Lecture 16 · 11.6.2021 · 10:15–12:00 via Zoom

Some applications of martingale theory (II)

Kakutani theorem

Theorem. *(Kakutani product martingale theorem) Let $(X_n)_{n \geqslant 1}$ be a sequence of independent, positive and mean 1 random variables. Let $M_0 = 1$ and define for all $n \geqslant 1$*

$$M_n = M_0 X_1 \cdots X_n.$$

Then $(M_n)_{n \geqslant 0}$ is a positive martingale and $M_n \to M_\infty$ a.s. as $n \to \infty$. Let $a_n := \mathbb{E}[X_n^{1/2}]$, then $a_n \in (0,1]$ and two things can happen:

a) If $\prod_{n=1}^\infty a_n > 0$ then $M_n \to M_\infty$ in L^1 and $\mathbb{E}[M_\infty] = 1$;

b) If $\prod_{n=1}^\infty a_n = 0$ then $M_n \to 0$ a.s.

Remark. This theorem make complete the picture of our previous counterexample of a positive martingale which converges to zero. In particular give a dichotomy on the possible behaviours. In case a) we have also $M_n = \mathbb{E}[M_\infty | \mathscr{F}_n]$.

Proof. It is easy to see that $(M_n)_{n \geqslant 0}$ is a martingale by independence of the factors in the product (exercise: alway check adapatedness, integrability and then the martingale property). Since $(M_n)_{n \geqslant 0}$ is a positive supermartingale we know that it converges a.s. to a positive limit $M_\infty \in L^1$. We have by Cauchy–Schwarz (or Jensen's)

$$a_n = \mathbb{E}[X_n^{1/2}] \leqslant \{\mathbb{E}[(X_n^{1/2})^2]\}^{1/2} \{\mathbb{E}[(1)^2]\}^{1/2} = 1,$$

and if $a_n = 0$ then $X_n^{1/2} = 0$ a.s. and therefore we have a contradiction so we established that $a_n \in (0,1]$.

Let now $N_0 = 1$ and

$$N_n = \frac{M_n^{1/2}}{\prod_{k=1}^n a_k} = \prod_{k=1}^n \frac{X_k^{1/2}}{\mathbb{E}[X_k^{1/2}]}, \qquad n \geqslant 1.$$

Then for the same reasons as above the stoch. process $(N_n)_{n \geqslant 0}$ is a positive martingale which again converges to a limit almost surely. We call the limit N_∞.

Case a) When $\prod_{k=1}^n a_k \to L > 0$, we have

$$\sup_n \mathbb{E}[N_n^2] = \sup_n \frac{\mathbb{E}[M_n]}{\prod_{k=1}^n (a_k)^2} = \frac{1}{\inf_n \prod_{k=1}^n (a_k)^2} < \infty$$

since $\prod_{k=1}^n (a_k)^2 > 0$ for all n and $\prod_{k=1}^n (a_k)^2 \to L^2 > 0$ as $n \to \infty$ therefore $\inf_n \prod_{k=1}^n (a_k)^2 > 0$. We discover that in this case $(N_n)_{n \geqslant 0}$ is a martingale bounded in L^2 and as consequence it converges not only a.s. but also in L^2 towards its limit $N_\infty \in L^2$.

Recall Doob's inequality applied to the running maximum $N_n^* = \sup_n N_n$:

$$\|N_n^*\|_{L^p} \leqslant \frac{p}{p-1} \|N_n\|_{L^p}$$

which gives (by Fatou's lemma and by the above estimation)

$$\|N_\infty^*\|_{L^2} = \left\| \liminf_n N_n^* \right\|_{L^2} \underset{\text{Fatou}}{\leqslant} \sup_n \|N_n^*\|_{L^2} \underset{\text{Doob's}}{\leqslant} 2 \sup_n \|N_n\|_{L^2} < \infty.$$

So we have that

$$N_\infty^* := \sup_{n \geqslant 0} N_n \in L^2$$

and therefore

$$M_\infty^* := \sup_n M_n = \sup_n \left[N_n^2 \prod_{k=1}^n (a_k)^2 \right] \leqslant \sup_n [N_n^2] = \left(\sup_n N_n \right)^2 \in L^1,$$

so the stochastic process $(M_n)_{n \geqslant 0}$ is uniformly bounded by an L^1 random variable M_∞^*. This implies in particular that $(M_n)_{n \geqslant 0}$ is a uniformly integrable martingale and therefore it converges a.s. and in L^1 to its limit M_∞.

Case b) We still have that almost surely

$$M_\infty = \lim_n M_n = \lim_n \left[N_n^2 \prod_{k=1}^n (a_k)^2 \right] = \left[\lim_n N_n^2 \right] \left[\lim_n \prod_{k=1}^n (a_k)^2 \right] = \underbrace{N_\infty^2}_{< \infty a.s.} \underbrace{\left[\lim_n \prod_{k=1}^n (a_k)^2 \right]}_{=0} = 0$$

almost surely. □

We can use Kakutani's theorem to study a questions in statistics, i.e. the likelyhood ratio test.

Assume you have a family on i.i.d (real for simplicity) observations $(X_n)_{n \geqslant 1}$ and that we want to test the null hypothesis

- $\boldsymbol{H_0}$: the observations are drawn from the probability law μ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, (i.e. $\mathrm{Law}(X_n) = \mu$)

against the alternative hypothesis

- $\boldsymbol{H_1}$: the observations are drawn from the probability law ν, (i.e. $\mathrm{Law}(X_n) = \nu$) with $\mu \neq \nu$.

We assume that ν is absolutely continuous wrt. μ with density $f \geqslant 0$, i.e.

$$\mathrm{d}\nu = f \, \mathrm{d}\mu$$

which I remind you that means that for all $A \in \mathscr{B}(\mathbb{R})$

$$\nu(A) = \int_A f(x) \mu(\mathrm{d}x).$$

The test protocol goes as follows: we consider the quantity (a statistics)

$$T_n = \sum_{k=1}^n \log f(X_k)$$

and observe that the stochastic process $(M_n)_{n\geqslant 1}$ defined as

$$M_n = e^{T_n} = \prod_{k=1}^{n} f(X_k), \qquad M_0 = 1.$$

is a martingale wrt. the filtration $(\mathscr{F}_n)_{n\geqslant 0}$ of the process $(X_n)_{n\geqslant 1}$ and the probability $\mathbb{P}$ which assing to every r.v. X_n the law μ, i.e. $\mathbb{P}(X_n \in A) = \mu(A)$ for all $A \in \mathscr{B}(\mathbb{R})$. Indeed it is adapted, integrable and

$$\mathbb{E}[M_{n+1}|\mathscr{F}_n] = \mathbb{E}[f(X_{n+1})M_n|\mathscr{F}_n] = \mathbb{E}[f(X_{n+1})|\mathscr{F}_n]M_n = \mathbb{E}[f(X_{n+1})]M_n = \underbrace{\int f(x)\mu(\mathrm{d}x)}_{\int \mathrm{d}\nu = 1} M_n = M_n.$$

We have also that $\mathbb{E}[M_n] = \prod_{k=1}^{n} \mathbb{E}[f(X_k)] = 1$.

Now we want to apply Kakutani theorem to determine what happens to M as $n \to \infty$. We compute therefore for all $n \geqslant 1$ we have

$$a_n = \mathbb{E}[f(X_n)^{1/2}] = \mathbb{E}[f(X_1)^{1/2}]$$

By Jensen's inequality (or Cauchy–Schwarz) we have $\mathbb{E}[f(X_1)^{1/2}] \leqslant \mathbb{E}[f(X_1)] = 1$ with **strict inequality** when $f(X_n)$ is not a.s. constant. In our case since $\mu \neq \nu$ we have that f is not a μ-a.s. constant function and therefore that $f(X_n)$ is not an $\mathbb{P}$-a.s. constant r.v.. This gives us that $0 < a_1 < 1$ and also that

$$\prod_{k=0}^{n} a_k = (a_1)^n \to 0$$

as $n \to \infty$. Therefore we are in case b) of Kakutani's theorem and we conclude that $M_n \to 0$ a.s.. Otherwise said we have that $T_n \to -\infty$ a.s.

Consider now the more general situation where the $(X_n)_{n\geqslant 1}$ are independent but not necessarily identically distributed.

Let $\mathbb{P}$ be the probability under which each of them has distribution μ while we let $\mathbb{Q}$ to the probability under which each of them as law ν_n with $\mathrm{d}\nu_n = f_n \mathrm{d}\mu$ for some sequence $(f_n)_n$ of densities.

Let $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$ and consider the filtration $(\mathscr{F}_n)_{n\geqslant 0}$ with $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and fix the measure space $(\Omega, \mathscr{F}_\infty = \sigma(\mathscr{F}_n : n \geqslant 0))$ and consider $\mathbb{P}, \mathbb{Q}$ as probability measures on this measure space.

In this situation Kakutani's theorem give necessary and sufficient conditions under which $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. Indeed note that

$$\mathrm{d}\mathbb{Q}|_{\mathscr{F}_n} = Z_n \mathrm{d}\mathbb{P}|_{\mathscr{F}_n}$$

where $Z_n = \prod_{k=1}^{n} f(X_k)$. Indeed for all $n \geqslant 1$ and any bounded measurable function $\varphi: \mathbb{R}^n \to \mathbb{R}$ we have

$$\mathbb{E}_{\mathbb{Q}}[\varphi(X_1, \ldots, X_n)] = \int \varphi(x_1, \ldots, x_n)\nu_1(\mathrm{d}x_1)\cdots\nu_n(\mathrm{d}x_n)$$

$$= \int \varphi(x_1, \ldots, x_n) f_1(x_1)\cdots f_n(x_n)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n)$$

$$= \mathbb{E}_{\mathbb{P}}[\varphi(X_1, \ldots, X_n) f_1(X_1)\cdots f_n(X_n)] = \mathbb{E}_{\mathbb{P}}[\varphi(X_1, \ldots, X_n) Z_n].$$

If

$$\prod_{k=1}^{\infty} \mathbb{E}_{\mathbb{P}}[f(X_k)^{1/2}] = \prod_{k=1}^{\infty} \int_{\mathbb{R}} f_k(x)^{1/2} \mu(\mathrm{d}x) > 0, \tag{1}$$

then the limit $Z_n \to Z_\infty$ exists a.s. and in L^1 by Kakutani's theorem and we have that $Z_n = \mathbb{E}_{\mathbb{P}}[Z_\infty | \mathscr{F}_n]$. Therefore for any $n \geqslant 0$ and any $A \in \mathscr{F}_n$ we have

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A Z_n] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A \mathbb{E}_{\mathbb{P}}[Z_\infty | \mathscr{F}_n]] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A Z_\infty]$$

This means that the probability measure $\tilde{\mathbb{Q}}$ defined by $\mathrm{d}\tilde{\mathbb{Q}} = Z_\infty \mathrm{d}\mathbb{P}$ is such that for any $n \geqslant 1$ and any $A \in \mathscr{F}_n$ we have

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A Z_\infty] = \mathbb{E}_{\tilde{\mathbb{Q}}}[\mathbb{1}_A] = \tilde{\mathbb{Q}}(A).$$

By Dinkin's $\pi - \lambda$ theorem the two measures coincide also on the σ-algebra generated by all the $(\mathscr{F}_n)_{n \geqslant 0}$, that is on $\mathscr{F}_\infty$. So they are equal as measures on $(\Omega, \mathscr{F}_\infty)$. We conclude that

$$\mathrm{d}\mathbb{Q} = Z_\infty \mathrm{d}\mathbb{P}.$$

On the other hand, if $\mathbb{Q}$ is absolutely continuous to $\mathbb{P}$ then there exist a positive measurable function H on $(\Omega, \mathscr{F}_\infty)$ such that $\mathrm{d}\mathbb{Q} = H \mathrm{d}\mathbb{P}$. Therefore we have as above that $Z_n = \mathbb{E}[H | \mathscr{F}_n]$ if the condition (1) is violated, that is if the infinite product is 0 then by Kakutani's theorem we have $Z_n \to 0$ and therefore that $H = 0$ which is not possible.

We just proved the following important corollary of Kakutani's theorem

Corollary. *In the conditions above we have*

$$\mathbb{Q} \ll \mathbb{P} \Leftrightarrow \prod_{k=1}^{\infty} \int_{\mathbb{R}} f_k(x)^{1/2} \mu(\mathrm{d}x) > 0.$$

Remark. In particular we have seen that if the sequence $(X_n)_{n \geqslant 0}$ is i.i.d. under both $\mathbb{Q}$ and $\mathbb{P}$ then the two measures are either equal or mutually singular (i.e. $\mathbb{Q} \not\ll \mathbb{P}$ and $\mathbb{P} \not\ll \mathbb{Q}$).

Example. Take $X_n \in \{0, 1\}$ (Bernoulli random variables) with $\mu = \mathrm{Ber}(1/2)$ and $\nu_k \sim \mathrm{Ber}(p_k)$ this defines $\mathbb{P}$ and $\mathbb{Q}$. Now we have

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1, \qquad \nu_k = (1 - p_k)\delta_0 + p_k \delta_1 = f_k(0)\frac{1}{2}\delta_0 + f_k(1)\frac{1}{2}\delta_1 = f_k \mu$$

with $f_k(0) = 2(1 - p_k)$ and $f_k(1) = 2p_k$. So we have

$$\prod_{k=1}^{\infty} \int_{\mathbb{R}} f_k(x)^{1/2} \mu(\mathrm{d}x) = \prod_{k=1}^{\infty} \int_{\mathbb{R}} f_k(x)^{1/2} \left[\frac{\delta_0(\mathrm{d}x) + \delta_1(\mathrm{d}x)}{2}\right] = \prod_{k=1}^{\infty} \frac{[(2(1 - p_k))^{1/2} + (2p_k)^{1/2}]}{2}$$

$$= \prod_{k=1}^{\infty} \left[\frac{(1 - p_k)^{1/2} + p_k^{1/2}}{2^{1/2}}\right]$$

and to have this positive we need $p_k \to \frac{1}{2}$ as $k \to \infty$. Exercise, let $p_k = \frac{1}{2} + \frac{1}{2}k^{-\alpha}$ and find conditions on $\alpha > 0$ to guarantee $\mathbb{Q} \ll \mathbb{P}$. One expect that α has to be large enough for the Kakutani criterion to be satisfied.
