Lecture 15 · 8.6.2021 · 14:15–16:00 via Zoom

Some applications of martingale theory

We will consider now some applications of martingale theory to some problem in probability theory.

Backwards martingales

Let $\mathbb{Z}_- = \{-n: n \in \mathbb{N}\}$ and $(\mathcal{G}_n)_{n \leqslant 0}$ an increasing sequence of σ-algebras indexed by $\mathbb{Z}_-$. This means that $\mathcal{G}_n \subseteq \mathcal{G}_m$ for $n \leqslant m \leqslant 0$.

We define a backwards martingale $(X_n)_{n \leqslant 0}$ as a process indexed by $\mathbb{Z}_-$, adapted to $(\mathcal{G}_n)_{n \leqslant 0}$ with $X_0 \in L^1$ and that for all $n \leqslant -1$ we have

$$\mathbb{E}[X_{n+1} | \mathcal{G}_n] = X_n.$$

By the tower property of conditional expectations we have that $X_n = \mathbb{E}[X_0 | \mathcal{G}_n]$ and therefore $X_n \in L^1$ automatically. Moreover the family $(X_n)_{n \leqslant 0}$ is uniformly integrable (bcs. it is given by conditional expectations of X_0).

Theorem. *Let $X_\bullet$ be a backwards martingale with $X_0 \in L^p$ for some $p \geqslant 1$. Then*

$$X_n \to X_{-\infty} = \mathbb{E}[X_0 | \mathcal{G}_{-\infty}]$$

as $n \to -\infty$. The convergence is almost surely and in L^p. Here $\mathcal{G}_{-\infty} = \cap_{n \leqslant 0} \mathcal{G}_n$.

Proof. (Sketch). The proof is similar to the (forwards) martingale case. In particular it uses a variant of Doobs' upcrossing inequality for backwards martingales, indeed note that for any fixed $N \geqslant 0$ the process $(Y_k = X_{-N+k})_{k=0,\ldots,N}$ is a (forward) martingale wrt the filtration $(\tilde{\mathcal{F}}_k = \mathcal{G}_{-N+k})_{k=0,\ldots,N}$ so one can apply to it the usual upcrossing inequality. The role of endpoint in the upcrossing inequality is played by the time zero. The rest of the proof can similarly adapted. (I leave the details to you, or maybe look in the book of Williams). □

The strong law of large numbers

Definition. *The* ***tail σ-algebra $\mathcal{T}$*** *of a stochastic process $(X_n)_{n \geqslant 0}$ is defined as*

$$\mathcal{T} = \cap_{n \geqslant 1} \mathcal{T}_n$$

where $\mathcal{T}_n = \sigma(X_k : k \geqslant n)$. It contains all the events which do not depend on any finite subset of the process $(X_n)_{n \geqslant 0}$, e.g. what happens "eventually".

Example. For any $\lambda \in \mathbb{R}$, the event

$$\{\omega \in \Omega : X_k(\omega) \geqslant \lambda \text{ infinitely often in } k\} = \cap_{k \geqslant 1} \underbrace{\cup_{n \geqslant k} \{X_n \geqslant \lambda\}}_{\in \mathcal{T}_k} \in \mathcal{T}.$$

Similarly the random variables $\limsup_k X_k$ and $\liminf_k X_k$ are both $\mathcal{T}$ measurable, as is the event that $\lim_k X_k$ exists.

Theorem. *(Komogorov 0-1 law) Let* $X_\bullet = (X_n)_{n\geqslant 0}$ *be a family of* $(\mathbb{P}-)$*independent random variables (not necessarily identically distributed). Then* $\mathcal{T}$ *of* $X_\bullet$ *is trivial, i.e. if* $A \in \mathcal{T}$ *then* $\mathbb{P}(A) \in \{0, 1\}$*. That is any event in* $\mathcal{T}$ *is either impossible or almost sure (wrt.* $\mathbb{P}$*).*

Proof. Take $A \in \mathcal{T}$. Let $\mathcal{G}_n = \sigma((X_k)_{k=0,\dots,n})$ and

$$Z_n = \mathbb{E}[\mathbb{1}_A | \mathcal{G}_n]$$

for all $n \geqslant 0$. Then $(Z_n)_{n\geqslant 0}$ is a UI (forwards) martingale for the filtration $(\mathcal{G}_n)_{n\geqslant 0}$, moreover $\mathcal{G}_n$ is independent of $\mathcal{T}_{n+1} = \sigma(X_k: k \geqslant n+1) \supseteq \mathcal{T}$, so $A \in \mathcal{T}_{n+1}$ and therefore $\mathbb{1}_A$ is independent of $\mathcal{G}_n$ and we have

$$Z_n = \mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A).$$

By UI martingale convergence we have that

$$Z_n \to Z_\infty = \mathbb{E}[\mathbb{1}_A | \mathcal{G}_\infty]$$

a.s. and in L^1. Now note that $\mathcal{G}_\infty \supseteq \mathcal{T}_n \supseteq \mathcal{T}$ and therefore we have also $\mathbb{E}[\mathbb{1}_A | \mathcal{G}_\infty] = \mathbb{1}_A$. We conclude that as $n \to \infty$ we have a.s.

$$\mathbb{P}(A) = Z_n \to Z_\infty = \mathbb{1}_A$$

and therefore that there exists an event $\mathcal{N}$ such that $\mathbb{P}(\mathcal{N}) = 0$ and

$$\mathbb{P}(A) = \mathbb{1}_A(\omega), \qquad \omega \notin \mathcal{N}.$$

So taking ω in $\mathcal{N}^c$ we conclude that $\mathbb{P}(A) \in \{0, 1\}$. □

Theorem. *(Strong law of large numbers) Let* $(X_n)_{n\geqslant 1}$ *be an i.i.d. random variables with* $X_1 \in L^1$*. Let* $S_n = X_1 + \cdots + X_n$ *for* $n \geqslant 1$*. Then*

$$\frac{S_n}{n} \to \mathbb{E}[X_1]$$

almost surely and in L^1*.*

This is a well known result which we will prove now using (backwards) martingales.

Proof. Let $\mathcal{G}_n := \sigma(S_n, X_{n+1}, X_{n+2}, \dots)$ for $n \geqslant 1$. Then $\mathcal{G}_m \subseteq \mathcal{G}_n$ for $m \geqslant n$: the family $(\mathcal{G}_n)_{n\geqslant 1}$ is decreasing. Let $\mathcal{F}_n = \mathcal{G}_{-n}$ for $n \leqslant -1$. Now $(\mathcal{F}_n)_{n\leqslant -1}$ is a filtration indexed by $\mathbb{Z}_{\leqslant -1}$. Let

$$Z_n := S_{-n}/(-n)$$

for $n \leqslant -1$, and observe that this is a backwards martingale wrt $(\mathcal{F}_n)_{n\leqslant -1}$, indeed for $n \leqslant -2$ (write $k = -n \geqslant 2$)

$$\begin{aligned}
\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = \mathbb{E}[Z_{-k+1} | \mathcal{F}_{-k}] = \frac{1}{k-1}\mathbb{E}[S_{k-1} | \mathcal{G}_k] &= \frac{1}{k-1}\mathbb{E}[S_{k-1} | S_k, X_{k+1}, X_{k+2}, \dots] \\
&= \frac{1}{k-1}\mathbb{E}[S_k - X_k | S_k, X_{k+1}, X_{k+2}, \dots] \\
&= \frac{1}{k-1}(S_k - \mathbb{E}[X_k | S_k, X_{k+1}, X_{k+2}, \dots]) \\
&= \frac{1}{k-1}(S_k - \mathbb{E}[X_k | S_k])
\end{aligned}$$

by independence. Now recall that by symmetry one has

$$\mathbb{E}[X_k|S_k] = \frac{1}{k}\mathbb{E}[X_1 + \cdots + X_k|S_k] = \frac{1}{k}\mathbb{E}[S_k|S_k] = \frac{S_k}{k},$$

from which we conclude that

$$\mathbb{E}[Z_{n+1}|\mathscr{F}_n] = \frac{1}{k-1}(S_k - \mathbb{E}[X_k|S_k]) = \frac{1}{k-1}\left(S_k - \frac{S_k}{k}\right) = \frac{S_k}{k} = Z_n.$$

This proves the martingale property. Moreover we have $Z_{-1} = X_1 \in L^1$. Therefore by the backwards martingale convergence theorem above we have that as $n \to -\infty$

$$Z_n \to Z_{-\infty} = \mathbb{E}[Z_{-1}|\mathscr{F}_{-\infty}] = \mathbb{E}[X_1|\mathscr{G}_\infty]$$

in L^1 and almost surely.

Let $\mathscr{T}_n = \sigma(X_n, X_{n+1}, \ldots)$ and $\mathscr{T} = \cap_{n\geqslant 1}\mathscr{T}_n$ the tail σ-algebra of $X_\bullet$. In particular

$$\mathscr{G}_n = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots) \supseteq \sigma(X_{n+1}, X_{n+2}, \ldots) = \mathscr{T}_{n+1}$$

and note now that

$$\mathscr{G}_\infty = \cap_{n\geqslant 1}\mathscr{G}_n \supseteq \cap_{n\geqslant 1}\mathscr{T}_{n+1} = \mathscr{T}.$$

Morever we have also that

$$Z_{-\infty} = \lim_{n\to\infty} S_n/n = \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \lim_{n\to\infty} \frac{X_k + \cdots + X_n}{n}$$

is measurable wrt. to $\mathscr{T}_k$ for any $k \geqslant 1$ and therefore measurable wrt. $\mathscr{T}$ (of $X_\bullet$) so we conclude that

$$Z_{-\infty} = \mathbb{E}[Z_{-\infty}|\mathscr{T}] = \mathbb{E}[\mathbb{E}[X_1|\mathscr{G}_\infty]|\mathscr{T}] = \mathbb{E}[X_1|\mathscr{T}].$$

However by Kolmogorov 0-1 law the tail σ-field $\mathscr{T}$ is trival because the X are i.i.d. and this means that $\mathbb{E}[X_1|\mathscr{T}] = \mathbb{E}[X_1]$ (exercise). We conclude that $S_n/n \to \mathbb{E}[X_1]$ a.s. and in L^1 as claimed. □

Remark 1. Note that for any $k \geqslant 1$

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \underbrace{\lim_{n\to\infty} \frac{X_1 + \cdots + X_k}{n}}_{=0} + \lim_{n\to\infty} \frac{X_k + \cdots + X_n}{n} = \lim_{n\to\infty} \frac{X_k + \cdots + X_n}{n}$$
