Lecture 9 · 11.5.2021 · 14:15–16:00 via Zoom

Last friday: stochastic process $(X_n)_{n\geqslant 0}$, $(\mathscr{F}_n)_{n\geqslant 0}$ filtration (e.g. increasing family of σ-algebras), adpted and previsible processes, and lastly the concept of stopping time. All of this in discrete time, i.e. when the time index n takes value in $\mathbb{N}$ or $\mathbb{N}_0 = \mathbb{Z}^+$.

Recall a r.v. $T:\Omega \to \mathbb{N}_0^* = \mathbb{N}_0 \cup \{+\infty\}$ is a stopping time (wrt. to a given filtration $(\mathscr{F}_n)_{n\geqslant 0}$) iff for all $n \in \mathbb{N}_0^*$ we have $\{T \leqslant n\} \in \mathscr{F}_n$ where $\mathscr{F}_\infty = \sigma(\mathscr{F}_n : n \geqslant 0)$.

Remark. We have always that $\{T \leqslant +\infty\} = \Omega \in \mathscr{F}_\infty$, moreover if $\{T \leqslant n\} \in \mathscr{F}_n$ for all $n \in \mathbb{N}_0$ then

$$\{T = +\infty\} = \cap_{n\geqslant 0}\{T \geqslant n\} \in \sigma((\mathscr{F}_n)_{n\geqslant 0}) \in \mathscr{F}_\infty.$$

The σ-algebra $\mathscr{F}_T$ of the stopping time T is defined as

$$\mathscr{F}_T := \{A \in \mathscr{F} : A \cap \{T \leqslant n\} \in \mathscr{F}_n \text{ for all } n \in \mathbb{N}_0^*\}.$$

(Exercise: prove that it is indeed a σ-algebra)

Proposition. *Let S, T two stopping times*

a) *If $S \leqslant T$ (i.e. pointwise for all $\omega \in \Omega$) then $\mathscr{F}_S \subseteq \mathscr{F}_T$;*

b) *$S \wedge T := \min(S,T)$ and $S \vee T := \max(S,T)$ are again stopping times and*

$$\mathscr{F}_{S\wedge T} = \mathscr{F}_T \cap \mathscr{F}_S, \qquad \mathscr{F}_{T\vee S} = \sigma(\mathscr{F}_T \cup \mathscr{F}_S);$$

c) *If $(X_n)_{n\geqslant 0}$ is an adapted process and $T < \infty$, then*

$$X_T \hat{\in} \mathscr{F}_T$$

where $X_T(\omega) := X_{T(\omega)}(\omega)$ is the r.v. representing the process X observed at the random time T. (Note that we look at the process $(X_n)_{n\geqslant 0}$ as the function $X: \mathbb{N}_0 \times \Omega \to \mathbb{R}$ such that $X(n,\omega) = X_n(\omega)$, it is easy to show that this function is measurable from $\mathbb{N}_0 \times \Omega$ to $\mathbb{R}$ because $\mathbb{N}_0$ is countable, then $X_T(\omega) = X(T(\omega),\omega)$ is again a measurable function as composition of measurable functions)

d) *A r.v. Z is $\mathscr{F}_T$-measurable iff the process $(Z_n := Z\,\mathbb{1}_{\{T=n\}})_{n\in\mathbb{N}_0^*}$ is adapted. In this case we have the relation $Z = Z_T$.*

Proof. Exercise. □

All these properties justify the notation $\mathscr{F}_T$ for the σ-algebra generated by stopping time T.

Example.

- If $T(\omega) = n$ for all $\omega \in \Omega$, then T is a stopping time. In particular every deterministic time n is a stopping time.

- If $(E, \mathscr{E})$ is a measure space and $A \in \mathscr{E}$ and $(X_n)_{n \geqslant 0}$ is an adapted process with values in $(E, \mathscr{E})$ then the <u>entrance time</u> in A for $(X_n)_{n \geqslant 0}$ defined as

  $$T_A := \inf \{n \geqslant 0 : X_n \in A\} : \Omega \to \mathbb{N}_0^*$$

  with $\inf(\emptyset) = +\infty$, is a stopping time. Indeed note that

  $$\{T_A \leqslant n\} = \cup_{k=0,\ldots,n} \underbrace{\{X_k \in A\}}_{\in \mathscr{F}_k} \in \sigma(\mathscr{F}_0, \ldots, \mathscr{F}_n) = \mathscr{F}_n$$

  for all $n \in \mathbb{N}_0$. So T_A is indeed a stopping time.

- Let $(X_n)_{n \geqslant 0}$ an adapted real valued process then

  $$T = \inf \{n \geqslant 0 : X_{n+1} \geqslant 100\}$$

  it is not necessarily a stopping time since in general $\{T = n\} \in \mathscr{F}_{n+1} \nsubseteq \mathscr{F}_n$. In general is just a random time, i.e. a random variable $\Omega \to \mathbb{N}_0^*$.

Using stopping times one can prove an interesting "impossibility" theorem.

Theorem. *(Wald's identity) Let $(X_n)_{n \geqslant 1}$ an i.i.d. sequence of integrable real valued r.v.s. Let T be stopping time for the filtration $(\mathscr{F}_n)_{n \geqslant 0}$ generated by the $(X_n)_{n \geqslant 1}$ (i.e. $\mathscr{F}_n = \mathscr{F}_n^X = \sigma(X_1, \ldots, X_n)$ with $\mathscr{F}_0 = \{\emptyset, \Omega\}$). Let*

$$S_n = X_1 + \cdots + X_n$$

for $n \geqslant 1$ with $S_0 = 0$. Then the process $(S_n)_{n \geqslant 0}$ is adapted to $(\mathscr{F}_n)_{n \geqslant 0}$ and if $\mathbb{E}[T] < \infty$ (i.e. T is an integrable stopping time) then the r.v. S_T is integrable and

$$\mathbb{E}[S_T] = \mathbb{E}[T]\, \mathbb{E}[X_1]$$

in particular if $\mathbb{E}[X_1] = 0$ then $\mathbb{E}[S_T] = 0$.

Remark. This theorem can be interpreted as follows: in a fair game (i.e. a game with average gain $\mathbb{E}[X_n] = 0$ at every round) with independent repeated trials any reasonable strategy (modeled by a stopping time T) give zero average gain.

Remark. Note that if $T = n$ then by linearity we have

$$\mathbb{E}[S_n] = \mathbb{E}[X_1 + \cdots + X_n] = n\, \mathbb{E}[X_1].$$

So Wald's identity says that this results also hold for general stopping times replacing n with $\mathbb{E}[T]$ on the r.h.s.

Remark. Let us consider the stopping time

$$T = \inf \{n \geqslant 0 : S_n \geqslant S_0 + 100\},$$

i.e. my strategy to stop is to quit the game when I gained 100 euros. On the set $\{T < \infty\}$ we have

$$S_T \geqslant S_0 + 100$$

so if $T<\infty$ a.s. we could expect that $\mathbb{E}[S_T] \geqslant \mathbb{E}[S_0+100] \geqslant 100$ even if $\mathbb{E}[X_1]=0$ (for example). We have a problem here, since the theorem make us expect that $\mathbb{E}[S_T]=0$. In order not to have a contradiction we are bound to conclude that $\mathbb{E}[T]=+\infty$, i.e. the stopping time is not integrable.

Therefore we see from this example that the integrability hypothesis on T is essential.

Remark. The integrability hypothesis on T is essential from a technical point view since it guarantees that S_T is an integrable random variable as claimed. We will see it in the proof.

Proof. The first thing to check is integrability of S_T:

$$S_T(\omega) = \sum_{n\geqslant 1} X_n(\omega)\mathbb{1}_{n\leqslant T(\omega)}, \qquad T(\omega) = \sum_{n\geqslant 1} \mathbb{1}_{n\leqslant T(\omega)}$$

then

$$\mathbb{E}[|S_T|] \leqslant \mathbb{E}\left[\sum_{n\geqslant 1} |X_n|\,\mathbb{1}_{n\leqslant T}\right] \underset{\substack{\text{Fubini}\\ \text{or}\\ \text{mon.conv.}}}{=\!=\!=} \sum_{n\geqslant 1} \mathbb{E}[|X_n|\,\mathbb{1}_{n\leqslant T}]$$

Now note that $\{n\leqslant T\}\in\mathscr{F}_{n-1}$ and that X_n is independent of $\mathscr{F}_{n-1}=\sigma(X_1,\dots,X_{n-1})$. Therefore $|X_n|$ is independent of $\mathbb{1}_{n\leqslant T}$ and we have

$$\sum_{n\geqslant 1} \mathbb{E}[|X_n|\,\mathbb{1}_{n\leqslant T}] \underset{\text{indep}}{=\!=\!=} \sum_{n\geqslant 1} \mathbb{E}[|X_n|\,]\,\mathbb{E}[\mathbb{1}_{n\leqslant T}] \underset{\text{ident.distr.}}{=\!=\!=} \mathbb{E}[|X_1|\,]\sum_{n\geqslant 1} \mathbb{E}[\mathbb{1}_{n\leqslant T}]$$

$$\underset{\substack{\text{Fubini}\\ \text{or}\\ \text{mon.conv.}}}{=\!=\!=} \mathbb{E}[|X_1|\,]\mathbb{E}\left[\sum_{n\geqslant 1} \mathbb{1}_{n\leqslant T}\right] \underset{\text{def of } T}{=\!=\!=} \mathbb{E}[|X_1|\,]\mathbb{E}[T] < \infty.$$

This shows that S_T is integrable. A similar computation now using Fubini–Tonelli shows that

$$\mathbb{E}[S_T] = \mathbb{E}\left[\sum_{n\geqslant 1} X_n\mathbb{1}_{n\leqslant T}\right] \underset{\text{Fub−Ton}}{=\!=\!=} \sum_{n\geqslant 1} \mathbb{E}[X_n\mathbb{1}_{n\leqslant T}] = \sum_{n\geqslant 1} \mathbb{E}[X_n]\,\mathbb{E}[\mathbb{1}_{n\leqslant T}]$$

$$= \mathbb{E}[X_1]\sum_{n\geqslant 1} \mathbb{E}[\mathbb{1}_{n\leqslant T}] = \mathbb{E}[X_1]\mathbb{E}[T]$$

where all the exchange of integrals and summations are justified via Fubini–Tonelli by the integrability assumptions and the computation above with the absolute values. □

This theorem shows that sums $(S_n)_{n\geqslant 0}$ of i.i.d r.v. which are integrable and with mean zero satisfy

$$\mathbb{E}[S_T] = S_0$$

for all integrable stopping times T.

A natural question then is to characterise the class $\mathscr{M}$ of stochastic processes $(X_n)_{n\geqslant 0}$ which are adapted, integrable (i.e. $X_n\in L^1(\mathbb{P})$ for all $n\geqslant 0$) and such that

$$\mathbb{E}[X_T] = \mathbb{E}[X_0], \tag{1}$$

for all almost surely bounded stopping times T. A stopping time is almost surely bounded if $T\in L^\infty(\mathbb{P})$, i.e. it exists a constant $K<\infty$ such that $\mathbb{P}(|X|<K)=1$.

If we interpret one of such stochastic processes as the total gain in a game, then it represents a fair game where there are no winning (or losing) stopping strategies.

Eq. (1) give a "global" characterisation of these "fair games". Then we can relate this to a "local" point of view which characterise the behaviour of the process at every time. The local char. is easier to check.

Remark. Note that if T is a.s. bounded and $(X_n)_{n\geqslant 0}$ is integrable then also X_T is integrable, indeed

$$X_T = \sum_{n\geqslant 1} X_n \mathbb{1}_{T=n},$$

and therefore

$$|X_T| \leqslant \sum_{n\geqslant 1} |X_n| \mathbb{1}_{T=n} \underset{a.s.}{=\!=} \sum_{n=1}^{K} |X_n| \mathbb{1}_{T=n} \leqslant \sum_{n=1}^{K} |X_n| \in L^1(\mathbb{P})$$

where K is any number such that $T \leqslant K$ a.s. and $\sum_{n=1}^{K} |X_n|$ is a finite sum of integrable r.v. and therefore is integrable:

$$\mathbb{E}|X_T| \leqslant \mathbb{E}\sum_{n=1}^{K} |X_n| \leqslant \sum_{n=1}^{K} \mathbb{E}[|X_n|] < \infty.$$

Lemma. *An adapted and integrable process $(X_n)_{n\geqslant 0}$ satisfies (1) iff for all $n \geqslant 0$ we have*

$$\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n.$$

We will do the proof on friday.
