's take $X_\infty = 0$ when the sequence do not converges. Let $Z_n = \mathbb{E}[X_\infty|\mathscr{F}_n]$ and consider

$$\|X_n - Z_n\|_{L^p} = \|X_n - \mathbb{E}[X_\infty|\mathscr{F}_n]\|_{L^p} = \|\mathbb{E}[X_m|\mathscr{F}_n] - \mathbb{E}[X_\infty|\mathscr{F}_n]\|_{L^p} \leqslant \|X_m - X_\infty\|_{L^p}$$

which is true for all $m \geqslant n$ by the martingale property of $X_\bullet$ and by the contractivity of the cond. exp. in L^p. Now we just take $m \to \infty$ to see that $\|X_m - X_\infty\|_{L^p} \to 0$ and therefore that $\|X_n - Z_n\|_{L^p} = 0$ for all n which gives us that $X_n = \mathbb{E}[X_\infty|\mathscr{F}_n]$.

c) ⇒ a). Easy again by contractivity of the cond. exp. in L^p:

$$\sup_{n \geqslant 0} \|X_n\|_{L^p} = \sup_{n \geqslant 0} \|\mathbb{E}[X_\infty|\mathscr{F}_n]\|_{L^p} \leqslant \|X_\infty\|_{L^p} < \infty.$$

□

Corollary. *Let $(X_n)_{n \geqslant 0}$ a martingale closed in L^p (i.e. $X_n = \mathbb{E}[Z|\mathscr{F}_n]$ for some $Z \in L^p$). Then*

$$X_n \to X_\infty = \mathbb{E}[Z|\mathscr{F}_\infty]$$

almost surely and in L^p.

> Recall that $\mathscr{F}_\infty = \sigma(\mathscr{F}_n : n \geqslant 0)$, the smallest σ-algebra which contains all the $\mathscr{F}_n$. In general is not true that $\mathscr{F} = \mathscr{F}_\infty$, $\mathscr{F}_\infty$ could be strictly smaller than $\mathscr{F}$. Example: take $\mathscr{F}_n = \mathscr{G} \subset \mathscr{F}$ for all $n \geqslant 0$ then $\mathscr{F}_\infty = \mathscr{G} \neq \mathscr{F}$.

Proof. By the previous theorem we know that $X_n \to X_\infty = \lim_n X_n$ a.s. and in L^p and moreover that $X_n = \mathbb{E}[X_\infty|\mathscr{F}_n]$. Let $A \in \mathscr{F}_n \subseteq \mathscr{F}_\infty$ for some $n \geqslant 0$, then by def. of cond. exp. we have

$$\mathbb{E}[\mathbb{1}_A(X_\infty - Z)] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[(X_\infty - Z)|\mathscr{F}_n]] = \mathbb{E}[\mathbb{1}_A(X_n - X_n)] = 0.$$

Therefore we have

$$\mathbb{E}[\mathbb{1}_A X_\infty] = \mathbb{E}[\mathbb{1}_A Z] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[Z|\mathscr{F}_\infty]]$$

for all $A \in \cup_{n \geqslant 0} \mathscr{F}_n$. Now note that $\Pi = \cup_{n \geqslant 0} \mathscr{F}_n$ is a π-system which generates $\mathscr{F}_\infty = \sigma(\cup_{n \geqslant 0} \mathscr{F}_n)$. Then the family $\Lambda = \{A \in \mathscr{F} : \mathbb{E}[\mathbb{1}_A X_\infty] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[Z|\mathscr{F}_\infty]]\}$ is also easily seen to be a λ-system such that $\Pi \subseteq \Lambda$. Then by Dynkin's $\pi - \lambda$ theorem we have $\Pi \subseteq \sigma(\Pi) \subseteq \Lambda$ so we have that the equality is true for all $A \in \mathscr{F}_\infty$.

The last ingredient given by the fact that X_∞ if $\mathscr{F}_\infty$ measurable, this come easily form the fact that $X_n \hat{\in} \mathscr{F}_\infty$ for all $n \geqslant 0$ and that $L^+ = \text{limsup}_n X_n \hat{\in} \mathscr{F}_\infty$ and $L^- = \text{liminf}_n X_n \hat{\in} \mathscr{F}_\infty$. Therefore $\{L^+ = L^-\} \in \mathscr{F}_\infty$ and as consequence $\hat{X}_\infty := L^+ \mathbb{1}_{\{L^+ = L^-\}}$ is $\mathscr{F}^\infty$ measurable and $\hat{X}_\infty = X_\infty$ a.s. So we can actually choose X_∞ to be $\mathscr{F}_\infty$ measurable and conclude that

$$X_\infty = \mathbb{E}[Z|\mathscr{F}_\infty], \qquad a.s.$$

using the above equality. □

Uniformly integrable martingales

We now understand quite well the case of martingales in L^p for $p > 1$. What happens when $p = 1$? We already know that boundedness in L^1 is not enough for closedness in L^1. It turns out that the right property in this case is uniform integrability.

Recall that

- A family $(Y_\alpha)_\alpha$ is UI iff for any $\varepsilon > 0$ there exists $L > 0$ such that
$$\sup_\alpha \mathbb{E}[|Y_\alpha| \mathbb{1}_{|Y_\alpha|>L}] < \varepsilon.$$
- A UI family $(Y_\alpha)_\alpha$ is also bounded in L^1:
$$\sup_\alpha \mathbb{E}[|Y_\alpha|] = \sup_\alpha \mathbb{E}[|Y_\alpha| \mathbb{1}_{|Y_\alpha|\leqslant L}] + \sup_\alpha \mathbb{E}[|Y_\alpha| \mathbb{1}_{|Y_\alpha|>L}] \leqslant L + \varepsilon < \infty.$$
- A family of r.v. bounded in L^p is automatically uniformly integrable (see the exercise sheet on uniform integrability).
- The family $(\mathbb{E}[Y|\mathscr{G}])_{\mathscr{G}\subseteq\mathscr{F}}$ of conditional expectations of a given L^1 random variable Y is also a UI family.

Theorem. *Let $(X_n)_{n\geqslant 0}$ be a martingale, then the following are equivalent statements:*

a) $(X_n)_{n\geqslant 0}$ is uniformly integrable;

b) $X_n \to X_\infty$ almost surely and in L^1;

c) There exists $Z \in L^1$ such that $X_n = \mathbb{E}[Z|\mathscr{F}_n]$ for all $n \geqslant 0$ (i.e. $X_\bullet$ is closed);

Proof. a) ⇒ b). From UI we deduce that $X_\bullet$ is bounded in L^1 and therefore by the submartingale convergence theorem we have $X_n \to X_\infty$ a.s. and that $X_\infty \in L^1$.

From UI and almost sure convergence we deduce that $X_n \to X_\infty$ converges in L^1. (This is the key point where we use UI!!!).

b) ⇒ c). The argument we used in L^p works also in L^1: define $Z_n := \mathbb{E}[X_\infty|\mathscr{F}_n]$ and observe that for all $m \geqslant n$, as $m \to \infty$ we have

$$\|X_n - Z_n\|_{L^1} = \|X_n - \mathbb{E}[X_\infty|\mathscr{F}_n]\|_{L^1} = \|\mathbb{E}[X_m|\mathscr{F}_n] - \mathbb{E}[X_\infty|\mathscr{F}_n]\|_{L^1} \leqslant \|X_m - X_\infty\|_{L^1} \to 0$$

and therefore $X_n = Z_n$.

c) ⇒ a). This is a basic property of the family of conditional expectations $(X_n)_{n\geqslant 0} = (\mathbb{E}[Z|\mathscr{F}_n])_{n\geqslant 0} \subseteq (\mathbb{E}[Z|\mathscr{G}])_{\mathscr{G}\subseteq\mathscr{F}}$ which is UI. □

Lemma. *If $(X_n)_{n\geqslant 0}$ is a UI supermartingale (resp. submartingale) then $X_n \to \infty$ almost sure and in L^1 and moreover $\mathbb{E}[X_\infty|\mathscr{F}_n] \leqslant X_n$ (resp. $\mathbb{E}[X_\infty|\mathscr{F}_n] \geqslant X_n$) for all $n \geqslant 0$. This means the supermartingale property (resp. submartingale) can be extended to the index set $\mathbb{N}^* = \mathbb{N} \cup \{+\infty\}$.*

Proof. (Exerice using argument as above). □

Definition. *If X is a UI martingale, then it is natural to define for any stopping time T (not necessarily finite)*

$$X_T = \sum_{n\geqslant 0} X_n \mathbb{1}_{T=n} + X_\infty \mathbb{1}_{T=\infty},$$

where $X_\infty = \lim_n X_n$ and $X_n = \mathbb{E}[X_\infty|\mathscr{F}_n]$.

We have then the following extension of the optimal stopping theorem.

> **Theorem.** *(Opt. Stop. for UI martingales) Let $(X_n)_{n\in\mathbb{N}^*}$ be a UI martingale and $S \leqslant T$ two stopping times, then $X_T, X_S \in L^1$ and*
>
> $$\mathbb{E}[X_T|\mathscr{F}_S] = X_S.$$
>
> *(In particular $X_T = \mathbb{E}[X_\infty|\mathscr{F}_T]$)*

Proof. Note that

$$|X_T| \leqslant \sum_{n\geqslant 0} |X_n| \mathbb{1}_{T=n} + |X_\infty| \mathbb{1}_{T=\infty}$$

and

$$\mathbb{E}[|X_T|] \leqslant \sum_{n\geqslant 0} \mathbb{E}[|X_n| \mathbb{1}_{T=n}] + \mathbb{E}[|X_\infty| \mathbb{1}_{T=\infty}]$$

$$\leqslant \sum_{n\geqslant 0} \mathbb{E}[|\mathbb{E}[X_\infty|\mathscr{F}_n]| \mathbb{1}_{T=n}] + \mathbb{E}[|X_\infty| \mathbb{1}_{T=\infty}]$$

$$\underset{\text{Jensen}}{\leqslant} \sum_{n\geqslant 0} \mathbb{E}[\mathbb{E}[|X_\infty||\mathscr{F}_n] \mathbb{1}_{T=n}] + \mathbb{E}[|X_\infty| \mathbb{1}_{T=\infty}]$$

$$\leqslant \sum_{n\geqslant 0} \mathbb{E}[|X_\infty| \mathbb{1}_{T=n}] + \mathbb{E}[|X_\infty| \mathbb{1}_{T=\infty}] \underset{\text{Fubini}}{=} \mathbb{E}\left[|X_\infty| \left(\sum_{n\geqslant 0} \mathbb{1}_{T=n} + \mathbb{1}_{T=\infty}\right)\right] = \mathbb{E}[|X_\infty|] < \infty.$$

Moreover for $A \in \mathscr{F}_T$ we have

$$\mathbb{E}[X_\infty \mathbb{1}_A] = \sum_{n\geqslant 0} \mathbb{E}[X_\infty \mathbb{1}_{A\cap\{T=n\}}] + \mathbb{E}[X_\infty \mathbb{1}_{A\cap\{T=\infty\}}]$$

$$= \sum_{n\geqslant 0} \mathbb{E}[\mathbb{E}[X_\infty|\mathscr{F}_n] \mathbb{1}_{A\cap\{T=n\}}] + \mathbb{E}[X_\infty \mathbb{1}_{A\cap\{T=\infty\}}]$$

$$= \sum_{n\geqslant 0} \mathbb{E}[X_n \mathbb{1}_{A\cap\{T=n\}}] + \mathbb{E}[X_\infty \mathbb{1}_{A\cap\{T=\infty\}}] = \mathbb{E}[X_T \mathbb{1}_A],$$

so in particular we have proven that $X_T = \mathbb{E}[X_\infty|\mathscr{F}_T]$. Then it is easy to see that ($\mathscr{F}_S \subseteq \mathscr{F}_T$)

$$\mathbb{E}[X_T|\mathscr{F}_S] = \mathbb{E}[\mathbb{E}[X_\infty|\mathscr{F}_T]|\mathscr{F}_S] = \mathbb{E}[X_\infty|\mathscr{F}_S] = X_S.$$

□
