Radiative transport limit of the random Schrödinger equation

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Born 03.06.1994 in Saratov, Russia

21st September 2017

Master's Thesis Mathematics

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1 Introduction

We consider the Schroedinger equation

$$i\partial_t\phi(t,x) - \Delta_x\phi(t,x) = \varepsilon^{\frac{1-\gamma}{2}} V\Big(\frac{t}{\varepsilon^\gamma}, x\Big)\phi(t,x) \tag{1}$$

where ε is a small parameter, and $\gamma \in [0,1)$. The interaction of V with ϕ is small (of order $\varepsilon^{\frac{1-\gamma}{2}}$), which means to see it influence the wave function significantly we must consider large scales. Let $\phi_\varepsilon(t,x) = \phi\big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\big)$.

We scale time and space in the same way, so we can leave the average velocity of the wave invariant.

Then ϕ_ε satisfies

$$i\varepsilon\phi_\varepsilon(t,x) - \varepsilon^2\Delta_x\phi_\varepsilon(t,x) = \varepsilon^{\frac{1-\gamma}{2}} V\Big(\frac{t}{\varepsilon^{1+\gamma}}, \frac{x}{\varepsilon}\Big)\phi_\varepsilon(t,x). \tag{2}$$

We want to study properties of ϕ_ε in the limit $\varepsilon \to 0$, or in other words we want to study the macroscopic properties of the system behaving microscopically according to (1). Let us give some heuristics on what we should expect: Let us imagine for the moment that V is a smooth bump constant in time, and $\gamma = 0$. If we write $H = \Delta_x - \varepsilon^{\frac{1}{2}}V$ scattering theory tells us that a wave scattering at V splits into two parts: The free part that propagated as if the obstacle was not there, and the scattering part. More precisely:

$$e^{it\mathrm{H}}\varphi = e^{it\Delta_x}\varphi + \varphi_{\mathrm{sc}}$$

such that $\|\varphi_{\mathrm{sc}}\| = \mathcal{O}(\varepsilon)$. So the probability to scatter at time t is $\mathcal{O}(\varepsilon^1)$, which means the probability to scatter up to time t is $\mathcal{O}(\varepsilon t)$. Since $t \in \mathcal{O}\big(\frac{1}{\varepsilon}\big)$ the number of collisions (scatterings) will remain bounded. This means we expect a kinetic equation in the limit, since for a diffusive or hydrodynamic limit infinite collisions would be required. Kinetic equations are posed for the space-momentum density, however $|\phi(t,x)|^2$ is only a space density while the Fourier transform $\big|\hat{\phi}(t,x)\big|^2$ gives the momentum density. By the Heisenberg uncertainty principle there is no space momentum density in quantum mechanics however the Wigner transform will be a suitable replacement for us:

It is defined by:

$$W_\phi(t,x,k) = \int e^{ik\cdot y}\phi\Big(t, x - \frac{y}{2}\Big)\bar{\phi}\Big(t, x + \frac{y}{2}\Big)\mathrm{d}y$$

We will drop the ϕ in the future. Appropriate for us will be a rescaled version of this, defined by

$$W_\varepsilon(t,x,k) = \int e^{ik\cdot y}\phi_\varepsilon\Big(t, x - \varepsilon\frac{y}{2}\Big)\bar{\phi_\varepsilon}\Big(t, x + \varepsilon\frac{y}{2}\Big)\mathrm{d}y$$

The advantage of the Wigner-transform is that it satisfies a closed equation, provided that the wave function satisfies the Schroedinger equation. By product rule

$$\begin{aligned}\partial_t W_\varepsilon(t,x,k) &= -\frac{i}{\varepsilon}\int e^{ik\cdot y}\Big(\varepsilon\Delta_x\phi_\varepsilon\Big(t,\,x-\varepsilon\frac{y}{2}\Big)+\varepsilon^{\frac{1-\gamma}{2}}V\Big(\frac{t}{\varepsilon^{1+\gamma}},\,\frac{x}{\varepsilon}-\varepsilon\frac{y}{2}\Big)\phi_\varepsilon\Big(t,\,x-\varepsilon\frac{y}{2}\Big).\Big)\bar{\phi_\varepsilon}\Big(t,\\ &\quad x+\varepsilon\frac{y}{2}\Big)\mathrm{d}y\\ &+\frac{i}{\varepsilon}\int e^{ik\cdot y}\phi_\varepsilon\Big(t,\,x-\varepsilon\frac{y}{2}\Big)\Big(\varepsilon\Delta_x\bar{\phi_\varepsilon}\Big(t,\,x+\varepsilon\frac{y}{2}\Big)+\varepsilon^{\frac{1-\gamma}{2}}V\Big(\frac{t}{\varepsilon^{1+\gamma}},\,\frac{x}{\varepsilon}+\varepsilon\frac{y}{2}\Big)\bar{\phi_\varepsilon}\Big(t,\\ &\quad x+\varepsilon\frac{y}{2}\Big)\Big)\mathrm{d}y\\ &= -\frac{i}{\varepsilon}\int e^{ik\cdot y}\Big(\Delta_y\phi_\varepsilon\Big(t,\,x-\varepsilon\frac{y}{2}\Big)+\varepsilon^{\frac{1-\gamma}{2}}V\Big(\frac{t}{\varepsilon^{1+\gamma}},\,\frac{x}{\varepsilon}-\varepsilon\frac{y}{2}\Big)\phi_\varepsilon\Big(t,\,x-\varepsilon\frac{y}{2}\Big).\Big)\bar{\phi_\varepsilon}\Big(t,\\ &\quad x+\varepsilon\frac{y}{2}\Big)\mathrm{d}y\\ &+\frac{i}{\varepsilon}\int e^{ik\cdot y}\phi_\varepsilon\Big(t,\,x-\varepsilon\frac{y}{2}\Big)\Big(\Delta_y\bar{\phi_\varepsilon}\Big(t,\,x+\varepsilon\frac{y}{2}\Big)+\varepsilon^{\frac{1-\gamma}{2}}V\Big(\frac{t}{\varepsilon^{1+\gamma}},\,\frac{x}{\varepsilon}+\varepsilon\frac{y}{2}\Big)\bar{\phi_\varepsilon}\Big(t,\,x+\\ &\quad \varepsilon\frac{y}{2}\Big)\Big)\mathrm{d}y\\ &= -k\cdot\nabla_x W_\varepsilon(t,x,k)+\frac{i}{\varepsilon^{\frac{1+\gamma}{2}}}\int e^{ik\cdot y}\Big(V\Big(\frac{t}{\varepsilon^{1+\gamma}},\frac{x}{\varepsilon}+\varepsilon\frac{y}{2}\Big)-V\Big(\frac{t}{\varepsilon^{1+\gamma}},\frac{x}{\varepsilon}-\varepsilon\frac{y}{2}\Big)\Big)\phi_\varepsilon\Big(t,\\ &\quad x-\varepsilon\frac{y}{2}\Big)\bar{\phi_\varepsilon}\Big(t,x+\varepsilon\frac{y}{2}\Big)\mathrm{d}y\end{aligned}$$

If we denote by $\hat{V}$ the Fourier transform of V we have, by the Fourier inversion formula

$$V\Big(\frac{t}{\varepsilon^{1+\gamma}},\frac{x}{\varepsilon}+\varepsilon\frac{y}{2}\Big)=\int e^{-ip\frac{x}{\varepsilon}-p\frac{y}{2}}\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p\Big)$$

Plugging this in we get

$$\begin{aligned}&\int V\Big(\frac{t}{\varepsilon^{1+\gamma}},\frac{x}{\varepsilon}+\varepsilon\frac{y}{2}\Big)\phi_\varepsilon\Big(t,x-\varepsilon\frac{y}{2}\Big)\bar{\phi_\varepsilon}\Big(t,x+\varepsilon\frac{y}{2}\Big)\mathrm{d}y\\ &=\int\int e^{ik\cdot y}e^{-ip\frac{x}{\varepsilon}-p\frac{y}{2}}\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p\Big)\phi_\varepsilon\Big(t,x-\varepsilon\frac{y}{2}\Big)\bar{\phi_\varepsilon}\Big(t,x+\varepsilon\frac{y}{2}\Big)\mathrm{d}y\\ &=\int e^{-ip\frac{x}{\varepsilon}}\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p\Big)W\Big(t,x,k-\frac{p}{2}\Big)\end{aligned}$$

Which implies the equation

$$\partial_t W_\varepsilon(t,\,x,\,k)+k\cdot\nabla_x W_\varepsilon(t,\,x,\,k)=\frac{i}{\varepsilon^{\frac{1+\gamma}{2}}}\int e^{-i\frac{p\cdot x}{\varepsilon}}\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\,\mathrm{d}p\Big)\Big(W_\varepsilon\Big(t,\,x,\,k-\frac{p}{2}\Big)-W_\varepsilon\Big(t,\,x,\,k+\frac{p}{2}\Big)\Big)\tag{3}$$

In this thesis we will study solutions of this equation with initial condition

$$W_\varepsilon(0,x,k)=W_{0,\varepsilon}(x,k)$$

such that for every $\lambda\in C_c^\infty(\mathbb{R}^{2n})$

$$\lim_{\varepsilon\to 0}\langle W_{0,\varepsilon}(x,k),\lambda\rangle_{L^2(\mathbb{R}^{2n})}=\langle W_0(x,k),\lambda\rangle_{L^2(\mathbb{R}^{2n})}$$

for some weak limit $W_0\in L^2(\mathbb{R}^{2n})$. Our main theorem will be the following:

Theorem 1. *Let W_ε be a uniformly bounded sequence in $C_b(\mathbb{R},L^2(\mathbb{R}^{2n}))$ such that $W_\varepsilon(0,\cdot,\cdot)$ as above and , W_ε satisfies equation 3. Then W_ε converges in probability, uniformly in time and weakly in space to some function W which satisfies the equation*

$$\partial_t W(t,x,k)+k\cdot\nabla_x W(t,x,k)=\int\sigma(p,k)(W(t,x,k-p)-W(t,x,k))\tag{4}$$

Where σ will depend on the distribution of V in a manner we will make more precise later.

Let us say a few things about the structure of this thesis. We will mainly follow the proof proof of [5]. In Sections 2,3 we develop some tools. Then in section 4 we prove that the W_ε is tight, and in section 5 we characterize the accumulation points of the sequence as solutions of (4). Finally in section 6 we prove that (4) has a unique solution which implies convergence of the whole sequence to that deterministic limit. Our contribution to the proof in [5] is that we prove path wise estimates for the operators in section 3 and 4. Even though it is not necessary to prove the final statement we believe it provides an interesting proof of concept for how modulation spaces can be used to obtain path-wise estimates for the operator norm of random operators. We also believe that the convergence of (1) is interesting not only conceptually but numerically as well. In such a context almost sure estimates on the necessary operators might be useful.

Let us introduce some notation

I would like to take this moment to thank Professor Gubinelli for his excellent supervision of my thesis.The time and effort he put into this thesis goes far beyond anything that can be reasonably expected.

2 Modulation spaces

In this section we give a brief exposition of modulation spaces, and prove how they can be used to get almost sure bounds on norms of random operators. This section mainly follows [6]. For a more detailed exposition we also refer to [6].

Definition 2. *Fix a Schwartz function $\varphi \neq 0$,which we will call the window. We define the short time Fourier transform, or STFT for a function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and set $z = (z_1, z_2) \in \mathbb{R}^{2d}$,*

$$V_\varphi f(z) = \int e^{i y \cdot z_2} f(y) \varphi(y - z_1) \mathrm{d}y = \widehat{f(\cdot)\varphi(\cdot - z_1)}(z_2)$$

The STFT satisfies the following orthogonality relations:

Lemma 3. *Let φ_1, φ_2 be Schwartz-functions and $f_1, f_2 \in L^2(\mathbb{R}^d)$ then*

$$\int V_{\varphi_2} f_1(z) V_{\varphi_1} f_2(z) \mathrm{d}z = \int f_1(y) f_2(y) \mathrm{d}y \int \varphi_1(z_1) \varphi_2(z_1) \mathrm{d}z_1 = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^N)} \langle \varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}^d)}$$

In particular the STFT is an isometry $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$, provided that $\|\varphi\|_{L^2} = 1$

Proof. By Parservals identity

$$\begin{aligned} \int V_{\varphi_2} f_1(z) V_{\varphi_1} f_2(z) \mathrm{d}z &= \int\int \mathrm{d}z_1 \mathrm{d}z_2 \widehat{f_1(\cdot)\varphi_1(\cdot - z_1)}(z_2) \widehat{f_2(\cdot)\varphi_2(\cdot - z_1)}(z_2) \\ &= \int\int f_1(y) \varphi_1(y - z_1) f_2(y) \varphi_2(y - z_1) \mathrm{d}y \mathrm{d}z_1 \\ &= \int f_1(y) f_2(y) \mathrm{d}y \int \varphi_1(z_1) \varphi_2(z_1) \mathrm{d}z_1 \end{aligned}$$

□

The reason the STFT is interesting to us is the following fact: Introduce the notation $\varphi_z(\cdot) = e^{i(\cdot)\cdot z_2} \varphi(\cdot - z_1)$. Then we have the following

Lemma 4. *Let φ_1, φ_2 be Schwarz-functions, if $f \in L^2$ the*

$$f(\cdot) = \frac{1}{\langle \varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}^N)}} \int V_{\varphi_1} f(z) \varphi_z(\cdot) \mathrm{d}z$$

Where the Integral is to be interpreted as a Bochner Integral with values in $L^2(\mathbb{R}^N)$.

Proof. *Set*

$$\tilde{f} = \frac{1}{\langle\varphi_1, \varphi_2\rangle_{L^2(\mathbb{R}^N)}} \int V_{\varphi_1} f\,(z)\varphi_{2,z}(\cdot)\mathrm{d}z$$

This is well defined, since $V_{\varphi_1} f\,(z)$ *is in* L^2. *For a function* $g \in L^2$ *we can compute*

$$\begin{aligned} \langle\, \tilde{f}, g\rangle &= \frac{1}{\langle\varphi_1, \varphi_2\rangle_{L^2(\mathbb{R}^N)}} \int V_{\varphi_1} f\,(z)\langle\varphi_{2,z}, g\rangle_{L^2}\mathrm{d}z \\ &= \frac{1}{\langle\varphi_1, \varphi_2\rangle_{L^2(\mathbb{R}^N)}} \int V_{\varphi_1} f\,(z) V_{\varphi_2} g(z)\mathrm{d}z \\ &= \langle f, g\rangle \end{aligned}$$

Which proves the statement. □

Much like the Fourier inversion formula this lemma enables us to write a function f as a superposition of the functions φ_z. However the advantage is that φ_z are significantly better behaved, then in the fourier transform case since the fourier inversion formula writes uses superposition of $e^{ip\cdot x}$ which has no decay.

Now we are ready to introduce modulation spaces. Modulation spaces are associated with the STFT in the same way that Sobolev spaces are associated with the Fourier transform.

Definition 5. *We say that a function* f *is in the space* $M_{p,q}^s$ *if*

$$\|f\|_{M_{p,q}^s} = \left(\int \left(\int |V_\varphi f(z)|^p \langle z\rangle^{ps} dz_1\right)^{\frac{q}{p}} \mathrm{d}z_2\right)^{\frac{1}{q}}$$

is finite.Here $\langle\rangle$ *denotes the Japanse Bracket, which is defined by* $\langle a\rangle = (1 + a^2)^{\frac{1}{2}}$

It can be proven that $M_{p,q}^s$ *is a Banachspace, and it is independent of* φ *up to equivalence of norms.*

The reason we are interested in modulation spaces is the following lemma:

Lemma 6. *Assume we have random operator* A_ω , $\omega \in \mathbb{P}$, *mapping* $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ *such that*

$$\mathbb{E}|\langle A\varphi_w, \varphi_z\rangle|^2 \leq \langle w\rangle^a K(z - w)$$

with $H \in L^1(\mathbb{R}^{2N})$,$a \in \mathbb{R}$ *then* A *is almost surely bounded from* $M_{2,2}^s$ *to* M_{L^2} *provided that* $2s > d + a$, *such that* $\mathbb{E}\big[\,\|A\|^2_{M_{2,2}^s \to M_{L^2}}\big] \leq C\|K\|_{L^1(\mathbb{R}^{2N})}$

Proof.

$$\begin{aligned} |\langle Af, g\rangle|^2 &= \left|\int\int V_\varphi f(w) V_\varphi f(z)\langle A\varphi_w, \varphi_z\rangle \mathrm{d}z\mathrm{d}w\right|^2 \\ &\leq \left(\int\int |V_\varphi f(w) V_\varphi f(z)|\,|\langle A\varphi_w, \varphi_z\rangle|\mathrm{d}z\mathrm{d}w\right) \end{aligned}$$

Applying Hoelders inequality we get

$$\begin{aligned} &\left(\int\int |V_\varphi f(w)\, V_\varphi g(z)|\,|\langle A\varphi_w, \varphi_z\rangle|\mathrm{d}z\mathrm{d}w\right)^2 \\ \leq &\int |V_\varphi f(w)|^2 \langle w\rangle^{2s+a}\mathrm{d}w \int \langle w\rangle^{-2s-a}\left(\int |V_\varphi g(z)||\langle A\varphi_w, \varphi_z\rangle|\mathrm{d}z\right)^2 \mathrm{d}w \\ \leq &\int |V_\varphi f(w)|^2 \langle w\rangle^{2s+a}\mathrm{d}w \int |V_\varphi g(z)|^2 \mathrm{d}z \int \langle w\rangle^{-2s} |\langle A\varphi_w, \varphi_z\rangle|^2 \mathrm{d}z\mathrm{d}w \end{aligned}$$

Taking expectation we get

$$\mathbb{E}\|A\|_{M^s_{2,2}\to L^2} = \mathbb{E}\sup_{f,g} |\langle Af, g\rangle| \leq \int \langle w\rangle^{2s}\mathbb{E}|\langle A\varphi_w, \varphi_z\rangle|^2 \mathrm{d}z\mathrm{d}w \leq C\|K\|_{L^1(\mathbb{R}^{2N})}$$

where the supremum is taken over f with $\|f\|_{M^s_{2,2}} \leq 1$ and g with $\|g\|_{L^2} \leq 1$ □

3 Construction of the random potential

This section mainly follows [1].

Definition 7. *Let ν be a σ-finite measure on $\mathbb{R}^d$ and let $\mathcal{B}$ be the collection of Borel measurable sets of finite ν-Measure, and $\mathbb{P}$ be a probability space. We say that a map $\mathcal{W}: \mathbb{P} \mapsto \mathbb{R}^{\mathcal{B}}$ is a Gaussian ν-noise, if for any $A, B \in \mathcal{B}$*

$$\mathcal{W}(A) \sim \mathcal{N}(0, \nu(A))$$

$$\mathcal{W}(A \cup B) = \mathcal{W}(A) + \mathcal{W}(B) \text{ if } A, B \text{ are disjoint}$$

$$\mathcal{W}(A), \mathcal{W}(B) \text{ are independent if } A, B \text{ are disjoint}$$

Lemma 8. *For every σ-finite measure ν there exists a Gaussian ν-noise.*

Proof. We need to specify a correlation function on $\mathcal{B}\times\mathcal{B}$,which will determine the finite dimensional distributions, then the statement will follow by the Kolmogorov extension theorem. Define

$$C(A, B) = \nu(A \cap B)$$

A necessary and sufficient condition is for C to be positive semi-definite, meaning that for any A_i∈B $\alpha_i \in \mathbb{R}$

$$\sum_{i,j} \alpha_i C(A_i, A_j)\alpha_j \geq 0$$

$$\sum_{i,j} \alpha_i C(A_i, A_j)\alpha_j = \sum_{i,j} \alpha_i \alpha_j \int \mathbb{1}_{A_i}\mathbb{1}_{\mathrm{Aj}}\mathrm{d}\nu = \int \left(\sum_i \alpha_i \mathbb{1}_{A_i}\right)^2 \mathrm{d}\nu \geq 0$$

□

We now want to construct a random field from a gaussian noise. To do that we need to give meaning to the expression

$$\int f(\lambda)\mathcal{W}(d\lambda)$$

for some function $f \in L^2(\mathbb{R}^N)$.

Proposition 9. *Let ν be σ-finite measure on $\mathbb{R}^d$.Let $S(\mathbb{R}^d, \nu)$ be the space of simple functions(measurable functions with support of finite measure taking finitely many values). We claim that the map*

$$S(\mathbb{R}^d, \nu) \mapsto L^2(\mathbb{P})$$

$$\sum a_i \mathbb{1}_{A_i} \to \sum a_i W(A_i)$$

extends to an isometry

$$L^2(\mathbb{R}^d, \nu) \mapsto L^2(\mathbb{P})$$

Proof. We prove that the map preserves inner products. The statement of the theorem then follows by density of simple functions. By Fubini's theorem

$$\mathbb{E}\big[\sum a_i \mathcal{W}(A_i) \sum b_j \mathcal{W}(B_j)\big] = \sum_{i,j} a_i b_j \mathbb{E}[\mathcal{W}(A_i)\mathcal{W}(B_j)]$$

$$\sum_{i,j} a_i b_j \mathbb{E}[\mathcal{W}(A_j)\mathcal{W}(B_i)] = \sum_{i,j} a_i b_j \nu(A_i \cap B_j) = \sum_{i,j} a_i b_j \int \mathbb{1}_{A_i}\mathbb{1}_{\mathrm{Bj}} \mathrm{d}\nu$$

$$\sum_{i,j} a_i a_j \int \mathbb{1}_{A_i}\mathbb{1}_{\mathrm{Aj}} \mathrm{d}\nu = \int \sum a_i \mathbb{1}_{A_i} \sum b_j \mathbb{1}_{\mathrm{Bj}} \mathrm{d}\nu$$

□

Now we are ready to define our random field. Choose ν to be absolutely continuous with density $\frac{\hat{R}(p)g(p)}{g(p)^2+\omega^2} \in L^1(\mathbb{R}^d \times \mathbb{R})$. We will specify the assumptions on $\hat{R}(p), g(p)$ below. For $t \geq 0, x \in \mathbb{R}^d$ we set

$$V(t,x) = \int e^{i\omega t} e^{i x \cdot p} \mathcal{W}(\mathrm{d}p\mathrm{d}\omega)$$

For V we have the correlation functions:

$$\begin{aligned} \mathbb{E}[V(t,x)V(s,y)] &= \int e^{i(x-y)\cdot p} e^{i(t-s)\omega} \frac{\hat{R}(p)g(p)}{g(p)^2+\omega^2} \mathrm{d}p\mathrm{d}\omega \\ &= \int e^{i(x-y)\cdot p} e^{-g(p)|t-s|} \hat{R}(p) \mathrm{d}p\mathrm{d}\omega \end{aligned}$$

We also introduce the fields $\hat{V}(t, \mathrm{d}p)$

given by

$$\hat{V}(t, \mathrm{d}p) = \int_{\mathbb{R}} e^{i\omega t} \mathcal{W}(\mathrm{d}\omega)$$

$$\mathbb{E}\big[\hat{V}(t,p_2)\hat{V}(s,\mathrm{d}p_1)\big] = e^{-g(p)|t-s|}\hat{R}(p_1)\delta(p_1+p_2)$$

$$\mathbb{E}\Big[\hat{V}(t,p_2)\overline{\hat{V}(s,\mathrm{d}p_1)}\Big] = e^{-g(p)|t-s|}\hat{R}(p_1)\delta(p_1-p_2)$$

We will need the following

Proposition 10. *Let $\mathcal{F}_t = \sigma\big(\hat{V}(s,\cdot), s \leq t\big)$ be the σ-algebra generated by $\big(\hat{V}(s,\cdot), s \leq t\big)$. We have*

$$\mathbb{E}\big[\hat{V}(t+h,\cdot)|\mathcal{F}_t\big] = e^{-g(p)h}\hat{V}(t,\cdot)$$

$$\begin{aligned} \mathbb{E}\big[\big\langle \hat{V}(t+h,\cdot),\varphi\big\rangle\big\langle \hat{V}(t+h,\cdot),\psi\big\rangle|\mathcal{F}_t\big] &= \mathbb{E}\big[\big\langle \hat{V}(t+h,\cdot),\varphi\big\rangle|\mathcal{F}_t\big] \\ &\times \mathbb{E}\big[\big\langle \hat{V}(t+h,\cdot),\psi\big\rangle|\mathcal{F}_t\big] \\ &+ \int \varphi(p)\psi(-p)\hat{R}(p)\big(1-e^{-g(p)h}\big)\mathrm{d}p \end{aligned}$$

Proof. This follows [4]. Define $Y = \hat{V}(t+h,\mathrm{d}p) - e^{-g(p)}\hat{V}(t,\mathrm{d}p)$. Then Y and $\hat{V}(t)$ are centered gaussian variables and they are uncorrelated since

$$\mathbb{E}\big[\big\langle \hat{V}(t),\varphi\big\rangle\langle Y,\psi\rangle\big] = \int \varphi(p)\psi(-p)\hat{R}(p)\big(e^{-g(p)h} - e^{-g(p)h}\big)\mathrm{d}p = 0$$

Which means they are independent. This means that

$$\mathbb{E}\big[\hat{V}(t+h,\mathrm{d}p)|\hat{V}(t)\big] = \mathbb{E}\big[Y\big|\hat{V}(t)\big] + \mathbb{E}\big[e^{-g(p)}\hat{V}(t,\mathrm{d}p)\big|\hat{V}(t)\big] = e^{-g(p)}\hat{V}(t,\mathrm{d}p)$$

Now we want to prove by induction that for any $i\geq 1$

$$\mathbb{E}\big[\hat{V}(t_{i+1},\mathrm{d}p)|\hat{V}(t_i),\cdots,\hat{V}(t_1)\big]=e^{-g(p)(t_{i+1}-t_i)}$$

We assume we have already proven

$$\mathbb{E}\big[\hat{V}(t_i,\mathrm{d}p)|\hat{V}(t_{i-1}),\cdots,\hat{V}(t_1)\big]=e^{-g(p)(t_i-t_{i-1})}\hat{V}(t_{i-1},\mathrm{d}p)$$

Then write again $Y=\hat{V}(t_{i+1},\mathrm{d}p)-e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\mathrm{d}p)$. Again Y and $\hat{V}(t_i)$ are independent so

$$\begin{aligned}\mathbb{E}\big[\hat{V}(t_{i+1},\mathrm{d}p)|\hat{V}(t_i),\cdots,\hat{V}(t_1)\big] &= \mathbb{E}\big[Y|\hat{V}(t_i),\cdots,\hat{V}(t_1)\big]\\ &+ \mathbb{E}\big[e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\mathrm{d}p)|\hat{V}(t_i),\cdots,\hat{V}(t_1)\big]\\ &= \mathbb{E}\big[Y|\hat{V}(t_{i-1}),\cdots,\hat{V}(t_1)\big]\\ &+ e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\mathrm{d}p)\\ &= e^{-g(p)(t_{i+1}-t_{i-1})}\hat{V}(t_{i-1},\mathrm{d}p)\\ &- e^{-g(p)(t_{i+1}-t_i)}e^{-g(p)(t_i-t_{i-1})}\hat{V}(t_{i-1},\mathrm{d}p)\\ &+ e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\mathrm{d}p)\\ &= e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\mathrm{d}p)\end{aligned}$$

Now our claim follows by induction.

For the second part we proceed similarly: We will to show that

$$\begin{aligned}\mathbb{E}[\langle\hat{V}(t_{i+1},\ \cdot),\ \varphi\rangle\langle\hat{V}(\breve{t_{i+1}},\ \cdot),\ \psi\rangle|\hat{V}(t_i),\ \cdots,\\ \hat{V}(t_1)] &= \mathbb{E}[\langle\hat{V}(t_i,\cdot),\varphi\rangle|\hat{V}(t_i),\cdots,\hat{V}(t_1)]\\ &\times \mathbb{E}[\langle\hat{V}(\tilde{t}_{i+1},\cdot),\psi\rangle|\hat{V}(t_i),\cdots,\hat{V}(t_1)]\\ &+ \int \varphi(p)\psi(-p)\hat{R}(p)\big(e^{g(p)(\tilde{t}_{i+1}-t_{i+1})} -\\ &e^{-g(p)(\tilde{t}_{i+1}-\tilde{t}_i)}-e^{-g(p)(t_{i+1}-t_i)}\big)\mathrm{d}p\end{aligned}$$

We write again $Y=\hat{V}(t_{i+1},\mathrm{d}p)-e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\mathrm{d}p)$, $\tilde{Y}=\hat{V}(\tilde{t}_{i+1},\mathrm{d}p)-e^{-g(p)(\tilde{t}_{i+1}-t_i)}\hat{V}(t_i,\mathrm{d}p)$, then $Y,\tilde{Y}$ are independent from $\hat{V}(t_i,\mathrm{d}p)$, so

$$\mathbb{E}\big[\langle\hat{V}(t_{i+1},\cdot),\varphi\rangle\langle\hat{V}(\breve{t_{i+1}},\cdot),\psi\rangle|\hat{V}(t_i)\big]=\mathbb{E}\big[\langle Y,\varphi\rangle\langle\tilde{Y},\psi\rangle\big]+\int \mathrm{d}p e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\mathrm{d}p)\varphi(p)+\int e^{-g(p)(\tilde{t}_{i+1}-t_i)}\psi(p)\hat{V}(t_i,\mathrm{d}p)$$

$$\mathbb{E}\big[\langle Y,\varphi\rangle\langle\tilde{Y},\psi\rangle\big]=\int \mathrm{d}p\hat{R}(p)\Big(\big(e^{g(p)(\tilde{t}_{i+1}-t_{i+1})}-e^{-g(p)(\tilde{t}_{i+1}-\tilde{t}_i)}-e^{-g(p)(t_{i+1}-t_i)}\Big)$$

Now we want to generalize this to

$$\mathbb{E}\big[\langle\hat{V}(t_{i+1},\cdot),\varphi\rangle\langle\hat{V}(\breve{t_{i+1}},\cdot),\psi\rangle|\hat{V}(t_i)\hat{V}(t_{i-1})\cdots\hat{V}(t_1)\big].$$

Assume we have already proven

$$\mathbb{E}\big[\langle\hat{V}(t_{i+1},\ \cdot),\ \varphi\rangle\langle\hat{V}(\breve{t_{i+1}},\ \cdot),\ \psi\rangle|\hat{V}(t_{i-1}),\ \cdots,\ \hat{V}(t_1)\big]=\mathbb{E}\big[\langle\hat{V}(t_{i+1},\ \cdot),\ \varphi\rangle|\hat{V}(t_{i-1}),\ \cdots,\ \hat{V}(t_1)\big]\times \mathbb{E}\big[\langle\hat{V}(\tilde{t}_{i+1},\ \cdot),\ \psi\rangle|\hat{V}(t_{i-1}),\ \cdots,\ \hat{V}(t_1)\big]+\int\varphi(p)\psi(-p)\hat{R}(p)\big(e^{g(p)(\tilde{t}_{i+1}-t_{i+1})}-e^{-g(p)(\tilde{t}_{i+1}-\tilde{t}_{i-1})}-e^{-g(p)(t_{i+1}-t_{i-1})}\big)\mathrm{d}p \tag{5}$$

Now by the definition of $Y,\tilde{Y}$ and the fact that $Y,\tilde{Y}$ are independent of $V(t_i)$

$$\mathbb{E}\big[\langle\hat{V}(t_{i+1},\cdot),\ \varphi\rangle\langle\hat{V}(\breve{t_{i+1}},\cdot),\ \psi\rangle|\hat{V}(t_i)\hat{V}(t_{i-1}),\cdots,\hat{V}(t_1)\big]=\mathbb{E}\big[\langle\hat{V}(t_{i+1},\cdot),\ \varphi\rangle|\hat{V}(t_i),\ \cdots,\ \hat{V}(t_1)\big]\times \mathbb{E}\big[\langle\hat{V}(\tilde{t}_{i+1},\cdot),\psi\rangle|\hat{V}(t_{i-1}),\cdots,\hat{V}(t_1)\big]+\mathbb{E}\big[\langle Y,\varphi\rangle\langle\tilde{Y},\psi\rangle|\hat{V}(t_{i-1}),\cdots,\hat{V}(t_1)\big]$$

By definition of $Y, \tilde{Y}$ we have

$\mathbb{E}\big[\langle Y,\ \varphi\rangle\langle\tilde{Y},\ \psi\rangle|\hat{V}(t_{i-1}),\ \cdots,\ \hat{V}(t_1)\big] = \mathbb{E}\big[\big\langle\hat{V}(t_{i+1},\ \cdot),\ \varphi\big\rangle,\ \big\langle\hat{V}(\tilde{t}_{i+1},\ \cdot),\ \psi\big\rangle|\hat{V}(t_{i-1}),\ \cdots,\ \hat{V}(t_1)\big] - \mathbb{E}\big[\big\langle e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\ \cdot),\ \varphi\big\rangle,\ \big\langle\hat{V}(\tilde{t}_{i+1},\ \cdot),\ \psi\big\rangle|\hat{V}(t_{i-1}),\ \cdots,\ \hat{V}(t_1)\big] - \mathbb{E}\Big[\big\langle\hat{V}(t_i,\ \cdot),\ \varphi\big\rangle,\ \big\langle e^{-g(p)(\tilde{t}_{i+1}-t_i)}\hat{V}(t_{i+},\ \cdot),\ \psi\big\rangle|\hat{V}(t_{i-1}),\ \cdots,\ \hat{V}(t_1)\Big] + \mathbb{E}\Big[\big\langle e^{-g(p)(t_{i+1}-t_i)}\hat{V}(t_i,\ \cdot),\ \varphi\big\rangle,\ \big\langle e^{-g(p)(\tilde{t}_{i+1}-t_i)}\hat{V}(t_i,\cdot),\psi\big\rangle|\hat{V}(t_{i-1}),\cdots,\hat{V}(t_1)\Big] = \mathrm{I}+\mathrm{II}$

Where I contains the terms coming from the conditional expectations on the r.h.s of (5), and II contains the deterministic terms coming from the r.h.s of (5).

$\mathrm{I} = \big\langle e^{-g(p)(t_{i+1}-t_{i-1})}\hat{V}(t_{i-1},\ \cdot),\ \varphi\big\rangle\big\langle e^{-g(p)(\tilde{t}_{i+1}-t_{i-1})}\hat{V}(t_{i-1},\ \cdot),\ \psi\big\rangle - \big\langle e^{-g(p)(t_{i+1}-t_i)}e^{-g(p)(t_i-t_{i-1})}\hat{V}(t_{i-1},\ \cdot),\ \varphi\big\rangle\big\langle e^{-g(p)(\tilde{t}_{i+1}-t_{i-1})}\hat{V}(t_{i-1},\ \cdot),\ \psi\big\rangle - \big\langle e^{-g(p)(t_{i+1}-t_{i-1})}\hat{V}(t_{i-1},\ \cdot),\ \varphi\big\rangle\big\langle e^{-g(p)(\tilde{t}_{i+1}-t_i)}e^{-g(p)(t_i-t_{i-1})}\hat{V}(t_{i-1},\ \cdot),\ \psi\big\rangle + \big\langle e^{-g(p)(t_{i+1}-t_i)}e^{-g(p)(t_i-t_{i-1})}\hat{V}(t_{i-1},\cdot),\varphi\big\rangle\big\langle e^{-g(p)(\tilde{t}_{i+1}-t_i)}e^{-g(p)(t_i-t_{i-1})}\hat{V}(t_{i-1},\cdot),\psi\big\rangle = 0$

$$\mathrm{II} = \int \mathrm{d}p\hat{R}(p)\varphi(p)\psi(-p)h(t_{i+1}.\tilde{t_{i+1}}, t_i, t_{i-1}, p)$$

With

$h(t_{i+1}.\tilde{t_{i+1}},\ t_i,\ t_{i-1},\ p) = e^{-g(p)(\tilde{t}_{i+1}-t_{i+1})} - e^{-g(p)(\tilde{t}_{i+1}-t_{i-1})}e^{-g(p)(t_{i+1}-t_{i-1})} - e^{-g(p)(t_{i+1}-t_i)}\big(e^{g(p)(\tilde{t}_{i+1}-t_i)} - e^{-g(p)(\tilde{t}_{i+1}-t_{i-1})}e^{-g(p)(t_i-t_{i-1})}\big) - e^{-g(p)(\tilde{t}_{i+1}-t_i)}\big(e^{g(p)(t_{i+1}-t_i)} - e^{-g(p)(t_{i+1}-t_{i-1})}e^{-g(p)(t_i-t_{i-1})}\big) + e^{g(p)(\tilde{t}_{i+1}-t_i)}e^{-g(p)(t_{i+1}-t_i)}\big(1-e^{-2g(p)(t_i-t_{i-1})}\big)$

$$= e^{-g(p)(\tilde{t}_{i+1}-t_{i+1})} - e^{-g(p)(\tilde{t}_{i+1}-t_{i-1})}e^{-g(p)(t_{i+1}-t_{i-1})}$$

Now the we complete the proof by induction.

□

In the sequel we will assume that for a large A of our choosing

$$\int \sum_{i=1}^{A} \frac{\hat{R}(p)}{g(p)^i}|p|^i \mathrm{d}p < \infty$$

$$\int \hat{R}(p)\langle p\rangle^A \mathrm{dp} < \infty$$

4 Tightness

We consider the equation

$$\begin{aligned} \frac{\partial}{\partial t}W_\varepsilon(t,x,k) + k\cdot\nabla_x W_\varepsilon(t,x,k) &= \frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\int \hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, p\Big)e^{i\frac{\mathrm{px}}{\varepsilon}}\Big(W\Big(t,x,k-\frac{p}{2}\Big) + W\Big(t,x,k+\frac{p}{2}\Big)\Big)\mathrm{d}p \\ &=: \frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\mathcal{K}W \end{aligned} \tag{6}$$

In the weak form this equation reads

$$\langle W_\varepsilon, \lambda\rangle(t) - \langle W_\varepsilon, \lambda\rangle(s) = \int_0^t \left\langle W_\varepsilon, k\cdot\nabla_x\lambda(t,x,k) + \frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\mathcal{K}\lambda(t,x,k)\right\rangle(s)\mathrm{ds} \tag{7}$$

We want to prove tightness of the process. First let us note that it is enough to prove tightness of $\langle W_\varepsilon, \lambda\rangle$. We know that W_ε is uniformly bounded in $L^2(\mathbb{R}^{2d})$, as it comes from a solution of the Schroedinger equation. We can say W_ε is a process in the space $E = \{f \in L^2(\mathbb{R}^{2d}) |\, \|f\|_{L^2} \leq C\}$ which is a compact metrizable space, equipped with the weak topology.

In this section

Proposition 11. *A process X_ε is tight in $C([0,\infty), E)$ if and only if $\langle X_\varepsilon, \lambda\rangle$ is tight in $C([0,\infty), \mathbb{C})$ for every $\lambda \in C_c^\infty(\mathbb{R}^{2d})$.*

Proof. We only prove that if $\langle X_\varepsilon, \lambda\rangle$ is tight in $C([0,\infty), \mathbb{C})$ for every $\lambda \in C_c^\infty(\mathbb{R}^{2d})$ then X_ε is tight in $C([0,\infty), E)$, the other direction is obvious. Recall the Arzela-Ascoli theorem: For a complete metric space (E, d) , $M \subseteq C([0,\infty), E)$ is compact if and only if M is bounded in $C([0,\infty), E)$ and

$$\lim_{\delta\to 0} \sup_{f\in M} \sup_{|s-t|\leq\delta} d(f(t), f(s)) = 0$$

In our case we can choose $d(x, y) = \sum_{n\in\mathbb{N}} \frac{1}{2^n}(\langle x - y, \lambda_n\rangle$ with λ_n a sequence of $C_c^\infty(\mathbb{R}^{2d})$ functions dense in E. By assumption for every n there exists a set $K_n \subseteq C([0,\infty), \mathbb{C})$ such that $\mathbb{P}(\langle X_\varepsilon, \lambda\rangle \in K_n) \geq 1 - \frac{1}{2^n}\delta$ and K_n compact. Writing $\tilde{K}_n = \{f \in C([0,\infty), K) |\, \langle f, \lambda_n\rangle \in K_n\}$ and taking $K = \cap_{n\in\mathbb{N}} \tilde{K}_n$ we get that $P(X_\varepsilon \in K) \geq 1 - \delta$ and that K is compact by Arzela-Ascoli. □

We will prove tightness of $\langle W_\varepsilon, \lambda\rangle$ using the following

Proposition 12. *Let $x_\varepsilon(t)$ be a sequence of stochastic processes with generators $\mathcal{A}^\varepsilon$, such that $\cap_\varepsilon \mathcal{D}(\mathcal{A}^\varepsilon)$ with continuous paths running up to time T. Assume that there exists a dense subset $\hat{C}$ of $C_b(\mathbb{R})$, such that $\hat{C}^2 \subseteq \hat{C}$, in other words $\hat{C}$ contains all squares of functions in $\hat{C}$, $\hat{C} \subseteq \mathcal{D}(\mathcal{A}^\varepsilon)$, and for every $f \in \hat{C}$, there exists a stochastic process $f_\varepsilon(t)$ such that*

$$\{\mathcal{A}^\varepsilon(f_\varepsilon)(t), \varepsilon > 0, t \leq T\}$$

is uniformly integrable, and

$$\lim_{\varepsilon} \mathbb{E}\left(\sup_{t\leq T} |f_\varepsilon(t) - f(x^\varepsilon(t))|\right) = 0 \text{ for every } \delta > 0$$

Then $f(x_\varepsilon)$ is tight for every $f \in \hat{C}$, which implies that x_ε is tight

To prove this we will need the following

Lemma 13. *Let y^ε be a sequence of stochastic processes with continuous paths running up to time T satisfying*

$$\lim_{K\to\infty} \limsup_{\varepsilon} \mathbb{P}\left(\sup_{t\leq T} y^\varepsilon \geq K\right) = 0$$

such that

$$\lim_{\delta\to 0} \limsup_{\varepsilon} \sup_{\tau} \mathbb{E}[\,|y^\varepsilon(\tau+\delta) - y^\varepsilon(\tau)|^2] = 0$$

where $\sup_\tau$ is the supremum over all stopping times $\tau \leq T$.

Then y^ε is tight in $C([0,T], \mathbb{C})$.

A proof of this lemma can be found in [3] Theorem 8.6 and Lemma 10.1.

Proof. Denote by $f_\varepsilon, f_\varepsilon^2$ the sequences associated with f, f^2 respectively which make the generator bounded. Then

$$|f(x^\varepsilon(t+u)) - f(x^\varepsilon(t))|^2 = f_\varepsilon^2(t+u) - f_\varepsilon^2(t) - 2f(x^\varepsilon(t))(f^\varepsilon(t+u) - f^\varepsilon(t)) + T_\varepsilon$$

With

$$T_\varepsilon = f^2(t+u) - f_\varepsilon^2(t+u) + 2f(x^\varepsilon(t))\,(f(x^\varepsilon(t+u)) - f^\varepsilon(t+u) + f(x^\varepsilon(t)) - f^\varepsilon(t))$$

Define $d^2(x,y)=\min\{1,|x-y|^2\}$

Now

$\mathbb{E}d^2(f(x^\varepsilon(\tau+u)), f(x^\varepsilon(\tau))) \leq K\mathbb{E}|\mathbb{E}_\tau f^{2,\varepsilon}(x^\varepsilon(\tau+u)) - f^{2,\varepsilon}(x^\varepsilon(\tau))| + K\mathbb{E}|\mathbb{E}_\tau f^\varepsilon(x^\varepsilon(\tau+u)) - f^\varepsilon(x^\varepsilon(\tau))| + K\mathbb{E}T_\varepsilon = K\mathbb{E}\left|\mathbb{E}_\tau\int_\tau^{\tau+u} \mathcal{A}^\varepsilon f^{2,\varepsilon}(t)\mathrm{d}t\right| + K\mathbb{E}\left|\mathbb{E}_\tau\int_\tau^{\tau+u} \mathcal{A}^\varepsilon f^\varepsilon(t)\mathrm{d}t\right| + K\mathbb{E}T_\varepsilon$

Now all terms on the right hand side, tend to 0 for $u\to 0$ uniformly in τ, which means we can apply the lemma.

□

First we compute the generator of the process: We denote by $\mathbb{E}_t^\varepsilon$ the conditional expectation with respect to

$$\mathcal{A}^\varepsilon f = \lim_{h\to 0}\frac{\mathbb{E}_t^\varepsilon[f(\langle W_\varepsilon,\lambda\rangle(t+h)) - f(\langle W_\varepsilon,\lambda\rangle(t))]}{h} = f'(\langle W_\varepsilon,\lambda\rangle(t))\left\langle W_\varepsilon, k\cdot\nabla_x\lambda + \frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\mathcal{K}\lambda\right\rangle(t)$$

so

$$f(\langle W_\varepsilon,\lambda\rangle(t)) - f(\langle W_\varepsilon,\lambda\rangle(0)) - \int_0^t f'(\langle W_\varepsilon,\lambda\rangle(s))\left\langle W_\varepsilon, k\cdot\nabla_x\lambda + \frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\mathcal{K}\lambda\right\rangle(t)\mathrm{d}s$$

is a martingale.

To apply our proposition we want to add a perturbation $f_{1,\varepsilon}$ such that $f_{1,\varepsilon}\to 0$ for $\varepsilon\to 0$ and $\mathcal{A}^\varepsilon(f+f_{1,\varepsilon})$ is uniformly bounded. To achieve this we obviously want $\mathcal{A}^\varepsilon f_{1,\varepsilon}$ to cancel $\frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\mathcal{K}\lambda$. To find such a function we make the ansatz

$$f_{1,\varepsilon} = \varepsilon^\alpha f'(\langle W_\varepsilon,\lambda\rangle(t))\langle W_\varepsilon,\mathcal{K}_{1,\varepsilon}\lambda\rangle(t)$$

With

$$\mathcal{K}_{1,\varepsilon}\lambda = \int a_\varepsilon(x,k,p)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},p\Big)e^{i\frac{\mathrm{px}}{\varepsilon}}\Big(\lambda\Big(t,x,k-\frac{p}{2}\Big)+\lambda\Big(t,x,k+\frac{p}{2}\Big)\Big)\mathrm{d}p$$

We will choose a_ε and α later in a suitable fashion. We then see that

$$\begin{aligned}\frac{f_{1,\varepsilon}(\langle W_\varepsilon,\lambda\rangle(t+h)) - f_{1,\varepsilon}(\langle W_\varepsilon,\lambda\rangle(t))}{h} &= \varepsilon^\alpha\frac{f'(\langle W_\varepsilon,\lambda\rangle(t+h)) - f'(\langle W_\varepsilon,\lambda\rangle(t))}{h}\langle W_\varepsilon,\mathcal{K}_{1,\varepsilon}\lambda\rangle(t)\\ &+ \varepsilon^\alpha f'(\langle W_\varepsilon,\lambda\rangle(t))\left\langle\frac{W_\varepsilon(t+h,\cdot,\cdot)-W_\varepsilon(t,\cdot,\cdot)}{h},\mathcal{K}_{1,\varepsilon}\lambda(t)\right\rangle\\ &+ \varepsilon^\alpha f'(\langle W_\varepsilon,\lambda\rangle\left\langle W_\varepsilon(t,\cdot),\frac{\mathcal{K}_{1,\varepsilon}\lambda(t+h,\cdot,\cdot)-\mathcal{K}_{1,\varepsilon}\lambda(t,\cdot,\cdot)}{h}\right\rangle\end{aligned}$$

Applying $\mathbb{E}_t$ and taking the limit $h\to 0$ we get, by using the equation and the definition of V that

$\mathcal{A}^\varepsilon f_{1,\varepsilon} = \varepsilon^\alpha f''(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon,\mathcal{K}_{1,\varepsilon}\lambda\rangle(t) + \varepsilon^\alpha f'(\langle W_\varepsilon,\lambda\rangle(t))\left\langle W_\varepsilon, k\cdot\nabla_x\mathcal{K}_{1,\varepsilon}\lambda + \frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\mathcal{K}\mathcal{K}_{1,\varepsilon}\lambda\right\rangle(t) + \varepsilon^\alpha f'(\langle W_\varepsilon,\lambda\rangle(t))\left\langle W_\varepsilon, \int\frac{a_\varepsilon(x,k,p)g(p)}{\varepsilon^{1+\gamma}}\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p\right)\Big(\lambda\Big(t,x,k-\frac{p}{2}\Big)+\lambda\Big(t,x,k+\frac{p}{2}\Big)\Big)\right\rangle$

Let us note that

$k\cdot\nabla_x\mathcal{K}_{1,\varepsilon}\lambda = \int\frac{a_\varepsilon(x,k,p)}{\varepsilon}ik\cdot pe^{i\frac{p\cdot x}{\varepsilon}}\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p\right)\Big(\lambda\Big(t,x,k-\frac{p}{2}\Big)+\lambda\Big(t,x,k+\frac{p}{2}\Big)\Big) + \int a_\varepsilon(x,k,p)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},p\Big)e^{i\frac{\mathrm{px}}{\varepsilon}}\Big(k\cdot\nabla_x\lambda\Big(t,x,k-\frac{p}{2}\Big)+k\cdot\nabla_x\lambda\Big(t,x,k+\frac{p}{2}\Big)\Big)\mathrm{d}p$

The first term must remain bounded since it has a f'' and cannot give cancellations, so we choose $\alpha = \frac{1+\gamma}{2}$. Then we have two terms of higher order which have to cancel $\frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\mathcal{K}\lambda$. More precisely we must have

$$\int \frac{a_\varepsilon(x,k,p)}{\varepsilon^{\frac{1+\gamma}{2}}} \varepsilon^\gamma i k \cdot p e^{i\frac{p\cdot x}{\varepsilon}} \hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p\right)\Big(\lambda\Big(t, x, k-\frac{p}{2}\Big) + \lambda\Big(t, x, k+\frac{p}{2}\Big)\Big) + \int \frac{a_\varepsilon(x,k,p)g(p)}{\varepsilon^{\frac{1+\gamma}{2}}} e^{i\frac{p\cdot x}{\varepsilon}} \hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p\right)\Big(\lambda\Big(t, x, k-\frac{p}{2}\Big) + \lambda\Big(t, x, k+\frac{p}{2}\Big)\Big) = \frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\int \hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p\Big) e^{i\frac{\mathrm{px}}{\varepsilon}}\Big(W\Big(t,x,k-\frac{p}{2}\Big) + W\Big(t,x,k+\frac{p}{2}\Big)\Big)$$

Which is satisfied if $a_\varepsilon(x,k,p) = \frac{1}{g(p)+\varepsilon^\gamma i k\cdot p}$.

This leads to

$$\mathcal{A}^\varepsilon(f+f_{1,\varepsilon}) = f'(\langle W_\varepsilon,\lambda\rangle(t))\langle W_\varepsilon, k\cdot\nabla_x\lambda + \mathcal{K}\mathcal{K}_{1,\varepsilon}\lambda\rangle(t) + f''(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon,\mathcal{K}\lambda\rangle\langle W_\varepsilon,\mathcal{K}_{1,\varepsilon}\lambda\rangle(t) + \varepsilon^{\frac{1+\gamma}{2}} f'(\langle W_\varepsilon,\lambda\rangle(t))\langle W_\varepsilon,\mathcal{K}_{1,\varepsilon}(k\cdot\nabla_x\lambda)\rangle(t) + \varepsilon^{\frac{1+\gamma}{2}} f''(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon, k\cdot\nabla_x\lambda\rangle \tag{8}$$

To prove uniform integrability of the generator we will prove the following lemmas

Lemma 14. *There exists random variables $C_1, C_2, C_3 \in L^2(\mathbb{P})$, such that almost surely*

$$\begin{aligned}\|\mathcal{K}\|_{M_{2,2}^{d+1}\mapsto L^2} &\leq C_{1,\varepsilon}\\ \|\mathcal{K}_{1,\varepsilon}\|_{M_{2,2}^{d+1}\mapsto L^2} &\leq C_{2,\varepsilon}\\ \|\mathcal{K}\mathcal{K}_{1,\varepsilon}\|_{M_{2,2}^{d+1}\mapsto L^2} &\leq C_{3,\varepsilon}\end{aligned}$$

and $\mathbb{E}[C_{1,\varepsilon}^2 + C_{2,\varepsilon}^2 + C_{3,\varepsilon}^2]$ is bounded uniformly in ε.

Proof. We will estimate the Gaussian random variables $\langle\varphi_z, \mathcal{K}\varphi_w\rangle$ in $L^2(\mathbb{P})$, then invoke Lemma 6.

$$\begin{aligned}\mathbb{E}[|\langle\varphi_z,\mathcal{K}\varphi_w\rangle|^2] &= \int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\ \varphi_z(x,k)\varphi_z(y,l)\int \mathbb{E}\Big[\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p\Big)\overline{\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}q\Big)}\Big] e^{i\frac{\mathrm{px}}{\varepsilon}} e^{-i\frac{\mathrm{qy}}{\varepsilon}}\Big(\varphi_w\Big(x, k-\frac{p}{2}\Big) - \varphi_w\Big(x,k+\frac{p}{2}\Big)\Big)\Big(\varphi_w\Big(y, l-\frac{q}{2}\Big) - \varphi_w\Big(y, l+\frac{q}{2}\Big)\Big)\\ &= \int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\varphi_z(x,k)\varphi_z(y,l)\int \mathrm{d}p\hat{R}(p)\, e^{i\frac{\mathrm{px}}{\varepsilon}} e^{-i\frac{\mathrm{py}}{\varepsilon}}\Big(\varphi_w\Big(x,k-\frac{p}{2}\Big) - \varphi_w\Big(x, k+\frac{p}{2}\Big)\Big)\Big(\varphi_w\Big(y,l-\frac{p}{2}\Big) - \varphi_w\Big(y,l+\frac{p}{2}\Big)\Big)\\ &= \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\ \Big|\varphi_z(x,k)\varphi_z(y,l)\int \mathrm{d}p\hat{R}(p)\ \Big(\varphi_w\Big(x,k-\frac{p}{2}\Big) - \varphi_w\Big(x,k+\frac{p}{2}\Big)\Big)\Big(\varphi_w\Big(y,l-\frac{p}{2}\Big) - \varphi_w\Big(y,l+\frac{p}{2}\Big)\Big)\Big|\\ &\lesssim \int \hat{R}(p)\langle p\rangle^{2A}\langle z_1-w_1\rangle^{-2A}\langle z_2-w_2\rangle^{-2A}\end{aligned}$$

On the other hand by the hand we have by Parservals identity we have

$$\begin{aligned}&\int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\varphi_z(x,k)\varphi_z(y,l)\int \mathrm{d}p\hat{R}(p)\ e^{i\frac{\mathrm{px}}{\varepsilon}} e^{-i\frac{\mathrm{py}}{\varepsilon}}\Big(\varphi_w\Big(x,k-\frac{p}{2}\Big) - \varphi_w\Big(x,k+\frac{p}{2}\Big)\Big)\Big(\varphi_w\Big(y,l-\frac{p}{2}\Big) - \varphi_w\Big(y,l+\frac{p}{2}\Big)\Big)\\ &= \int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\widehat{\varphi_z(x,k)}\widehat{\varphi_z(y,l)}\int \mathrm{d}p\hat{R}(p)\Big(\widehat{\varphi_{w-(0,\frac{p}{2},\frac{p}{\varepsilon},0)}}(y,l) - \widehat{\varphi_{w-(0,-\frac{p}{2},\frac{p}{\varepsilon},0)}}(y,l)\Big)\\ &\lesssim \int R(p)\Big\langle z_3-w_3-\frac{p}{\varepsilon}\Big\rangle^{-2A}\langle z_4-w_4\rangle^{-2A}\langle p\rangle^{2A}\end{aligned}$$

Interpolating we get that with $K(z-w)=\int \mathrm{d}pR(p)\langle z_1-w_1\rangle^{-A}\langle z_2-w_2\rangle^{-A}\left\langle z_3-w_3-\frac{p}{\varepsilon}\right\rangle^{-A}\langle z_4-w_4\rangle^{-A}$ and $\int \mathrm{d}wK(z-w)$ uniformly bounded in ε

$$\mathbb{E}[|\langle\varphi_z,\mathcal{K}\varphi_w\rangle|^2]\leq K(z-w)$$

Similarly for $\mathcal{K}_{1,\varepsilon}$ we get

$$\begin{aligned}
\mathbb{E}[|\langle\varphi_z,\mathcal{K}_{1,\varepsilon}\varphi_w\rangle|^2] &= \int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\varphi_z(x,\ k)\varphi_z(y,\ l)\int \frac{1}{g(p)+\varepsilon^{\gamma}ik\cdot p}\mathbb{E}\Big[\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p\Big)\overline{\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}q\Big)}\Big]e^{i\frac{\mathrm{px}}{\varepsilon}}e^{-i\frac{\mathrm{qy}}{\varepsilon}}\Big(\varphi_w\Big(x,\,k-\frac{p}{2}\Big)-\varphi_w\Big(x,\,k+\frac{p}{2}\Big)\Big)\Big(\varphi_w\Big(y,l-\frac{q}{2}\Big)-\varphi_w\Big(y,l+\frac{q}{2}\Big)\Big)\\
&= \int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\varphi_z(x,\ k)\varphi_z(y,\ l)\int \mathrm{d}p\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}ik\cdot p}\ e^{i\frac{\mathrm{px}}{\varepsilon}}e^{-i\frac{\mathrm{py}}{\varepsilon}}\Big(\varphi_w\Big(x,\, k-\frac{p}{2}\Big)-\varphi_w\Big(x,k+\frac{p}{2}\Big)\Big)\Big(\varphi_w\Big(y,l-\frac{p}{2}\Big)-\varphi_w\Big(y,l+\frac{p}{2}\Big)\Big)\\
&\leq \int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\Big|\varphi_z(x,\ k)\varphi_z(y,\ l)\int \mathrm{d}p\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}ik\cdot p}\Big(\varphi_w\Big(x,\,k-\frac{p}{2}\Big)- \varphi_w\Big(x,k+\frac{p}{2}\Big)\Big)\Big(\varphi_w\Big(y,l-\frac{p}{2}\Big)-\varphi_w\Big(y,l+\frac{p}{2}\Big)\Big)\Big|\\
&\lesssim \int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\Big|\varphi_z(x,\ k)\varphi_z(y,\ l)\int \mathrm{d}p\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}ik\cdot p}\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\varphi_w(x, k-tp)\,\mathrm{d}t\Big(\varphi_w\Big(y,l-\frac{p}{2}\Big)-\varphi_w\Big(y,l+\frac{p}{2}\Big)\Big)\Big|\\
&\lesssim \langle w_4\rangle\langle z_1-w_1\rangle^{-A}\langle z_2-w_2\rangle^{-A}
\end{aligned}$$

on the other hand we have integrating by parts

$$\begin{aligned}
&\int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\varphi_z(x,\ k)\varphi_z(y,\ l)\int \mathrm{d}p\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}ik\cdot p}\Big(\varphi_w\Big(x,\ k-\frac{p}{2}\Big)-\varphi_w\Big(x,\ k+\frac{p}{2}\Big)\Big)\Big(\varphi_w\Big(y,l-\frac{p}{2}\Big)-\varphi_w\Big(y,l+\frac{p}{2}\Big)\Big)\\
&= \int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\varphi(x-z_1,\ k-z_2)\varphi(y-z_1,\ l-z_2)\int \mathrm{d}p e^{i\left(w_3-z_3-\frac{p}{\varepsilon}\right)\cdot x}e^{i(w_4-z_4)\cdot k}\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}ik\cdot p}\Big(\varphi\Big(x-w_1,\,k-\frac{p}{2}-w_2\Big)-\varphi\Big(x-w_1,\,k+\frac{p}{2}-w_2\Big)\Big)\Big(\varphi\Big(y-w_1,l-\frac{p}{2}-w_2\Big)-\varphi\Big(y-w_1,l+\frac{p}{2}-w_2\Big)\Big)\\
&= \int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,e^{i\left(w_3-z_3-\frac{p}{\varepsilon}\right)\cdot y}e^{i(w_4-z_4)\cdot l}\varphi(x-z_1,\ k-z_2)\varphi(y-z_1,\ l-z_2)\int \mathrm{d}p\frac{1}{\left|w_3-z_3-\frac{p}{\varepsilon}\right|^A}\frac{1}{|w_4-z_4|^A}\Delta_x^A e^{i\left(w_3-z_3-\frac{p}{\varepsilon}\right)\cdot x}\Delta_k^A e^{i(w_4-z_4)\cdot k}\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}ik\cdot p}\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}il\cdot p}\Big(\varphi\Big(x-w_1,k-\frac{p}{2}-w_2\Big)-\varphi\Big(x-w_1,k+\frac{p}{2}-w_2\Big)\Big)\Big(\varphi\Big(y-w_1,l-\frac{p}{2}-w_2\Big)-\varphi\Big(y-w_1,l+\frac{p}{2}-w_2\Big)\Big)\\
&= \int\int\int\int\int \mathrm{d}p\frac{1}{\left|w_3-z_3-\frac{p}{\varepsilon}\right|^A}\frac{1}{|w_4-z_4|^A}e^{i\left(w_3-z_3-\frac{p}{\varepsilon}\right)\cdot x}e^{i(w_4-z_4)\cdot k}e^{i\left(w_3-z_3-\frac{p}{\varepsilon}\right)\cdot y}e^{i(w_4-z_4)\cdot l}\varphi(y-z_1,\ l-z_2)\Big(\varphi\Big(y-w_1,\ l-\frac{p}{2}-w_2\Big)-\varphi\Big(y-w_1,\ l+\frac{p}{2}-w_2\Big)\Big)\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}il\cdot p}\Delta_x^A\Delta_k^A\left(\frac{\hat{R}(p)}{g(p)+\varepsilon^{\gamma}ik\cdot p}\ \varphi(x-z_1,\ k-z_2)\varphi\Big(x-w_1,\ k-\frac{p}{2}-w_2\Big)-\varphi\Big(x-w_1,k+\frac{p}{2}-w_2\Big)\right)\mathrm{d}x\mathrm{d}k\mathrm{d}y\mathrm{d}l
\end{aligned}$$

$$= \int\iiint \mathrm{d}p \frac{1}{|w_3 - z_3 - \frac{p}{\varepsilon}|^A} \frac{1}{|w_4 - z_4|^A} \Bigg| \varphi(y - z_1, l - z_2)\Big(\varphi\Big(y - w_1, l - \frac{p}{2} - w_2\Big) - \varphi\Big(y - w_1, l + \frac{p}{2} - w_2\Big)\Big) \frac{\hat{R}(p)}{g(p) + \varepsilon^\gamma i l \cdot p} \Delta_x^A \Delta_k^A \Bigg(\frac{\hat{R}(p)}{g(p) + \varepsilon^\gamma i k \cdot p} \varphi(x - z_1, k - z_2)\Big(\varphi\Big(x - w_1, k - \frac{p}{2} - w_2\Big) - \varphi\Big(x - w_1, k + \frac{p}{2} - w_2\Big)\Big)\Bigg)\Bigg| \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k$$

$$\lesssim \int \mathrm{d}p \frac{1}{|w_3 - z_3 - \frac{p}{\varepsilon}|^A} \frac{1}{|w_4 - z_4|^A} \Bigg(\frac{\hat{R}(p)}{|g(p)|}|p| + \frac{\hat{R}(p)}{|g(p)|^{2A+1}}|p|^{2A+1} \Bigg) \langle z_1 - w_1\rangle^{-A} \langle z_2 - w_2\rangle^{-A}$$

By combining this we the above estimate we get

$$\mathbb{E}[|\langle \varphi_z, \mathcal{K}_{1,\varepsilon}\varphi_w\rangle|^2] \lesssim \int \mathrm{d}p \Big\langle z_3 - w_3 - \frac{p}{\varepsilon}\Big\rangle^{-A} \langle z_4 - w_4\rangle^{-A} \Bigg(\frac{\hat{R}(p)}{|g(p)|}|p| + \frac{\hat{R}(p)}{|g(p)|^{2A+1}}|p|^{2A+1} \Bigg) \langle z_1 - w_1\rangle^{-A} \langle z_2 - w_2\rangle^{-A}$$

We have $\mathcal{K}\mathcal{K}_{1,\varepsilon} f(x, k) = \int \int \hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_1\Big) \hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_2\Big) e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}} \Bigg(\frac{1}{g(p_2) - i\varepsilon^\gamma(k - \frac{p_1}{2})p_2} \Big(f\Big(x, k - \frac{p_1}{2} - \frac{p_2}{2}\Big) - f\Big(x, k - \frac{p_1}{2} + \frac{p_2}{2}\Big)\Big) - \Big(\frac{1}{g(p_2) - i\varepsilon^\gamma(k + \frac{p_1}{2})p_2} \Big(f\Big(x, k + \frac{p_1}{2} - \frac{p_2}{2}\Big) - f\Big(x, k + \frac{p_1}{2} - \frac{p_2}{2}\Big)\Big)\Big)\Bigg)$

So

$$\begin{aligned}
\mathbb{E}[|\langle \varphi_z, \mathcal{K}\mathcal{K}_{1,\varepsilon}\varphi_w\rangle|^2] &= \int\int\int\int \varphi_z(x, k)\varphi_z(y, l) \int\int\int e^{-i\frac{(q_1+q_2)\cdot y}{\varepsilon}} e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}} \mathbb{E}\Big[\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}q_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}q_2\Big) \overline{\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_2\Big)}\Big] \Bigg(\frac{1}{g(q_2) - i\varepsilon^\gamma(l - \frac{q_1}{2})q_2} \Big(\varphi_w\Big(y, l - \frac{q_1}{2} - \frac{q_2}{2}\Big) - \varphi_w\Big(y, l - \frac{q_1}{2} + \frac{q_2}{2}\Big)\Big) - \Big(\frac{1}{g(q_2) - i\varepsilon^\gamma(l + \frac{q_1}{2})q_2} \Big(\varphi_w\Big(y, l + \frac{q_1}{2} - \frac{q_2}{2}\Big) - \varphi_w\Big(y, l + \frac{q_1}{2} + \frac{q_2}{2}\Big)\Big)\Big)\Bigg) \Bigg(\frac{1}{g(p_2) - i\varepsilon^\gamma(k - \frac{p_1}{2})p_2} \Big(\varphi_w\Big(x, k - \frac{p_1}{2} - \frac{p_2}{2}\Big) - \varphi_w\Big(x, k - \frac{p_1}{2} + \frac{p_2}{2}\Big)\Big) - \Big(\frac{1}{g(p_2) - i\varepsilon^\gamma(k + \frac{p_1}{2})p_2} \Big(\varphi_w\Big(x, k + \frac{p_1}{2} - \frac{p_2}{2}\Big) - \varphi_w\Big(x, k + \frac{p_1}{2} + \frac{p_2}{2}\Big)\Big)\Big)\Bigg) \mathrm{d}x\mathrm{d}k\mathrm{d}y\mathrm{d}l \\
&= \int\int \mathrm{d}x\mathrm{d}k \int\int \mathrm{d}y\mathrm{d}l\, \varphi_z(x, k)\varphi_z(y, l) \int\int \mathrm{d}p\mathrm{d}q \hat{R}(p)\hat{R}(q) \Bigg(\frac{1}{g(q) - i\varepsilon^\gamma(l - \frac{q}{2})q} (\varphi_w(y, l - q) - \varphi_w(y, l)) - \Bigg(\frac{1}{g(q_2) - i\varepsilon^\gamma(l + \frac{q}{2})q_2} (\varphi_w(y, l) - \varphi_w(y, l + q))\Bigg)\Bigg) \Bigg(\frac{1}{g(p_2) - i\varepsilon^\gamma(k - \frac{p}{2})p} (\varphi_w(x, k - p) - \varphi_w(x, k)) - \Bigg(\frac{1}{g(p_2) - i\varepsilon^\gamma(k + \frac{p_1}{2})p_2} (\varphi_w(x, k) - \varphi_w(x, k + p))\Bigg)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\ \varphi_z(x,k)\varphi_z(y,l)\int\int \mathrm{d}p_1\mathrm{d}p_2\hat{R}(p_1)\hat{R}(p_2)e^{i\frac{(p_1+p_2)\cdot(x-y)}{\varepsilon}}\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(l-\frac{p_1}{2}\big)p_2}\Big(\varphi_w\Big(y,l-\frac{p_1}{2}-\frac{p_2}{2}\Big)-\varphi_w\Big(y,l-\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(l+\frac{p_1}{2}\big)p_2}\Big(\varphi_w\Big(y,l+\frac{p_1}{2}-\frac{p_2}{2}\Big)-\varphi_w\Big(y,l+\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg)\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k-\frac{p_1}{2}\big)p_2}\Big(\varphi_w\Big(x,k-\frac{p_1}{2}-\frac{p_2}{2}\Big)-\varphi_w\Big(x,k-\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}\Big(\varphi_w\Big(x,k+\frac{p_1}{2}-\frac{p_2}{2}\Big)-\varphi_w\Big(x,k+\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg)\\
&+\int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\ \varphi_z(x,k)\varphi_z(y,l)\int\int \hat{R}(p_1)\hat{R}(p_2)\mathrm{d}p_1\mathrm{d}p_2e^{i\frac{(p_1+p_2)\cdot(x+y)}{\varepsilon}}\Bigg(\frac{1}{g(p_1)-i\varepsilon^\gamma\big(l-\frac{p_2}{2}\big)p_1}\Big(\varphi_w\Big(y,l-\frac{p_1}{2}-\frac{p_2}{2}\Big)-\varphi_w\Big(y,l-\frac{p_2}{2}+\frac{p_1}{2}\Big)\Big)-\Bigg(\frac{1}{g(p_1)-i\varepsilon^\gamma\big(l+\frac{p_2}{2}\big)p_1}\Big(\varphi_w\Big(y,l+\frac{p_2}{2}-\frac{p_1}{2}\Big)-\varphi_w\Big(y,l+\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg)\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k-\frac{p_1}{2}\big)p_2}\Big(\varphi_w\Big(x,k-\frac{p_1}{2}-\frac{p_2}{2}\Big)-\varphi_w\Big(x,k-\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}\Big(\varphi_w\Big(x,k+\frac{p_1}{2}-\frac{p_2}{2}\Big)-\varphi_w\Big(x,k+\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg)\\
&=\mathrm{I}+\mathrm{II}+\mathrm{III}
\end{aligned}
$$

Now we proceed with each term similarly to the previous computations:

$$
\begin{aligned}
I&=\int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\varphi(x-z_1,k-z_2)\varphi_z(y-z_1,l-z_2)e^{ix\cdot z_3}e^{ik\cdot z_4}\int\int \mathrm{d}p\mathrm{d}q\hat{R}(p)\hat{R}(q)\Bigg(\frac{1}{g(q)-i\varepsilon^\gamma\big(l-\frac{q}{2}\big)q}(\varphi_w(y,l-q)-\varphi_w(y,l))-\Bigg(\frac{1}{g(q_2)-i\varepsilon^\gamma\big(l+\frac{q}{2}\big)q_2}(\varphi_w(y,l)-\varphi_w(y,l+q))\Bigg)\Bigg)\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k-\frac{p}{2}\big)p}(e^{-ix\cdot w_3}e^{-i(k-p)\cdot w_4}\varphi(x-w_1,k-p-w_2)-e^{-ix\cdot w_3}e^{-ik\cdot w_4}(x-w_1,k-w_4))-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}(e^{-ik\cdot w_4}e^{-ix\cdot w_3}\varphi_w(x,k)-e^{-i(k+p)\cdot w_4}e^{-ix\cdot w_3}\varphi_w(x,k+p))\Bigg)\Bigg)\\
&=\int\int \mathrm{d}x\mathrm{d}k\int\int \mathrm{d}y\mathrm{d}l\,\varphi(x-z_1,k-z_2)\varphi_z(y-z_1,l-z_2)\frac{1}{|w_3-z_3|^A}\frac{1}{|w_4-z_4|^A}\Delta_k^A\Delta_x^Ae^{ix\cdot(z_3-w_3)}e^{ik\cdot(z_4-w_4)}\int\int \mathrm{d}p\mathrm{d}q\hat{R}(p)\hat{R}(q)\Bigg(\frac{1}{g(q)-i\varepsilon^\gamma\big(l-\frac{q}{2}\big)q}(\varphi_w(y,l-q)-\varphi_w(y,l))-\Bigg(\frac{1}{g(q_2)-i\varepsilon^\gamma\big(l+\frac{q}{2}\big)q_2}(\varphi_w(y,l)-\varphi_w(y,l+q))\Bigg)\Bigg)\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k-\frac{p}{2}\big)p}(e^{ip\cdot w_4}\varphi(x-w_1,k-p-w_2)-(x-w_1,k-w_4))-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}(\varphi(x,k)-e^{-ip\cdot w_4}\varphi(x,k+p))\Bigg)\Bigg)
\end{aligned}
$$

$$= \iiiint \varphi(x-z_1,k-z_2)\varphi_z(y-z_1,l-z_2)\frac{1}{|w_3-z_3|^A}\frac{1}{|w_4-z_4|^A}\Delta_k^A\Delta_x^A e^{ix\cdot(z_3-w_3)}e^{ik\cdot(z_4-w_4)}\iint \mathrm{d}p\mathrm{d}q\hat{R}(p)\hat{R}(q)\Bigg(\frac{1}{g(q)-i\varepsilon^\gamma\big(l-\frac{q}{2}\big)q}(\varphi_w(y,l-q)-\varphi_w(y,l))-\Bigg(\frac{1}{g(q_2)-i\varepsilon^\gamma\big(l+\frac{q}{2}\big)q_2}(\varphi_w(y,l)-\varphi_w(y,l+q))\Bigg)\Bigg)\Delta_k^A\Delta_x^A\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k-\frac{p}{2}\big)p}\big(e^{ip\cdot w_4}\varphi(x-w_1,k-p-w_2)-(x-w_1,k-w_4)\big)-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}(\varphi(x,k)-e^{-ip\cdot w_4}\varphi(x,k+p))\Bigg)\Bigg)\mathrm{d}x\mathrm{d}k\mathrm{d}y\mathrm{d}l$$

$$\lesssim \iint \mathrm{d}p\mathrm{d}q\sum_{k=1}^{A}\frac{\hat{R}(p)}{|g(p)|^k}|p|^k\frac{\hat{R}(q)}{|g(q)|^k}|q|^k\frac{1}{|w_3-z_3|^A}\frac{1}{|w_4-z_4|^A}\langle z_1-w_1\rangle^{-A}\langle z_2-w_2\rangle^{-A}\langle p\rangle^{2A}\langle q\rangle$$

Taking absolute values right away we get

$$\mathrm{I}\lesssim\iint \mathrm{d}p\mathrm{d}q\frac{\hat{R}(p)}{|g(p)|}|p|\frac{\hat{R}(q)}{|g(q)|}|q|\langle z_1-w_1\rangle^{-A}\langle z_2-w_2\rangle^{-A}\langle p\rangle^{2A}\langle w_3\rangle^2$$

so in total

$$\mathrm{I}\lesssim\iint \mathrm{d}p\mathrm{d}q\sum_{k=1}^{A}\frac{\hat{R}(p)}{|g(p)|^k}|p|^k\frac{\hat{R}(q)}{|g(q)|^k}|q|^k\langle z_1-w_1\rangle^{-A}\langle z_2-w_2\rangle^{-A}\langle z_3-w_3\rangle^{-A}\langle z_4-w_4\rangle^{-A}\langle p\rangle^{2A}\langle q\rangle^{2A}\langle w_3\rangle^2$$

By the same computations we get for the other terms :

$$\mathrm{II}\lesssim\iint \mathrm{d}p\mathrm{d}q\sum_{k=1}^{A}\frac{\hat{R}(p)}{|g(p)|^k}|p|^k\frac{\hat{R}(q)}{|g(q)|^k}|q|^k\langle z_1-w_1\rangle^{-A}\langle z_2-w_2\rangle^{-A}\Big\langle z_3-w_3-\frac{p}{\varepsilon}\Big\rangle^{-A}\langle z_4-w_4\rangle^{-A}\langle p\rangle^{2A}\langle q\rangle^{2A}\langle w_3\rangle^2$$

$$\mathrm{III}\lesssim\iint \mathrm{d}p\mathrm{d}q\sum_{k=1}^{A}\frac{\hat{R}(p)}{|g(p)|^k}|p|^k\frac{\hat{R}(q)}{|g(q)|^k}|q|^k\langle z_1-w_1\rangle^{-A}\langle z_2-w_2\rangle^{-A}\Big\langle z_3-w_3-\frac{p}{\varepsilon}\Big\rangle^{-A}\langle z_4-w_4\rangle^{-A}\langle p\rangle^{2A}\langle q\rangle^{2A}\langle w_3\rangle^2$$

□

So now we are ready to prove :

Proposition 15. $\mathcal{A}^\varepsilon(f+f_{1,\varepsilon})$ *is uniformly integrable for any* $f\in C^3(\mathbb{R})$.

Proof. We have that

$$\begin{aligned}\mathcal{A}^\varepsilon(f+f_{1,\varepsilon}) &= f'(\langle W_\varepsilon,\lambda\rangle(t))\langle W_\varepsilon,k\cdot\nabla_x\lambda+\mathcal{K}\mathcal{K}_{1,\varepsilon}\lambda\rangle(t)\\ &+ f''(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon,\mathcal{K}\lambda\rangle\langle W_\varepsilon,\mathcal{K}_{1,\varepsilon}\lambda\rangle(t)\\ &+ \varepsilon^{\frac{1+\gamma}{2}}f'(\langle W_\varepsilon,\lambda\rangle(t))\langle W_\varepsilon,\mathcal{K}_{1,\varepsilon}(k\cdot\nabla_x\lambda)\rangle(t)\\ &+ \varepsilon^{\frac{1+\gamma}{2}}f''(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon,k\cdot\nabla_x\lambda\rangle\end{aligned}$$

We will prove that this bounded in $L^2(\mathbb{P})$ so it is uniformly integrable. The only term we have to worry about is

$$f''(\langle W_\varepsilon, \lambda\rangle)\langle W_\varepsilon, \mathcal{K}\lambda\rangle\langle W_\varepsilon, \mathcal{K}_{1,\varepsilon}\lambda\rangle(t)$$

since for the others the bound follows from the uniform boundedness of W_ε and Lemma 14. For the remaining term we remark that

$$\mathbb{E}|\langle\varphi_z\otimes\varphi_{\bar{z}}, \mathcal{K}\otimes\mathcal{K}_{1,\varepsilon}\,\varphi_w\otimes\varphi_{\bar{w}}\rangle| = |\langle\varphi_z, \mathcal{K}_1\varphi_w\rangle\langle\varphi_{\bar{z}}, \mathcal{K}_{1,\varepsilon}\varphi_{\bar{w}}\rangle|$$

As $\hat{V}$ is a gaussian field $\langle\varphi_z, \mathcal{K}\varphi_w\rangle, \langle\varphi_{\bar{z}}, \mathcal{K}_{1,\varepsilon}\varphi_{\bar{w}}\rangle$ are gaussian variables so by hypercontractivity

$$\begin{aligned}\mathbb{E}|\langle\varphi_z, \mathcal{K}\varphi_w\rangle\langle\varphi_{\bar{z}}, \mathcal{K}_{1,\varepsilon}\varphi_{\bar{w}}\rangle|^2 &\leq (\mathbb{E}|\langle\varphi_z, \mathcal{K}\varphi_w\rangle|^4)^{\frac{1}{2}}(\mathbb{E}|\langle\varphi_{\bar{z}}, \mathcal{K}_{1,\varepsilon}\varphi_{\bar{w}}\rangle|^4)^{\frac{1}{2}}\\ &\leq C\mathbb{E}|\langle\varphi_z, \mathcal{K}\varphi_w\rangle|^2\mathbb{E}|\langle\varphi_{\bar{z}}, \mathcal{K}_{1,\varepsilon}\varphi_{\bar{w}}\rangle|^2\end{aligned}$$

which can be estimated in terms of $K\,(z-w)K(\bar{z}-\bar{w})$ with K like in the proof of Lemma 14. Now invoking Lemma 6 shows that $\mathcal{K}\otimes\mathcal{K}_{1,\varepsilon}$ is almost surely bounded with the operator norm uniformly bounded in $L^2\,(\mathbb{P})$, which completes the proof. □

5 Identification of the Limit

To identify the limit we need to get rid of the randomness seen in to do this we have to correctors once again. First we want to find a corrector $f_{3,\varepsilon}$ such that $f_{32\varepsilon} = \mathcal{O}(\varepsilon^{1+\gamma})$ and $\mathcal{A}^\varepsilon f_{2,\varepsilon} = f'(\langle W_\varepsilon, \lambda\rangle(t))\langle W_\varepsilon, -\mathcal{K}\mathcal{K}_{1,\varepsilon}\lambda + D_\varepsilon\lambda\rangle(t) + \mathcal{O}\Big(\varepsilon^{\frac{1+\gamma}{2}}\Big)$ with some deterministic operator D_ε such that we can more easily compute the limit of D_ε.

Recall that we have

$$\begin{aligned}\mathcal{K}\mathcal{K}_{1,\varepsilon}f(x,k) = &\iint\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_2\Big)e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}}\Bigg(\frac{1}{g(p_2) - i\varepsilon^\gamma\big(k-\frac{p_1}{2}\big)p_2}\Big(f\Big(x, k-\frac{p_1}{2}-\\ &\frac{p_2}{2}\Big) - f\Big(x, k-\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big) - \Bigg(\frac{1}{g(p_2) - i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}\Big(f\Big(x, k+\frac{p_1}{2}-\frac{p_2}{2}\Big) - f\Big(x,\\ &k+\frac{p_1}{2}-\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg)\end{aligned}$$

So for following the same principle as before we make the ansatz

$$f_{2,\varepsilon}(t) = \varepsilon^{1+\gamma}f'(\langle W_\varepsilon, \lambda\rangle(t))\langle W_\varepsilon, H_{1,\varepsilon}\lambda\rangle(t)$$

with

$$\begin{aligned}\mathcal{H}_{1,\varepsilon}\lambda = &\int\int a_\varepsilon(k,\ p)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\ \mathrm{d}p_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\ \mathrm{d}p_2\Big)e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}}\Bigg(\frac{1}{g(p_2) - i\varepsilon^\gamma\big(k-\frac{p_1}{2}\big)p_2}\Big(f\Big(x,\ k\ -\\ &\frac{p_1}{2}-\frac{p_2}{2}\Big) - f\Big(x, k-\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big) - \Bigg(\frac{1}{g(p_2) - i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}\Big(f\Big(x, k+\frac{p_1}{2}-\frac{p_2}{2}\Big) - f\Big(x,\\ &k+\frac{p_1}{2}-\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg)\end{aligned}$$

$$\lim_{h\to 0}\frac{\mathbb{E}_\varepsilon^t[f_{2,\varepsilon}(t+h) - f_{2,\varepsilon}(t)]}{h} = \lim_{h\to 0}\varepsilon^{1+\gamma}\frac{f'(\langle W_\varepsilon, \lambda\rangle(t+h)) - f'(\langle W_\varepsilon, \lambda\rangle(t))}{h}\langle W_\varepsilon, \mathcal{H}_{1,\varepsilon}\lambda\rangle(t)$$

$$+ \lim_{h\to 0} \varepsilon^{1+\gamma} f'(\langle W_\varepsilon, \lambda\rangle \left\langle \frac{\mathbb{E}_\varepsilon^t[W_\varepsilon(t+h,\cdot,\cdot)-W_\varepsilon(t,\cdot,\cdot)]}{h}, \mathcal{K}_{1,\varepsilon}\lambda(t)\right\rangle$$
$$+ \lim_{h\to 0} \varepsilon^{1+\gamma} f'(\langle W_\varepsilon, \lambda\rangle(t)) \left\langle W_\varepsilon(t,\cdot,\cdot), \frac{\mathcal{H}_{1,\varepsilon}\lambda(t+h,\cdot,\cdot)-\mathcal{H}_{1,\varepsilon}\lambda(t,\cdot,\cdot)}{h}\right\rangle$$
$$= \varepsilon^{1+\gamma} f'(\langle W_\varepsilon, \lambda\rangle(t)) \langle W_\varepsilon, \mathcal{H}_{1,\varepsilon}\lambda\rangle(t)$$
$$+ \lim_{h\to 0} \varepsilon^{1+\gamma} f'(\langle W_\varepsilon, \lambda\rangle(t)) \left\langle W_\varepsilon, k\cdot\nabla_x \mathcal{H}_{1,\varepsilon}\lambda + \frac{1}{\varepsilon^{\frac{1+\gamma}{2}}}\mathcal{K}\mathcal{H}_{1,\varepsilon}\lambda\right\rangle$$
$$+ \lim_{h\to 0} \varepsilon^{1+\gamma} f'(\langle W_\varepsilon, \lambda\rangle(t)) \left\langle W_\varepsilon(t,\cdot,\cdot), \frac{\mathbb{E}_\varepsilon^t[\mathcal{H}_{1,\varepsilon}\lambda(t+h,\cdot,\cdot)-\mathcal{H}_{1,\varepsilon}\lambda(t,\cdot,\cdot)]}{h}\right\rangle$$

By the definition of V we have

$$\lim_{h\to 0} \frac{\mathbb{E}_\varepsilon^t[\mathcal{H}_{1,\varepsilon}\lambda(t+h,\cdot,\cdot)-\mathcal{H}_{1,\varepsilon}\lambda(t,\cdot,\cdot)]}{h} = \lim_{h\to 0} \iint a_\varepsilon(k, p_1, p_2)\left(\frac{\mathbb{E}_\varepsilon^t\left[\hat{V}\left(\frac{t+h}{\varepsilon^{1+\gamma}},\mathrm{d}p_1\right)\hat{V}\left(\frac{t+h}{\varepsilon^{1+\gamma}},\mathrm{d}p_2\right)\right]-\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_1\right)\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_2\right)}{h}\right)e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}}$$
$$\left(\frac{1}{g(p_2)-i\varepsilon^\gamma\left(k-\frac{p_1}{2}\right)p_2}\left(\lambda\left(x,k-\frac{p_1}{2},-\frac{p_2}{2}\right)-\lambda\left(x,k-\frac{p_1}{2}+\frac{p_2}{2}\right)\right)-\left(\frac{1}{g(p_2)-i\varepsilon^\gamma\left(k+\frac{p_1}{2}\right)p_2}\left(\lambda\left(x,k+\frac{p_1}{2}-\frac{p_2}{2}\right)-\lambda\left(x,k+\frac{p_1}{2}+\frac{p_2}{2}\right)\right)\right)\right)$$
$$=\varepsilon^{1+\gamma}\lim_{h\to 0}\iint a_\varepsilon(x,k,p_1,p_2)e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}}$$
$$\left(\frac{e^{\frac{-g(p_1)h-g(p_2)h}{\varepsilon^{1+\gamma}}}\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_1\right)\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_2\right)-\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_1\right)\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_2\right)+\hat{R}(p_1)\left(1-e^{\frac{-2g(p_1)h}{\varepsilon^{1+\gamma}}}\right)\delta(p_1+p_2)}{h}\right)$$
$$\left(\frac{1}{g(p_2)-i\varepsilon^\gamma\left(k-\frac{p_1}{2}\right)p_2}\left(\lambda\left(x,k-\frac{p_1}{2}-\frac{p_2}{2}\right)-\lambda\left(x,k-\frac{p_1}{2}+\frac{p_2}{2}\right)\right)-\left(\frac{1}{g(p_2)-i\varepsilon^\gamma\left(k+\frac{p_1}{2}\right)p_2}\left(f\left(x,k+\frac{p_1}{2}-\frac{p_2}{2}\right)-f\left(x,k+\frac{p_1}{2}+\frac{p_2}{2}\right)\right)\right)\right)$$
$$= -\iint a_\varepsilon(x, k, p_1, p_2)(g(p_1)+g(p_2))e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}}\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_1\right)\hat{V}\left(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_2\right)\left(\frac{1}{g(p_2)-i\varepsilon^\gamma\left(k-\frac{p_1}{2}\right)p_2}\left(\lambda\left(x,k-\frac{p_1}{2}-\frac{p_2}{2}\right)-\lambda\left(x,k-\frac{p_1}{2}+\frac{p_2}{2}\right)\right)-\left(\frac{1}{g(p_2)-i\varepsilon^\gamma\left(k+\frac{p_1}{2}\right)p_2}\left(\lambda\left(x, k+\frac{p_1}{2}-\frac{p_2}{2}\right)-\lambda\left(x,k+\frac{p_1}{2}+\frac{p_2}{2}\right)\right)\right)\right)+\int \mathrm{d}p\hat{R}(p)a_\varepsilon(x,k,p,-p)g(p)\left(\frac{1}{g(p)-i\varepsilon^\gamma\left(k-\frac{p}{2}\right)p}(f(x,k-p)-f(x,k))-\left(\frac{1}{g(p)-i\varepsilon^\gamma\left(k+\frac{p}{2}\right)p}(f(x,k)-f(x,k-p))\right)\right)$$

We observe that in the case $\gamma=0$ we also have another $\mathcal{O}(1)$ term coming from $\varepsilon^{1+\gamma}\langle W_\varepsilon,k\cdot\nabla_x\mathcal{H}_{1,\varepsilon}\lambda\rangle$

as

$$\varepsilon^{1+\gamma}k\cdot\nabla_x\mathcal{H}_{1,\varepsilon}\lambda = \iint k\cdot\nabla_x a_\varepsilon(k,p)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_2\Big)e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}}\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k-\frac{p_1}{2}\big)p_2}\Big(\lambda\Big(x,k-\frac{p_1}{2}-\frac{p_2}{2}\Big)-\lambda\Big(x,k-\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}\Big(\lambda\Big(x,k+\frac{p_1}{2}-\frac{p_2}{2}\Big)-\lambda\Big(x,k+\frac{p_1}{2}-\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg) = \iint a_\varepsilon(k,p_1,p_2)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_2\Big)\frac{i(p_1+p_2)}{\varepsilon}e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}}\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k-\frac{p_1}{2}\big)p_2}\Big(\lambda\Big(x,k-\frac{p_1}{2}-\frac{p_2}{2}\Big)-\lambda\Big(x,k-\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}\Big(\lambda\Big(x,k+\frac{p_1}{2}-\frac{p_2}{2}\Big)-\lambda\Big(x,k+\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg)+\mathcal{H}_{1,\varepsilon}(k\cdot\nabla_x\lambda)$$

This means we want

$$(-\varepsilon^\gamma ik\cdot(p_1+p_2)+(g(p_1)+g(p_2)))a_\varepsilon(k,p_1,p_2)=1\Rightarrow a_\varepsilon(k,p_1,p_2)=\frac{1}{g(p_1)+g(p_2)-\varepsilon^\gamma ik\cdot(p_1+p_2)}$$

To conclude that we have a suitable corrector we have to prove bounds on $\mathcal{H}_{1,\varepsilon},\mathcal{K}\mathcal{H}_{1,\varepsilon}$. That will be the content of the following lemmas

Lemma 16. *There exist random variables $C_{4,\varepsilon}>0$ such that $C_{4,\varepsilon}\in L^2(\mathbb{P})$ and $\sup_\varepsilon\mathbb{E}[C_{4,\varepsilon}^2]<\infty$ while*

$$\|\mathcal{H}_{1,\varepsilon}\lambda\|_{L^2}\leq C_{4,\varepsilon}\|\lambda\|_{\mathcal{M}^A}$$

Proof. We again estimate $\mathbb{E}[|\langle\varphi_z,\mathcal{H}_{1,\varepsilon}\varphi_w\rangle|^2]$

$$\begin{aligned}\mathbb{E}[|\langle\varphi_z,\mathcal{H}_{1,\varepsilon}\varphi_w\rangle|^2] = &\iiiint\varphi_z(x,k)\overline{\varphi_z(y,l)}\iiiint e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}}\\ &\mathbb{E}\Big[\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}q_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}q_2\Big)\overline{\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}},\mathrm{d}p_2\Big)}\Big]\\ &\frac{1}{g(p_1)+g(p_2)-\varepsilon^\gamma ik\cdot(p_1+p_2)}\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k-\frac{p_1}{2}\big)p_2}\Big(\overline{\varphi_w}\Big(x,k-\frac{p_1}{2}-\frac{p_2}{2}\Big)-\overline{\varphi_w}\Big(x,k-\frac{p_1}{2}+\frac{p_2}{2}\Big)\Big)-\Bigg(\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k+\frac{p_1}{2}\big)p_2}\Big(\overline{\varphi_w}\Big(x,k+\frac{p_1}{2}-\frac{p_2}{2}\Big)-\overline{\varphi_w}\Big(x,k+\frac{p_1}{2}-\frac{p_2}{2}\Big)\Big)\Bigg)\Bigg)\\ &\frac{1}{g(q_1)+g(q_2)-\varepsilon^\gamma il\cdot(q_1+q_2)}e^{i\frac{(q_1+q_2)\cdot x}{\varepsilon}}\\ &\Bigg(\frac{1}{g(q_2)-i\varepsilon^\gamma\big(l-\frac{q_1}{2}\big)q_2}\Big(\varphi_w\Big(y,l-\frac{q_1}{2}-\frac{q_2}{2}\Big)-\varphi_w\Big(y,l-\frac{q_1}{2}+\frac{q_2}{2}\Big)\Big)-\Bigg(\frac{1}{g(q_2)-i\varepsilon^\gamma\big(l+\frac{q_1}{2}\big)q_2}\Big(\varphi_w\Big(y,l+\frac{q_1}{2}-\frac{q_2}{2}\Big)-\varphi_w\Big(y,l+\frac{q_1}{2}+\frac{q_2}{2}\Big)\Big)\Bigg)\Bigg)\mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\end{aligned}$$

$$
\begin{aligned}
= & \iiiint \varphi_z(x,k)\overline{\varphi_z(y,l)} \iiiint e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}} \\
& \mathbb{E}\Big[\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}q_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}q_2\Big)\overline{\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_2\Big)}\Big] \\
& \frac{1}{g(p_1)+g(p_2)-\varepsilon^{\gamma} i k\cdot(p_1+p_2)} e^{-i\frac{(p_1+p_2)\cdot x}{\varepsilon}} \\
& \Bigg(\frac{1}{g(p_2)-i\varepsilon^{\gamma}\big(k-\frac{p_1}{2}\big)p_2}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_2\cdot\nabla_k\overline{\varphi_w}\Big(x,\ k+\frac{p_1}{2}-t\frac{p_2}{2}\Big)\mathrm{d}t - \\
& \Bigg(\frac{1}{g(p_2)-i\varepsilon^{\gamma}\big(k+\frac{p_1}{2}\big)p_2}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_2\cdot\nabla_k\overline{\varphi_w}\Big(x,k-\frac{p_1}{2}+t\frac{p_2}{2}\Big)\mathrm{d}t\Bigg)\Bigg) \\
& \frac{1}{g(q_1)+g(q_2)-\varepsilon^{\gamma} i k\cdot(q_1+q_2)} e^{i\frac{(q_1+q_2)\cdot x}{\varepsilon}} \\
& \Bigg(\frac{1}{g(q_2)-i\varepsilon^{\gamma}\big(l-\frac{q_1}{2}\big)q_2}\Big(\varphi_w\Big(y,\ l-\frac{q_1}{2}-\frac{q_2}{2}\Big)-\varphi_w\Big(y,\ l-\frac{q_1}{2}+ \\
& \frac{q_2}{2}\Big)\Big)-\Bigg(\frac{1}{g(q_2)-i\varepsilon^{\gamma}\big(l+\frac{q_1}{2}\big)q_2}\Big(\varphi_w\Big(y,\ l+\frac{q_1}{2}-\frac{q_2}{2}\Big)-\varphi_w\Big(y,\ l+\frac{q_1}{2}+ \\
& \frac{q_2}{2}\Big)\Big)\Bigg)\Bigg)\mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k \\
= & \iiint \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\,\varphi_z(x,k)\overline{\varphi_z(y,l)} \iiiint e^{i\frac{(p_1+p_2)\cdot x}{\varepsilon}} \\
& \mathbb{E}\Big[\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}q_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}q_2\Big)\overline{\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_1\Big)\hat{V}\Big(\frac{t}{\varepsilon^{1+\gamma}}, \mathrm{d}p_2\Big)}\Big] \\
& \frac{1}{g(p_1)+g(p_2)-\varepsilon^{\gamma} i k\cdot(p_1+p_2)} e^{-i\frac{(p_1+p_2)\cdot x}{\varepsilon}} \\
& \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_1\nabla_k\Big(\frac{1}{g(p_2)-i\varepsilon^{\gamma}(k-s p_1)\cdot p_2} p_2\cdot\nabla_k\overline{\varphi_w}(x,\ k-s p_1+ \\
& t p_2)\mathrm{d}t\mathrm{d}s\Big)\frac{1}{g(q_1)+g(q_2)-\varepsilon^{\gamma} i l\cdot(q_1+q_2)} e^{-i\frac{(q_1+q_2)\cdot x}{\varepsilon}} \\
& \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} q_1\nabla_k\Big(\frac{1}{g(q_2)-i\varepsilon^{\gamma}(l-s q_1)\cdot q_2} q_2\cdot\nabla_k\overline{\varphi_w}(x,l-s q_1+t q_2)\mathrm{d}t\mathrm{d}s\Big) \\
= & \iiiint \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\,\varphi_z(x,\ k)\overline{\varphi_z(y,l)} \iiiint e^{-i\frac{(p_1+p_2)\cdot x}{\varepsilon}}\big(\hat{R}(p_1)\hat{R}(q_1)\delta(p_1+ \\
& p_2)\delta(q_1+q_2)+\hat{R}(p_1)\hat{R}(p_2)\delta(p_1-q_1)\delta(p_2-q_2)+\hat{R}(p_1)\hat{R}(p_2)\delta(p_1-q_2)\delta(q_1- \\
& p_2)\big)\frac{1}{g(p_1)+g(p_2)-\varepsilon^{\gamma} i k\cdot(p_1+p_2)} \\
& \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_1\nabla_k\Big(\frac{1}{g(p_2)-i\varepsilon^{\gamma}(k-s p_1)\cdot p_2} p_2\cdot\nabla_k\overline{\varphi_w}(x,\ k-s p_1+ \\
& t p_2)\mathrm{d}t\mathrm{d}s\Big)\frac{1}{g(q_1)+g(q_2)-\varepsilon^{\gamma} i l\cdot(q_1+q_2)} e^{i\frac{(q_1+q_2)\cdot y}{\varepsilon}} \\
& \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} q_1\nabla_k\Big(\frac{1}{g(q_2)-i\varepsilon^{\gamma}(l-s q_1)\cdot q_2} q_2\cdot\nabla_k\overline{\varphi_w}(y,l-s q_1+t q_2)\mathrm{d}t\mathrm{d}s\Big) \\
= & \mathrm{I}+\mathrm{II}+\mathrm{III}
\end{aligned}
$$

□

So

$$
\begin{aligned}
\mathrm{I} \;=\; & \iiiint \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\varphi_z(x,k)\overline{\varphi_z(y,l)}\iint \mathrm{d}p\mathrm{d}q\hat{R}(p)\hat{R}(q)\frac{1}{2g(q)g(p)} \\
& \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p\nabla_k\Big(\frac{1}{g(p)-i\varepsilon^{\gamma}(k-sp)\cdot p}p \cdot \nabla_k\overline{\varphi_w}(x, k - sp + \\
& tp)\mathrm{d}t\mathrm{d}s\Big)\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} q\nabla_k\Big(\frac{1}{g(q)-i\varepsilon^{\gamma}(k-sq)\cdot q}q\cdot\nabla_k\overline{\varphi_w}(y,l-sq+tq)\mathrm{d}t\mathrm{d}s\Big) \\
= & \int\int\int\int \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\; e^{z_3\cdot x}e^{z_4\cdot k}\varphi(x - z_1, k - z_2)\overline{\varphi_z(y,l)}\int \\
& \int \mathrm{d}p\mathrm{d}q\hat{R}(p)\hat{R}(q)\frac{1}{2g(q)g(p)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p\nabla_k\Big(\frac{1}{g(p)-i\varepsilon^{\gamma}(k-sp)\cdot p}p \cdot \\
& \nabla_k e^{-w_3\cdot x}e^{-w_4 k}e^{w_4(sp-tp)}\varphi(x - w_1, k - sp + tp - \\
& w_2)\mathrm{d}t\mathrm{d}s\Big)\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} q\nabla_k\Big(\frac{1}{g(q)-i\varepsilon^{\gamma}(k-sq)\cdot q}q\cdot\nabla_k\overline{\varphi_w}(y,l-sq+tq)\mathrm{d}t\mathrm{d}s\Big) \\
= & \int\int\int \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\varphi(x - z_1, k - \\
& z_2)\overline{\varphi_z(y,l)}\frac{1}{|(z_3-w_4)|^{2A}}\frac{1}{|(z_3-w_4)|^{2A}}\Delta_x^A\Delta_k^A e^{(z_3-w_3)\cdot x}e^{(z_4-w_4)\cdot k}\int \\
& \mathrm{d}p\mathrm{d}q\hat{R}(p)\hat{R}(q)\frac{1}{2g(q)g(p)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p\nabla_k\Big(\frac{1}{g(p)-i\varepsilon^{\gamma}(k-sp)\cdot p}p\cdot\nabla_k e^{w_4(sp-tp)}\varphi(x-w_1,k- \\
& sp+tp-w_2)\mathrm{d}t\mathrm{d}s\Big)\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} q\nabla_k\Big(\frac{1}{g(q)-i\varepsilon^{\gamma}(k-sq)\cdot q}q\cdot\nabla_k\overline{\varphi_w}(y,l-sq+tq)\mathrm{d}t\mathrm{d}s\Big) \\
+ & \int\int\int\int \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\varphi(x - z_1, k - \\
& z_2)\overline{\varphi_z(y,l)}\frac{1}{|(z_3-w_4)|^{2A}}\frac{1}{|(z_3-w_4)|^{2A}}\Delta_x^A\Delta_k^A e^{(z_3-w_3)\cdot x}e^{(z_4-w_4)\cdot k}\int \\
& \mathrm{d}p\mathrm{d}q\hat{R}(p)\hat{R}(q)\frac{1}{2g(q)g(p)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot z_4\Big(\frac{1}{g(p)-i\varepsilon^{\gamma}(k-sp)\cdot p}p\cdot z_4\; e^{w_4(sp-tp)}\varphi(x - w_1, \\
& k-sp+tp-w_2)\mathrm{d}t\mathrm{d}s\Big)\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} q\nabla_k\Big(\frac{1}{g(q)-i\varepsilon^{\gamma}(k-sq)\cdot q}q\cdot\nabla_k\overline{\varphi_w}(y,l-sq+tq)\mathrm{d}t\mathrm{d}s\Big) \\
+ & \int\int\int \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\varphi(x - z_1, k - \\
& z_2)\overline{\varphi_z(y,l)}\frac{1}{|(z_3-w_4)|^{2A}}\frac{1}{|(z_3-w_4)|^{2A}}\Delta_x^A\Delta_k^A e^{(z_3-w_3)\cdot x}e^{(z_4-w_4)\cdot k}\int \\
& \mathrm{d}p\mathrm{d}q\hat{R}(p)\hat{R}(q)\frac{1}{2g(q)g(p)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p\nabla_k\Big(\frac{1}{g(p)-i\varepsilon^{\gamma}(k-sp)\cdot p}\; p\cdot z_4\; e^{w_4(sp-tp)}\varphi(x-w_1,k- \\
& sp+tp-w_2)\mathrm{d}t\mathrm{d}s\Big)\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} q\nabla_k\Big(\frac{1}{g(q)-i\varepsilon^{\gamma}(l-sq)\cdot q}q\cdot\nabla_k\overline{\varphi_w}(y,l-sq+tq)\mathrm{d}t\mathrm{d}s\Big)
\end{aligned}
$$

and after integrating by parts and using product rule we get

$$
\mathrm{I} \lesssim \frac{1}{|(z_3-w_4)|^{2A}}\frac{1}{|(z_3-w_4)|^{2A}}\langle z_1-w_1\rangle^{-A}\langle z_2-w_2\rangle^{-A}\bigg(\int \mathrm{d}p\hat{R}(p)\langle p\rangle^{A}\sum_{i=1}^{2A}\left|\frac{p}{g(p)}\right|^{i}\bigg)^2
$$

and taking absolute values straight away we get

$$\mathrm{I} \lesssim \langle z_3 - w_3\rangle^{-A}\langle z_4 - w_4\rangle^{-A}\langle z_1 - w_1\rangle^{-A}\langle z_2 - w_2\rangle^{-A}\left(\int \mathrm{d}p\hat{R}\,(p)\langle p\rangle^A\sum_{i=1}^{2A}\left|\frac{p}{g(p)}\right|^i\right)^2$$

To estimate II we proceed similarly

$$\mathrm{II} = \int\int\int\int \mathrm{d}y\mathrm{d}l\mathrm{d}x\mathrm{d}k\ \varphi_z(x, k)\overline{\varphi_z(y,l)}\int\int\int\int e^{-i\frac{(p_1+p_2)\cdot x}{\varepsilon}}\hat{R}(p_1)\hat{R}(p_2)\frac{1}{g(p_1)+g(p_2)-\varepsilon^\gamma ik\cdot(p_1+p_2)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_1\nabla_k\left(\frac{1}{g(p_2)-i\varepsilon^\gamma(k-sp_1)\cdot p_2}p_2\cdot\nabla_k\overline{\varphi_w}(x,k-sp_1+tp_2)\mathrm{d}t\mathrm{d}s\right)\frac{1}{g(p_1)+g(p_2)-\varepsilon^\gamma il\cdot(p_1+p_2)}e^{i\frac{(p_1+p_2)\cdot y}{\varepsilon}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_1\nabla_k\left(\frac{1}{g(p_2)-i\varepsilon^\gamma(l-sp_1)\cdot p_2}p_2\cdot\nabla_k\overline{\varphi_w}(y,l-sp_1+tp_2)\mathrm{d}t\mathrm{d}s\right)$$

Proceeding as with I we get

$$\mathrm{II} \lesssim \langle z_1 - w_1\rangle^{-A}\langle z_2 - w_2\rangle^{-A}\langle z_4 - w_4\rangle^{-A}\int \mathrm{d}p\hat{R}(p)\langle p\rangle^A\sum_{i=1}^{2A}\left|\frac{p}{g(p)}\right|^i\left\langle z_3 - w_3 - \frac{p}{\varepsilon}\right\rangle^{-A}\int \mathrm{d}q\hat{R}\,(q)\langle q\rangle^A\sum_{i=1}^{2A}\left|\frac{q}{g(q)}\right|^i$$

and by the same reasoning

$$\mathrm{III} \lesssim \langle z_1 - w_1\rangle^{-A}\langle z_2 - w_2\rangle^{-A}\langle z_4 - w_4\rangle^{-A}\int \mathrm{d}p\hat{R}(p)\langle p\rangle^A\sum_{i=1}^{2A}\left|\frac{p}{g(p)}\right|^i\left\langle z_3 - w_3 - \frac{p}{\varepsilon}\right\rangle^{-A}\int \mathrm{d}q\hat{R}\,(q)\langle q\rangle^A\sum_{i=1}^{2A}\left|\frac{q}{g(q)}\right|^i$$

This gives bounds on $\mathcal{H}_{1,\varepsilon}$. To bound the $\mathcal{KH}_{1,\varepsilon}$ we compute

$$\mathcal{KH}_{1,\varepsilon}f = \int\int\int \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3 e^{-i\frac{(p_1+p_2+p_3)\cdot x}{\varepsilon}}\hat{V}(t,\ \mathrm{d}p_1)\hat{V}(t,\ \mathrm{d}p_2)\hat{V}(t,\ \mathrm{d}p_3)\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_3\cdot\nabla_k\frac{1}{g(p_1)+g(p_2)-\varepsilon^\gamma i\,(k-up_3)\cdot(p_1+p_2)}p_1\nabla_k\left(\frac{1}{g(p_2)-i\varepsilon^\gamma(k-up_3-sp_1)\cdot p_2}p_2\cdot\nabla_k\overline{f_w}(x,\ k-up_3-sp_1+tp_2)\mathrm{d}t\mathrm{d}s\mathrm{d}u\right)$$

$$\mathbb{E}[|\langle\varphi_z,\ \mathcal{KH}_{1,\varepsilon}\varphi_w\rangle|^2] = \int\int\int\int \mathrm{d}x\mathrm{d}k\mathrm{d}y\mathrm{d}l\varphi_z(x, k)\overline{\varphi_z(y,l)}\int\int\int \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\int\int\int \mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3 e^{-i\frac{(p_1+p_2+p_3)\cdot x}{\varepsilon}}\mathbb{E}\Big[\hat{V}(t,\ \mathrm{d}p_1)\hat{V}(t,\ \mathrm{d}p_2)\hat{V}(t,\ \mathrm{d}p_3)\overline{\hat{V}(t,\mathrm{d}q_1)\hat{V}(t,\mathrm{d}q_2)\hat{V}(t,\mathrm{d}q_3)}\Big]\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} p_3\cdot\nabla_k\frac{1}{g(p_1)+g(p_2)-\varepsilon^\gamma i\,(k-up_3)\cdot(p_1+p_2)}p_1\nabla_k\left(\frac{1}{g(p_2)-i\varepsilon^\gamma(k-up_3-sp_1)\cdot p_2}p_2\cdot\nabla_k\overline{\varphi_w}(x,\ k-up_3-sp_1+tp_2)\mathrm{d}t\mathrm{d}s\mathrm{d}u\right)e^{i\frac{(p_1+p_2+p_3)\cdot y}{\varepsilon}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} q_3\cdot\nabla_k\frac{1}{g(q_1)+g(q_2)-\varepsilon^\gamma i\,(l-uq_3)\cdot(q_1+q_2)}p_1\cdot\nabla_k\left(\frac{1}{g(q_2)-i\varepsilon^\gamma(l-uq_3-sq_1)\cdot q_2}q_2\cdot\nabla_k\varphi_w(x,k-uq_3-sq_1+tq_2)\mathrm{d}t\mathrm{d}s\mathrm{d}u\right)$$

Now we proceed as before

We need to choose the other corrector such that $\mathcal{A}^\varepsilon f_{3,\varepsilon} = -f''(\langle W_\varepsilon, \lambda\rangle)\langle W_\varepsilon, \mathcal{K}\lambda\rangle\langle W_\varepsilon, \mathcal{K}_{1,\varepsilon}\lambda\rangle(t) = -f''(\langle W_\varepsilon, \lambda\rangle)\langle W_\varepsilon \otimes W_\varepsilon, \mathcal{K}\lambda \otimes \mathcal{K}_{1,\varepsilon}\lambda\rangle$

We make the ansatz

$$f_{3,\varepsilon} = \varepsilon^{1+\gamma} f''(\langle W_\varepsilon, \lambda\rangle)\langle W_\varepsilon \otimes W_\varepsilon, \mathcal{H}_{2,\varepsilon}\lambda \otimes \lambda\rangle$$

with

$$\mathcal{H}_{2,\varepsilon}\lambda_1 \otimes \lambda_2(x) = \int a_\varepsilon(k, p_1, p_2)\hat{V}(t, dp_1)\hat{V}(t, \mathrm{d}p_2)e^{-i\frac{p_1\cdot x_1 + p_2\cdot x_2}{\varepsilon}} \frac{1}{g(p_2) + \varepsilon^\gamma k_2\cdot p_2}\Big(\lambda_1\Big(x_1, k_1 - \frac{p_1}{2}\Big) - \lambda_1\Big(x_1, k_1 + \frac{p_1}{2}\Big)\Big)\Big(\lambda_2\Big(x_2, k_2 - \frac{p_2}{2}\Big) - \lambda_2\Big(x_2, k_2 + \frac{p_2}{2}\Big)\Big)$$

We compute

$$\begin{aligned}
f_{3\varepsilon}(t+h) - f_{3,\varepsilon}(t) &= (f''(\langle W_\varepsilon, \lambda\rangle(t+h)) - f''(\langle W_\varepsilon, \lambda\rangle(t)))\langle W_\varepsilon \otimes W_\varepsilon, \mathcal{H}_{2,\varepsilon}\lambda \otimes \lambda\rangle(t)\\
&+ f''(\langle W_\varepsilon, \lambda\rangle)(\langle W_\varepsilon(t+h) \otimes W_\varepsilon(t), \mathcal{H}_{2,\varepsilon}\lambda(t) \otimes \lambda(t)\rangle - \langle W_\varepsilon(t) \otimes W_\varepsilon(t),\\
&\quad \mathcal{H}_{2,\varepsilon}\lambda(t) \otimes \lambda(t)\rangle)\\
&+ f''(\langle W_\varepsilon, \lambda\rangle)(\langle W_\varepsilon(t) \otimes W_\varepsilon(t+h), \mathcal{H}_{2,\varepsilon}\lambda(t) \otimes \lambda(t)\rangle - \langle W_\varepsilon(t) \otimes W_\varepsilon(t),\\
&\quad \mathcal{H}_{2,\varepsilon}\lambda(t) \otimes \lambda(t)\rangle)\\
&+ f''(\langle W_\varepsilon, \lambda\rangle)(\langle W_\varepsilon(t+h) \otimes W_\varepsilon(t), \mathcal{H}_{2,\varepsilon}\lambda(t+h) \otimes \lambda(t+h)\rangle - \langle W_\varepsilon(t) \otimes W_\varepsilon(t),\\
&\quad \mathcal{H}_{2,\varepsilon}\lambda(t) \otimes \lambda(t)\rangle)
\end{aligned}$$

Taking expectation and the limit, we get

$$\begin{aligned}
\mathcal{A}^\varepsilon f_{3,\varepsilon}(t) &= \varepsilon^{1+\gamma} f'''(\langle W_\varepsilon, \lambda\rangle(t))\langle W_\varepsilon \otimes W_\varepsilon, \mathcal{H}_{2,\varepsilon}\lambda \otimes \lambda\rangle\\
&+ \varepsilon^{\frac{1+\gamma}{2}}\langle W_\varepsilon \otimes W_\varepsilon, (\mathbb{1} \otimes \mathcal{K})\mathcal{H}_{2,\varepsilon}\lambda \otimes \lambda\rangle + \varepsilon^{\frac{1+\gamma}{2}}\langle W_\varepsilon \otimes W_\varepsilon, (\mathcal{K} \otimes \mathbb{1})\mathcal{H}_{2,\varepsilon}\lambda \otimes \lambda\rangle\\
&+ \varepsilon^{1+\gamma} f''(\langle W_\varepsilon, \lambda\rangle)\langle W_\varepsilon \otimes W_\varepsilon, k_1 \cdot \nabla_{x_1}\mathcal{H}_{2,\varepsilon}\lambda \otimes \lambda\rangle\\
&+ \varepsilon^{1+\gamma} f''(\langle W_\varepsilon, \lambda\rangle)\langle W_\varepsilon \otimes W_\varepsilon, k_2 \cdot \nabla_{x_2}\mathcal{H}_{2,\varepsilon}\lambda \otimes \lambda\rangle\\
&+ \varepsilon^{1+\gamma}\lim_{h\to 0}\frac{1}{h}\mathbb{E}_\varepsilon^t f''(\langle W_\varepsilon, \lambda\rangle)(\langle W_\varepsilon(t) \otimes W_\varepsilon(t), \mathcal{H}_{2,\varepsilon}\lambda(t+h) \otimes \lambda(t+h)\rangle - \langle W_\varepsilon(t) \otimes W_\varepsilon(t),\\
&\quad \mathcal{H}_{2,\varepsilon}\lambda(t) \otimes \lambda(t)\rangle)
\end{aligned}$$

Now

$$\begin{aligned}
&= \lim_{h\to 0}\frac{\mathbb{E}_\varepsilon^t \mathcal{H}_{2,\varepsilon}\lambda(t+h) \otimes \lambda(t+h) - \mathcal{H}_{2,\varepsilon}\lambda(t) \otimes \lambda(t)}{h}\\
&= \lim_{h\to 0}\int a_\varepsilon(k, p_1, p_2)e^{-\frac{p_1\cdot x_1 + p_2\cdot x_2}{\varepsilon}}\frac{\mathbb{E}_\varepsilon^t \hat{V}(t+h, dp_1)\hat{V}(t+h, \mathrm{d}p_2) - \hat{V}(t, dp_1)\hat{V}(t, \mathrm{d}p_2)}{h}\\
&\quad \frac{1}{g(p_2) + \varepsilon^\gamma k_1\cdot p_1}\Big(\lambda\Big(x_1, k_1 - \frac{p_1}{2}\Big) - \lambda_1\Big(x_1, k_1 + \frac{p_1}{2}\Big)\Big)\Big(\lambda\Big(x_2, k_2 - \frac{p_2}{2}\Big) - \lambda_2\Big(x_2, k_2 + \frac{p_2}{2}\Big)\Big)\\
&= \int a_\varepsilon(k, p_1, p_2)e^{-\frac{p_1\cdot x_1 + p_2\cdot x_2}{\varepsilon}}\Bigg(\frac{e^{-g(p_1)h - g(p_2)h}\hat{V}(t, dp_1)\hat{V}(t, \mathrm{d}p_2) - \hat{V}(t, dp_1)\hat{V}(t, \mathrm{d}p_2)}{h} +\\
&\quad \frac{(e^{-g(p_1)h} - 1)\delta(p_1 + p_2)}{h}\Bigg)\frac{1}{g(p_2) + \varepsilon^\gamma k_1\cdot p_1}\Big(\lambda\Big(x_1, k_1 - \frac{p_1}{2}\Big) - \lambda_1\Big(x_1, k_1 + \frac{p_1}{2}\Big)\Big)\Big(\lambda\Big(x_2, k_2 -\\
&\quad \frac{p_2}{2}\Big) - \lambda_2\Big(x_2, k_2 + \frac{p_2}{2}\Big)\Big)
\end{aligned}$$

$$= \varepsilon^{-1-\gamma}\int -a_\varepsilon(k,p_1,p_2)\,(g(p_1)+g(p_2))e^{-\frac{p_1\cdot x_1+p_2\cdot x_2}{\varepsilon}}\frac{1}{g(p_2)+\varepsilon^\gamma k_1\cdot p_1}\Big(\lambda\Big(x_1,k_1-\frac{p_1}{2}\Big)-\lambda_1\Big(x_1, k_1+\frac{p_1}{2}\Big)\Big)\Big(\lambda\Big(x_2,k_2-\frac{p_2}{2}\Big)-\lambda_2\Big(x_2,k_2+\frac{p_2}{2}\Big)\Big)+\varepsilon^{-1-\gamma}D_2\lambda\otimes\lambda$$

with D_2 a deterministic operator. We have

$$k_1\cdot\nabla_{x_1}\mathcal{H}_{2,\varepsilon}\lambda\otimes\lambda = -\varepsilon^{-1}i\int a_\varepsilon(k, p_1, p_2)\hat{V}(t, d\,p_1)\hat{V}(t, \mathrm{d}p_2)\; k_1\cdot p_1\; e^{-i\frac{p_1\cdot x_1+p_2\cdot x_2}{\varepsilon}}\frac{1}{g(p_2)+i\varepsilon^\gamma k_1\cdot p_1}\Big(\lambda_1\Big(x_1,\ k_1-\frac{p_1}{2}\Big)-\lambda_1\Big(x_1,\ k_1+\frac{p_1}{2}\Big)\Big)\Big(\lambda_2\Big(x_2,\ k_2-\frac{p_2}{2}\Big)-\lambda_2\Big(x_2,k_2+\frac{p_2}{2}\Big)\Big)$$

$$k_2\cdot\nabla_{x_2}\mathcal{H}_{2,\varepsilon}\lambda\otimes\lambda = -\varepsilon^{-1}i\int a_\varepsilon(k, p_1, p_2)\hat{V}(t, d\,p_1)\hat{V}(t, \mathrm{d}p_2)\; k_2\cdot p_2\; e^{-\frac{p_1\cdot x_1+p_2\cdot x_2}{\varepsilon}}\frac{1}{g(p_2)+i\varepsilon^\gamma k_1\cdot p_1}\Big(\lambda_1\Big(x_1,k_1-\frac{p_1}{2}\Big)-\lambda_1\Big(x_1,k_1+\frac{p_1}{2}\Big)\Big)\Big(\lambda_2\Big(x_2,k_2-\frac{p_2}{2}\Big)-\lambda_2\Big(x_2,k_2+\frac{p_2}{2}\Big)\Big)$$

So we want

$$(g(p_1)+g(p_2)+i\varepsilon^\gamma(k_1\cdot p_1+k_2\cdot p_2))a_\varepsilon(k,p_1,p_2)=1$$

Provided this is satisfied we will have

$$\begin{aligned}\mathcal{A}^\varepsilon f_{3,\varepsilon}(t) &= -f''(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon,\mathcal{K}\lambda\rangle\langle W_\varepsilon,\mathcal{K}_{1,\varepsilon}\lambda\rangle(t)\\ &+ f''(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon\otimes W_\varepsilon, D_{2,\varepsilon}\lambda\otimes\lambda\rangle+\mathcal{O}\Big(\varepsilon^{\frac{1+\gamma}{2}}\Big)\end{aligned}$$

Once we have proved

Lemma 17. *There exist random variables $C_{5,\varepsilon}>0$ such that and $\sup_\varepsilon\mathbb{E}[C_{5,\varepsilon}^2]<\infty$ while*

$$\|\mathcal{H}_{2,\varepsilon}\lambda_1\otimes\lambda_2\|_{L^2}\le C_{5,\varepsilon}\|\lambda_1\|_{M_{2,2}^A}\|\lambda_2\|_{M_{2,2}^A}$$

Proof.

$$\begin{aligned}\mathbb{E}|\langle\varphi_z\otimes\varphi_{\tilde z},\mathcal{H}_{2,\varepsilon}\varphi_w\otimes\varphi_{\tilde w}\rangle|^2 =& \iiiint\varphi_z(x_1,k_1)\varphi_{\tilde z}(x_2,k_2)\overline{\varphi_z(y_1,l_1)\varphi_{\tilde z}(y_2,l_2)}\iiiint\\ &\mathbb{E}\Big[\hat{V}(t,\mathrm{d}p_1)\hat{V}(t,\mathrm{d}p_2)\overline{\hat{V}(t,\mathrm{d}q_1)\hat{V}(t,\mathrm{d}q_2)}\Big]e^{-i\frac{p_1\cdot x_1+p_2\cdot x_2}{\varepsilon}}\\ &\frac{1}{g(p_1)+g(p_2)+i\varepsilon^\gamma(k_1\cdot p_1+k_2\cdot p_2)}\frac{1}{g(p_2)+i\varepsilon^\gamma k_1\cdot p_1}\\ &\overline{\Big(\varphi_w\Big(x_1,k_1-\frac{p_1}{2}\Big)-\varphi_w\Big(x_1,k_1+\frac{p_1}{2}\Big)\Big)}\\ \times&\overline{\Big(\varphi_{\tilde w}\Big(x_2,k_2-\frac{p_2}{2}\Big)-\varphi_{\tilde w}\Big(x_2,k_2+\frac{p_2}{2}\Big)\Big)}\\ &e^{i\frac{q_1\cdot y_2+q_2\cdot y_2}{\varepsilon}}\frac{1}{g(q_1)+g(q_2)+i\varepsilon^\gamma(l_1\cdot q_1+l_2\cdot q_2)}\\ &\frac{1}{g(q_2)+i\varepsilon^\gamma l_1\cdot q_1}\Big(\varphi_w\Big(y_1,\ l_1-\frac{q_1}{2}\Big)-\varphi_w\Big(y_1,\ l_1+\frac{q_1}{2}\Big)\Big)\Big(\varphi_{\tilde w}\Big(y_2,\ l_2-\frac{q_2}{2}\Big)-\varphi_{\tilde w}\Big(y_2,l_2+\frac{q_2}{2}\Big)\Big)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}l_1\mathrm{d}l_2\end{aligned}$$

$$
\begin{aligned}
= & \iiiint\iiiint \varphi_z(x_1,k_1)\varphi_{\tilde z}(x_2,k_2)\overline{\varphi_z(y_1,l_1)\varphi_{\tilde z}(y_2,l_2)} \\
& \iiiint \mathbb{E}\Big[\hat V(t,\mathrm{d}p_1)\hat V(t,\mathrm{d}p_2)\overline{\hat V(t,\mathrm{d}q_1)\hat V(t,\mathrm{d}q_2)}\Big]e^{-i\frac{p_1\cdot x_1+p_2\cdot x_2}{\varepsilon}} \\
& \frac{1}{g(p_1)+g(p_2)+i\varepsilon^\gamma(k_1\cdot p_1+k_2\cdot p_2)}\frac{1}{g(p_2)+i\varepsilon^\gamma k_1\cdot p_1} \\
& \overline{\int p_1\cdot\nabla_{k_2}\varphi_w\Big(x_1,k_1-t_1\frac{p_1}{2}\Big)\mathrm{d}t_1\int p_2\cdot\nabla_{k_2}\varphi_{\tilde w}\Big(x_2,k_2-t_2\frac{p_2}{2}\Big)\mathrm{d}t_2} \\
& e^{i\frac{q_1\cdot y_2+q_2\cdot y_2}{\varepsilon}} \\
& \frac{1}{g(q_1)+g(q_2)+i\varepsilon^\gamma(l_1\cdot q_1+l_2\cdot q_2)}\int q_1\cdot\nabla_{l_1}\varphi_w\Big(y_1,l_1-t_1\frac{q_1}{2}\Big)\mathrm{d}t_1 \\
& \int q_2\cdot\nabla_{l_2}\varphi_{\tilde w}\Big(y_2,l_2-t_2\frac{q_2}{2}\Big)\mathrm{d}t_2\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}l_1\mathrm{d}l_2 \\
= & \iiiint\iiiint \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}l_1\mathrm{d}l_2 \\
& \iiint\iiint \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}q_1\mathrm{d}q_2 \\
& \big(\hat R(p_1)\hat R(q_1)\delta(p_1+p_2)\delta(q_1+q_2)+\hat R(p_1)\hat R(p_2)\delta(p_1-q_1)\delta(p_2- \\
& q_2)+\hat R(p_1)\hat R(p_2)\delta(p_1-q_2)\delta(q_1-p_2)\big) \\
& \frac{1}{\left|z_3-w_3-\frac{p_1}{\varepsilon}\right|^{2A}}\frac{1}{|z_4-w_4|^{2A}}\frac{1}{\left|\tilde z_3-\tilde w_3-\frac{p_2}{\varepsilon}\right|^{2A}}\frac{1}{|\tilde z_4-\tilde w_4|^{2A}} \\
& \Delta^A_{x_1}\Delta^A_{x_2}\Delta^A_{k_1}\Delta^A_{k_2}\Big(e^{-i\frac{p_1\cdot x_1+p_2\cdot x_2}{\varepsilon}}e^{iz_3\cdot x_1}e^{-iw_3\cdot x_1}e^{iz_4\cdot k_1}e^{-iw_4k_1} \\
& e^{i\tilde z_3\cdot x_2}e^{-i\tilde w_3\cdot x_2}e^{i\tilde z_4\cdot k_2}e^{-i\tilde w_4k_2}\Big)\varphi(x_1,\quad k_1)\varphi(x_2, \\
& k_2)\frac{1}{\left|z_3-w_3-\frac{q_1}{\varepsilon}\right|^{2A}}\frac{1}{|z_4-w_4|^{2A}}\frac{1}{\left|\tilde z_3-\tilde w_3-\frac{q_2}{\varepsilon}\right|^{2A}}\frac{1}{|\tilde z_4-\tilde w_4|^{2A}} \\
& \overline{\begin{array}{l}\Delta^A_{y_1}\Delta^A_{y_2}\Delta^A_{l_1}\Delta^A_{l_2}\Big(e^{i\frac{q_1\cdot y_1+q_2\cdot y_2}{\varepsilon}}e^{-iz_3\cdot y_1}e^{iw_3\cdot y_1}e^{-iz_4\cdot l_1}e^{iw_4l_1} \\ e^{-i\tilde z_3\cdot y_2}e^{i\tilde w_3\cdot y_2}e^{-i\tilde z_4\cdot l_2}e^{i\tilde w_4l_2}\Big)\end{array}} \\
& \overline{\varphi(y_1-z_1,l_1-z_2)\varphi(y_2-\tilde z_1,l_2-\tilde z_2)} \\
& \overline{\frac{1}{g(p_1)+g(p_2)+i\varepsilon^\gamma(k_1\cdot p_1+k_2\cdot p_2)}} \\
& \frac{1}{g(q_1)+g(q_2)+i\varepsilon^\gamma(l_1\cdot q_1+l_2\cdot q_2)}\frac{1}{g(q_2)+i\varepsilon^\gamma k_2\cdot q_2} \\
& \int_{-\frac12}^{\frac12}\Big(e^{-iw_4\cdot t_1\frac{q_1}{2}}q_1\cdot\nabla_{l_1}\varphi\Big(y_1-w_1,l_1-t_1\frac{q_1}{2}-w_2\Big)+e^{-iw_4\cdot t_1\frac{q_1}{2})}q_1\cdot \\
& w_4\,\varphi\Big(y_1-w_1,l_1-t_1\frac{q_1}{2}-w_2\Big)\Big)\mathrm{d}t_1\int\Big(e^{-i\tilde w_4\cdot t_2\frac{q_2}{2}}q_2\cdot\nabla_{l_2}\varphi\Big(y_2-\tilde w_1, \\
& l_2-t_2\frac{q_2}{2}-\tilde w_2\Big)+e^{-i\tilde w_4\cdot t_2\frac{q_2}{2}}q_2\cdot\tilde w_4\,\varphi\Big(y_2-\tilde w_1,l_2-t_2\frac{q_2}{2}-\tilde w_2\Big)\Big)\mathrm{d}t_2 \\
\times & \int_{-\frac12}^{\frac12}\Big(e^{iw_4\cdot t_1\frac{p_1}{2}}p_1\cdot\nabla_{k_1}\varphi\Big(x_1-w_1,\,k_1-t_1\frac{p_1}{2}-w_2\Big)+e^{iw_4\cdot t_1\frac{p_1}{2}}p_1\cdot \\
& w_4\,\varphi\Big(x_1-w_1,k_1-t_1\frac{p_1}{2}-w_2\Big)\Big)\mathrm{d}t_1 \\
\times & \frac{1}{g(p_2)+i\varepsilon^\gamma k_2\cdot p_2}\int_{-\frac12}^{\frac12}\Big(e^{-i\tilde w_3\cdot x_2}e^{i\tilde w_4\cdot t_2\frac{p_2}{2}}p_2\cdot\nabla_{k_2}\varphi\Big(x_2-\tilde w_1,\,k_2- \\
& t_2\frac{p_2}{2}-\tilde w_2\Big)+e^{-i\tilde w_4k_2\cdot t_2\frac{p_2}{2}}p_2\cdot\tilde w_4\,\varphi\Big(x_2-\tilde w_1,k_2-t_2\frac{p_2}{2}-\tilde w_2\Big)\Big)\mathrm{d}t_2 \\
\times & \int_{-\frac12}^{\frac12}\Big(e^{-iw_4\cdot t_1\frac{q_1}{2}}q_1\cdot\nabla_{l_1}\varphi\Big(y_1-w_1,l_1-t_1\frac{q_1}{2}-w_2\Big)+e^{-iw_4\cdot t_1\frac{q_1}{2})}q_1\cdot \\
& w_4\,\varphi\Big(y_1-w_1,l_1-t_1\frac{q_1}{2}-w_2\Big)\Big)\mathrm{d}t_1
\end{aligned}
$$

$$\times \int \Big(e^{-i\tilde{w}_4\cdot t_2\frac{q_2}{2}} q_2\cdot\nabla_{l_2}\varphi\Big(y_2-\tilde{w}_1, l_2-t_2\frac{q_2}{2}-\tilde{w}_2\Big)+e^{-i\tilde{w}_4\cdot t_2\frac{q_2}{2}} q_2\cdot \tilde{w}_4\,\varphi\Big(y_2-\tilde{w}_1, l_2-t_2\frac{q_2}{2}-\tilde{w}_2\Big)\Big)\mathrm{d}t_2$$
$$= \mathrm{I}+\mathrm{II}+\mathrm{III}$$

Integrating by parts we get for term I

$$\mathrm{I} = \iiiint\!\!\iint \mathrm{d}p_1\mathrm{d}q_1\big(\hat{R}(p_1)\hat{R}(q_1)\frac{1}{|z_3-w_3-\frac{p_1}{\varepsilon}|^{2A}}\frac{1}{|z_4-w_4|^{2A}}\frac{1}{|\tilde{z}_3-\tilde{w}_3+\frac{p_1}{\varepsilon}|^{2A}}\frac{1}{|\tilde{z}_4-\tilde{w}_4|^{2A}}$$
$$\Big(e^{-i\frac{p_1\cdot x_1-p_1\cdot x_2}{\varepsilon}}e^{iz_3\cdot x_1}e^{-iw_3\cdot x_1}e^{iz_4\cdot k_1}e^{-iw_4k_1}e^{i\tilde{z}_3\cdot x_2}e^{-i\tilde{w}_3\cdot x_2}e^{i\tilde{z}_4\cdot k_2}e^{-i\tilde{w}_4k_2}\Big)$$
$$\times\ \Delta_{x_1}^A\Delta_{x_2}^A\Delta_{k_1}^A\Delta_{k_2}^A\Big(\varphi(x_1,\ k_1)\varphi(x_2,\ k_2)\frac{1}{g(p_1)+g(p_1)+i\varepsilon^\gamma(k_1\cdot p_1-k_2\cdot p_1)}\frac{1}{g(p_1)-i\varepsilon^\gamma k_2\cdot p_1}\Big(\int e^{iw_4\cdot t_1\frac{p_1}{2}}p_1\cdot\nabla_{k_1}\varphi\Big(x_1-w_1,\ k_1-t_1\frac{p_1}{2}-w_2\Big)+e^{iw_4\cdot t_1\frac{p_1}{2}}p_1\cdot w_4\ \varphi\Big(x_1-w_1,\ k_1-t_1\frac{p_1}{2}-w_2\Big)\Big)\mathrm{d}t_1\Big(\int e^{i\tilde{w}_3\cdot x_2}e^{i\tilde{w}_4\cdot t_2\frac{p_1}{2}}p_2\cdot\nabla_{k_2}\varphi\Big(x_2-\tilde{w}_1,\ k_2+t_1\frac{p_1}{2}-\tilde{w}_2\Big)+e^{i\tilde{w}_4k_2\cdot t_2\frac{p_1}{2}}p_2\cdot\tilde{w}_4\ \varphi\Big(x_2-\tilde{w}_1, k_2-t_2\frac{p_2}{2}-\tilde{w}_2\Big)\mathrm{d}t_2\Big)$$
$$\times\ \frac{1}{|z_3-w_3-\frac{q_1}{\varepsilon}|^{2A}}\frac{1}{|z_4-w_4|^{2A}}\frac{1}{|\tilde{z}_3-\tilde{w}_3+\frac{q_1}{\varepsilon}|^{2A}}\frac{1}{|\tilde{z}_4-\tilde{w}_4|^{2A}}$$
$$\times\ \Big(e^{i\frac{q_1\cdot y_1-q_1\cdot y_2}{\varepsilon}}e^{-iz_3\cdot y_1}e^{iw_3\cdot y_1}e^{-iz_4\cdot l_1}e^{iw_4l_1}e^{-i\tilde{z}_3\cdot y_2}e^{i\tilde{w}_3\cdot y_2}e^{-i\tilde{z}_4\cdot l_2}e^{i\tilde{w}_4l_2}\Big)$$
$$\times\ \Delta_{y_1}^A\Delta_{y_2}^A\Delta_{l_1}^A\Delta_{l_2}^A\Big(\overline{\varphi(y_1-z_1,l_1-z_2)\varphi(y_2-\tilde{z}_1,l_2-\tilde{z}_2)}\frac{1}{g(q_1)+g(q_1)+i\varepsilon^\gamma(l_1\cdot q_1-l_2\cdot q_1)}\frac{1}{g(q_1)-i\varepsilon^\gamma k_2\cdot q_1}\Big(\Big(\int e^{-iw_4\cdot t_1\frac{q_1}{2}}q_1\cdot\nabla_{l_1}\varphi\Big(y_1-w_1,l_1-t_1\frac{q_1}{2}-w_2\Big)+e^{-iw_4\cdot t_1\frac{q_1}{2})}q_1\cdot w_4\ \varphi\Big(y_1-w_1,\ l_1-t_1\frac{q_1}{2}-w_2\Big)\mathrm{d}t_1\Big)\ \Big(\int e^{-i\tilde{w}_4\cdot t_2\frac{q_1}{2}}q_2\cdot\nabla_{l_2}\varphi\Big(y_2-\tilde{w}_1, l_2-t_1\frac{q_2}{2}-\tilde{w}_2\Big)+e^{i\tilde{w}_4\cdot t_2\frac{q_1}{2}}q_2\cdot\tilde{w}_4\ \varphi\Big(y_2-\tilde{w}_1,l_2-t_2\frac{q_2}{2}-\tilde{w}_2\Big)\mathrm{d}t_2\Big)\ \Big)$$

Now we know that the first $2A$ derivatives of

$$\frac{1}{g(p_1)+g(p_2)+i\varepsilon^\gamma(k_1p_1-k_2\cdot p_1)}\text{ and }\frac{1}{g(p_1)-i\varepsilon^\gamma k_2\cdot p_1}$$

are bounded by

$$C\Big(\sum_{i=1}^{2A}\Big|\frac{1}{g(p_1)}\Big|^i|p_1|^i\Big)$$

Which implies

$$\mathrm{I}\lesssim\Big(\sum_{i=1}^{2A}\Big|\frac{1}{g(p_1)}\Big|^i|p_1|^i+\sum_{i=1}^{2A}\Big|\frac{1}{g(p_2)}\Big|^i|p_2|^i\Big)\langle z_1-w_1\rangle^{-2A}\langle z_2-w_2\rangle^{-2A}\langle p_1\rangle^{2A}\langle\tilde{z}_1-\tilde{w}_1\rangle^{-2A}\langle\tilde{z}_2-\tilde{w}_2\rangle^{-2A}\langle p_2\rangle^{2A}$$

So in total we get

$$\mathrm{I}\lesssim\int \mathrm{d}p\hat{R}(p_1)\Big(\sum_{i=1}^{2A}\Big|\frac{1}{g(p_1)}\Big|^i|p_1|^i\langle p_1\rangle^{4A}\int \mathrm{d}q\hat{R}(q_1)\Big(\sum_{i=1}^{2A}\Big|\frac{1}{g(q_1)}\Big|^i|q_1|^i\Big)\langle q_1\rangle^{4A}\langle w_4\rangle^2\langle\tilde{z}_1-\tilde{w}_1\rangle^{-4A}\langle\tilde{z}_2-\tilde{w}_2\rangle^{-4A}\langle z_1-w_1\rangle^{-4A}\langle z_2-$$

$$w_2\rangle^{-4A}\frac{1}{\left|z_3-w_3-\frac{p_1}{\varepsilon}\right|^{2A}}\frac{1}{|z_4-w_4|^{2A}}\frac{1}{\left|\tilde{z}_3-\tilde{w}_3-\frac{p_2}{\varepsilon}\right|^{2A}}\frac{1}{|\tilde{z}_4-\tilde{w}_4|^{2A}}\frac{1}{\left|z_3-w_3-\frac{q_1}{\varepsilon}\right|^{2A}}$$
$$\frac{1}{|z_4-w_4|^{2A}}\frac{1}{\left|\tilde{z}_3-\tilde{w}_3-\frac{q_2}{\varepsilon}\right|^{2A}}\frac{1}{|\tilde{z}_4-\tilde{w}_4|^{2A}}$$

If we do not integrate by parts we also obtain the estimate

$$\mathrm{I} \lesssim \int \mathrm{d}p\hat{R}(p_1)\left(\sum_{i=1}^{2A}\left|\frac{1}{g(p_1)}\right|^i|p_1|^i\langle p_1\rangle^{4A}\int \mathrm{d}q\hat{R}(q_1)\left(\sum_{i=1}^{2A}\left|\frac{1}{g(q_1)}\right|^i|q_1|^i\right)\langle q_1\rangle^{4A}\langle w_4\rangle^2\langle\tilde{w}_4\rangle^2\langle\tilde{z}_1-\right.$$
$$\tilde{w}_1\rangle^{-4A}\langle\tilde{z}_2-\tilde{w}_2\rangle^{-4A}\langle z_1-w_1\rangle^{-4A}\langle z_2-w_2\rangle^{-4A}\left\langle\tilde{z}_3-\tilde{w}_3+\frac{p_1}{\varepsilon}\right\rangle^{-2A}\left\langle z_3-w_3-\frac{p_1}{\varepsilon}\right\rangle^{-2A}\left\langle\tilde{z}_3-\right.$$
$$\left.\tilde{w}_3-\frac{q_1}{\varepsilon}\right\rangle^{-2A}\left\langle\tilde{z}_3-\tilde{w}_3+\frac{q_1}{\varepsilon}\right\rangle^{-2A}$$

In total we get get

$$\mathrm{I} \lesssim \int \mathrm{d}p\hat{R}(p_1)\left(\sum_{i=1}^{2A}\left|\frac{1}{g(p_1)}\right|^i|p_1|^i\langle p_1\rangle^{4A}\int \mathrm{d}q\hat{R}(q_1)\left(\sum_{i=1}^{2A}\left|\frac{1}{g(q_1)}\right|^i|q_1|^i\right)\langle q_1\rangle^{4A}\langle w_4\rangle^2\langle\tilde{w}_4\rangle^2\langle\tilde{z}_1-\right.$$
$$\tilde{w}_1\rangle^{-4A}\langle\tilde{z}_2-\tilde{w}_2\rangle^{-4A}\langle z_1-w_1\rangle^{-4A}\langle z_2-w_2\rangle^{-4A}\left\langle z_3-w_3-\frac{p_1}{\varepsilon}\right\rangle^{-2A}\left\langle\tilde{z}_3-\tilde{w}_3+\frac{p_1}{\varepsilon}\right\rangle^{-2A}\langle z_4-$$
$$w_4\rangle^{-4A}\left\langle\tilde{z}_3-\tilde{w}_3-\frac{q_1}{\varepsilon}\right\rangle^{-2A}\left\langle\tilde{z}_3-\tilde{w}_3+\frac{q_1}{\varepsilon}\right\rangle^{-2A}\langle\tilde{z}_4-\tilde{w}_4\rangle^{-4A}$$

For the terms II and III we can proceed in the same way. □

To conclude we need to prove bounds on $\mathcal{KH}_{1,\varepsilon}$.

Lemma 18. *There exist random variables $C_{5,\varepsilon}$, $C_{6,\varepsilon}\in L^2(\mathbb{P},\mathbb{R}_{>0})$ such that $C_{4,\varepsilon}\in L^2(\mathbb{P})$ and $\sup_\varepsilon\mathbb{E}[C_{5,\varepsilon}^2+C_{6,\varepsilon}^2]<\infty$ while*

$$\begin{aligned}\|(\mathbb{1}\otimes\mathcal{K})\mathcal{H}_{2,\varepsilon}\lambda_1\otimes\lambda_2\|_{L^2} &\leq C_{6,\varepsilon}\|\lambda_1\|_{\mathcal{M}^A}\|\lambda_2\|_{\mathcal{M}^A}\\ \|(\mathcal{K}\otimes\mathbb{1})\mathcal{H}_{2,\varepsilon}\lambda\otimes\lambda\|_{L^2} &\leq C_{7,\varepsilon}\|\lambda_1\|_{\mathcal{M}^A}\|\lambda_2\|_{\mathcal{M}^A}\end{aligned}$$

Proof. We only proof the first statement, the proof of the second one is the same.

$$\begin{aligned}(\mathbb{1}\otimes\mathcal{K})\mathcal{H}_{2,\varepsilon}\lambda_1\otimes\lambda_2 &= \iint\hat{V}(t,\mathrm{d}p_1)\hat{V}(t,\mathrm{d}p_2)\hat{V}(t,\mathrm{d}p_3)e^{-i\frac{p_1\cdot x_1+(p_2+p_3)\cdot x_2}{\varepsilon}}\\ &\times\left(\lambda_1\left(x_1,k_1-\frac{p_1}{2}\right)-\lambda_1\left(x_1,k_1+\frac{p_1}{2}\right)\right)\\ &\times\left(\frac{1}{g(p_1)+g(p_2)+i\varepsilon^\gamma\left(k_1\cdot p_1+\left(k_2-\frac{p_3}{2}\right)\cdot p_2\right)}\frac{1}{g(p_2)+i\varepsilon^\gamma\left(k_2-\frac{p_3}{2}\right)\cdot p_2}\right.\\ &\quad\left(\lambda_2\left(x_2,k_2-\frac{p_3}{2}-\frac{p_2}{2}\right)-\lambda_2\left(x_2,k_2-\frac{p_3}{2}+\frac{p_2}{2}\right)\right)\\ &\quad-\frac{1}{g(p_1)+g(p_2)+i\varepsilon^\gamma\left(k_1\cdot p_1+\left(k_2+\frac{p_3}{2}\right)\cdot p_2\right)}\\ &\quad\left.\frac{1}{g(p_2)+i\varepsilon^\gamma\left(k_2+\frac{p_3}{2}\right)\cdot p_2}\left(\lambda_2\left(x_2,k_2+\frac{p_3}{2}-\frac{p_2}{2}\right)-\lambda_2\left(x_2,k_2+\frac{p_3}{2}+\frac{p_2}{2}\right)\right)\right)\\ &= H_+\lambda_1\otimes\lambda_2-H_-\lambda_1\otimes\lambda_2\end{aligned}$$

We prove bounds on H_+ the bounds on H_- work in the same way. In the following the sum $\sum$ is over all partitions σ of $\{1,2,...,6\}$ into disjoint pairs $\{i_\pi^\sigma, j_\pi^\sigma\}$ and we use the notations $p_i = -q_{i-3}$ for $i > 3$.

$$\begin{aligned}
\mathbb{E}|\langle\varphi_z\otimes\varphi_{\tilde z}, H_+\varphi_w\otimes\varphi_{\tilde w}\rangle|^2 = & \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \mathrm{d}x_1\mathrm{d}k_1\mathrm{d}y_1\mathrm{d}l_1\mathrm{d}x_2\mathrm{d}k_2\mathrm{d}y_2\mathrm{d}l_2 \\
& \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \varphi_z(x_1,k_1)\varphi_{\tilde z}(x_2,k_2)\overline{\varphi_z(y_1,l_1)\varphi_{\tilde z}(y_2,l_2)} \\
\times\ & \mathbb{E}\Big[\overline{\hat V(t,\mathrm{d}p_1)\hat V(t,\mathrm{d}p_2)\hat V(t,\mathrm{d}p_3)}\hat V(t,\mathrm{d}q_1)\hat V(t,\mathrm{d}q_2)\hat V(t,\mathrm{d}q_3)\Big] \\
\times\ & e^{-i\frac{p_1\cdot x_1+(p_2+p_3)\cdot x_2}{\varepsilon}}\Big(\overline{\varphi_w\Big(x_1,k_1-\frac{p_1}{2}\Big)}-\overline{\varphi_w\Big(x_1,k_1+\frac{p_1}{2}\Big)}\Big) \\
\times\ & \frac{1}{g(p_1)+g(p_2)-i\varepsilon^\gamma\big(k_1\cdot p_1+\big(k_2-\frac{p_3}{2}\big)\cdot p_2\big)}\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k_2-\frac{p_3}{2}\big)\cdot p_2} \\
& \Big(\overline{\varphi_{\tilde w}\Big(x_2,k_2-\frac{p_3}{2}-\frac{p_2}{2}\Big)}-\overline{\varphi_{\tilde w}\Big(x_2,k_2-\frac{p_3}{2}+\frac{p_2}{2}\Big)}\Big)e^{i\frac{q_1\cdot y_1+(q_2+q_3)\cdot y_2}{\varepsilon}} \\
\times\ & \frac{1}{g(q_1)+g(q_2)+i\varepsilon^\gamma\big(l_1\cdot q_1+\big(l_2-\frac{q_3}{2}\big)\cdot q_2\big)}\frac{1}{g(q_2)+i\varepsilon^\gamma\big(l_2-\frac{q_3}{2}\big)\cdot q_2} \\
\times\ & \Big(\varphi_w\Big(y_1,l_1-\frac{q_1}{2}\Big)-\varphi_w\Big(y_1,l_1+\frac{p_1}{2}\Big)\Big)\Big(\varphi_{\tilde w}\Big(y_2,l_2-\frac{q_3}{2}-\frac{q_2}{2}\Big)- \\
& \varphi_{\tilde w}\Big(y_2,l_2-\frac{q_3}{2}+\frac{q_2}{2}\Big)\Big) \\
= & \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \mathrm{d}x_1\mathrm{d}k_1\mathrm{d}y_1\mathrm{d}l_1\mathrm{d}x_2\mathrm{d}k_2\mathrm{d}y_2\mathrm{d}l_2 \\
& \int\int\int\int\int\int \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4\mathrm{d}p_5\mathrm{d}p\Big(\sum_\sigma\prod_{\pi=1}^3\hat R(p_{i_\pi^\sigma})\delta\big(p_{i_\pi^\sigma}+ \\
& p_{j_\pi^\sigma}\big)\Big)\varphi_z(x_1,k_1)\varphi_{\tilde z}(x_2,k_2)\overline{\varphi_z(y_1,l_1)\varphi_{\tilde z}(y_2,l_2)}\,e^{-i\frac{p_1\cdot x_1+(p_2+p_3)\cdot x_2}{\varepsilon}} \\
\times\ & \Big(\overline{\varphi_w\Big(x_1,k_1-\frac{p_1}{2}\Big)}-\overline{\varphi_w\Big(x_1,k_1+\frac{p_1}{2}\Big)}\Big) \\
\times\ & \frac{1}{g(p_1)+g(p_2)-i\varepsilon^\gamma\big(k_1\cdot p_1+\big(k_2-\frac{p_3}{2}\big)\cdot p_2\big)}\frac{1}{g(p_2)-i\varepsilon^\gamma\big(k_2-\frac{p_3}{2}\big)\cdot p_2} \\
\times\ & \Big(\overline{\varphi_{\tilde w}\Big(x_2,k_2-\frac{p_3}{2}-\frac{p_2}{2}\Big)}-\overline{\varphi_{\tilde w}\Big(x_2,k_2-\frac{p_3}{2}+\frac{p_2}{2}\Big)}\Big) \\
\times\ & e^{i\frac{q_1\cdot y_1+(q_2+q_3)\cdot y_2}{\varepsilon}}\frac{1}{g(q_1)+g(q_2)+i\varepsilon^\gamma\big(l_1\cdot q_1+\big(l_2-\frac{q_3}{2}\big)\cdot q_2\big)} \\
\times\ & \frac{1}{g(q_2)+i\varepsilon^\gamma\big(l_2-\frac{q_3}{2}\big)\cdot q_2}\Big(\varphi_w\Big(y_1,l_1-\frac{q_1}{2}\Big)-\varphi_w\Big(y_1,l_1+\frac{p_1}{2}\Big)\Big) \\
\times\ & \Big(\varphi_{\tilde w}\Big(y_2,l_2-\frac{q_3}{2}-\frac{q_2}{2}\Big)-\varphi_{\tilde w}\Big(y_2,l_2-\frac{q_3}{2}+\frac{q_2}{2}\Big)\Big)
\end{aligned}$$

Now for every term we can show, by a slight modification of the proof of Lemma 17, that each of the terms is bounded by

$$\int\int\int\int\int\int \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4\mathrm{d}p_5\mathrm{d}p_6\prod_{\pi=1}^3\hat R(p_{i_\pi^\sigma})\delta\big(p_{i_\pi^\sigma}+p_{j_\pi^\sigma}\big)\langle w_4\rangle^2\langle\tilde w_4\rangle\prod_{i=1}^6\Bigg(\sum_{i=1}^{2A}\left|\frac{1}{g(p_i)}\right|^i|p_i|^i\langle p_i\rangle^{2A}\Bigg)\langle\tilde z_1-\tilde w_1\rangle^{-4A}\langle\tilde z_2-\tilde w_2\rangle^{-4A}\langle z_1-w_1\rangle^{-4A}\langle z_2-w_2\rangle^{-4A}\Big\langle\tilde z_3-\tilde w_3-\frac{p_2}{\varepsilon}-\frac{p_3}{\varepsilon}\Big\rangle^{-2A}=\langle w\rangle^2\langle\tilde w\rangle^2K_{\sigma,\varepsilon}(z,w,\tilde z,\tilde w)$$

It is not hard to check that K satisfies the requirements of Lemma 6. □

From the established bounds we now know that

$\mathcal{A}^\varepsilon(f+f_{1,\varepsilon}+f_{2,\varepsilon}+f_{3,\varepsilon})=f'(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon,k\cdot\nabla_x\lambda+D_{1,\varepsilon}\lambda\rangle+f''(\langle W_\varepsilon,\lambda\rangle)\langle W_\varepsilon\otimes W_\varepsilon, D_{2,\varepsilon}\lambda\otimes\lambda\rangle+\mathcal{O}\Big(\varepsilon^{\frac{1+\gamma}{2}}\Big)$

With $D_{1,\varepsilon}, D_{2,\varepsilon}$ given by

$$D_{1,\varepsilon}\lambda(x,k) = \int \mathrm{d}p\hat{R}(p)g(p)\frac{1}{g(p)+g(p)-\varepsilon^{\gamma}ik\cdot(p-p)}\left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k-\frac{p}{2}\right)\cdot p}(\lambda(x,k-p)-\lambda(x,k)) - \left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k+\frac{p}{2}\right)\cdot p}(\lambda(x,k)-\lambda(x,k+p))\right)\right) = \frac{1}{2}\int \mathrm{d}p\hat{R}(p)\left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k-\frac{p}{2}\right)\cdot p}(\lambda(x,k-p)-\lambda(x,k))\right) - \left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k+\frac{p}{2}\right)\cdot p}(\lambda(x,k)-\lambda(x,k+p))\right)$$

And

$$D_{2,\varepsilon}\lambda\otimes\lambda = \int \mathrm{d}p e^{i\frac{p\cdot(x_2-x_1)}{\varepsilon}}\frac{g(p)}{2g(p)-i\varepsilon^{\gamma}p\cdot(k_1-k_2)}\,\frac{1}{g(p)-i\varepsilon^{\gamma}k_1\cdot p}\left(\lambda\left(x_1,k_1-\frac{p}{2}\right)-\lambda\left(x_1,k_1+\frac{p}{2}\right)\right)\left(\lambda\left(x_2,k_2-\frac{p}{2}\right)-\lambda\left(x_2,k_2+\frac{p}{2}\right)\right)$$

We need to compute the limits of $D_{1,\varepsilon}, D_{2,\varepsilon}$ to be able to identify the limiting equation.

Lemma 19. *Define $D\lambda$ by*

$$D\lambda = \int \mathrm{d}p\,\frac{\hat{R}(p)}{g(p)}(\lambda(x,k-p)-\lambda(x,k)) \quad \text{if} \quad \gamma>0$$
$$D\lambda = \int \mathrm{d}p\,\hat{R}(p)\frac{g(p)}{g(p)^2+\left(\left(k+\frac{p}{2}\right)\cdot p\right)^2}(\lambda(x,k-p)-\lambda(x,k)) \quad \text{if} \quad \gamma=0$$

We claim that

$$\lim_{\varepsilon\to 0}\|D_{1,\varepsilon}\lambda - D\lambda\|_{L^2(\mathbb{R}^{2d})} = 0$$

Proof. In the case $\gamma=0$ this is an elementary computation, as $D_{1,\varepsilon}$ does not depend on ε anymore. In the case $\gamma>0$ we can write

$$D_{1,\varepsilon}\lambda(x,k) = \frac{1}{2}\int \mathrm{d}p\hat{R}(p)\left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k-\frac{p}{2}\right)\cdot p}\left(\int_0^1 p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right) - \left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k+\frac{p}{2}\right)\cdot p}\left(\int_0^1 p\cdot\nabla_k\lambda(x,k+tp)\mathrm{d}t\right)\right)\right)$$

Now we have by dominated convergence that

$$\lim_{\varepsilon\to 0}\int \mathrm{d}p\hat{R}(p)\left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k-\frac{p}{2}\right)\cdot p}\left(\int_0^1 p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right) - \left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k+\frac{p}{2}\right)\cdot p}\left(\int_0^1 p\cdot\nabla_k\lambda(x,k+tp)\mathrm{d}t\right)\right)\right) = \int \mathrm{d}p\,\frac{\hat{R}(p)}{g(p)}(\lambda(x,k-p)-\lambda(x,k))$$

Since

$$\left|\hat{R}(p)\left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k-\frac{p}{2}\right)\cdot p}\left(\int_0^1 p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right) - \left(\frac{1}{g(p)-i\varepsilon^{\gamma}\left(k+\frac{p}{2}\right)\cdot p}\left(\int_0^1 p\cdot\nabla_k\lambda(x,k+tp)\mathrm{d}t\right)\right)\right)\right| \leq \hat{R}(p)\left(\frac{1}{g(p)}\left(\int_0^1 |p|\cdot|\nabla_k\lambda(x,k-tp)|\mathrm{d}t\right) - \left(\frac{1}{g(p)}\left(\int_0^1 |p|\cdot|\nabla_k\lambda(x,k+tp)|\mathrm{d}t\right)\right)\right)$$

which is in $L^1(\mathbb{R}^n, \mathrm{d}p)$. The statement follows by applying dominated convergence again, since

$$\int \mathrm{d}p\hat{R}(p)\left(\frac{1}{g(p)}\left(\int_0^1 |p|\cdot|\nabla_k\lambda(x,k-tp)|\mathrm{d}t\right) - \left(\frac{1}{g(p)}\left(\int_0^1 |p|\cdot|\nabla_k\lambda(x,k+tp)|\mathrm{d}t\right)\right)\right)$$

is also in $L^2(\mathbb{R}^{2n}, \mathrm{d}x\mathrm{d}k)$ by Hoelder,Fubini and translation invariance of the integral.

□

Lemma 20.

$$\lim_{\varepsilon\to 0}\|D_{2,\varepsilon}\lambda\|_{L^2(\mathbb{R}^{2d})}=0$$

Proof. Define

$$\Phi_\varepsilon(x_1, k_1, x_2, k_2, p) = \frac{g(p)}{2g(p) - i\varepsilon^\gamma p\cdot(k_1-k_2)}\,\frac{1}{g(p)-i\varepsilon^\gamma k_1\cdot p}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\lambda(x, k-tp)\mathrm{d}t\right)\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\lambda(x, k-tp)\mathrm{d}t\right)$$

and define

$$\Phi_0(x_1,k_1,x_2,k_2,p)==\frac{g(p)}{2g(p)-p\cdot(k_1-k_2)}\,\frac{1}{g(p)-ik_1\cdot p}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right)\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right)\text{ if }\gamma=0$$

$$\Phi_0(x_1,k_1,x_2,k_2,p)=\frac{1}{2}\,\frac{1}{g(p)}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right)\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right)\text{if }\gamma>0$$

Then again using dominated convergence just as before, we know that for every $(x_1,k_1,x_2,k_2)\in\mathbb{R}^{4d}$

$$\Phi_\varepsilon(x_1,k_1,x_2,k_2,p)\to\Phi_0(x_1,k_1,x_2,k_2,p)\ \text{ in }\ L^1(\mathbb{R}^d,\mathrm{d}p)$$

$$\|D_{2,\varepsilon}\lambda\|_{L^2(\mathbb{R}^{2d})}=\iint \mathrm{d}x_1\mathrm{d}k_1\mathrm{d}x_2\mathrm{d}k_2\left|\int \mathrm{d}pe^{i\frac{p\cdot(x_2-x_1)}{\varepsilon}}\frac{g(p)}{2g(p)-i\varepsilon^\gamma p\cdot(k_1-k_2)}\,\frac{1}{g(p)-\varepsilon^\gamma k_1\cdot p}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right)\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} p\cdot\nabla_k\lambda(x,k-tp)\mathrm{d}t\right)\right|^2=\iint \mathrm{d}x_1\mathrm{d}k_1\mathrm{d}x_2\mathrm{d}k_2\left|\int \mathrm{d}pe^{i\frac{p\cdot(x_2-x_1)}{\varepsilon}}\Phi_\varepsilon(x_1,k_1,x_2,k_2,p)\right|$$

By the triangle inequality and the Riemann-Lebesgue lemma for every $(x_1,k_1,x_2,k_2)\in\mathbb{R}^{4d}$

$$\left|\int \mathrm{d}pe^{i\frac{p\cdot(x_2-x_1)}{\varepsilon}}\Phi_\varepsilon(x_1, k_1, x_2, k_2, p)\right|\leq\left|\int \mathrm{d}pe^{i\frac{p\cdot(x_2-x_1)}{\varepsilon}}\Phi_0(x_1, k_1, x_2, k_2, p)\right|+\|\Phi_\varepsilon(x_1, k_1, x_2, k_2, p)-\Phi_0(x_1,k_1,x_2,k_2,p)\|_{L^1(\mathbb{R}^d,\mathrm{d}p)}\to 0$$

Now applying dominated convergence again completes the proof. □

Now let $f_\varepsilon(t)=f(\langle W_\varepsilon(t),\lambda\rangle)+f_{1,\varepsilon}(t)+f_{2,\varepsilon}(t)+f_{3,\varepsilon}(t)$. By a classic result from probability theory we know that the process

$$M_{\varepsilon,f,\lambda}(t)=f^\varepsilon(t)-f^\varepsilon(0)-\int_0^t\mathcal{A}^\varepsilon f^\varepsilon(u)\mathrm{d}u$$

is a martingale. By tightness we know W_ε has a sub sequence(not relabeled) convergent in probability in the weak topology $L^2(\mathrm{d}x\mathrm{d}k)$, with limit W.

Assuming that $f\in C_b^3(\mathbb{R})$ We have derived that

$$\mathcal{A}^\varepsilon f^\varepsilon(u)=f'(\langle W_\varepsilon(u),\lambda\rangle)\langle W_\varepsilon(u),k\cdot\nabla_x\lambda+D_{1,\varepsilon}\lambda\rangle+f''(\langle W_\varepsilon(u),\lambda\rangle)\langle W_\varepsilon(u)\otimes W_\varepsilon(u),D_{2,\varepsilon}\lambda\otimes\lambda\rangle+\mathcal{O}\left(\varepsilon^{\frac{1+\gamma}{2}}\right)\to f'(\langle W(u),\lambda\rangle)\langle W(u),D\lambda\rangle$$

Where the convergence holds in $L^1(\mathbb{P})$ and locally uniformly in time.

So

$$\mathbb{E}\left[\sup_{t\in[0,T]}|M_{\varepsilon,f,\lambda}(t)-M_{f,\lambda}(t)|\right]\to 0$$

Where

$$M_{f,\lambda} = f\left(\langle W(t), \lambda\rangle\right) - f(\langle W(s), \lambda\rangle) - \int_0^t f'(\langle W(u), \lambda\rangle)\langle W(u), k\cdot\nabla_x\lambda + D_1\lambda\rangle \mathrm{d}u$$

This implies that $M_{f,\lambda}$ is a martingale. If we choose $f\in C_b^3(\mathbb{R})$ such that $f(v)=v$ for $|v|\leq \|W(0)\|_{L^2(\mathbb{R}^{2d},\mathrm{d}x\mathrm{d}k)}\|\lambda\|_{L^2(\mathbb{R}^{2n},\mathrm{d}x\mathrm{d}k)}$ we get that

$$M_\lambda = \langle W(t), \lambda\rangle - \langle W(s), \lambda\rangle - \int_s^t \langle W(u), k\cdot\nabla_x\lambda + D_1\lambda\rangle \mathrm{d}u$$

is a martingale, with quadratic variation 0 which implies $M_\lambda=0$ (follows for example from [2] Theorem 3.33)

6 Uniqueness of the limiting equation

We have established that any accumulation point of $\langle W_\varepsilon, \lambda\rangle$ satisfies the equation

$$\langle W(t), \lambda\rangle - \langle W(s), \lambda\rangle = \int_s^t \langle W(u), k\cdot\nabla_x\lambda + D_1\lambda\rangle \mathrm{d}u \tag{9}$$

Where in the case $\gamma>0$

$$D_1\lambda(x,k) = \int \frac{\hat{R}(p)}{g(p)}(\lambda(x,k-p)-\lambda(x,k))\mathrm{d}p$$

and for $\gamma=0$

$$D_1\lambda(x,k) = \int \frac{\hat{R}(p)g(p)}{g^2(p)+\left(\left(k-\frac{p}{2}\right)\cdot p\right)^2}(\lambda(x,k-p)-\lambda(x,k))\mathrm{d}p$$

As we already have tightness of the sequence, we will get convergence of the full sequence to the equation, provided that we can show uniqueness. If W solves (9) then $\tilde{W}(t,x,k)=W(t,x-tk,k)$ solves the equation

$$\left\langle\tilde{W}(t), \lambda\right\rangle - \left\langle\tilde{W}(s), \lambda\right\rangle = \int_s^t \left\langle\tilde{W}(u), D_{1,u}\lambda\right\rangle \mathrm{d}u \tag{10}$$

With

$$D_{1,u}\lambda(x,k) = \int \frac{\hat{R}(p)}{g(p)}(\lambda(x+up,k-p)-\lambda(x,k))\mathrm{d}p$$

$$D_{1,u}\lambda(x,k) = \int \frac{\hat{R}(p)g(p)}{g^2(p)+\left(\left(k-\frac{p}{2}\right)\cdot p\right)^2}(\lambda(x+up,k-p)-\lambda(x,k))\mathrm{d}p$$

respectively.

We will prove that (10) has a unique solution, which will imply the same for (9), assuming that $\frac{\hat{R}(p)}{g(p)}\in L^1(\mathbb{R}^n)$.

Proposition 21. *Equation (10) has unique solution for initial data $W_0\in L^2(\mathbb{R}^{2d})$, given by*

$$\tilde{W}(t,x,k) = W_0 + \sum_{n\geq 1}\int_{\Delta_n} D_{1,u_1}D_{1,u_2}\ldots D_{1,u_n}W_0 \tag{11}$$

Where Δ_n is the simplex $\{(s_1,s_2,s_2,\ldots,s_n)|\, s_n\leq s_{n-1}\leq\ldots\leq s_1\}$

Proof. We first note that $D_{1,u}$ is a bounded operator on L^2: By Jensen's inequality

$$\|D_{1,u}f(x,k)\|_{L^2}^2 = \int\int \mathrm{d}x\mathrm{d}k\left(\int \frac{\hat{R}(p)}{g(p)}(\lambda(x+up,k-p)-\lambda(x,k))\mathrm{d}p\right)^2$$

$\int\int \mathrm{d}x\mathrm{d}k\left(\int \frac{\hat{R}(p)}{g(p)}(\lambda(x+up,k-p)-\lambda(x,k))\mathrm{d}p\right)^2 \leq \left\|\frac{\hat{R}(p)}{g(p)}\right\|_{L^1(\mathbb{R}^n)}\int\int\int \mathrm{d}x\mathrm{d}k\mathrm{d}p\frac{\hat{R}(p)}{g(p)}((\lambda(x+up,k-p)-\lambda(x,k))^2\mathrm{d}p \leq 4\left\|\frac{\hat{R}(p)}{g(p)}\right\|_{L^1(\mathbb{R}^n)}^2 \|\lambda\|_{L^2(\mathbb{R}^{2n})}^2$

the same proof works for $\gamma = 0$, since $\left|\frac{\hat{R}(p)g(p)}{g^2(p) + ((k - \frac{p}{2}) \cdot p)^2}\right| \leq \frac{\hat{R}(p)}{g(p)}$. Since we have this bound we know that weak solutions also solve the strong formulations

$$\tilde{W}(t,x,k) = W_0 + \int_0^t D_{1,u}\tilde{W}(u,x,k)\mathrm{d}t$$

Let $c \leq \frac{1}{8\left\|\frac{\hat{R}(p)}{g(p)}\right\|^2_{L^1(\mathbb{R}^n)}} \leq \frac{1}{2\|D_{1,u}\|_{L^2\to L^2}}$. Then

$$\sup_{t\leq c}\left\|\tilde{W}(t,x,k)\right\|_{L^2} \leq \|W_0\|_{L^2} + \left\|\int_0^t D_{1,u}\tilde{W}(u,x,k)\mathrm{d}t\right\|_{L^2} \leq \|W_0\|_{L^2} + 4t\left\|\frac{\hat{R}(p)}{g(p)}\right\|^2_{L^1(\mathbb{R}^n)} \sup_{t\leq c}\left\|\tilde{W}(t,x,k)\right\|_{L^2}$$

Which implies

$$\sup_{t\leq c}\left\|\tilde{W}(t,x,k)\right\|_{L^2} \lesssim \|W_0\|_{L^2}$$

This implies uniqueness.

Boundedness of $D_{1,t}$ also implies convergence of the r.h.s of (11), and we can easily see that it is a solution.

□

We are now interested in the uniqueness of solutions of equation (10) if $\frac{\hat{R}(p)}{g(p)} \notin L^1$. Unfortunately we can only treat the case $\gamma > 0$ here. Consider the Operator

$$D_{N,u}\lambda(x,k) = \int_{|p|>\frac{1}{N}} \frac{\hat{R}(p)}{g(p)}(\lambda(x+up, k-p) - \lambda(x,k))\mathrm{d}p$$

Then with the same computation as above we can check that $D_{N,u}$ is bounded on L^2. We can also see that $D_{N,u}$ commutes with $D_{1,t}$, this fact is what makes the $\gamma > 0$ easier then the $\gamma = 0$ case.

We can prove that the equation

$$\partial_t\hat{W}_N(t,x,k) = D_{N,t}\hat{W}_N(t,x,k)$$

has a unique solution with initial data $\hat{W}_0$ given by

$$\hat{W}(t,x,k) = \hat{W}_0 + \sum_{n\geq 1}\int_{\Delta_n} D_{N,u_1}D_{N,u_2}\ldots D_{N,u_n}\hat{W}_0$$

Now let us consider the function $\lambda(t,x,k) = \hat{W}(T-t,x,k)$. Obviously $\lambda(T,x,k) = \hat{W}_0(x,k)$.

Now we are ready to prove uniqueness for the limiting equation:

Proposition 22. *Assume that but* $\frac{\hat{R}(p)}{g(p)}|p| \in L^1$ *and* $\gamma > 0$. *Then the equation*

$$\langle\tilde{W}(t),\lambda\rangle - \langle\tilde{W}(s),\lambda\rangle = \int_s^t \langle\tilde{W}(u), D_{1,u}\lambda\rangle\mathrm{d}u$$

has at most one solution in $C([0,T], L^2(\mathbb{R}^{2n}))$.

Proof. From the equation and the product rule, we get that

$$\langle\tilde{W}(t),\lambda(t)\rangle - \langle\tilde{W}(s),\lambda(s)\rangle = \int_s^t \langle\tilde{W}(u), \partial_u\lambda(u) + D_{1,u}\lambda(u)\rangle\mathrm{d}u$$

provided that $\lambda \in C^1([0,T], L^2(\mathbb{R}^{2n}))$.

Now we choose $\lambda(t,x,k) = \hat{W}_N(T-t,x,k)$ as above.$\lambda \in C^1([0,T], L^2(\mathbb{R}^{2n}))$ because

$$\partial_t\lambda(t) = -\partial_t\hat{W}_N(T-t,x,k) = -D_{N,T-t}\hat{W}_N(T-t,x,k) = \sum_{n\geq 1}\int_{\Delta_n} D_{N,u_1}D_{N,u_2}\ldots D_{N,u_n}D_{N,T-t}\hat{W}_0$$

Furthermore we have that

$$D_{1,u}\lambda(u)=\sum_{n\geq 1}\int_{\Delta_n} D_{N,u_1}D_{N,u_2}...D_{N,u_n}D_{1,u}\hat{W}_0\in C^0([0,T],L^2(\mathbb{R}^{2n}))$$

Provided that $\hat{W}_0\in W^{1,2}(\mathbb{R}^{2n})$, since $D_{1,t}$ is bounded from $W^{1,2}\to L^2$ and $D_{N,u}$ is bounded on L^2. In total we get

$$\langle\tilde{W}(T),\lambda(T)\rangle-\langle\tilde{W}(0),\lambda(0)\rangle=\int_0^T\langle\tilde{W}(u),(D_{1,u}-D_{N,u})\lambda(u)\rangle \mathrm{d}u$$

Now

$$\begin{aligned}(D_{1,u}-D_{N,u})\lambda(u) &= \int_{|p|<\frac{1}{N}}\frac{\hat{R}(p)}{g(p)}(\lambda(x+up,k-p)-\lambda(x,k))\\ &= \int_{|p|<\frac{1}{N}}\frac{\hat{R}(p)}{g(p)}\int_0^1 p\cdot(u\nabla_x+\nabla_k)(\lambda(x+tup,k-\mathrm{tp}))\mathrm{d}t\end{aligned}$$

By Fubini's theorem we can estimate

$$\|(D_{1,u}-D_{N,u})f\|_{L^2(\mathbb{R}^{2d})}\leq(1+|u|)\left\|\int_{|p|<\frac{1}{N}}\frac{\hat{R}(p)}{g(p)}|p|\right\|_{L^1}\|f\|_{W^{1,2}(\mathbb{R}^{2d})}$$

We can again commute and get

$$(D_{1,u}-D_{N,u})\lambda(u)=\sum_{n\geq 1}\int_{\Delta_n}D_{N,u_1}D_{N,u_2}...D_{N,u_n}(D_{1,u}-D_{N,u})\hat{W}_0$$

Which implies that $\|(D_{1,u}-D_{N,u})\lambda(u)\|_{L^2}\to 0$ as $N\to\infty$ locally uniformly in time. So from the equation we get that

$$\langle\tilde{W}(T),\widehat{W_0}\rangle=\langle\tilde{W}(T),\lambda(T)\rangle=\langle\tilde{W}(0),\lambda(0)\rangle$$

since $\hat{W}_0$ is an arbitrary function in $W^{1,2}(\mathbb{R}^{2n})$, this implies uniqueness. $\square$

In conclusion we have proven the following theorem:

Theorem 23. *Let W_ε be the solution to (4) with intial condition $W_{\varepsilon,0}\to W_0$ weakly in $L^2(\mathbb{R}^{2d})$. Then W_ε is tight in the space $C([0,\infty),E)$, where E is the unit ball in L^2 equipped with the weak topology. By Prokhorov's theorem theorem it has a subsequnce convergent in probabiltiy. Every such accumulation point of W_ε satisfies (9). In the case $\gamma>0$ (9) has a unique solution under our assumptions, which implies convergence of the whole sequence. If in addition*

$$\frac{\hat{R}(p)}{g(p)}\in L^1(\mathbb{R}^d)$$

Then the solution to (9) is also unique in the case $\gamma=0$.

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