Stochastic Wave-packet Reduction Models and Connections Between Them

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A thesis submission to the Mathematical Institute

for the degree of MSc in Mathematical and Theoretical Physics

University of Oxford

May 2024

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Abstract

Since its creation in the early 20th century, quantum mechanics has triumphed numerous experimental tests making it one of the most successful theories in modern physics. Nonetheless, it seems to draw a distinct line between intrinsically quantum behaviour and the classical limit. This dissertation will delve into the elements of dynamical reduction theories and their response to the measurement problem in the context of the GRW (Ghirardi, Rimini, and Weber) model. The reduction mechanism is shown to suppress the off-diagonal terms in the density matrix and reduce the state vector specifically forbidding quantum superpositions of macroscopically distinguishable states. We then compactify the mechanism by deriving the general form of the stochastic evolution equation needed to induce a continuous reduction, and almost sure convergence to one of the eigenstates of a pre-determined operator is shown to be true under a change of probability measure. To illustrate a unified description of dynamical trajectories, reduction rates for microscopic and macroscopic bodies are analysed by using the mass proportional CSL (Continuous Spontaneous Localisation) model leading to results compatible to both classical and quantum observations. We conclude by revealing the connections between the discrete (GRW) and continuous (CSL) collapse models and highlighting open problems with the present spontaneous collapse theories.

1 | Introduction

In this paper we address one of the most longstanding difficulties quantum theory. Namely there is a contradiction between observing quantum superpositions of states predicted by the linear Schrödinger equation, and the microscopically objective pointer states we observe in a measurement apparatus or classical phase space trajectories [1]. In orthodox interpretations, the wave-packet reduction postulate was created as a remedy to this problem by incorporating a transitional mechanism from pure states to statistical mixtures. The act of observation projects the state onto one of the observable's eigenbasis in a stochastic manner, introducing new implications such as the role of the observer dynamically altering the system - coined the *measurement problem* outlined in section 2. From a dynamical standpoint, two types of evolution are established: i) linear deterministic Schrödinger evolution and ii) non-linear, stochastic state vector reduction of quantum particles.

Of course, one can accept this dualistic description of natural phenomena by simply switching the technique used to describe microscopic and macroscopic objects. However, this means giving up on a program containing a unified derivation of the behaviour of all particles and systems, in addition to setting an ambiguity on how to describe certain mesoscopic systems that border these two regimes. Dynamical reduction models present a way to unify these two evolutions of dynamical systems by incorporating stochastic collapse to the linear Schrödinger equation. This stochastic term suppresses linear superpositions of non-localised states, and aims to present a way to deduce the dynamics of macroscopic objects from their microscopic, quantum components.

After further motivating the need of these models, we introduce the first reduction theory that was developed: GRW (Ghirardi, Rimini, Weber) and demonstrate the collapse mechanism, this will highlight underlying assumptions about these sets of theories. The later proposed CSL model (Continuous Spontaneous Localisation) is analysed at a deeper level, the stochastic

Schrödinger equation is found by starting at a general form with an additional stochastic term and adding constraints such as norm-squared preservation and effective convergence onto eigenstates over time. The model can be further improved by accounting for indistinguishable particles in quantum mechanics and making the collapse proportional to the mass. At the end of section 4 we outline flaws in the theory before exploring the connections between GRW and CSL in section 5.

2| Quantum Mechanical Formalism

A brief overview of the measurement problem is given, which illustrates the key difficulties associated with describing quantum measurement within the quantum mechanical formalism.

2.1 Von Neumann Measurement Scheme

Consider a microscopic system $\mathcal{S}$ whose basis vectors $\{|o_n\rangle\}$ from obervable O span the whole Hilbert space $\mathcal{H}_\mathcal{S}$. Now let $\mathcal{S}$ interact with a measurement apparatus $\mathcal{A}$ which has a set of mutually orthogonal basis vectors $\{|a_n\rangle\}$ spanning a Hilbert space $\mathcal{H}_\mathcal{A}$. $|a_n\rangle$ correspond to macroscopically distinguishable pointer states[1] according to the measurement of observable O. Finally, we assume the interaction between systems $\mathcal{S}$ and $\mathcal{A}$ is linear, that is, governed by Schrödinger evolution, and there is a perfect correlation between the initial state of $\mathcal{S}$ and the final state of $\mathcal{A}$.

$$|o_n\rangle \otimes |a_0\rangle \qquad \xrightarrow[\text{evolution}]{\text{time}} \qquad |o_n\rangle \otimes |a_n\rangle \tag{2.1}$$

where the measurement apparatus was set to be in initial ready position $|a_0\rangle$, by reading off the measurement instrument we can easily discover that the microscopic system $\mathcal{S}$ was in state $|o_n\rangle$. Now let $\mathcal{S}$ be in a superposition $\sum_n c_n|o_n\rangle$ of basis states, the Hilbert product space

[1]These macroscopically distinguishable states are read off the screen of the measuring instrument with no ambiguity.

$\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A}$ evolves according to

$$\Big(\sum_n c_n|o_n\rangle\Big) \otimes |a_0\rangle \qquad \xrightarrow[\text{evolution}]{\text{time}} \qquad \sum_n c_n|o_n\rangle \otimes |a_n\rangle. \tag{2.2}$$

The right hand side is found to be in a superposition of entangled states between the system and the apparatus, this does not align with the one distinct pointer state observed in reality. The quantum mechanical formalism suggests that this is only the pre-measurement state and the collapse postulate induces a measurement to be one of terms in the superposition $|o_n\rangle \otimes |a_n\rangle$ with non-zero c_n. As mentioned before, this incorporates a very unsatisfactory feature of quantum mechanics, adding a separate dynamical process to the evolution of quantum particles. Moreover, it becomes ambiguous when someone asks; who causes, and at what time does Schrödinger evolution to halt and a spontaneous collapse mechanism to take over[2] [2]. This defines the first part of the measurement problem, termed the *'problem of definite outcomes'* as without the reduction postulate it is difficult to interpret such a superposition[3], onto the definite pointer positions that are perceived as a result of an experimental measurement.

The second part of the measurement problem goes as follows: The right-hand side of Eq. 2.2 is an entangled state of the apparatus and the measured system, this is not an eigenstate of the observable O - which is embedded solely in space $\mathcal{H}_\mathcal{S}$. In the standard interpretation of quantum mechanics an observable corresponding to a physical quantity has a definite value if and only if the system is in an eigenstate of the observable. This suggests that the states $|a_n\rangle$ do not have definite values associated to them as they are not eigenstates of O thus they are not uniquely defined[4] [3]. A two-qubit example is given in appendix A to illustrate this second part of the measurement problem, termed *'the preferred-basis problem'*.

[2]Von-Neumann himself as well as many other prominent scientists at the time were lead to the conclusion that consciousness was the non-linear entity that halted unitary evolution of quantum objects, and allowed them to evolve into classical statistical mixtures.

[3]One can otherwise can interpret the measurement as the observer entangling themselves with an eigenstate of the apparatus, however, this only extends the superposition to a larger system and no point is defined where the superposition terminates.

[4]For any choice of system states $|o_n\rangle$ we can find a corresponding set of apparatus states $|a_n\rangle$ and vice versa.

3| GRW Model

3.1 The Reduction Mechanism

For Quantum Mechanics with Spontaneous Localisations (QMSL), we no longer model measurement as an external intervention of the system but as a dynamical process which localises the wavefunction in the position space basis. The principles of spontaneous localisation models are as follows:

1. Each particle constituting to a n-particle wavefunction $|\psi\rangle$ will independently experience a spontaneous localisation according to Poisson process $N_t \sim \mathcal{P}(\lambda t)$ characterised by mean rate λ.

2. When there is no spontaneous localisation, the wave-function will evolve under unitary Schrödinger evolution.

3. For a localisation, the state vector undergoes the transition

$$|\psi\rangle \longrightarrow \frac{|\psi_x^i\rangle}{\sqrt{\langle\psi_x^i|\psi_x^i\rangle}}, \tag{3.1}$$

where $|\psi_x^i\rangle = L_x^i|\psi\rangle$. L_x^i is a norm-reducing, positive, self-adjoint, linear operator on particle i in the n-particle Hilbert space $\mathcal{H}$.

4. The spatial probability density for a localisation to occur is determined by the quantity:

$$P_i(x) = \|\langle\psi_x^i|\psi_x^i\rangle\|, \tag{3.2}$$

requiring that, in d spatial dimensions,

$$\int dx^d \, (L_x^i)^\dagger L_x^i = 1. \tag{3.3}$$

5. The localisation operator for GRW has Gaussian form

$$L_x^i = \Big(\frac{1}{\pi r_C^2}\Big)^{\frac{3}{4}} e^{-\frac{(q_i - x)^2}{2r_C^2}}, \tag{3.4}$$

q_i being the position operator for particle i.

QMSL models enforce the localisation onto the position basis because we observe macroscopically distinct position states in classical physics. Theories later developed, such as the CSL model generalise this approach by allowing the localisation process to be chosen in any basis. There are two new parameters introduced in QMSL models, the collapse rate λ and the localisation width r_C. For our purposes these values should be treated as constants of nature and are universal for all particles and systems. These constants were chosen by GRW to be $\lambda \simeq 10^{-16}$ s^{-1}, $r_C \simeq 10^{-5}$, λ should be a value low enough such that the probability for individual particles to spontaneously collapse is negligible, and high enough such that macroscopic systems of many particles exhibit frequent localisation ($N\lambda \sim 10^7$ s^{-1} for an Avogadro number N of particles[1]).

We show a simple example of the localisation process mechanism by considering the a superposition of two Gaussians in a one dimensional wave function

$$\psi(x) = \frac{1}{\mathcal{N}}\Big[e^{-\frac{1}{2\sigma^2}(x+a)^2} + e^{-\frac{1}{2\sigma^2}(x-a)^2}\Big], \tag{3.5}$$

$\mathcal{N}$ is a normalisation constant. For effective localisation we let $\sigma \gg r_C$ and $r_C \ll a$, the width of the Gaussians are large compared to the localisation width, whereas the distance between Gaussians are spatially far compared to the localisation width. After some time has passed, the system will experience a hitting from operator L_x^i, let us say that principle 4 chose the localisation at position a such that.

$$\psi(x) \longrightarrow \psi_a(x) = \frac{1}{\mathcal{N}_a}\Big[e^{-\frac{(x-a)^2}{2r_C^2}} e^{-\frac{1}{2\sigma^2}(x+a)^2} + e^{-\frac{1}{2}(\frac{1}{\sigma^2}+\frac{1}{r_C^2})(x-a)^2}\Big]. \tag{3.6}$$

Because $\sigma \gg r_C$, we can ignore the $\frac{1}{r_C^2}$ term in the second exponential. As $r_C \ll a$, the former multiplicative factor in the first term is heavily suppressed when $x = -a$ and the latter is suppressed when $x = a$. This allows $\psi_a(x)$ to model a localised Gaussian wavefunction about a, squashing the superposition held by $\psi(x)$ previously. Postulate 4 enables us to sample from

[1]For the GRW model, the original authors show that one if particle in a macroscopic body localises then the whole body will localise, - termed the amplification mechanism. [1]

a probability distribution that assigns more weight to the frequency of localisations around the peaks $x = a$ and $x = -a$, this provides more weight to states which have one term in the superposition suppressed much more than the other.

3.2 Statistical Operator Formalism

To show how the dynamical reduction mechanism transforms pure states to statistical mixtures, we adopt the statistical operator formalism $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ [4]. When a pure wave-vector $|\psi\rangle$ undergoes a localisation about coordinate x, it is not known where the state has localised to, only a probability distribution of where the localisation can be found is given.

$$\begin{aligned} |\psi\rangle\langle\psi| \quad \longrightarrow \quad & \int d^3x P(\mathbf{x}) \frac{|\psi_{\mathbf{x}}\rangle\langle\psi_{\mathbf{x}}|}{\|\langle\psi_{\mathbf{x}}|\psi_{\mathbf{x}}\rangle\|} \\ = & \int d^3x \, L^i_{\mathbf{x}} |\psi\rangle\langle\psi| L^i_{\mathbf{x}} \equiv T[|\psi\rangle\langle\psi|] \end{aligned} \tag{3.7}$$

Postulates 1 and 2 suggest there is a probability λdt of the hitting (map T) to occur in the time interval dt, and probability $1 - \lambda dt$ for ρ to evolve according to the standard Liouville-von Neumann equation.

$$\rho(t + dt) = (1 - \lambda dt)\Big[\rho(t) - \frac{i}{\hbar}[H, \rho(t)]\Big] + \lambda dt \, T[\rho(t)], \tag{3.8}$$

finding the derivative by taking the limit

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] - \lambda(\rho(t) - T[\rho(t)]). \tag{3.9}$$

This describes the time evolution of a particle subject to localisation frequency λ. For the case of a free particle in one-dimension x under Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{d}{dx^2}$, the matrix elements of the density matrix are

$$\begin{aligned} \frac{\partial}{\partial t}\langle x'|\rho(t)|x''\rangle &= iH\langle x'|\rho(t)|x''\rangle - i\langle x'|\rho(t)|x''\rangle H - \lambda\Big[\langle x'|\rho(t)|x''\rangle - \langle x'|T[\rho(t)]|x''\rangle\Big] \\ &= \frac{i\hbar}{2m}\Big[\frac{\partial}{\partial x'^2} - \frac{\partial}{\partial x''^2}\Big]\langle x'|\rho(t)|x''\rangle - \lambda\Big[1 - e^{\frac{1}{4r_c^2}(x'-x'')}\Big]\langle x'|\rho(t)|x''\rangle. \end{aligned} \tag{3.10}$$

Postulate 5 was implemented in the last step, the density matrix under map T is

$$\langle x'|T[\rho]|x''\rangle = e^{-\frac{1}{4r_C^2}(x'-x'')^2}\langle x'|\rho|x''\rangle, \tag{3.11}$$

to check that $T[\rho]$ preserves density matrix properties we can easily find $\langle x'|T[\rho]|x'\rangle = \langle x'|\rho|x'\rangle$, diagonal entries of the density matrix are not affected by the imposed localisation. As a corollary the trace and positivity is preserved under this mapping:

$$\frac{d}{dt}\int dx^3\,\langle x|\rho|x\rangle = \frac{d}{dt}\mathrm{Tr}\rho = 0. \tag{3.12}$$

Another thing to highlight is Eq 3.9 takes the form of a generator for a quantum dynamical semi-group (Eq B.5 with $\sum_i A_i A_i^\dagger = \mathbb{I}$ and $T[\rho] = \sum_i A_i \rho A_i^\dagger$). The generator is a particular Lindblad generator with Lindblad operators that satisfy the sum rule Eq. 3.3. More information about quantum dynamical semi-groups can be found in Appendix B.

4 | Continuous Spontaneous Localisation

4.1 Stochastic Processes in Hilbert Space

There are some important remarks to be made regarding the GRW theory to model many particle systems.

1. *preferred basis* - Why is the localisation only to happen in the position basis and not another basis such as momentum?

2. *indistinguishable particles* - The dynamics does not preserve the symmetry character of wavefunctions describing systems of identical particles.

3. *amplification mechanism* - How do systems of many particles collapse at higher rates than microscopic systems.

4. *compactness* - The reduction mechanism of GRW are a set of principles which replace the wave-packet reduction postulate. A compact equation to get to Eq. 3.9 is not presented.

Ideas of spontaneous localisation from G. Ghirardi, A. Rimini, T. Webber were borrowed by P.

Pearle to develop a continuous physical collapse theory using the Itô formalism [5]. This was established into the Continuous Spontaneous Localisation Model (CSL) by Ghirardi, Pearle and Rimini [6].

4.1.1 Linear Itô formula

The construction in [6] is followed to get to the stochastic differential equation describing the evolution of the state vector $|\psi(t=0)\rangle$ including a continuous dynamical reduction. A general linear differential equation of the Itô form is postulated

$$d|\psi\rangle = \Big[Cdt + \mathbf{A}\cdot d\mathbf{B}_t\Big]|\psi\rangle \tag{4.1}$$

where C is an operator, $\mathbf{A} = \{A_i\}$ is a set of operators and $\mathbf{B}_t = \{B_i\}$ is a set of real Wiener processes each defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfying

$$\mathbb{E}[dB_i] = 0, \qquad \mathbb{E}[dB_i dB_j] = \gamma\delta_{ij}dt, \tag{4.2}$$

γ being a real positive constant. After time t, the resulting state vector $\psi_\omega(t)$ is produced, dictated by a realisation of Wiener process $B_t(\omega)$ and evolution of Eq 4.1. We consider an ensemble Ω of normalised state vectors $\{\psi_\omega(t)/\|\psi_\omega(t)\|, \forall\,\omega \in \Omega\}$ termed the *raw* ensemble under measure $\mathbb{P}(d\omega)$, the important feature of the raw ensemble is that each state vector $\psi_\omega(t)$ is tied to each realisation $B_t(\omega)$ with one-to-one weighting. We require that the norm squared of the state vector is conserved over the sample space $d\mathbb{E}[\||\psi\rangle\|^2] = \mathbb{E}[d\||\psi\rangle\|^2] = 0$, utilising the Itô formula[1]

$$\begin{aligned} d\||\psi\rangle\|^2 &= d(\langle\psi|\psi\rangle) = \langle d\psi|\psi\rangle + \langle\psi|d\psi\rangle + \langle d\psi|d\psi\rangle \\ &= \langle\psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi\rangle\cdot d\mathbf{B} + \langle\psi|(C+C^\dagger)|\psi\rangle dt + \langle\psi|\mathbf{A}^\dagger\cdot\mathbf{A})|\psi\rangle\gamma dt \end{aligned} \tag{4.3}$$

It follows that the following condition must be satisfied

$$C + C^\dagger = -\gamma\mathbf{A}^\dagger\cdot\mathbf{A}. \tag{4.4}$$

[1] notation $|d\psi\rangle = d|\psi\rangle$ is used and any order of dt above first order is not considered

Applying this condition to Eq. 4.1 and denoting $-\frac{i}{\hbar}H$ the anti-Hermitian part of C, the resulting linear stochastic differential equation becomes

$$d|\psi_t(\omega)\rangle = \left[-\frac{i}{\hbar}H\,dt + \mathbf{A}\cdot d\mathbf{B} - \frac{\gamma}{2}\mathbf{A}^\dagger\cdot\mathbf{A}\,dt\right]|\psi_t(\omega)\rangle. \tag{4.5}$$

The first term denotes the standard Schrödinger evolution whereas the last two terms contribute to the collapse mechanism. We now have the remaining part of Eq. 4.3 which we will write as

$$d\||\psi\rangle\|^2 = 2\mathbf{R}\cdot d\mathbf{B}\,\||\psi\rangle\|^2 \tag{4.6}$$

where $\mathbf{R} = \frac{1}{2}\langle\psi|(\mathbf{A}+\mathbf{A}^\dagger)|\psi\rangle/\||\psi\rangle\|^2$. Although the average process $B_t(\omega)$ over all $\omega\in\Omega$ in time interval $(0,\,t)$ conserves the norm squared of ψ, Eq. 4.6 tells us that a given ω does not necessarily do so, amounting to an *unnormalised* realisation $|\psi_\omega(t)\rangle$. This is accepted in the raw ensemble as every wave function in the ensemble is then normalised, but this alone will not lead to the satisfactory localisations we need, an additional precept is equipped which will give a weighting to each ψ_ω inducing localisations onto the eigenspace of A_i.

As a motivation, consider an unnormalised wave function $\psi(x)$ in Hilbert space $L^2(\mathbb{R})$ s.t. $\int_{\mathbb{R}}\|\psi(x)\|^2\,dx<\infty$. If one knew the state $\psi(x)$ then, for every dx_i they could find $\psi(x_i)\,dx$ and then normalise it, but this would tell us nothing about the initial state $\psi(x)$ and we would end up with uniformly dense, normalised wave functions on the real line. Instead, in standard QM procedure all possible elements $x\in\mathbb{R}$ are first considered by the wave function in a so called 'quantum superposition', each element consists of the possible realisation $\psi(x)$ if it were to be found at position x. The continuous distribution of realisations (whose values are $\psi(x)$) is then normalised such that $\int_{\mathbb{R}}\|\psi(x)\|^2\,dx=1$ to make a continuous probability amplitude ready for the Born rule. The measure $\|\psi(x)\|^2\,dx$ will assign more weight to increments $\psi(x_i)\,dx$ (eigenvalues x_i) where $\psi(x_i)$ takes larger amplitudes, allowing for a higher frequency that measurements are observed in that increment[2] [7].

[2]We have described the Born rule in such a way such that the logic can be suitably mapped from space $x\in\mathbb{R}$ to space $\omega\in\Omega$.

Similarly, utilising the raw ensemble will lead to a continuous array of normalised wave functions, distributed with the same weights as elements $\mathbb{P}(d\omega)$, onto the eigenmanifold of A_i. This is no good for assigning more weight to those $\psi_\omega(t)$ with larger amplitudes w.r.t. A_i and the convergence of paths onto eigenspaces of A_i. So, following the motivation above, a quantum superposition[3] is considered for the wave function at time $t = 0$ with amplitudes (w.r.t A_i) $\psi_\omega(t)$ - the possible wave functions at time t. To normalise amplitudes over the sample space we set the sum of the norms squared to 1

$$\int_\Omega \| |\psi_\omega(t)\rangle \|^2 \mathbb{P}(d\omega) = 1. \tag{4.7}$$

A new probability measure

$$\mathbb{Q}(d\omega) = \| |\psi_\omega(t)\rangle \|^2 \mathbb{P}(d\omega) \tag{4.8}$$

is adopted to account for this precept[4]. By extension of the Born rule, the normalised weights/probability amplitudes correspond to the probability/frequency that $|\psi_\omega(t)\rangle$ is measured over repeated experiments. For this reason we call the ensemble of normalised wave functions $\{|\psi_\omega(t)/\|\psi_\omega(t)\|, \forall \omega \in \Omega\}$ under probability measure $\mathbb{Q}(d\omega)$ to be the *physical* ensemble[5].

When evaluating observables under the physical ensemble, e.g. for operator $\hat{O}_{\delta t}$ which acts on system at time δt, satisfying $0 \leq \delta t \leq t$

$$\int_\Omega \hat{O}_{\delta t}(\omega) \underbrace{\| |\psi_\omega(t)\rangle \|^2 \mathbb{P}(d\omega)}_{\mathbb{Q}(d\omega)} = \mathbb{E}[\hat{O}]_t \tag{4.9}$$

One finds when working under measure $\mathbb{Q}(d\omega)$, that the expected value of $\hat{O}$ may be dependent on the state of the wave function at an arbitrary later time t even if the operator acted on the system at previous time δt, $0 \leq \delta t \leq t$. This is unphysical as it suggests future knowledge of the state vector contained in the information of a measurement, but we shall show that causality is not violated if the the Wiener process $Y_t = \| |\psi_\omega(t)\rangle \|^2$ is a continuous-time martingale with

[3]We consider a superposition of states $\psi_\omega(t)$ as Eq. 4.5 is a linear differential equation.

[4]This is also a generalisation of postulate 4 from the GRW model stating the probability of a hit taking place is proportional to the squared norm of the wave function after the hit.

[5]The elements of the sample space Ω and Filtration $\mathcal{F}$ for the physical ensemble are the same as the ones for the raw ensemble, but measure $\mathbb{Q}(\omega)$ is now considered.

respect to stochastic process B_t.

$$\mathbb{E}[Y_t \,|\, \{B_\tau,\, \tau \geq 0\}] = Y_\tau, \qquad \forall\, \tau \leq t \tag{4.10}$$

That is, the best estimate for ψ_t is $\psi_{\delta t}$ [8]. This property can be proved by understanding that Eq. 4.7 implies $\mathbb{E}[Y_t] = 1$, and from 4.6

$$Y_t = \exp\Big(\int_0^t 2R_s\, dB_s - 2\gamma \int_0^t R_s R_s\, ds \Big) = \exp \mathcal{Y}_t \tag{4.11}$$

with

$$d\mathcal{Y}_t = 2R_t\, dB_t - 2\gamma R_t R_t\, dt + \frac{1}{2}\big(4\gamma R_t R_t\big)\, dt. \tag{4.12}$$

Remembering that $dY_t = d\mathcal{Y}_t \exp(\mathcal{Y}_t)$ and $\mathbb{E}[\exp(\mathcal{Y}_t)] = 1$, we can obtain the expression

$$Y_t - Y_{\delta t} = 2 \int_{\delta t}^t R_s \exp(\mathcal{Y}_s)\, dB_s \tag{4.13}$$

then Y_t satisfies the local martingale property $\mathbb{E}[Y_t - Y_{\delta t} \,|\, \{B_{\delta t},\, \delta t \geq 0\}] = 0$ for stopping time t. We have thus shown that observable quantities under measure $\mathbb{Q}$ will not depend on the state of the wave function after the measurement and Eq. 4.9 becomes

$$\int_\Omega \hat{O}_t(\omega)\, \| |\psi_\omega(t)\rangle \|^2 \mathbb{P}(d\omega) = \mathbb{E}[\hat{O}_t] \tag{4.14}$$

evaluated for any $t > 0$.

4.1.2 Non-linear Itô formula

By finding elements in the physical ensemble, we could evaluate wave functions evolved through Eq. 4.5 in interval $(0, t)$ by following the procedure of using $\mathbb{Q}$ to find a probability density for observing the normalised wave functions

$$|\chi_\omega(t)\rangle = \frac{|\psi_\omega(t)\rangle}{\| |\psi_\omega(t)\rangle \|} \tag{4.15}$$

This is one perfectly viable method of evaluating wave functions evolving under CSL [9]. Alternatively, we could compactify these two prescriptions into one SDE by defining a new

stochastic process that incorporates the weighting prescription $\mathbb{Q}$, then write this new process in terms of standard BM defined in Eq. 4.5. This can be accomplished by calculating the physical probabilities under $\mathbb{Q}$ every Δt time steps, and taking the limit $\Delta t \to 0$. from Eqs 4.8 and 4.13, the probability under $\mathbb{Q}$ that random variable $\Delta_i B$ is in increment dx_i is given by

$$\begin{aligned}\mathbb{Q}(\Delta_i B \in dx_i) &= (1 + 2R_t \Delta_i B)\mathbb{P}(\Delta_i B \in dx_i) \\ &= (1 + 2R_t x)\frac{1}{\sqrt{2\pi\gamma\Delta t}} e^{-\frac{x^2}{2\gamma\Delta t}}.\end{aligned} \tag{4.16}$$

In the limit $\Delta t \to 0$, $\Delta_i B \to 0$ we can take the first term in exponential form as in Eq. 4.11

$$\mathbb{Q}(\Delta_i B \in dx_i) = \frac{1}{\sqrt{2\pi\gamma\Delta t}} e^{-\frac{x^2}{2\gamma\Delta t} + 2R_t x} = \frac{1}{\sqrt{2\pi\gamma\Delta t}} e^{\frac{(x - 2R_t\gamma\Delta t)^2}{2\gamma\Delta t}} \tag{4.17}$$

to find a new Gaussian random variable $d\tilde{B} = dB - 2\gamma R_t dt$. The collection of Gaussians $\tilde{B}_t$ is not a Brownian motion under $\mathbb{P}$ because of the drift term $\mathbb{E}[d\tilde{B}] = 2\gamma R dt$, we shall see it is a Brownian motion under $\mathbb{Q}$ from the following theorem [10], shown to be true in Appendix C.

Theorem (Girsanov) Let $d\tilde{B}_i = B_i - h(t_i)dt$ be a stochastic process with dB_i defined above (Eqs. 4.2) and a drift, then $d\tilde{B}_i$ under a measure $\mathbb{Q}$ defined as

$$d\mathbb{Q} = \exp\Big(\int_0^T h(s_i)\, dB_i(s) - \frac{1}{2}\int_0^T h(s_i)^2\, ds \Big) d\mathbb{P} \tag{4.18}$$

is a Brownian motion for a functional $h(s)$ with natural filtration $\mathcal{F}_s$.

Hence the process

$$\tilde{B}_t = B_t - \int_0^t 2\gamma R_\tau d\tau \tag{4.19}$$

is a Brownian motion under $\mathbb{Q}$. Expressing the processes B_t and $\tilde{B}_t = B_t - 2\gamma R_t dt$ on the change in the normalised wave functions[6] $|\chi\rangle$ and $|\phi\rangle$ (Eq. 4.15)

$$\begin{aligned} d|\chi\rangle &= \Big[-\frac{i}{\hbar} H\, dt + \gamma\Big(-\frac{1}{2}\mathbf{A}^\dagger \cdot \mathbf{A} - \mathbf{A}\cdot\mathbf{R} + \frac{3}{2}\mathbf{R}\cdot\mathbf{R}\Big) dt + (\mathbf{A} - \mathbf{R}) \cdot d\mathbf{B}\Big] |\chi\rangle \\ d|\phi\rangle &= \Big[-\frac{i}{\hbar} H\, dt + \gamma\Big(-\frac{1}{2}\mathbf{A}^\dagger \cdot \mathbf{A} - \mathbf{A}\cdot\mathbf{R} + \frac{3}{2}\mathbf{R}\cdot\mathbf{R}\Big) dt + (\mathbf{A} - \mathbf{R}) \cdot d\tilde{\mathbf{B}}\Big] |\phi\rangle \end{aligned} \tag{4.20}$$

The equation for the process $|\phi\rangle$ under $\mathbb{P}$ will be driven by the same Brownian motion as

[6] from Eq. 4.15, Eq. 4.20 can be shown to be true using Itô's rules.

that $|\chi\rangle$ under $\mathbb{Q}$. They are equivalent in that sense, but obey different laws when both are considered under the same probability measure. Thus, we can deduce that $|\chi\rangle$ under $\mathbb{Q}$ obeys the same law as $|\phi\rangle$ under $\mathbb{P}$.

$$\begin{aligned} d|\phi\rangle &= \Big[-\frac{i}{\hbar} H\, dt + \gamma\Big(-\frac{1}{2}\mathbf{A}^\dagger \cdot \mathbf{A} - \mathbf{A}\cdot\mathbf{R} + \frac{3}{2}\mathbf{R}\cdot\mathbf{R}\Big)\, dt + (\mathbf{A}-\mathbf{R})\cdot\,(d\mathbf{B} - 2\gamma R_t\, dt)\Big]|\phi\rangle \\ &= \Big[-\frac{i}{\hbar} H\, dt - \frac{\gamma}{2}\Big(-\mathbf{A}^\dagger\cdot\mathbf{A} - 2\mathbf{A}\cdot\mathbf{R} + \mathbf{R}\cdot\mathbf{R}\Big)\, dt + (\mathbf{A}-\mathbf{R})\cdot\, d\mathbf{B}\Big]|\phi\rangle \end{aligned} \tag{4.21}$$

For self-adjoint A_i the evolution equation is

$$\begin{aligned} d|\phi\rangle &= \Big[-\frac{i}{\hbar} H\, dt - \frac{1}{2}\gamma(\mathbf{A}-\mathbf{R})^2\, dt + (\mathbf{A}-\mathbf{R})\cdot d\mathbf{B}\Big]|\phi\rangle \\ \mathbf{R} &= \langle\phi|\mathbf{A}|\phi\rangle. \end{aligned} \tag{4.22}$$

We have now arrived at a non-linear stochastic differential equation describing the evolution of a normalised wave function under spontaneous localisation. Eq. 4.22 tells us about the evolution of ϕ when it is under a continuous norm-squared weighting of elements in the sample space of Gaussian random variables and renormalisation. We shall show how Eq. 4.22 under $\mathbb{P}$ and/or Eq. 4.5 under the prescripted measure $\mathbb{Q}$ lead to localisations on the eigenspace of the self-adjoint, commuting operators A_i in Section 4.1.4.

4.1.3 The Statistical Operator

The statistical operator is constructed by finding the average outer product of normalised vectors $|\phi_\omega(t)\rangle$ for all $\omega \in \Omega$ under measure $\mathbb{Q}$. Calculating

$$\rho = \int_\Omega \frac{|\psi_\omega\rangle}{\||\psi_\omega\rangle\|}\,\frac{\langle\psi_\omega|}{\|\langle\psi_\omega|\|}\,\||\psi_\omega\rangle\|^2\,\mathbb{P}(d\omega) = \mathbb{E}[|\psi\rangle\langle\psi|]_{\mathbb{P}} \tag{4.23}$$

is found to be equal to the average outer product of the unnormalised wavefuctions over $\mathbb{P}$. To find the Liouville-von Neumann equation of the CSL model, the differential of the outer product $|\psi\rangle\langle\psi|$ is calculated using Itô's formula:

$$\begin{aligned} d\big(|\psi\rangle\langle\psi|\big) &= -\frac{i}{\hbar}\big[H, |\psi\rangle\langle\psi|\big]dt + \big\{A_\alpha dB_\alpha, |\psi\rangle\langle\psi|\big\}dt - \frac{\gamma}{2}\big\{A_\beta^\dagger A_\beta, |\psi\rangle\langle\psi|\big\}dt \\ &\quad + dB_\epsilon A_\epsilon|\psi\rangle\langle\psi|A_\eta^\dagger\, dB_\eta - \frac{i}{\hbar}\big[H|\psi\rangle, \langle\psi|A_\mu\big]\, dt\, dB_\mu - \frac{\gamma}{2}\big\{A_\nu|\psi\rangle, \langle\psi|A_\sigma^\dagger A_\sigma\big\}\, dt\, dB_\nu, \end{aligned} \tag{4.24}$$

here, $\{\cdot,\cdot\}$ denotes the anti-commutator. Then, orders of dt greater than 1 are neglected and the expectation value is calculated

$$\begin{aligned}\frac{d}{dt}\rho &= \frac{d}{dt}\mathbb{E}[|\psi\rangle\langle\psi|]_{\mathbb{P}} \\ &= -\frac{i}{\hbar}\big[H,\rho\big] - \frac{\gamma}{2}\big\{A_\beta^\dagger A_\beta,\rho\big\} + \mathbb{E}\big[dB_\epsilon A_\epsilon|\psi\rangle\langle\psi|A_\eta^\dagger dB_\eta\big].\end{aligned} \tag{4.25}$$

With relations 4.2 and one obtains

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}\big[H,\rho(t)\big] + \gamma\mathbf{A}\rho(t)\cdot\mathbf{A}^\dagger - \frac{\gamma}{2}\big\{\mathbf{A}^\dagger\cdot\mathbf{A},\rho(t)\big\} \tag{4.26}$$

arriving at the evolution equation for the statistical operator of normalised wave function ϕ. Just as the GRW model, Eq 4.26 takes the form of the Lindblad master equation which is the generator for the quantum dynamical semigroup. This is a reassuring result: from using Stochastic processes in Hilbert space, we have come to the same results as the ones obtained in open quantum systems in contact with their surroundings [11].

4.1.4 State Vector Reduction

Now we shall show how the modified Schrödinger equation of the norm preserving type (Eq. 4.22 in one dimension ($\mathbf{A} = \{\hat{A}\}$) induces a continuous evolution onto the eigenstates of a self-adjoint operator $\hat{A}$ that commutes with the Hamiltonian $\hat{H}$, we follow similar methods from [12].

Using the Itô product rule to find the expectation value $G_t = \langle\phi_t|\hat{G}|\phi_t\rangle$ of an observable $\hat{G}$ acting on the Hilbert space of the system $\mathcal{H}$.

$$\begin{aligned}dG_t = -i\langle\phi_t|\big[\hat{G},\hat{H}\big]|\phi_t\rangle\,dt + \gamma\langle\phi_t|\Big(\hat{A}\hat{G}\hat{A} - \frac{1}{2}\big\{\hat{A}^2,\hat{G}\big\}\Big)|\phi_t\rangle\,dt \\ +\langle\phi_t|\big\{(\hat{G}-G_t),\,(\hat{A}-A_t)\big\}|\phi_t\rangle\,dB_t\end{aligned} \tag{4.27}$$

It follows that G_t is a stochastic process with the first drift term relating to the Ehrenfest theorem [13] and the second term of the Lindblad type resulting from the diffusive dynamics of the state vector. The last term describes the volatility of the stochastic process G_t and is

given by the expected value of the covariance of $\hat{G}$ and $\hat{A}$. If $\hat{G}$ commutes with $\hat{H}$ and $\hat{A}$, then

$$dG_t = 2\big(\langle\phi_t|\hat{G}\hat{A}|\phi_t\rangle - G_t A_t\big)\, dB_t. \tag{4.28}$$

In the case that $\hat{G}$ is the operator $\hat{A}$, the process for expectation value of $\hat{A}$: A_t is

$$dA_t = 2V_t dB_t, \qquad V_t = \langle\phi_t|\big(\hat{A} - A_t\big)^2|\phi_t\rangle \tag{4.29}$$

where V_t is the stochastic process for the variance of A_t. V_t has the property that if ϕ_t is an eigenstate of the $\hat{A}$, then $V_t = 0$ for time t. As before we can show that A_t is a martingale by finding A_t

$$A_t = A_0 + 2\int_0^t V_\tau\, dB_\tau, \tag{4.30}$$

and taking the conditional mean of the process at time t minus a previous time s. A_0 is the initial expectation value of $\hat{A}$ with the state $|\phi_{t=0}\rangle$.

$$\mathbb{E}[A_t - A_s \,|\, \{A_s, s \geq 0\}] = 0, \qquad \forall\, t \geq s \tag{4.31}$$

Where A_t is adapted to the natural filtration $\mathcal{F}_t$. Furthermore, the process V_t will always be bounded by the eigenvalues of $\hat{A}$ acting on complete Hilbert spaces[7], if $\hat{A}$ has eigenvalues $\{\lambda_k, k = 0, 1 \ldots n\}$ then $|A_t|$ is bounded by $\max(|\lambda_-|, \lambda_+|)$ implying[8] that $\int_0^t V_u^2 du < \infty$. Thus A_t satisfies the requirements to be a martingale with respect to Wiener process B_t, this is expected as an operator $\hat{A}$ which commutes with the Hamiltonian, is expected to be a conserved quantity, a requirement made by standard Schrödinger dynamics.

We now focus on the dynamics of the mass variance V_t, written as $V_t = A_t^{(2)} - (A_t)^2$ where notation $A_t^{(2)} = \langle\phi_t|\hat{A}^2|\phi_t\rangle$. Itô's formula gives

$$dV_t = dA_t^{(2)} - 2A_t\, dA_t - (dA_t)^2 \tag{4.32}$$

[7]Later on we associate $\hat{A}$ with the second quantised mass density operator $\hat{\mathcal{M}}(x)$, the eigenvalues of which are also bounded by the *square root* of the correlation function $g(x)$ which defines them.

[8]λ_- and λ_+ are the lowest and highest eigenvalues of $\hat{H}$ respectively.

From Eq. 4.28 we deduce that $dA_t^{(2)} = 2\big(A_t^{(3)} - A_t A_t^{(2)}\big)\, dB_t$ to obtain

$$dV_t = -4\gamma V_t^2\, dt + 2\beta_t\, dB_t \tag{4.33}$$

in terms of the skewness of the eigenvalue distribution at time[9] t, $\beta_t = \langle\phi_t|(\hat{A} - A_t)^3|\phi_t\rangle$. As before

$$V_t = V_0 - 4\gamma \int_0^t V_\tau^2\, d\tau + 2\int_0^t \beta_\tau\, dB_\tau \tag{4.34}$$

it follows that

$$\mathbb{E}[V_t\,|\,\{V_s, s \geq 0\}] = V_s - 4\gamma\mathbb{E}\Big[\int_s^t V_\tau^2\, d\tau\,\Big|\,\{V_s, s \geq 0\}\Big], \qquad \forall\, t \geq s \tag{4.35}$$

and Eq. 4.35 shows that $\mathbb{E}[V_t\,|\,\{V_s, s \geq 0\}] \leq V_s$ as from Eq. 4.29 we know the variance is a positive process. We have determined that V_s is a *supermartingale* with respect to B_t. By setting $s = 0$ for convenience, we can find $\partial_t V_t$ for the ensemble average of the variance of A_t

$$\begin{aligned} \mathbb{E}[V_t] &= V_0 - 4\gamma\mathbb{E}\Big[\int_s^t V_\tau^2\, d\tau\Big] \\ \frac{d\mathbb{E}[V_t]}{dt} &= -4\gamma\mathbb{E}[V_t]^2\left(1 + \frac{\big(V_t - \mathbb{E}[V_t]\big)^2}{\mathbb{E}[V_t]^2}\right). \end{aligned} \tag{4.36}$$

By Integration, the expression

$$\mathbb{E}[V_t] = \frac{V_0}{1 + 4\gamma V_0(t + \xi_t)} \tag{4.37}$$

is obtained where $\xi_t = \int_0^t \frac{(V_s - \mathbb{E}[V_s])^2}{\mathbb{E}[V_s]^2}\, ds$, and we can deduce the following expression: $\mathbb{E}[V_t] \leq \frac{V_0}{1+4\gamma V_0 t}$. Hence, $\mathbb{E}[V_t]$ converges to zero almost surely to zero $\lim_{t\to\infty}\mathbb{E}[V_t] = 0$. Therefore, the dynamical process Eq. 4.22 induces a collapse of the wave function to one of the eigenstates on the eigenmanifold of operator $\hat{A}$.

4.1.5 Redution Probability

Now we analyse the probability that initial state vector $|\phi_{t=0}\rangle$ will converge to each eigenstate of the operator $\hat{A}$. Let us consider the orthogonal projection operators P_α which span the subspace $\mathcal{H}_n$ of the eigenstates of $\hat{A}$ with eigenvalues a_α s.t. $\hat{A}P_\alpha = P_\alpha\hat{A} = a_\alpha P_\alpha$ and $\hat{A} = \sum_{\alpha=1}^n a_\alpha \hat{P}_\alpha$.

[9] $\beta_t = A_t^{(3)} - 3A_t A_t^{(2)} + 2(A_t)^3$

In the case of non-degenerate eigenvalues we have $\hat{P}_\alpha = |\alpha\rangle\langle\alpha|$ such that the expected value of finding $|\phi_{t=0}\rangle$ in eigenstate $|\alpha\rangle$ is $P_{\alpha 0} = \langle\phi_{t=0}|\hat{P}_\alpha|\phi_{t=0}\rangle$, this is equal to the probability of finding $|\phi_{t=0}\rangle$ in eigenstate $|\alpha\rangle$, $P_{\alpha 0} = \pi_\alpha$ if $\sum_{\alpha=1}^{n} \hat{P}_\alpha = 1$. Now let us sate the process

$$P_{\alpha t} = \langle\phi_t|\hat{P}_\alpha|\phi_t\rangle \tag{4.38}$$

which, since the projection commutes with both $\hat{H}$ and $\hat{A}$, $P_{\alpha t}$ can be written in the form of Eq. 4.28 and be shown to satisfy the following relation

$$dP_{\alpha t} = 2P_{\alpha t}(a_\alpha - A_t)dB_t. \tag{4.39}$$

From inspection $P_{\alpha t}$ will fluctuate unless $2P_{\alpha t} = 0$ or $|\phi_t\rangle$ is an eigenstate of $\hat{A}$ in that case $A_t = a_\alpha$.

$$P_{\alpha t} = P_{\alpha 0} \exp\Big(2\int_0^t (a_\alpha - A_\tau)\, dB_\tau - 2\gamma \int_0^t (a_\alpha - A_\tau)^2\, d\tau\Big) \tag{4.40}$$

The following theorem [14] was used to obtain Eq. 4.40.

Theorem (Doléans-Dade exponential) Let σ_t be any bounded $\mathcal{F}_t$ adapted process, and W_t be a standard Wiener process, then the solution to the differential equation $dX_t = \sigma_t X_t dW_t$ for $X_0 > 0$ is

$$X_t = X_0 \exp\Big(\int_0^t \sigma_\tau\, dB_\tau - \frac{1}{2}\int_0^t \sigma_\tau^2\, d\tau\Big). \tag{4.41}$$

If Novikov's condition holds [15]

$$\mathbb{E}\Big[\exp\big(\frac{1}{2}\int_0^t \sigma_\tau^2\, d\tau\big)\Big] < \infty, \tag{4.42}$$

then $P_{\alpha\tau}$, $0 \le \tau \le t$ is a martingale for t up to infinity, and we can write $\mathbb{E}[P_{\alpha\infty}] = P_{\alpha 0}$. $P_{\alpha\infty}$ is the the ensemble average value for the projection operator when processes A_t have either reached their terminal value. $P_{\alpha\tau}$ will take a value of 1 for eigenvalues $A_t = a_\alpha$ and zero otherwise. Hence we can deduce that $\mathbb{E}[P_{\alpha\infty}]$ is the probability of reaching a state $|\phi_{t=0}\rangle$ in eigenstate $|\alpha\rangle$

$$\mathbb{E}[P_{\alpha\infty}] = \pi_\alpha. \tag{4.43}$$

We can conclude that the stochastic model gives us correct results for the transition probability from initial state $|\phi_0\rangle$ to non-degenerate[10] eigensate $|\alpha\rangle$. The interesting point here is that this is an assumption in standard quantum mechanics whereas here we have deduced it from martingale properties of the stochastic evolution equation.

4.2 Continuous Localisation on Physical Systems

4.2.1 Systems of Identical Particles and Choice of Basis

We now have a construction for the evolution of the state vector onto the eigenmanifolds of an arbitrary set of commuting operators A_i. To model a physically appropriate system[11], A_i are chosen to be the density operators $\hat{\mathcal{N}}_j(x_i)$ but weighted for mass with i indexing the dimension of the space $i = 1, 2, 3$ and j indexing the type of particle. The smeared mass density operator $\hat{\mathcal{M}}(\mathbf{x})$ represents the probability to find mass density at position $\mathbf{x}$, in terms of the creation and annihilation operators $a_j^\dagger(\mathbf{y}, s)$, $a_j(\mathbf{y}, s)$ for particles j of spin s and position $\mathbf{y}$ [16][12]. The creation and annihilation operators, and by extension the CSL stochastic differential equation are defined in a Fock space associated with the system.

$$\hat{\mathcal{M}}(\mathbf{x}) = \sum_j m_j \hat{\mathcal{N}}_j(\mathbf{x}), \qquad \hat{\mathcal{N}}_j(\mathbf{x}) = \sum_s \int d^3y\, g(\mathbf{y}-\mathbf{x}) a_j^\dagger(\mathbf{y}, s) a_j(\mathbf{y}, s) \tag{4.44}$$

m_j is the mass of particle type j. For the expectation value of the mass density operator in the normalised state $|\phi_t\rangle$ we write

$$\mathcal{M}_t(\mathbf{x}) = \langle\phi_t|\hat{\mathcal{M}}(\mathbf{x})|\phi_t\rangle, \qquad M = \int d^3x\, \mathcal{M}_t(\mathbf{x}). \tag{4.45}$$

[10] For the degenerate eigenvalue method see more information at [12]

[11] To make an appropriate choice the following conditions need to be satisfied: 1. macroscopic objects are localised in space in contrast to their microscopic counterparts 2. total energy increases arising from localisations must not be detectable 3. symmetry properties of identical particles must be preserved. However, Any set of commuting operators can be implemented for this technology.

[12] Pearle and Squires 1996 found that spontaneous collapse models become untenable by treating electrons and quarks equivalently, a mass proportional model was proposed to enable viable numerical values for parameters for r_C and γ

The quantity M represents the total mass of the system, and, unlike $\hat{\mathcal{M}}_t(\mathbf{x})$ is not a stochastic process because it sums over all eigenvalues $\mathbf{x}$. $g(\mathbf{y}-\mathbf{x})$ is a normalised, spherically symmetric, positive, real correlation function peaked around $\mathbf{y}=\mathbf{x}$, we will choose it to be the Gaussian

$$g(\mathbf{x}) = \Big(\frac{1}{2\pi r_C^2}\Big)^{3/2} e^{-\frac{1}{2r_C^2}\mathbf{x}^2}. \tag{4.46}$$

We interpret $g(\mathbf{x})$ to be the functional "square root" of the correlation function (Eq. 4.50) for a noise variable that couples locally to the mass density, and characterises the density at which the particle(s) can be found [17]: naturally $\int d^3x\, g(\mathbf{x}) = 1$. Particle eigenstates of $\hat{\mathcal{N}}_j(\mathbf{x})$ are also defined

$$|q,s\rangle = \mathbf{a}^\dagger(\mathbf{q}_1,s_1)\,\mathbf{a}^\dagger(\mathbf{q}_2,s_2)\,\ldots\,\mathbf{a}^\dagger(\mathbf{q}_n,s_n)\,|0\rangle \tag{4.47}$$

with corresponding eigenvalues $n_j(\mathbf{x}) = \sum_{k=1}^n g(\mathbf{q}_k-\mathbf{x})$, $\mathbf{q}_k$ are particles $k=1,\ldots,n$ of the j type in the position basis. Hence the eigenvalues for operator $\hat{\mathcal{M}}_t(\mathbf{x})$ are $m(\mathbf{x}) = \sum_j m_j n_j(\mathbf{x})$. The creation and annihilation operators will satisfy canonical commutation or anti-commutation relations depending on the spin of particles s_k. Substituting $\int d^3x\,\hat{\mathcal{M}}(x_i)$ for A_i in Eqs. 4.22 and 4.26, the non-linear and density operator evolution equations are as follows

$$d|\phi_t\rangle = \Big[-\frac{i}{\hbar}\hat{H}\,dt + \int d^3x\,\Big(\hat{\mathcal{M}}(\mathbf{x})-\mathcal{M}_t\Big)dB(\mathbf{x}) - \frac{\gamma}{2m_0^2}\int d^3x\,\Big(\hat{\mathcal{M}}(\mathbf{x})-\mathcal{M}_t\Big)^2 dt\Big]|\phi_t\rangle \tag{4.48a}$$

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}\big[\hat{H},\rho(t)\big] + \frac{\gamma}{m_0^2}\int d^3x\,\hat{\mathcal{M}}(\mathbf{x})\rho(t)\hat{\mathcal{M}}(\mathbf{x}) - \frac{\gamma}{2m_0^2}\big\{\hat{\mathcal{M}}^2(\mathbf{x}),\rho(t)\big\} \tag{4.48b}$$

m_0 is a reference mass which we will define to be the mass of the nucleon, this way the reduction rates for macroscopic objects become practically the same as the mass excluding CSL model - contributions from electrons bring negligible effects to the induced collapse[13]. The choice of operator $\hat{\mathcal{M}}(\mathbf{x})$ localises the system on the position basis whilst also accounting for indistinguishable particles through commutation/anti-commutation relations of bosons/fermions.

[13]The mass density CSL model is utilised here since it remains the best contender for a continuous dynamical collapse model whose parameters lie in the feasible range to describe both micro and macro dynamics [18].

4.2.2 Reduction for Microscopic Particles

Focusing our attention to a many nucleon system positioned in space, we look at the off-diagonal components of Eq. 4.48b [19]

$$\frac{\partial}{\partial t}\langle q', s'|\rho(t)|q'', s''\rangle = -\frac{i}{\hbar}\langle q', s'|[H, \rho(t)]|q'', s''\rangle - \Gamma\langle q', s'|\rho(t)|q'', s''\rangle \tag{4.49a}$$

$$\Gamma = \frac{\gamma}{2m_0^2}\sum_{ij} m_i m_j \Big[G(\mathbf{q}_i' - \mathbf{q}_j') + G(\mathbf{q}_i'' - \mathbf{q}_j'') - 2G(\mathbf{q}_i' - \mathbf{q}_j'')\Big] \tag{4.49b}$$

where

$$G(\mathbf{q}' - \mathbf{q}'') = \int d^3x\, g(\mathbf{q}' - \mathbf{x})g(\mathbf{q}'' - \mathbf{x}) = \Big(\frac{1}{4\pi r_C^2}\Big)^{3/2} e^{-\frac{1}{4r_C^2}(\mathbf{q}' - \mathbf{q}'')^2} \tag{4.50}$$

and the summation over all the particles in the system ij come from the eigenvalues $n(\mathbf{x})$. Note if we work in the eigenbasis of $\hat{\mathcal{N}}_{\text{nuc}}(\mathbf{x})$ the nucleon mass m_0 will cancel in Eq. 4.49b, integration variables were also changed $\mathbf{x} \to \mathbf{x} + \mathbf{q}''$. For a single nucleon Eq. 4.49a becomes

$$\frac{\partial}{\partial t}\langle \mathbf{q}'|\rho(t)|\mathbf{q}''\rangle = -\frac{i}{\hbar}\langle \mathbf{q}'|[H, \rho(t)]|\mathbf{q}''\rangle - \gamma\Big(\frac{1}{4\pi r_C^2}\Big)^{3/2}\Big[1 - e^{-\frac{1}{4r_C^2}(\mathbf{q}' - \mathbf{q}'')^2}\Big]\langle \mathbf{q}'|\rho(t)|\mathbf{q}''\rangle \tag{4.51}$$

which for $|q_i' - q_i''| \gg r_C$, $\lambda = \gamma\Big(\frac{1}{4\pi r_C^2}\Big)^{3/2}$ is equivalent to the GRW reduction rate Eq. 3.10 satisfying a map $M : \rho(t) \mapsto \rho(t + dt)$ which is completely positive and trace preserving. Ghirardi, Pearle, and Rimini [6] chose the following value for γ: $\gamma \sim 10^{-30}$ cm^3 s^{-1}, corresponding to a reduction rate $\Gamma \simeq \lambda_{\text{CSL}} \sim 2.2 \times 10^{-17}$ s^{-1} for superpositions separated by distances much greater than r_C.

4.2.3 Wavefunction Reduction for Macroscopic Objects

In the previous section, one can identify that the decay rate Γ is not only dependent on locally separated superpositions, but also the distances between constituents for one state. Let us consider an N nucleon macroscopic body in one dimension with a centre of mass Q and position operators q_i for particle i.

$$q_i = Q + \tilde{q}_i(r) \tag{4.52}$$

The relative coordinates $\tilde{q}_i$ all sum to zero so we have $N-1$ internal degrees of freedom[14] indicated by r and one degree of freedom for Q outlining the N variables for the coordinates which we will write q as a shorthand. These separated variables can be represented as a separable wavefunction $\psi(q,s)$ [20]

$$\psi(q,s) = \Psi(Q)\chi(r,s), \qquad \chi(r,s) = \Delta^{(S/A)}(r,s) \tag{4.53}$$

where (S/A) indicates symmetry or anti-symmetry[15] with respect to exchange of arguments (q_i, s_i) and (q_j, s_j). We assume that The body is rigid, so nucleons will oscillate about their equilibrium position r_0 and the wave function $\Delta(r,s)$ is sharply localised such that $\left|\tilde{q}_i'(r) - \tilde{q}_i'(r_0)\right| \ll r_C$. The macroscopic body is also in no superposition of any kind such that $\Psi(Q)$ and $\chi(q,s)$ are eigenstates of $\hat{\mathcal{M}}(x)$.

$$\hat{\mathcal{M}}(x)\Psi(Q)\chi(r,s) = \sum_{i=1}^{N} m_i \Big(\frac{1}{2\pi r_C^2}\Big)^{1/2} e^{-\frac{1}{2r_C^2}(Q+\tilde{q}_i(r)-x)^2} \Psi(Q)\Delta^{(S/A)}(r,s) \tag{4.54}$$

Under the assumption that macroscopic objects are roughly homogeneous within the parameter r_C, dense sums of Gaussians within parameter r_C will not vary much compared to the amplitude of $\Delta^{(S/A)}(r,s)$. We can therefore neglect all other internal variables different to r_0 since they add negligible contribution.

$$\begin{aligned} \hat{\mathcal{M}}(x)\Psi(Q)\chi(r,s) &= F(Q-x)\Psi(Q)\chi(r,s) \\ F(Q-x) &= \sum_{i=1}^{N} \Big(\frac{1}{2\pi r_C^2}\Big)^{1/2} e^{-\frac{1}{2r_C^2}(Q+\tilde{q}_i(r_0)-x)^2} \end{aligned} \tag{4.55}$$

Under the macroscopic assumption the internal degrees of freedom are assumed to be very localised about r_0, and operator $\hat{\mathcal{M}}(x)$ becomes an operator for the eigenfunction $\Psi(Q)$ only. Given also that the Hamiltonian $\hat{H}$ can be split into the centre of mass H_Q and internal motion

[14] For a d-dimensional system there will be $d(N-1)$ d.o.f, these d.o.f also include possible rotations of the macroscopic object, allowing further constraint of the system further by d, for simplicity we will only treat the system under translation invariance.

[15] Bosons are symmetric with respect to bosons and fermions, fermions are anti-symmetric wit respect to themselves.

acting in their respective state spaces $\hat{H} = H_Q + H_r$, we can write Eq. 4.5 as

$$\begin{aligned} d\Psi(x) &= \Big[-\frac{i}{\hbar} H_Q \, dt + \int dx \, F(Q-x) \, dB_t - \frac{\gamma}{2} \int dx \, F^2(Q-x) \, dt \Big] \Psi(x) \\ d\chi(x) &= \Big[-\frac{i}{\hbar} H_r \, dt \Big] \chi(x) \end{aligned} \tag{4.56}$$

We have shown that the center of mass and internal motion decouple in this approximation, and the reduction mechanism is only determined by superpositions of coordinate Q. Note that $\Psi(x)$ following Eq. 4.22 in the eigenstate of $\hat{\mathcal{M}}(x)$ is not subject to stochastic motion unlike a general wave function, to look at superpositions of eigenstates we turn to the elements of the density operator.

4.2.4 Statistical Operator reduction for Macroscopic Objects

Consider the vector $|q^L\rangle$ consisting of a set of particles forming a macroscopic body length D with positions $|q^L\rangle = \{q_i^L\}_{i=1}^N$, similarly the vector $|q^R\rangle$ is a rigid displacement Δ of the particles s.t. $q^R = \{q_i^L + \Delta\}_{i=1}^N$. Neglecting the free evolution in Eq. 4.49a, we consider the rate of reduction for a superposition of states $|q^L\rangle$ and $|q^R\rangle$ of the macroscopic body by analysing elements of the density matrix.

$$\frac{\partial}{\partial t} \langle q^L | \rho(t) | q^R \rangle = -\Gamma(q^L, q^R) \langle q^L | \rho(t) | q^R \rangle \tag{4.57}$$

Γ is the rate factor, off-diagonal elements get suppressed faster with larger Γ

$$\langle q^L | \rho(t) | q^R \rangle = e^{-\Gamma(q^L, q^R)} \langle q^L | \rho(0) | q^R \rangle. \tag{4.58}$$

The rigid body is defined to emulate atoms in a lattice such that the body consists of groups of n nucleons mass m_0 separated by a distance smaller than r_C [21], and N of these groups which are separated by a distance larger than r_C. Substituting centre of mass equations for the positions of each group of nucleons $q^L = \{Q + r_i, \, i = 1, 2, \ldots, N\}$ and $q^L = \{Q + r_i + \Delta, \, i = 1, 2, \ldots, N\}$ into

Γ from Eq. 4.49b

$$\begin{aligned}\Gamma(q^L, q^R) =& \frac{\gamma}{2m_0^2}\sum_i^N nm_i \sum_j^N nm_j \int dx \Big[g(Q + r_i - x)g(Q + r_j - x) \\ &+ g(Q + r_i + \Delta - x)g(Q + r_j + \Delta - x) - 2g(Q + r_i - x)g(Q + r_j + \Delta - x)\Big]\end{aligned} \tag{4.59}$$

Note the the groups of n nucleons within range $\ll r_C$ are treated as particles of mass nm_0. From formula 4.50 we can change integration variables in every term of the sum to find the correlation functions

$$\Gamma(q^L, q^R) = \Big(\frac{1}{4\pi r_C^2}\Big)^{1/2} \frac{\gamma n^2}{m_0^2} \sum_{i,j}^N m_i m_j \Big[e^{-\frac{1}{4r_C^2}(r_i - r_j)^2} - e^{-\frac{1}{4r_C^2}(\Delta + r_i - r_j)^2} \Big]. \tag{4.60}$$

Adler [17] considered two cases for the superposition of the body in different spatial states. Firstly, $\Delta > D$ such that there is no overlap between superposed bodies then the second term reduces to zero as $\Delta + r_i \neq r_j \ \forall i, j$. Additionally, the assumption that the groups of nucleons are further apart from each other than the correlation length, means have negligible contribution to the collapse rate $r_j - r_j \gg r_C$ if $i \neq j$, and we can introduce a Dirac delta δ_{r_i, r_j} above. Under these conditions our idealised rate of reduction becomes

$$\Gamma(q^L, q^R)\Big|_{\Delta \gg D} \simeq \lambda n^2 N \Big(\frac{m_p}{m_0}\Big)^2 \tag{4.61}$$

where $\lambda = \gamma\Big(\frac{1}{2\pi r_C^2}\Big)^{1/2}$ is the rate of collapse for one nucleon mass m_0 and m_p is the mass of one (or many) of the constituent particles that make up the body.

Secondly the superposition of two very close states $\Delta \ll r_C$ is considered, we again make the assumption that groups of nucleons do not induce each others collapse. This time the second term in Eq. 4.60 is not neglected and the reduction rate is approximated as

$$\begin{aligned}\Gamma(q^L, q^R)\Big|_{\Delta \ll r_C} &\simeq \Big(\frac{1}{2\pi r_C^2}\Big)^{1/2} \frac{\gamma n^2}{m_0^2} \sum_{ij}^N m_i m_j \Big[\delta_{r_i, r_j} - \delta_{r_i, r_j} e^{-\frac{1}{4r_C^2}(\Delta)^2} \Big] \\ &\simeq \lambda n^2 N \Big(\frac{m_p}{m_0}\Big)^2 \frac{\Delta^2}{2r_C^2}\end{aligned} \tag{4.62}$$

It is remarked that Adler's division of a macroscopic body into tight clusters of nucleons far ($\gg r_C$) from each other is idealised and should be considered as an estimate when $r_C \to 0$.

Ghirardi *et al* instead considered taking assuming a continuous mass distribution of a lattice of atoms Eq. 4.60 by taking the sum over all particles to be an integral in space[16].

$$\Gamma(q^L, q^R) = \frac{\lambda}{m_0^2}\int_0^L du \int_0^L dv\, \mu(u)\mu(v)\left(e^{-\frac{(u-v)^2}{4r_C^2}} - e^{\frac{(u-v-\Delta)^2}{4r_C^2}}\right) \tag{4.63}$$

Where if one replaces the discrete mass distribution $\mu(u) = \sum_i m_i\delta(u-q_i)$ we obtain back Eq. 4.60. Here we take constant density distribution $\mu(u) = M_0$, and the limit as $r_C \to 0$, only terms $v = u$ and $v = u - \Delta$ will conribute to the v integral

$$\begin{aligned}\lim_{r_C\to 0}\Gamma(q^L, q^R) &= \frac{\lambda M_0^2}{m_0^2}\int_0^D du \int_0^D dv\left(e^{-\frac{(u-v)^2}{4r_C^2}} - e^{-\frac{(u-v-\Delta)^2}{4r_C^2}}\right)\\ &= \frac{\gamma M_0^2}{m_0^2}\int_0^D du \int_0^D dv\left(\delta(u-v) - \delta(u-v-\Delta)\right)\\ &= \frac{\gamma M_0^2}{m_0^2}\int_0^D du\left(H(u) - H(u-\Delta)\right)\end{aligned} \tag{4.64}$$

$H(x)$ is the Heaviside step function $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ otherwise, $\Delta > 0$. From observation under this limit the rate Γ increases linearly as Δ does, until a point is reached where no density lies in the intersection of the two volumes, from there the collapse rate will stay constant. The rate in this limit becomes

$$\lim_{r_C\to 0}\Gamma(q^L, q^R) = \gamma D_{\text{out}}\left(\frac{M_0}{m_0}\right)^2 \tag{4.65}$$

$\frac{M_0}{m_0}$ is the number density of the constituent nucleons[17] and D_{out} proportion of space the object occupies when the body is in position q^L that is outside the body when the centre of mass is positioned in q^R. For the one dimensional case it is easy to tell that this is the length minus the overlap compared to the total length of the body, but in higher dimensions this quantity will depend more on the geometry and orientations of the object. Note that the result obtained is slightly different than the result obtained by Ghirardi *et al* [6] who find a rate of $\Gamma = \gamma D_0 N_{\text{out}}$ for the continuous macroscopic body where D_0 is the density and N_{out} is the number of particles residing in space D_{out}. If we take $D_0 = \frac{M_0}{m_0}$ and $N_{\text{out}} = D_{\text{out}}\frac{M_0}{m_0}$ then our do results coincide,

[16] This is more in line with macroscopic treatments of rigid bodies since treating 10^{23} atoms individually is computationally exhausting.

[17] If the constituent particles are nucleons then the factor on the bottom cancels out, when this is not the case, the mass difference relative to a nucleon will contribute to the collapse.

although it is unclear how the authors define D_0.

Inter-atomic spacing in most materials is of order $\sim 10^{-10}$ m [22], significantly smaller than the correlation length predicted by collapse models[18] $r_C \sim 10^{-7}$ [18]. Hence, for a 3D lattice, spheres of radius r_C will contribute to the reduction rate of their neighbours, an important feature we have neglected in the above treatments. Eqs 4.61 and 4.65 both only consider the rate of a large body superposed with itself and does not consider the quantum superpositions of it's microscopic constituents. Therefore they outline a lower bound for the rate reduction of a microscopic body, it is true that more dense and homogeneous objects experience faster reduction rates which is an interesting feature of these theories. Given this fact, the minimum reduction rates for a macroscopic object with density $\sim 10^{23}$ nucleons per cm^3, are calculated to be $\Gamma \sim 10^{16}$ s^{-1} for CSL value for $\gamma \sim 10^{-30}$ cm^{3}s^{-1}. This is already a frequency that surpasses the resolution of modern instruments, hence macroscopic objects are effectively localised in collapse model descriptions.

4.2.5 Localisation in Phase Space

Arbitrarily precise locations can be obtained in phase space for classical dynamical objects, whereas this precision is irreducibly restricted by Heisenberg's uncertainty principle in quantum mechanics. We have recovered the localisation in position which satisfy the eigenvalues of the mass density operator

$$m_j(\mathbf{x}) = \sum_j m_j \sum_{k=1}^{n} g(\mathbf{q}_k - \mathbf{x}) \tag{4.66}$$

substituting the chosen Gaussian localisation spread $g(\mathbf{x})$ from 4.46 into the mass density operator Eq. 4.44, and ignoring spin for our purposes here

$$\hat{\mathcal{M}}(\mathbf{x}) = \sum_j m_j \int d^3y \left(\frac{1}{2\pi r_C^2}\right)^{3/2} e^{-\frac{(\mathbf{y}-\mathbf{x})^2}{2r_C^2}} a_j^\dagger(\mathbf{y}) a_j(\mathbf{y}). \tag{4.67}$$

[18] r_C predicted by Adler is roughly an order of magnitude larger $r_C \sim 10^{-6}$ than the one predicted by Ghirardi *et al*

To explore what happens to the momenta of our system, the momentum representation is obtained by Fourier transform of the mass density operator

$$\hat{\mathcal{M}}(\mathbf{x}) = \sum_j m_j \int \frac{d^3p}{(2\pi\hbar)^3} \int d^3q\, e^{-\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{x}}\, e^{-\frac{r_C^2}{2\hbar^2}\mathbf{q}^2}\, a_j^\dagger(\mathbf{p}+\mathbf{q})a(\mathbf{p}). \tag{4.68}$$

$a_j^\dagger(\mathbf{p})$ and $a_j(\mathbf{p})$ are the creation and annihilation operators for a particle type j with momentum $\mathbf{p}$. The form of Eq. is analogous to that of a particle under an interaction potential [23]

$$\tilde{V}(\mathbf{q}) = \int d^3y\, g(\mathbf{y})\, e^{-\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{x}} = e^{-\frac{r_C^2}{2\hbar^2}\mathbf{q}^2} \tag{4.69}$$

to which we see the position and momentum are tied[19]. An improbable position localisation far from the peak of $g(\mathbf{x})$, in infinitesimal time step dt, will correspond to a larger displacement momentum $\mathbf{q}$ induced by the external potential V. Hence the system 'borrows'[20] momenta $\mathbf{q}$ from the external Gaussian noise that induces localisations in position space, violating momentum conservation. Although the external contributions of momenta $\mathbf{q}$ are too small (standard deviation $\sigma_q = r_C/\hbar \sim 10^{-27}$ kg m s^{-1}) to be detectable by experiments, it is not a desirable property of the theory and consequently predicts perpetual increase of energy of a system in contact with surroundings [24] [25]. It is noted that Heisenberg's uncertainty is satisfied in this theory, with standard deviations in momenta and position $\sigma_x = r_C$, $\sigma_p = \hbar/r_C$ we can determine the uncertainty $\sigma_x\sigma_p = \hbar$ to be greater than $\hbar/2$. There have been efforts made to extend the CSL [26] and GRW [27] models to include dissipative effects which guarantee a finite energy during the systems entire evolution. For localisation in phase space, the CSL model does not aid in converting momenta to statistical mixtures. One could define the localisation operators A_i in Eq. 4.22 to have eigenstates of momenta, however, this would lead to linear superpositions of spatially distant states.

[19] We have used the property that the Fourier transform of a Gaussian is another Gaussian.

[20] We can interprete Eq. 4.68: A acting on a one particle state $|p\rangle$ with momentum $\mathbf{p}$, the potential term yields another one-particle state with momentum $\mathbf{p}+\mathbf{q}$, where $\mathbf{q}$ is the momentum transfer, with amplitude $\tilde{V}(\mathbf{q})$.

5| Connections between Spontaneous Localisation and Continuous collapse

We now come back to the GRW model and show how the CSL model can be derived by taking the infinite frequency limit. Consider the collapse operator $L_{\mathbf{b}}$ acting on state ψ at times dictated by the Poisson process $N_t \sim \mathcal{P}(\lambda t)$.

$$\psi \longrightarrow \psi_b = L_{\mathbf{b}}\psi = \Big(\frac{1}{\pi r_C^2}\Big)^{d/4} \exp\Big[-\frac{(A_i - b_i)^2}{2r_C^2}\Big]\psi \tag{5.1}$$

For a class of commuting, self-adjoint operators $\{A_i,\, i = 1, 2 \dots d\}$, and random variables $\{b_i,\, i = 1, 2 \dots d\}$. ψ_b is not normalised so we normalise the localised state such that the process

$$\begin{aligned} \phi \longrightarrow \phi_b &= \frac{\psi_b}{\|\psi_b\|^2} \\ \psi_b &= L_{\mathbf{b}}\phi = \Big(\frac{1}{\pi r_C^2}\Big)^{d/4} \exp\Big[-\frac{(A_i - b_i)^2}{2r_C^2}\Big]\phi \end{aligned} \tag{5.2}$$

is the one we consider. As per the 4th axiom of QMSL, the probability density for the occurrence of b_i is

$$P(b_i) = \|\psi_b\|^2, \qquad \Big(\frac{1}{\pi r_C^2}\Big)^{d/2} \int d^d b \, \exp\Big[-\frac{b_i^2}{\pi r_C^2}\Big] = 1 \tag{5.3}$$

such that the total probability of all possible hits happening across $\mathbf{b}$ is normalised. Between hits the state ϕ_t evolves through standard Schrödinger evolution which is unitary. Unlike the CSL model, whenever a hit occurs, the stochastic process induces a discontinuous jump in the evolution of the system, sometimes called a 'hit'. We showed that the statistical operator for such a system is of the form of Eq. 3.9 with

$$T[\rho] = \Big(\frac{1}{\pi r_C^2}\Big)^{d/2} \int d^d b \, \exp\Big[-\frac{(A_i - b_i)^2}{2r_C^2}\Big]\rho \exp\Big[-\frac{(A_i - b_i))^2}{2r_C^2}\Big] \tag{5.4}$$

rewriting

$$T[\rho] = \Big(\frac{1}{\pi r_C^2}\Big)^{d/2} e^{-\frac{1}{2r_C^2}A_i^2} \int d^d b \, \exp\Big(-\frac{b_i^2}{r_C^2}\Big) \exp\Big(\frac{A_i b_i}{r_C^2}\Big)\rho \exp\Big(\frac{A_i b_i}{r_C^2}\Big) \exp\Big(-\frac{b_i^2}{r_C^2}\Big) e^{-\frac{1}{2r_C^2}A_i^2} \tag{5.5}$$

and taking the limit $r_C^2 \to \infty$.

$$T[\rho] = \Big(\frac{1}{\pi r_C^2}\Big)^{d/2} e^{-\frac{1}{2r_C^2}A_i^2} \int d^d b \, \exp\Big(-\frac{b_i^2}{r_C^2}\Big)(1+\frac{A_i b_i}{r_C^2}+\frac{(A_i b_i)^2}{r_C^4})\rho(1+\frac{A_i b_i}{r_C^2}+\frac{(A_i b_i)^2}{r_C^4}) \exp\Big(-\frac{b_i^2}{r_C^2}\Big) e^{-\frac{1}{2r_C^2}A_i^2} \tag{5.6}$$

Multiplying out up to, and including $\mathcal{O}(1/r_C^4)$ and using the following identities

$$\Big(\frac{a}{\pi}\Big)^{d/2} \int d^d x \, e^{-ax^2} \, x = 0 \qquad \Big(\frac{a}{\pi}\Big)^{d/2} \int d^d x \, e^{-ax^2} \, x_i x_j = \frac{1}{2a}\delta_{ij}, \tag{5.7}$$

one finds one term of order zero and three terms of order one in $1/r_C^2$

$$T[\rho] = e^{-\frac{1}{2r_C^2}A_i^2}\Big[\rho + \frac{1}{2r_C^2}\sum_i (A_i \rho A_i + \frac{1}{2}\{A_i^2, \rho\}) + \mathcal{O}((1/r_C^2)^2)\Big] e^{-\frac{1}{2r_C^2}A_i^2}. \tag{5.8}$$

From

$$\exp\Big(-\frac{A_i^2}{r_C^2}\Big) = 1 - \frac{1}{2}\beta A_i^2 + \mathcal{O}(1/r_C^4), \tag{5.9}$$

$$T[\rho] = \rho + \frac{1}{2r_C^2}\Big[A_i \rho A_i - \frac{1}{2}\{A_i^2, \rho\}\Big] + \mathcal{O}(1/r_C^2)^2. \tag{5.10}$$

The second term is the result of the hitting process occurring with Poisson frequency λ in time. In the infinite frequency limit $\lambda \to \infty$ and $r_C^2 \to \infty$, $\frac{\lambda}{r_C^2} = \gamma$, substituting Eq. 5.10 into Eq. 3.9 we obtain

$$\frac{d\rho}{dt} = -i\big[H, \rho\big] + \gamma A_i \rho A_i - \frac{\gamma}{2}\{A_i^2, \rho\} \tag{5.11}$$

which, for self-adjoint A_i is the form of Eq. 4.26, the evolution of the statistical operator for the CSL model.

To find the connections on the wave function level we consider the wave function undergoing collapses during some time t.

$$|\psi_t\rangle = L_{\mathbf{b}}^{N_t}|\psi_0\rangle \tag{5.12}$$

where N_t is a Poisson random variable representing the number of localisations. Writing the Poisson process as $N_t = \sqrt{\lambda}B_t$ a Brownian motion with drift λt

$$L_{\mathbf{b}}^{N_t}|\psi_0\rangle = \Big[\underbrace{\Big(\frac{1}{2r_C^2}\Big)^{d/4} e^{-\frac{1}{2r_C^2}(A_i - b_i)^2}}_{=1}\Big]^{\sqrt{\lambda}B_t}|\psi_0\rangle \tag{5.13}$$

From GRW postulate 4 the localisation operators[1] are normalised s.t. $L_{\mathbf{b}}L_{\mathbf{b}} = 1$, In CSL this condition is relaxed hence we send the factor in front of the Gaussian operator to unity. We now take the high frequency limit: $\lambda \to \infty$, $r_C \to \infty$, $\frac{\sqrt{\lambda}}{2r_C^2} \to \sqrt{\gamma}$

$$|\psi_t\rangle = \Big[e^{-\frac{\sqrt{\lambda}}{2r_C^2}(A_i-b_i)^2 B_t}\Big]|\psi_0\rangle \quad \longrightarrow \quad \Big[e^{-\sqrt{\gamma}(A_i-b_i)^2 B_t}\Big]|\psi_0\rangle \tag{5.14}$$

and then take the limit $t \to 0$ so we consider the wave function after an infinitesimal time increment:

$$|\psi_{0+dt}\rangle = |\psi_0\rangle + d|\psi\rangle = \Big[e^{-\sqrt{\gamma}(A_i-b_i)^2 dB_t}\Big]|\psi_0\rangle \approx \Big[1 - \sqrt{\gamma}(A_i-b_i)^2\, dB_t + \frac{\gamma}{2}(A_i-b_i)^4\, dt\Big]|\psi_0\rangle. \tag{5.15}$$

The evolution equation is

$$d|\psi\rangle = \Big[-\sqrt{\gamma}(A_i-b_i)^2 dB_t + \gamma(A_i-b_i)^4\, dt\Big]|\psi_0\rangle, \tag{5.16}$$

which takes a similar form to Eq. 4.5 without the Schrödinger evolution and $A_i = (A_i - b_i)^2$. Note the Brownian term has drift hence this is equivalent to looking at Eq. 4.5 but under the physical probability measure $\mathbb{Q}$, the signs of the two terms above are also switched, this is still valid if the drift of B_t is negative $\mathbb{E}[dB_t] = -\sqrt{\lambda}dt$ and Eq. 4.4 is satisfied.

6 | Conclusion

In this paper we have explored two of the most prominent stochastic reduction models and found connections between them. After an overview of the measurement problem and the difficulties faced in the quantum to classical transition, we proposed how theories which incorporate a collapse into the dynamics of the wave function can provide satisfactory explanations to contradicting observations seen in quantum and classical experiments alike. Alongside an overview of the formerly created GRW model, the collapse mechanism was outlined for two superposed Gaussian wave-packets. This highlighted some fundamental changes that these theories assume such as the addition of two more fundamental constants λ and r_C, and the rejection of collapsed particles being point-like objects instead very localised wave packets.

[1]For our purposes we simplify the problem to only consider localisation at one point $\mathbf{b}$.

The latter formulated and more consistent CSL model was then explored, this model describes universal dynamics in a compact way by adding a stochastic term to the Schrödinger equation. From the general form of a evolutionary stochastic differential equation in Hilbert space, we clarified the steps needed to arrive at a norm preserving evolving wavefunction which, over time arrives at one of the eigenstates of arbitrary commuting operators A_i. A more rigorous proof of reduction was shown juxtaposed to the papers published by the original authors. Then, employing second quantised form the mass density operator was chosen to allow a reduction onto the position eigenstates of the system, and consideration of canonical (anti-)commutation relations between indistinguishable particles were taken. The rate of reduction was studied in the context of mass density, we found mean rates of microscopic bodies and approximate rates that bound the reduction of macroscopic trajectories.

In the last section the relations between our two theories were studied, first their density operator evolutions were shown to be equivalent in the infinite frequency limit, and that they both take the form of Lindblad generators. Secondly, connections were found between the Gaussian localisation operator occurring at Poisson times and the unnormalised evolution equation of CSL. We conclude that to align with experimental observations, more realistic reduction models need to be developed to stand amongst other interpretations of quantum mechanics. A major downfall of the CSL model is the localisations induce stochastic changes in momentum space resulting in violation of energy conservation. Additional dissipative terms can be added to account for this, however, they are argued to complicate the theory. A model fully describing localisations in phase space may be an instructive way forward to develop the CSL theory, providing a comprehensive and compact description to solve quantum and classical trajectories alike.

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A | Preferred Basis Problem

To illustrate the *preferred basis problem*, suppose our microscopic system $\mathcal{S}$ is a spin-qubit and our measuring aperatus $\mathcal{A}$ is a second spin-qubit, both represented in the σ_z observable eigenbasis $|z\pm\rangle$. when brought together they form an entangled spin state in a product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|z+\rangle_1|z-\rangle_2 - |z-\rangle_1|z+\rangle_2) \tag{A.1}$$

The state $|\psi\rangle$ can, however, be expressed in the spin basis of observable σ_x

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|x+\rangle_1|x-\rangle_2 - |x-\rangle_1|x+\rangle_2). \tag{A.2}$$

Now, suppose that particle 2 acts as a measuring device for the spin of particle 1. One can imply that system 2 has established a correlation between the z and the x spins of system 1, since the $|z\pm\rangle_2$ (equivenetly the $|x\pm\rangle_2$) eigenstate is not uniquley defined to correspond to any particular basis of particle 1. If such a correlation acts as a measurement, then our apparatus (system 2) has measured the x-basis spin as well as the z-basis spin of particle 1. This is an apparent contradiction to the quantum mechanical formalism as the observables σ_z and σ_x cannot be measured simultaneously due to the fact that they do not commute. Whats more, Eq. A.1 can be written in an infinite numuber of basis representations, so the pointer basis of apparatus $\mathcal{A}$ can tell nothing about which observable regarding the system $\mathcal{S}$ has been recorded. This is in obvious contradiction to measurement devices which are designed to measure a specific quantity of a system,

B| Quantum Dynamical Semigroup

To study quantum mechanics and its transition to the classical regime we need to be able to describe open quantum systems, these systems obey non-Hamiltonian evolutions, which are irreversible due to contact with a heat reservoir or the environment. Quantum dynamical or markov semigroups prove to be a useful tool when describing open quantum systems as finding the unitary dynamics for the system and the environment can be too complicated to evaluate microscopically due to the large number of degrees of freedom.

We introduce the Banach space $\mathcal{T}(\mathcal{H}_S)$ of trace-class operators with a trace norm $\|\cdot\|_1$, and the Banach space of all linear and bounded operators $\mathcal{B}(\mathcal{H}_S)$ with operator norm $\|\cdot\|_\infty$[1]. Operators a part of these spaces act on the Hilbert space of a system S consisting of a sub-system and it's environment $\mathcal{H}_S$. The single-parameter dynamical map Λ_t maps the Banach space to the Banach space

$$\Lambda_t \quad : \quad \mathcal{T}(\mathcal{H}_s) \to \mathcal{T}(\mathcal{H}_s) \tag{B.1}$$

for $t \geq 0$ and so density operators $\rho \in \mathcal{T}(\mathcal{H}_s)$ will evolve through time when we take the infinitesimal limit $t \to t + dt$. The quantum dynamical semigroup is a family of maps $\{\Lambda_t, t \geq 0\}$ such that:

1. Λ_t is a semi-positive dynamical map, i.e. it can be written as

$$\Lambda_t(\rho) = \sum_\alpha A_\alpha \rho A_\alpha^\dagger \geq 0 \qquad \forall \quad \rho \geq 0,\, t \geq 0$$

   where $A_\alpha \in \mathcal{B}(\mathcal{H}_S)$ and $\sum_\alpha A_\alpha A_\alpha\dagger = \mathbb{I}$

2. It is trace-preserving $\mathrm{Tr}[\Lambda_t(\rho)] = \mathrm{Tr}[\rho]$

3. Λ_t exhibits the Markov property $\Lambda_t \Lambda_s = \Lambda_{t+s}$

4. $\lim_{t\to 0} \|\Lambda_t(\rho) - \rho\|_1 = 0$.

[1] $\mathcal{B}(\mathcal{H}_S)$ is also an algebra with respect to multiplication of operators

As a result of the mathematical theory of one-parameter contracting semi-groups on Banach spaces [28], there exists a densely defined linear map L called a generator of a subgroup, such that

$$\frac{d}{dt}\rho_t = L\rho_t \tag{B.2}$$

where $\rho_t = \Lambda_t(\rho)$; $\rho \in \{\text{domain of } L\}$. For an isolated quantum system, A takes the form of standard Hamiltionian evolution

$$\Lambda_t(\rho) = e^{-\frac{i}{\hbar}Ht}\rho e^{\frac{i}{\hbar}Ht} \tag{B.3}$$

in such a case the generator becomes

$$L\rho = -\frac{i}{\hbar}[H,\rho] \tag{B.4}$$

or the quantum Louiville equation, for an open quantum system Lindblad showed [29] the most general generator of a quantum dynamical semigroup is of the form

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_i \gamma_i\Big(A_i\rho A_i^\dagger - \frac{1}{2}\Big\{A_i^\dagger A_i,\rho\Big\}\Big). \tag{B.5}$$

This is known as the Lindbladian and is the generator for a quantum Markov semigroup when it becomes uniformly continuous under the limit $\lim_{t\to 0}\|\Lambda_t(\rho)-\rho\|_1 = 0$. It is one of the general forms of a master equation describing open quantum systems with additional terms L_i and γ_i which help model dissipative effects into the environment. L_i are a set of jump operators describing how the environment acts on the subsystem and γ_i are damping coefficients modeling how strong the two systems interact with each-other, the higher the damping rate, the more non-reversible the evolution is.

C | Girsanov Theorem

Let us go back and look again at Girsanov's theorem Eq. 4.18. We have a measure $\mathbb{P}$ under which B is a standard Brownian motion (i.e. $\gamma = 1$ for simplicity) and we want to understand its law under a new measure of the form

$$\mathbb{Q} = \exp\left(\int_0^T h(s)B(s) - \frac{1}{2}\int_0^T h(s)^2 t\right)\mathbb{P} \tag{C.1}$$

Instead of looking at the Brownian motion B we approximate it on a uniform grid $\{0 = \tau_0, \dots, \tau_N = T\}$ of size $\Delta\tau = T/(N+1)$ and write B^N the piecewise linear approximations of the BM on this grid and let $\Delta B_i = B_{\tau_{i+1}} - B_{\tau_i}$. For a correlation function $F(B)$

$$\int F(B)\mathbb{Q}(B) = \int F(B)\exp\left(\int_0^T h(s)B(s) - \frac{1}{2}\int_0^T h(s)^2 s\right)\mathbb{P}(B) \tag{C.2}$$

we can approximate to

$$\approx \int F(B^N)\exp\left(\sum_{i=0}^{N-1} h(\tau_i)\Delta B_i - \frac{1}{2}\sum_{i=0}^{N-1} h(\tau_i)^2\Delta\tau\right)\prod_{i=0}^{N-1}\exp\left(-\frac{1}{2}\frac{\Delta B_i^2}{\Delta\tau}\right)(\Delta B_i) \tag{C.3}$$

$$= \int F(B^N)\prod_{i=0}^{N-1}\exp\left(-\frac{1}{2}\frac{\Delta B_i^2}{\Delta\tau} + h(\tau_i)\Delta B_i - \frac{1}{2}h(\tau_i)^2\Delta\tau\right)(\Delta B_i) \tag{C.4}$$

$$= \int F(B^N)\prod_{i=0}^{N-1}\exp\left(-\frac{1}{2}\frac{(\Delta B_i - h(\tau_i)\Delta\tau)^2}{\Delta\tau}\right)(\Delta B_i) \tag{C.5}$$

With the change of variables $\Delta\tilde{B}_i = \Delta B_i - h(\tau_i)\Delta\tau$ we have.

$$= \int F(B^N)\prod_{i=0}^{N-1}\exp\left(-\frac{1}{2}\frac{(\Delta\tilde{B}_i)^2}{\Delta\tau}\right)(\Delta\tilde{B}_i) \tag{C.6}$$

Showing that $\Delta\tilde{B}_i$ is a Brownian motion under $\mathbb{Q}$.

Note that the functional $h(\tau_i)$ depends on the ΔB_j only with $0 \leqslant j < i$, i.e.

$$h(\tau_i) = h(\tau_i, (\Delta B_j)_{j\in[0,i-1]})$$

so

$$\frac{\partial h(\tau_i)}{\partial \Delta B_i} = 0$$

and the Jacobian for the change of variables $\Delta B_i \to \Delta \tilde{B}_i$ is identically 1.