Lecture 17 · 15.6.2021 · 14:15–16:00 via Zoom

Information der Fachschaft: Dieses Jahr findet das Mathe-Sommerfest virtuell am Freitag, den 25.06, ab 18 c.t. statt. Alle aktuellen Informationen sind auf https://fsmath.uni-bonn.de/veranstaltungen-detail/events/sommerfest-331.html zu finden. Schaut vorbei!

Information from the Student Council: This year the Maths Summer Party will take place virtually on Friday, June 25, starting at 18 c.t.. Latest information can be found on https://fsmath.uni-bonn.de/events-detail/events/sommer-fest-kopie.html. Come by!

Some applications of martingale theory (III)

The Radon-Nikodým theorem via martingales

Let $(\Omega, \mathscr{F}, \mu)$ a measure space and $f: \Omega \to \mathbb{R}_+$ positive measurable function, then we can form the set function μ^f on $\mathscr{F}$ defined as

$$\mu^f(A) := \int_A f \mathrm{d}\mu.$$

In this case we write

$$\mathrm{d}\mu^f = f \mathrm{d}\mu$$

or that

$$f = \frac{\mathrm{d}\mu^f}{\mathrm{d}\mu}$$

which is called the Radon-Nikodým derivative of μ^f wrt. μ. It is well defined μ-almost everywhere.

This is actually a measure on $(\Omega, \mathscr{F})$ (prove it by using the properties of the Lebesgue integral) which has the following property: $\mu^f(A) = 0$ for all $A \in \mathscr{F}$ such that $\mu(A) = 0$.

Definition. *We say that a measure ν on $(\Omega, \mathscr{F})$ is absolutely continuous with respect to a measure μ iff $\nu(A) = 0$ for all $A \in \mathscr{F}$ such that $\mu(A) = 0$. (All the null sets of μ are also null sets of ν). In this case we write $\nu \ll \mu$.*

Theorem. *(Radon-Nikodým) If μ, ν are two σ-finite measures on $(\Omega, \mathscr{F})$ then the following two statements are equivalent:*

a) $\nu \ll \mu$;

b) $\mathrm{d}\nu = f \mathrm{d}\mu$ for some positive measurable function $f: \Omega \to \mathbb{R}_+$, which is unique μ-almost everywhere.

We want to propose a proof of this theorem which is based on the UI martingale convergence theorem.

Proof. b)⇒a) it is clear from the considerations above. Let us focus on the proof that a) implies b).

Reduction to finite measures. We start by noting that by σ-finiteness we can reduce the problem to a statement involving finite measures μ, ν (eventually only defined on sets of a partition of Ω). (Think about the details for exercise).

Reduction to $\nu \leqslant \mu$. The second reduction comes by considering only the case where $\nu \leqslant \mu$ (here I mean that $\nu(A) \leqslant \mu(A)$ for all $A \in \mathscr{F}$). This is possible because we can always take

$$\tilde{\mu} = \mu + \nu$$

in which case we have $\nu \leqslant \tilde{\mu}$ and we still have $\nu \ll \tilde{\mu}$. Moreorver we have $\mu \leqslant \tilde{\mu}$ and of course $\mu \ll \tilde{\mu}$. So provided we already proved the existence of the RN derivative for ν and μ wrt. $\tilde{\mu}$ we have that there exists functions f, g such that

$$\mathrm{d}\nu = g\,\mathrm{d}\tilde{\mu}, \qquad \mathrm{d}\mu = f\,\mathrm{d}\tilde{\mu}.$$

But now we also have that $\mu(\{f=0\}) = \int_{\{f=0\}} f\,\mathrm{d}\tilde{\mu} = 0$ which implies that

$$\tilde{\mu}(\{f=0\}) = \int_{\{f=0\}} \mathrm{d}\tilde{\mu} = \int_{\{f=0\}} \mathrm{d}\mu + \int_{\{f=0\}} \mathrm{d}\nu = \nu(\{f=0\}) = 0$$

since $\nu \ll \mu$ and $\mu(\{f=0\}) = 0$. Therefore $\mathrm{d}\mu = f\,\mathrm{d}\tilde{\mu} = (f + \mathbb{1}_{f=0})\,\mathrm{d}\tilde{\mu}$ and we have that $f + \mathbb{1}_{f=0} > 0$ $\tilde{\mu}$ almost everywhere so we can write

$$\mathrm{d}\nu = g\,\mathrm{d}\tilde{\mu} = \frac{g}{(f + \mathbb{1}_{f=0})}(f + \mathbb{1}_{f=0})\mathrm{d}\tilde{\mu} = \frac{g}{(f + \mathbb{1}_{f=0})}\mathrm{d}\mu$$

and we proved that ν has RN derivative

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \frac{g}{(f + \mathbb{1}_{f=0})}$$

wrt. to μ.

Now it remains to prove that the RN exists in the particular case when $\nu \leqslant \mu$ and μ is finite and $\mu \neq 0$. We can normalize μ so that $\mu(\Omega) = 1$ and we think of $(\Omega, \mathscr{F}, \mu)$ as a probability space.

Let $\mathscr{R}$ be the family of all paritions $\mathscr{P}$ of Ω made of measurable sets from $\mathscr{F}$. The family $\mathscr{R}$ is a partial order once endowed with the order relation $\mathscr{P} \leqslant \mathscr{Q}$ iff $\sigma(\mathscr{P}) \subseteq \sigma(\mathscr{Q})$ (concretely: the partition $\mathscr{Q}$ is finer than $\mathscr{P}$). Moreover it is a directed set for this relation meaning that for any $\mathscr{P}$, $\mathscr{Q} \in \mathscr{R}$ there exists another partition $\mathscr{W} \in \mathscr{R}$ such that $\mathscr{P} \leqslant \mathscr{W}$ and $\mathscr{Q} \leqslant \mathscr{W}$ (just take the parition formed by all the interections of elements in $\mathscr{P}$ and $\mathscr{Q}$).

For any $\mathscr{P} \in \mathscr{R}$ we define $f_{\mathscr{P}}: \Omega \to \mathbb{R}_+$ as

$$f_{\mathscr{P}} := \sum_{A \in \mathscr{P}} \frac{\nu(A)}{\mu(A)} \mathbb{1}_A$$

which simply means that $f_{\mathscr{P}}(\omega) = \frac{\nu(A)}{\mu(A)}$ for $\omega \in A$ and which is well defined by absolute continuity since $\nu(A) = 0$ when $\mu(A) = 0$, in which case we take $\nu(A) / \mu(A) = 0$. Since $\nu \leqslant \mu$ we also have that $f_{\mathscr{P}}: \Omega \to [0,1]$ and moreover that $f_{\mathscr{P}}$ is $\sigma(\mathscr{P})$ measurable.

The key property of these functions is that

$$\nu(A) = \int_A f_{\mathscr{P}}\mathrm{d}\mu$$

for all $A \in \sigma(\mathscr{P})$ (by a $\pi - \lambda$ argument). Moreover if $\mathscr{P} \leqslant \mathscr{Q}$ then for any $A \in \sigma(\mathscr{P})$

$$\int_A f_{\mathscr{P}}\mathrm{d}\mu = \int_A f_{\mathscr{Q}}\mathrm{d}\mu. \qquad (1)$$

(think about it). This implies that the family of random variables $(f_{\mathscr{P}})_{\mathscr{P}\in\mathscr{R}}$ is a martingale indexed by the directed set $\mathscr{R}$ with respect to the partial order $\geqslant$ (refinement of partitions).

Now the key point is that we can generalize the notion of martingales indexed by $\mathbb{N}$ to any directed set, and in particular to $\mathscr{R}$ with the natural notion of convergence which is similar to that for sequences indexed by $\mathbb{N}$. Indeed $(\mathbb{N},\geqslant)$ is just another directed set, in this respect. Moreover the martingales convergence theorems that we proved are also valid in this more general context and since $(f_{\mathscr{P}})_{\mathscr{P}\in\mathscr{R}}$ is a bounded martingale since $|f_{\mathscr{P}}|\leqslant 1$ we have that is uniformly integrable (and more than that in $L^p(\mu)$ for all $p>1$). Therefore it converges a.s. and in L^1 (at least) to a limit $g\in L^1(\mu)$. (Here convergence means that for every $\varepsilon>0$ there exists $\mathscr{Q}\in\mathscr{R}$ s.t. for all $\mathscr{P}\geqslant\mathscr{Q}$ we have $|f_{\mathscr{P}}-g|\leqslant\varepsilon$)

By passing to the limit in (1) we have that for all $\mathscr{P}\in\mathscr{R}$ and all $A\in\sigma(\mathscr{P})$ we have

$$\int_A f_{\mathscr{P}}\mathrm{d}\mu=\int_A g\mathrm{d}\mu.$$

But now if $A\in\mathscr{F}$ one can always consider the partition $\mathscr{P}=\{A,A^c\}$ and therefore have that

$$\nu(A)=\int_A f_{\mathscr{P}}\mathrm{d}\mu=\int_A g\mathrm{d}\mu$$

for all $A\in\mathscr{F}$. This proves that the measure ν can be written as $g\mathrm{d}\mu$ and that $g=\mathrm{d}\nu/\mathrm{d}\mu$. This closes the existence step.

Uniqueness. Let assume that we have another function $\hat{g}$ such that $\mathrm{d}\nu=\hat{g}\mathrm{d}\mu$. Then for any $C>0$:

$$\int \mathbb{1}_{\{C>\hat{g}>g>-C\}}(\hat{g}-g)\mathrm{d}\mu=\int \mathbb{1}_{\{C>\hat{g}>g>-C\}}\mathrm{d}\nu-\int \mathbb{1}_{\{C>\hat{g}>g>-C\}}\mathrm{d}\nu=0$$

which implies that $\mathbb{1}_{\{C>\hat{g}>g>-C\}}(\hat{g}-g)=0$ μ-a.s. and therefore that $\mu(\{C>\hat{g}>g>-C\})=0$. Since C is arbitrary this also implies $\mu(\{\hat{g}>g\})=0$. Then we can interchange the role of $\hat{g}$ and g to conclude that $\mu(\{\hat{g}<g\})=0$ and therefore that $\mu(\{\hat{g}\neq g\})=0$, i.e. $\hat{g}=g$ μ-almost everywhere.

□

> **Remark.** It is a useful exercise to fill out the details on how to extend Doobs' submartingale and supermartingale convergence theorems to martingales indexed by directed sets. Try it if you like.

Lemma. *Let μ,ν be σ-finite measures on $(\Omega,\mathscr{F})$ such that $\nu\ll\mu$. If $f:\Omega\to\mathbb{R}$ is a $\mathscr{F}$-measurable function which is in $L^1(\nu)$ then for all $A\in\mathscr{F}$:*

$$\int_A f\mathrm{d}\nu=\int_A f\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mathrm{d}\mu.$$

Proof. We may assume that μ is finite by σ-finiteness and that $f\geqslant 0$ by splitting it into positive and negative parts. By monotone convergence it is also possible to assume that f is bounded (indeed otherwise just consider $(f\wedge N)$ and then the take the monotone limit $N\to\infty$).

Now we prove that the theorem is true for bounded measurable functions. Let $\mathscr{H}$ be the family of all $\mathscr{F}$-measurable bounded functions g such that

$$\int_A g\mathrm{d}\nu=\int_A g\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mathrm{d}\mu.$$

Then $\mathscr{H}$ contains 1 (by definition of the RN derivative as we proved in the previous theorem). Moreover $\mathscr{H}$ is a vector space and it is closed by monotone limits (by monotone convergence in the l.h.s. and the r.h.s. separately). Therefore by the monotone class theorem the set $\mathscr{H}$ must contain all the $\mathscr{F}$ measurable bounded functions since it contains all the indicators of $\mathscr{F}$ measurable sets (again by the RN theorem):

$$\int_A \mathbb{1}_B \mathrm{d}\nu = \nu(B \cap A) = \int_{A \cap B} \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\mu = \int_A \mathbb{1}_B \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\mu.$$

□