Renormalization group methods for stochastic PDEs

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Abstract

Preface

Chapter 1 - Preliminaries on SSPDEs

We want to study the dynamic Φ^4_3 model, i.e. the equation given by

$$\partial_t \varphi = \Delta\varphi - \varphi^3 - r\varphi + \Xi \tag{1.1}$$

where $x \in \mathbb{T}^3$ is the torus, $t \in [0,\tau]$, $\tau > 0$ almost surely and Ξ is space-time white noise. This equation belongs to a class of nonlinear stochastic PDEs of the form

$$\partial_t u = \Delta u + V(u) + \Xi.$$

Since Ξ is a distribution, the equation is not well-defined as nonlinear functions of distributions are ill-defined objects. If one is interested in properties of these equations like scaling properties or stationary states, one uses regularized versions of these equations where the noise is replaced by a mollified version. The question we will adress is well-posedness of the equation without the regularization, i.e. we will study the limiting problem of the regularized versions. Since the solutions are expected to have weak regularity, setting up a solution theory is a difficult task and will rely on perturbative analysis. It will turn out that random fields occuring in this analysis will diverge. These kinds of divergencies are similar to problems in quantum field theory and we will use rigorous methods known from the theory of renormalization to deal with these divergencies. The restriction to three space dimensions comes from the assumption of sub-criticality which will be discussed below and, in physical language, should be thought of as super-renormalizability. The renormalization appears in the equation as an additive linear term with coefficient "$r = +\infty$".
In recent years, there have appeared several methods to deal with the renormalization problem. Most notably Hairer's theory of regularity structures and Gubinelli's, Imkeller's and Perkowski's theory of paracontrolled distributions. We will study the equation using the method of renormalization group introduced by Antti Kupiainen. For more details on the history of solutions, see below.

1.1 Space-time white noise

We start by defining the space-time white noise for any dimension $d > 0$ Ξ which is the indeterministic part of the equation. Formally, Ξ is a Gaussian random field on $\mathbb{R} \times \mathbb{T}^d$, i.e. Ξ is a Gaussian random variable on $\mathbb{R} \times \mathbb{T}^d$ with covariance

$$\mathbb{E}[\Xi(t,x)\Xi(t',x')] = \delta(t'-t)\delta(x'-x).$$

This holds true only formally, there is no coordinate process. Instead Ξ is a random distribution, i.e. a random element of $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^d)$ which is a centered Gaussian with covariance

$$\mathbb{E}[\Xi(\eta_1)\Xi(\eta_2)] = \int_{\mathbb{R}\times\mathbb{T}^d} \eta_1(t,x)\eta_2(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

We will need the scaling behaviour of space-time white noise to understand the scaling behaviour of the equation. Define for $\tau, \lambda > 0$ the scaling operator $s^{\tau,\lambda}$ by

$$s^{\tau,\lambda}\eta(t,x) = \tau\lambda^d\eta(\tau t, \lambda x).$$

If η is defined on the torus, then $s^{\tau,\lambda}\eta(t,x)$ is defined on the rescaled torus $\lambda^{-1}\mathbb{T}^d$. We want to give sense to the rescaled space-time white noise, i.e. a random distribution $\Xi_{\tau,\lambda}$ such that

$$(\Xi_{\tau,\lambda}, \eta) = (\Xi, s^{\tau,\lambda}\eta)$$

where $\eta \in \mathcal{S}'(\mathbb{R} \times \lambda^{-1}\mathbb{T}^d)$, i.e. $\Xi_{\tau,\lambda}$ is a random distribution on $\mathbb{R} \times \lambda^{-1}\mathbb{T}^d$. If Ξ were a function, we could just apply the adjoint and get $\Xi_{\tau,\lambda}(t,x) = \Xi(\tau^{-1}t, \lambda^{-1}x)$. Of course, $\Xi_{\tau,\lambda}$ is again a centered Gaussian with variance given by

$$\begin{aligned}\mathbb{E}[(\Xi_{\tau,\lambda}, \eta)^2] &= \mathbb{E}[(\Xi, s^{\tau,\lambda}\eta)^2] \\ &= \tau^2\lambda^{2d}\int_{\mathbb{R}\times\mathbb{T}^d} \eta(\tau t, \lambda x)^2 \,\mathrm{d}t\,\mathrm{d}x \\ &= \tau\lambda^d\int_{\mathbb{R}\times\lambda^{-1}\mathbb{T}^d} \eta(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

We conclude that if Ξ_λ is space-time white noise on $\mathbb{R} \times \lambda\mathbb{T}$, then $\Xi_{\tau,\lambda} \overset{d}{=} \tau^{-1/2}\lambda^{-d/2}\Xi_\lambda$.
We want to determine the regularity of space-time white noise. One fruitful choice is to measure space-time white noise in (parabolically scaled) Besov spaces $\mathcal{B}^\alpha_{\infty,\infty}$. Although later we will use negative index Sobolev spaces, to get a regularity theory for the linear equation, we will introduce the Besov spaces, see e.g. ??.
We will use the notation $\|(t,x) - (t',x')\|_\mathfrak{s} = |t-t'|^{\frac{1}{2}} + \sum_{i=1}^d |x_i - x_i'|$, the parabolic distance.

(1.1) Definition
For $r \in \mathbb{N}$ denote by B_r the set of smooth $\eta\colon \mathbb{R}^{d+1} \to \mathbb{R}$ supported on the unit ball w.r.t. $\|.\|_\mathfrak{s}$ and $\|\eta\|_{C^r} \leq 1$ where

$$\|\eta\|_{C^r} = \sup_{\alpha:|\alpha|\leq r} \|D^\alpha\eta\|_\infty.$$

Suppose $\alpha < 0$, define the space $\mathcal{C}^\alpha_\mathfrak{s}$ to be the set of all distributions $u \in \mathcal{S}'(\mathbb{R}^{d+1})$ such that for any compact set $K \subset \mathbb{R}^{d+1}$ it holds

$$\|u\|_{\mathcal{C}^\alpha_\mathfrak{s}(K)} = \sup_{(t,x)\in K} \sup_{\substack{\eta\in B_r \\ \lambda\in(0,1]}} \left|\frac{u(S^\lambda_{(t,x)}\eta)}{\lambda^\alpha}\right| < \infty$$

where we denote $S^\lambda_{(t,x)}\eta(t',x') = \lambda^{-d-2}\eta(\lambda^{-2}(t'-t), \lambda^{-1}(x'-x))$ and $r = \lceil -\alpha \rceil$. ◇

One can also define these spaces for $\alpha > 0$ and for $\alpha \in (0,1)$ they will agree with classical Hölder spaces. Since we are only going to need the spaces for $\alpha < 0$, we omit the discussion here.
The regularity of Ξ in terms of the $\mathcal{C}^\alpha_\mathfrak{s}$-spaces follows from a Kolmogorov type theorem. Its proof is basically the same as the one for the classical Kolmogorov theorem.

(1.2) Theorem
Assume we have a random distribution ξ on $\mathbb{R}^{d+1}$, that is a stochastic process indexed on $\mathcal{S}(\mathbb{R}^{d+1})$ $(\xi(\cdot))$ that is linear as a map $\mathcal{S}(\mathbb{R}^{d+1})$ to the space of random variables.
Fix $\alpha < 0$ and $p \geq 1$. Assume there is a constant C such that for any $(t,x) \in \mathbb{R}^{d+1}$ and for all $\eta \in \mathcal{S}(\mathbb{R}^{d+1})$ that are supported in the unit ball of $\mathbb{R}^{d+1}$ (in parabolic scaling) and satisfy $\|\eta\|_\infty \leq 1$, one has

$$\mathbb{E}[\left|\xi(S^\lambda_{(t,x)}\eta)\right|^p] \leq C\lambda^{\alpha p} \quad \text{for any } \lambda \in (0,1],$$

then there exists a random distribution $\tilde{\xi}$ such that for all $\eta \in \mathcal{S}(\mathbb{R}^{d+1})$ it is $\xi(\eta) = \tilde{\xi}(\eta)$ a.s. Furthermore, for any $\alpha' < \alpha - \frac{d+2}{p}$ and any compact subset $K \subset \mathbb{R}^{d+1}$, it holds

$$\mathbb{E}[\|\tilde{\xi}\|^p_{\mathcal{C}^\alpha_{\mathfrak{s}}(K)}] < \infty.$$

◇

We can compute that for the space-time white noise Ξ, it is

$$\mathbb{E}\left[\Xi(S^\lambda_{(t,x)}\eta)^2\right] \lesssim \lambda^{-d-2}.$$

In particular, $\Xi \in \mathcal{C}^\alpha_{\mathfrak{s}}$ for any $\alpha < -\frac{d}{2} - 1$.

1.2 Scaling behaviour and subcriticality

We want to study the scaling behaviour of the equation (1.1). Before we do this, we first identify the scaling behaviour of the linearized equation

$$\partial_t \varphi = \Delta\varphi + \Xi.$$

Define for $\lambda > 0$ and scaling exponents $\alpha, \beta, \gamma > 0$

$$\hat{\varphi}(t,x) = \lambda^\alpha \varphi(\lambda^\beta t, \lambda^\gamma x).$$

Then $\hat{\varphi}$ is a function on $\mathbb{R} \times \lambda^{-\beta}\mathbb{T}$. Furthermore, define

$$\hat{\Xi} = \lambda^{\beta/2}\lambda^{d\gamma/2}\Xi_{\lambda^\beta,\lambda^\gamma}.$$

We have already seen that $\hat{\Xi} = \Xi_{\lambda^\gamma}$ where the latter is space-time white noise on $\mathbb{R} \times \lambda^\gamma\mathbb{T}$. Inserting the scaling in the equation, it follows

$$\partial_t \hat{\varphi} = \lambda^{\beta-2\gamma}\Delta\hat{\varphi} + \lambda^{\alpha+\beta/2-d\gamma/2}\hat{\Xi}.$$

Therefore, set

$$\gamma = 1, \quad \beta = 2, \quad \alpha = \frac{d}{2} - 1$$

and we see that $\hat{\varphi} \overset{d}{=} \varphi_\lambda$ where φ_λ is the solution to the equation on the rescaled torus, thus the equation is scale invariant.

Applying the scaling $\hat{\varphi}(t,x) = \lambda^{d/2-1}\varphi(\lambda^2 t, \lambda x)$ to (1.1) without the renormalization, we obtain

$$\partial_t \hat{\varphi}(t,x) = \Delta\hat{\varphi} - \lambda^{4-d}\hat{\varphi}(t,x) + \hat{\Xi}.$$

In the limit $\lambda \to 0$, the prefactor in front of the nonlinear term vanishes only if the spatial dimension is strictly less than 4. This is called the subcritical regime. It means that the small-scale terms are described by the linearized problem and there, the nonlinearity is not very present.

1.3 Why renormalization?

We will study the equation in $d = 3$, i.e. we are in the subcritical regime. In $d = 3$, by Schauder theory φ will only be a distribution. Since nonlinear functions of distributions are in general not defined, we need to make sense of our concept of solution. The usual way to interpret nonlinear problems with irregular objects is to regularize the equation and study convergence properties of the solutions of the regularized equations. The problem that arises is that either these solutions fail to converge at all or converge to an uninteresting limit. It was shown that if we choose a smooth bump ρ on $\mathbb{R} \times \mathbb{R}^d$ and set

$$\rho_\delta(t,x) = \delta^{-2-d}\rho(\delta^{-2}t, \delta^{-1}x)$$

and regularize by mollification $\Xi^\delta = \Xi * \rho_\delta$, then the family of unique solutions φ_δ to

$$\partial_t\varphi_\delta = \Delta\varphi_\delta - \varphi_\delta^3 + \Xi^\delta$$

in $d = 2$ converges to the trivial limit. To obtain a non-trivial limit, we thus introduce renormalisation constants c_δ (that will also depend on the dimension) and solve

$$\partial_t\varphi_\delta = \Delta\varphi_\delta - \varphi_\delta^3 - c_\delta\varphi_\delta + \Xi^\delta.$$

The family of constants c_δ will diverge as $\delta \to 0$. We then study the limit of the renormalized solutions φ_δ as $\delta \to 0$ and we will show that with the correct choice of renormalization constants, these actually converge to a non-trivial limit. It may be remarked that the concept of solution is dependent on the choice of regularization and on the choice of the renormalization constants. This phenomenon already appears in space dimension $d = 0$, i.e. if we consider SDEs. It is a well-known fact that an explicit Euler scheme converges to solution in the Itô sense whereas mollification of the noise converges to a solution in the sense of Stratonovich.

1.4 Renormalization and physicality of solutions

The question arising from our new concept of solution is, how the renormalized solutions agree with the physical phenomenon described by the equation. One might think that by renormalizing the equations, the renormalized solutions lack to represent the physical phenomena they represent. In fact, it is exactly the other way round and there is strong evidence that for Φ^4_d, $d \leq 3$, the renormalized solutions are the physical solutions.

It is shown in ?? that for $d = 2$, that Φ^4_2 is the scaling limit of a discrete Ising-Kac model evolving according to Glauber dynamics. It turns out that the renormalisation constant has a natural interpretation as shift of the critical temperature. It also turns out that the scaling factors necessary to realize Φ^4_d can only be obtained in dimension $d \leq 3$ which is in agreement with the subcriticality condition.

1.5 Linear theory and Schauder estimates

We will review the linear theory and determine the regularity of solutions to the linear equation. Consider the linear stochastic heat equation

$$\begin{aligned}\partial_t u &= \Delta u + \Xi \\ u(0, \cdot) &= u_0\end{aligned}$$

on $\mathbb{R}_+ \times \mathbb{T}^d$. By Duhamel's principle, the solution is formally given by

$$u(t, x) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \Xi(s, x) \, \mathrm{d}s.$$

Here

$$e^{t\Delta} f(x) = \int_{\mathbb{T}^d} H(t, x - y) f(y) \, \mathrm{d}y$$

is the heat operator where the kernel H is given by

$$H(t, x) = \sum_{i \in \mathbb{Z}^d} \frac{1}{(4\pi t)^{-\frac{d}{2}}} \exp\left(\frac{-|x + i|^2}{4t}\right).$$

We have the following Schauder estimate

(1.3) Theorem (Schauder estimate)
Assume $f \in \mathcal{C}^\alpha_{\mathfrak{s}}((0, T) \times \mathbb{T}^d)$. Define

$$u(t, x) = \int_0^t e^{(t-s)\Delta} f(s, x) \, \mathrm{d}s,$$

intepreted in distributional sense. Then, if $\alpha \notin \mathbb{Z}$, it holds

$$\|u\|_{\mathcal{C}^{\alpha+2}_{\mathfrak{s}}((0,T)\times\mathbb{T}^d)} \lesssim \|f\|_{\mathcal{C}^{\alpha}_{\mathfrak{s}}((0,T)\times\mathbb{T}^d)}. \qquad \diamond$$

Especially, for u the solution of the linear equation, it holds that $u \in \mathcal{C}^\alpha_{\mathfrak{s}}((0, T) \times \mathbb{T}^d)$ for any $\alpha < -\frac{d}{2} + 1$ since $\Xi \in \mathcal{C}^\alpha_{\mathfrak{s}}((0, T) \times \mathbb{T}^d)$ for any $\alpha < -\frac{d}{2} - 1$. For $d = 1$, this shows that the solution will be a continuous function, for $d \geq 2$, the solution is only a distribution. Thus equation (1.1) is ill-defined since nonlinear functions of distributions are ill-defined objects and a solution theory relies on regularization and the study of the corresponding limiting problem.

1.6 History of solutions to the problem

Solutions to the Φ^4_3 model were already constructed before by Hairer who used his technique of regularity structures which is a way of perturbative renormalization. Another way of providing solutions was introduced by Gubinelli, Imkeller and Perkowski. They developed the theory of paracontrolled distributions and used it to construct local-in-time solutions.

We shall be interested in the third approach: Kupiainen developed an approach to the problem using the Wilsonian renormalization group analysis. Kupiainen constructs a solution to Φ^4_3 by regularizing the heat kernel via cutting-off the singularity at $t = 0$:

$$(G_\varepsilon f)(t) = \int_0^t (1 - \chi((t-s)/\varepsilon^2) e^{(t-s)\Delta} f(s)\,\mathrm{d}s.$$

The sequence of regularized solutions (φ_ε) to

$$\varphi_\varepsilon = G_\varepsilon(-\varphi_\varepsilon^3 - r_\varepsilon \varphi_\varepsilon + \Xi)$$

can then be studied by means of renormalization group analysis. That is, starting from scale λ^N for some $\lambda < 1$, one studies the flow of the effective potential.

Here we will use the same techniques, but we will use a different regularization. We will mollify the noise and leave the heat kernel unchanged. Choose a smooth bump ρ on $\mathbb{R} \times \mathbb{R}^d$ whose space-time integral is one and define

$$\rho_\varepsilon = \delta^{-5} \rho(\varepsilon^{-2} t, \varepsilon^{-1} x).$$

Introducing parabolic scaling for the mollifier will come in handy later. Then we define the regularized noise by

$$\Xi_\varepsilon = \rho_\varepsilon * \Xi.$$

Here $*$ means convolution in space and time. It may be remarked that this is the same setup as studied by Hairer in ??. Since Ξ^ε is smooth, we have no trouble in proving (local) existence and uniqueness of solutions of the regularized equations

$$\partial_t \varphi_\varepsilon = \Delta \varphi_\varepsilon - \varphi_\varepsilon^3 - r_\varepsilon \varphi_\varepsilon + \Xi_\varepsilon.$$

We now want to prove the following main theorem

(1.4) Theorem
For every $\varepsilon > 0$ there exists r_ε such that for almost all ω there exists $t(\Xi(\omega)) > 0$ such that (1.1) has a unique smooth solution φ_ε on $t \in [0, t(\Xi(\omega))]$, $x \in \mathbb{T}^3$ and there exists $\varphi \in \mathcal{D}'([0, t(\Xi)] \times \mathbb{T}^3)$ such that $\varphi_\varepsilon \to \varphi$ in distributions. Furthermore, the limit is independent of the chosen mollifier ρ. ◇

Chapter 2 - The setup of the RG method

Recall that we were studying the equation

$$\partial_t \varphi = \Delta \varphi - \varphi^3 - r\varphi + \Xi. \tag{2.1}$$

Let $\chi \geq 0$ be a smooth bump so that

$$\chi(t) = 1 \text{ for } t \in [0,1] \text{ and } \chi(t) = 0 \text{ for } t \in [2,\infty).$$

We define

$$(Gf)(t) = \int_0^t e^{(t-s)\Delta} f(s)\,\mathrm{d}s.$$

We define the regularization of the heat kernel with parameter $\varepsilon > 0$ by

$$(G_\varepsilon f)(t) = \int_0^t \left(1 - \chi\left(\frac{t-s}{\varepsilon^2}\right)\right) e^{(t-s)\Delta} f(s)\,\mathrm{d}s.$$

Note that we cut away the singularity of the heat kernel, i.e. we consider only $t - s \geq \varepsilon^2$.

Remark
Since $\left(1 - \chi\left(\frac{t-s}{\varepsilon^2}\right)\right) e^{(t-s)\Delta}$ is in $\mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$, $G_\varepsilon \Xi$ is a.s. smooth. ◇

Using Duhamel's principle together with the regularization of the heat kernel, we introduce the regularization scheme for solutions of (2.1) with initial data $\varphi(0) = 0$:

$$\varphi_\varepsilon = G_\varepsilon(-\varphi_\varepsilon^3 - r_\varepsilon \varphi + \Xi) \tag{2.2}$$

where r_ε will be chosen later ensuring that (2.2) has a unique solution φ_ε which converges as $\varepsilon \to 0$ to a non-trivial limit.

(2.1) Theorem
For every $\varepsilon > 0$ there exists r_ε such that for almost all ω there exists $t(\Xi(\omega)) > 0$ such that (2.2) has a unique smooth solution φ_ε on $t \in [0, t(\Xi(\omega))]$, $x \in \mathbb{T}^3$ and there exists $\varphi \in \mathcal{D}'([0,t(\Xi)] \times \mathbb{T}^3)$ such that $\varphi_\varepsilon \to \varphi$ in distributions. Furthermore, the limit is independent of the chosen cut-off χ. ◇

Remark
It is well-known, see ??, that the correct choice of the renormalization constant for this problem is given by

$$r_\varepsilon = \frac{m_1}{\varepsilon} + m_2 \log(\varepsilon) + m_3$$

where m_1 and m_3 depend explicitly on the choice of the cut-off χ and m_2 is universal. ◇

2.1 Effective equation

So consider the regularized problem

$$\varphi_\varepsilon = G_\varepsilon(-\varphi_\varepsilon^3 - r_\varepsilon\varphi_\varepsilon + \Xi) = G_\varepsilon(V_\varepsilon(\varphi_\varepsilon) + \Xi) \tag{2.3}$$

for $\varphi(t,x)$ given on $(t,x) \in [0,\tau] \times \mathbb{T}^3$ and where at scale ε the nonlinearity is given by

$$V_\varepsilon(\varphi)(t,x) = -\varphi^3(t,x) - r_\varepsilon\varphi(t,x)$$

for r_ε to be determined later. We want to study the limit of φ_ε as $\varepsilon \to 0$. Renormalization group techniques study a problem at multiple scales at the same time and pursue the question how a theory at scale ε, say, is influenced by the same theory at scale ε' for $\varepsilon' > \varepsilon$. This kind of description turns out to be fruitful in identifying the behaviour of the objects involved and mirrors the scale dependence of mathematical and physical objects.
We first describe how to obtain the solution at level ε in terms of the scales at $\varepsilon' > \varepsilon$. We decompose

$$G_\varepsilon = G_{\varepsilon'} + \Gamma_{\varepsilon,\varepsilon'}$$

where for $\mu < \eta$

$$\Gamma_{\mu,\eta} f(t) = \int_0^t \left(\chi\left(\frac{t-s}{\eta^2}\right) - \chi\left(\frac{t-s}{\mu^2}\right)\right) e^{(t-s)\Delta} f(s)\,\mathrm{d}s.$$

This term involves all time-scales $t-s$ between μ^2 and η^2. So we split the scales $t - s \geq \varepsilon^2$ into the time scales between ε^2 and ε'^2 and those greater than ε'^2. We decompose φ_ε now into φ'_ε which belongs to the latter part and $Z(\varphi')$ that solves a small scale equation

$$\varphi_\varepsilon = \varphi'_\varepsilon + Z(\varphi')$$

where $Z(\varphi'_\varepsilon)$ solves the fixed point equation

$$Z(\varphi'_\varepsilon) = \Gamma_{\varepsilon,\varepsilon'}(V_\varepsilon(\varphi'_\varepsilon + Z(\varphi'_\varepsilon)) + \Xi). \tag{2.4}$$

Then equation (2.3) will hold provided φ'_ε satisfies

$$\varphi'_\varepsilon = G_{\varepsilon'}(V_\varepsilon(\varphi_\varepsilon) + \Xi) = G_{\varepsilon'}(V'_\varepsilon(\varphi'_\varepsilon) + \Xi) \tag{2.5}$$

where the new effective potential V' is defined as

$$V'_\varepsilon(\varphi'_\varepsilon) = V_\varepsilon(\phi'_\varepsilon + Z(\phi'_\varepsilon)).$$

Together with (2.4), we see that V'_ε solves the fixed point problem

$$V'_\varepsilon(\cdot) = V_\varepsilon(\cdot + \Gamma_{\varepsilon,\varepsilon'}(V'_\varepsilon(\cdot) + \Xi)).$$

Since (2.5) is of the same form as (2.3) with the new effective potential V'_ε, we can iterate this procedure and obtain a flow where ε' increases to $\tau^{1/2}$ and τ is the time of existence for the

original equation.
We set this flow up step-by-step. We fix a number

$$\lambda < 1$$

that we will choose later and start with

$$\varepsilon = \lambda^N.$$

We will denote $V^{(N)} = V_{\lambda^N}$, i.e.

$$V_N^{(N)}(\varphi) = -\varphi^3 - r_{\lambda^N}\varphi.$$

Then, upon iteration, for $n < N$ we derive the effective potentials on scale λ^{n-1} from scale λ^n as solution to the fixed point problem

$$V_{n-1}^{(N)}(\cdot) = V_n^{(N)}(\cdot + \Gamma_{\lambda^n,\lambda^{n-1}}(V_{n-1}^{(N)}(\cdot) + \Xi)).$$

Set $\varphi_N = \varphi_{\lambda^N}$ the solution of

$$\varphi_N = G_{\lambda^N}(V_N^{(N)}(\varphi_N) + \Xi).$$

The solution to this equation can be iteratively constructed by

$$\begin{aligned} F_N^{(N)}(\varphi) &= \varphi, \\ F_{n-1}^{(N)}(\cdot) &= F_n^{(N)}(\cdot + \Gamma_{\lambda^n,\lambda^{n-1}}(V_{n-1}^{(N)}(\cdot) + \Xi)). \end{aligned}$$

Then

$$\varphi_N = F_n^{(N)}(\varphi_n)$$

where ϕ_n solves

$$\varphi_n = G_{\lambda^n}(V_n^{(N)}(\varphi_n) + \Xi).$$

What happens here is that we decompose the solution at level λ^N into a sum of solutions of small scale equations that involve scales between λ^m and λ^{m-1} for $n < m \leq N$ and the solution to an equation that involves all scales larger than λ^n.
Our aim will be to study the limits $V_n = \lim_{N\to\infty} V_n^{(N)}$ and $F_n = \lim_{N\to\infty} F_n^{(N)}$.

2.2 Rescaling the flow

For $\mu > 0$ define the parabolic scaling operator

$$s_\mu f(t,x) = \mu^{\frac{1}{2}} f(\mu^2 t, \mu x).$$

(2.2) Lemma
We have the following identities:

$$s_\mu \circ G \circ s_\mu^{-1} = \mu^2 G, \qquad s_\mu \circ G_\varepsilon \circ s_\mu^{-1} = \mu^2 G_{\frac{\varepsilon}{\mu}},$$
$$s_\mu \circ \Gamma_{\varepsilon,\varepsilon'} \circ s_\mu^{-1} = \mu^2 \Gamma_{\frac{\varepsilon}{\mu},\frac{\varepsilon'}{\mu}}, \qquad s_\mu \Xi \overset{d}{=} \mu^{-2}\Xi^{(\mu)}.$$

where by $\Xi^{(\mu)}$ we denote space-time white noise on the $\mathbb{R} \times \mu^{-1}\mathbb{T}^3$. ◇

Now we define the dimensionless variables. We rescale to $\mathbb{T}_n = \lambda^{-n}\mathbb{T}^3$. Define

$$v_n^{(N)} = \lambda^{2n} s_{\lambda^n} \circ V_n^{(N)} \circ s_{\lambda^n}^{-1}, \tag{2.6}$$
$$f_n^{(N)} = s_{\lambda^n} \circ F_n^{(N)} \circ s_{\lambda^n}^{-1}. \tag{2.7}$$

We furthermore define

$$\phi_n = s_{\lambda^n}\varphi_n$$

where we drop the superscript whenever the scaling is clear from the context. We remark that if φ_n is understood to involve the spatial scales from $[\lambda^n, 1]$, then $\phi_n = s_{\lambda^n}\varphi_n$ lives on the spatial scales $[1, \lambda^{-n}]$.
Setting $s = s_\lambda$ and using $s \circ s_{\lambda^n} = s_{\lambda^{n-1}} \circ s = s_{\lambda^n}$, we compute

$$v_{n-1}^{(N)}(\phi) = \lambda^{-2} s^{-1} v_n^{(N)} \left(s(\phi + \Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi) + \xi_{n-1}) \right) \tag{2.8}$$

and

$$f_{n-1}^{(N)}(\phi) = s^{-1} f_n^{(N)}(s(\phi + \Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi) + \xi_{n-1}))). \tag{2.9}$$

The analysis following in section 4 will be concerned with solving (2.10). We define the solution map $v_n^{(N)} \mapsto v_{n-1}^{(N)}$ to be

$$v_{n-1}^{(N)} = \mathcal{R}_n v_n^{(N)},$$

called the RG map. We then iterate this and look for limits as $N \to \infty$.
We will show that $v_n^{(N)}$ and then also $f_n^{(N)}$ converge in a suitable space as $N \to \infty$. Then reconstructing the solution φ_ε for $\varepsilon = \lambda^N$ via the flow of effective potential and the bookkeeping operator f, will show convergence of φ_ε to a non-trivial limit.
The three equation of importance here, are

$$v_{n-1}^{(N)}(\phi) = \lambda^{-2} s^{-1} v_n^{(N)} \left(s(\phi + \Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi) + \xi_{n-1}) \right), \tag{2.10}$$
$$f_{n-1}^{(N)}(\phi) = s^{-1} f_n^{(N)}(s(\phi + \Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi) + \xi_{n-1}))), \tag{2.11}$$
$$\phi_n = G_1(v_n^{(N)} + \xi_n)). \tag{2.12}$$

They involve the operators G_1 and $\Gamma_{\lambda,1}$ which are given by Schwartz-kernel so that they are infinitely smoothing and have fast decay in space-time. In particular, if we consider $\zeta = \Gamma_{\lambda,1}\xi_n$. Since this object involves only time scales between 1 and λ^2, whenever $|t - s| > 2\lambda^{-2}$, it holds

$$\mathbb{E}[\zeta(t,x)\zeta(s,y)] = 0$$

since the supports of the cut-offs are mutually disjoint. Also the covariance inherits the Gaussian decay in space from the integral kernel. These properties will help us to prove the stochastic bounds so that the analysis of the fixed point problem turns out to be relatively simple.

Remark
Due to the scaling, we role that $f_n^{(N)}$ plays, is to decompose the solution into terms that involve scales in between λ^2 and 1, i.e. we have $\mathcal{O}(1)$-contributions to the solution. Still, rescaling turns out to be only an analytic trick here. Later, we only need to deal only with the operators G_1 and $\Gamma_{\lambda,1}$ so that proving bounds is less messy and the proof of the fixed point is simpler. We could do the analysis without the scaling, adapting all the norms and bounds.
This differs from the use of renormalization group as, for example, in ?? where the rescaling into blocks of similar size is necessary to set up a flow whose fixed point behaviour we study.$\diamond$

2.3 Synopsis

Chapter 3 - Perturbative study and noise estimates

We now study the fixed point problem

$$v_{n-1}^{(N)}(\phi) = \lambda^{-2}s^{-1}v_n^{(N)}\left(s(\phi + \Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi) + \xi_{n-1})\right)$$

where

$$\xi_n = \lambda^{2n}s_{\lambda^n}\Xi$$

perturbatively up to second order. We do this in order to identify the "relevant" terms, i.e. those terms that will explode as $N \to \infty$ and thus we need to take care of by the renormalization constant.

3.1 First order perturbation

We define the linear flow map $\mathcal{L}_n = D\mathcal{R}_n(0)$ given by

$$(\mathcal{L}_n v)(\phi) = \lambda^{-2}s^{-1}v(s(\phi + \Gamma_{\lambda,1}\xi_{n-1})).$$

It holds

$$v_{n-1}^{(N)}(\phi) = (\mathcal{R}_n v_n^{(N)}))(\phi) = (\mathcal{L}_n v_n^{(N)})(\phi + \Gamma_{\lambda,1}v_{n-1}^{(N)}(\phi)).$$

We decompose

$$v_n^{(N)} = u_n^{(N)} + w_n^{(N)}$$

where $u_n^{(N)}$ is the linear part, i.e. the flow along $\mathbb{L}_n$:

$$u_{n-1} = \mathcal{L}_n u_n.$$

For general initial data u_N, the flow can be easily evaluated

$$u_n(\phi) = (\mathcal{L}_{n+1}\dots\mathcal{L}_N u_N)(\phi) = \lambda^{-2(N-n)}s^{n-N}u_N(s^{N-n}(\phi + \eta_n^{(N)})).$$

Here we define the stochastic field $\eta_n^{(N)}$ to be

$$\eta_n^{(N)} = \Gamma_{1,\lambda^{N-n}}\xi_n =: \Gamma_n^{(N)}\xi_n.$$

$\Gamma_n^{(N)}$ is given by

$$(\Gamma_n^{(N)}f)(t,x) = \int \Gamma_n^{(N)}(t,s,x,y)f(s,y)\,\mathrm{d}s\,\mathrm{d}y$$

where the integral kernel

$$\Gamma_n^{(N)}(t,s,x,y) = \chi_{N-n}(t-s)H_n(t-s,x-y)$$

is described in terms of the cut-off

$$\chi_{N-n}(s) = \chi(s) - \chi\left(\frac{s}{\lambda^{2(N-n)}}\right)$$

which is a smooth indicator on $[\lambda^{2(N-n)}, 2]$ and $H_n = e^{t\Delta}(x,y)$ is the heat kernel. Remark that the integrand in $\Gamma_n^{(N)}$ is thus supported in $\lambda^{2(N-n)} \le t-s \le 2$.

Remark
The initial data we want to use for the flow is given by $u_N(\phi) = -\lambda^N\phi^3 - \lambda^{2N}m_1\phi$. For this kind of initial data, the linearized flow is very simple. If $u_N = \phi^k$, we get

$$u_n = \lambda^{(N-n)(k-5)/2}(\phi + \eta_n^{(N)})^k. \tag{3.1}$$

Indeed, it can be easily computed that

$$\begin{aligned}\mathcal{L}_N\phi^k &= \lambda^{-2}s^{-1}(s(\phi + \Gamma_{\lambda,1}\xi_{N-1}))^k \\ &= \lambda^{(k-5)/2}(\phi + \Gamma_{\lambda,1}\xi_{N-1})^k \\ &= \lambda^{(k-5)/2}(\phi + \Gamma_{N-1}^{(N)}\xi_{N-1})^k \\ &= \lambda^{(k-5)/2}(\phi + \eta_{N-1}^{(N)})^k\end{aligned}$$

and using (2.2) together with $\eta_n^{(N)} = \Gamma_{1,\lambda^{(N-n)}}\xi_n$

$$\begin{aligned}\mathcal{L}_n(\phi + \eta_n^{(N)})^k &= \lambda^{-2}s^{-1}(s(\phi + \Gamma_{\lambda,1}\xi_{n-1}) + \eta_n^{(N)})^k \\ &= \lambda^{-2}s^{-1}(s(\phi + \Gamma_{\lambda,1}\xi_{n-1} + \Gamma_{\lambda^{(N-(n-1))},\lambda}\xi_{n-1}))^k \\ &= \lambda^{(k-5)/2}(\phi + \eta_{n-1}^{(N)})^k.\end{aligned}$$

Now as $N - n \to \infty$, since $\lambda < 1$ terms are relevant for $k < 5$, marginal for $k = 5$ and irrelevant for $k > 5$. ◇

The source of the first renormalization constant comes from the divergence of the covariance of $\eta_n^{(N)}$ as $N - n \to \infty$. Since

$$\begin{aligned}\mathbb{E}\left[\eta_n^{(N)}(t',x')\eta_n^{(N)}(t,x)\right] &= \int_0^t H_n(t'-t+2s, x'-x)\chi_{N-n}(t'-t+s)\chi_{N-n}(s)\,\mathrm{d}s \\ &=: C_{N-n}(t',t,x',x)\end{aligned}$$

the diagonal behaves

$$\mathbb{E}\left[\eta_n^{(N)}(t,x)^2\right] = \int_0^t H_n(2s,0)\chi_{N-n}(s)^2\,\mathrm{d}s$$

which diverges at $N - n \to \infty$.

In order to show that the limit is independent of the chosen cut-off χ with bounded C^1-norm, we need to analyse the dependence of the cut-off. I.e. given two cut-offs χ and χ' with bounded C^1-norm. We define a kernel

$$\Gamma_n'^{(N)}(t,s,x,y) = \chi'_{N-n}(t-s)H_n(t-s,x-y)$$

where we denote by

$$\chi'_{N-n}(s) = \chi(s) - \chi'(\lambda^{-2(N-n)}s).$$

We will vary χ' in order to study the cut-off dependence and also to study the N-dependence of the scheme, since the choice of $\chi'(s) = \chi(\lambda^{-2}s)$ gives $\Gamma_n^{(N)'} = \Gamma_n^{(N+1)}$.
We will now study the divergent behaviour of

$$\mathbb{E}\left[\eta_n^{(N)}(t,x)^2\right] = \int_0^t H_n(2s,0)\chi_{N-n}(s)^2 \, \mathrm{d}s$$

and its dependence on the cut-off.

(3.1) Lemma
Set

$$\rho = \int_0^\infty (8\pi s)^{-\frac{3}{2}}(1-\chi(s)^2)\, \mathrm{d}s.$$

It holds

$$\mathbb{E}\left[\eta_n^{(N)}(t,x)^2\right] = \lambda^{-(N-n)}\rho + \delta_n^{(N)}(t)$$

where it holds

$$|\delta_n^{(N)}(t)| \leq C(1+t^{-\frac{1}{2}}).$$

If we vary the cut-off, it is

$$|\delta_n^{(N)}(t) - \delta_n^{(N)'}(t)| \leq C\left(t^{-\frac{1}{2}}1_{[0,2\lambda^{2(N-n)}]} + e^{-c\lambda^{-2N}}\right)\|\chi-\chi'\|_\infty.$$

◇

Proof
Denote by

$$H(t,x) = e^{t\Delta}(0,x) = (4\pi s)^{-\frac{3}{2}}e^{-\frac{|x|^2}{4t}}$$

the heat kernel on $\mathbb{R}^3$, then the heat kernel H_n on $\mathbb{T}_n$ is given by

$$H_n(t,x) = \sum_{i\in\mathbb{Z}^3} H(t, x+\lambda^{-n}i).$$

Thus, we may write

$$\begin{aligned}
\mathbb{E}\left[\eta_n^{(N)}(t,x)^2\right] &= \int_0^t H_n(2s,0)\chi_{N-n}(s)^2\,\mathrm{d}s \\
&= \int_0^t \sum_{i\in\mathbb{Z}^3} H(2s,\lambda^{-n}i)\left(\chi(s)-\chi\left(\frac{s}{\lambda^{2(N-n)}}\right)\right)^2 \mathrm{d}s \\
&= \int_0^t \sum_{i\in\mathbb{Z}^3} H(2s,\lambda^{-n}i)\left(\chi(s)^2-\chi\left(\frac{s}{\lambda^{2(N-n)}}\right)^2\right) \mathrm{d}s
\end{aligned}$$

where we used that since $\chi(s) = 1$ for $s \in [0,1]$ and $\chi\left(\frac{s}{\lambda^{2(N-n)}}\right)$ is supported on $s \in [0, 2\lambda^{2(N-n)}] \subset [0,1]$, we get

$$\left(\chi(s)-\chi\left(\frac{s}{\lambda^{2(N-n)}}\right)\right)^2 = \chi(s)^2 - \chi\left(\frac{s}{\lambda^{2(N-n)}}\right)^2.$$

Separating the $i=0$ term from the sum, we write

$$\begin{aligned}\mathbb{E}\left[\eta_n^{(N)}(t,x)^2\right] &= \int_0^t H(2s,0)\left(\chi(s)^2-\chi\left(\frac{s}{\lambda^{2(N-n)}}\right)^2\right)\mathrm{d}s+\alpha(t)\\ &= \int_0^t (8\pi s)^{-\frac{3}{2}}\left(\chi(s)^2-\chi\left(\frac{s}{\lambda^{2(N-n)}}\right)^2\right)\mathrm{d}s+\alpha(t)\end{aligned}$$

with

$$\begin{aligned}|\alpha(t)| &= \left|\sum_{i\neq 0}\int_0^t H(2s,\lambda^{-n}i)\left(\chi(s)^2-\chi\left(\frac{s}{\lambda^{2(N-n)}}\right)^2\right)\mathrm{d}s\right|\\ &\leq \sum_{i\neq 0}\int_0^2 (8\pi s)^{-\frac{3}{2}} e^{\frac{-|i|^2}{4s\lambda^{2n}}}\,\mathrm{d}s\\ &\leq Ce^{-c\lambda^{-2n}}.\end{aligned}$$

Let χ' be another cut-off and let

$$\alpha'(t)=\sum_{i\neq 0}\int_0^t H(2s,\lambda^{-n}i)\left(\chi(s)^2-\chi'\left(\frac{s}{\lambda^{2(N-n)}}\right)^2\right)\mathrm{d}s.$$

Then, it holds

$$\begin{aligned}|\alpha(t)-\alpha'(t)| &= \left|\sum_{i\neq 0}\int_0^t H(2s,\lambda^{-n}i)\left(\chi'\left(\frac{s}{\lambda^{2(N-n)}}\right)^2-\chi'\left(\frac{s}{\lambda^{2(N-n)}}\right)^2\right)\right|\\ &\leq C\int_0^t s^{-\frac{3}{2}}\left|\chi'\left(\frac{s}{\lambda^{2(N-n)}}\right)^2-\chi'\left(\frac{s}{\lambda^{2(N-n)}}\right)^2\right| e^{\frac{-\lambda^{-2n}}{4s}}\,\mathrm{d}s\\ &\leq Ce^{-c\lambda^{-2N}}\|\chi-\chi'\|_\infty.\end{aligned}$$

If we denote

$$\begin{aligned}\beta(t,\varepsilon) &= \int_0^t (8\pi s)^{-\frac{3}{2}}\left(\chi(s)^2-\chi\left(\frac{s}{\varepsilon^2}\right)^2\right)\mathrm{d}s,\\ \beta'(t,\varepsilon) &= \int_0^t (8\pi s)^{-\frac{3}{2}}\left(\chi(s)^2-\chi'\left(\frac{s}{\varepsilon^2}\right)^2\right)\mathrm{d}s\end{aligned}$$

so that $\beta(t,\varepsilon)$ is the $i=0$ term for $\varepsilon=\lambda^{(N-n)}$. We compute by the change of variables s to $\varepsilon^2 s$

$$\begin{aligned}\beta'(\infty,\varepsilon) &= \int_0^\infty (8\pi s)^{-\frac{3}{2}}\left(\chi(s)^2-\chi'\left(\frac{s}{\varepsilon^2}\right)^2\right)\mathrm{d}s\\ &= \int_0^\infty (8\pi\varepsilon^2 s)^{-\frac{3}{2}}\left(\chi(\varepsilon^2 s)^2-\chi'(s)^2\right)\varepsilon^2\,\mathrm{d}s\\ &= \varepsilon^{-1}\int_0^\infty (8\pi s)^{-\frac{3}{2}}\left(\chi(\varepsilon^2 s)^2-\chi'(s)^2\right)\mathrm{d}s\\ &= \varepsilon^{-1}\int_0^\infty (8\pi s)^{-\frac{3}{2}}\left(1-\chi'(s)^2\right)\mathrm{d}s-\varepsilon^{-1}\int_0^\infty (8\pi s)^{-\frac{3}{2}}\left(1-\chi(\varepsilon^2 s)^2\right)\mathrm{d}s\\ &= \varepsilon^{-1}\rho'-\rho.\end{aligned}$$

Thus (using $\chi' = \chi$) we may write

$$\mathbb{E}\left[\eta_n^{(N)}(t,x)^2\right] = \lambda^{-(N-n)}\rho + \delta_n^{(N)}(t)$$

and using $\lambda^{-(N-n)}\rho = \beta(\infty, \lambda^{(N-n)}) + \rho$

$$\begin{aligned}\delta_n^{(N)}(t) &= \beta(t, \lambda^{(N-n)}) - \lambda^{-(N-n)}\rho + \alpha(t) \\ &= \alpha(t) - \gamma(t, \lambda^{(N-n)}) - \rho.\end{aligned}$$

Here we denote by

$$\begin{aligned}\gamma(t,\varepsilon) &= \beta(\infty,\varepsilon) - \beta(t,\varepsilon) \\ &= \int_t^\infty (8\pi s)^{-\frac{3}{2}} \left(\chi(s)^2 \chi\left(\frac{s}{\varepsilon^2}\right)^2\right) \mathrm{d}s \\ &\leq Ct^{-\frac{1}{2}}.\end{aligned}$$

Thus

$$|\delta_n^{(N)}(t)| \leq \rho + Ce^{-c\lambda^{-2n}} + Ct^{-\frac{1}{2}} \leq C(1 + t^{-\frac{1}{2}}).$$

The last part of the lemma follows from

$$\begin{aligned}\left|\gamma(t,\varepsilon) - \gamma'(t,\varepsilon)\right| &\leq \int_t^\infty (8\pi s)^{-\frac{3}{2}} \left|\chi\left(\frac{s}{\varepsilon^2}\right)^2 - \chi'\left(\frac{s}{\varepsilon^2}\right)^2\right| \mathrm{d}s \\ &\leq Ct^{-\frac{1}{2}} \|\chi - \chi'\|_\infty 1_{[0,\varepsilon]}(t).\end{aligned}$$

□

We fix the first renormalization constant to be

$$m_1 = -3\rho$$

taking care of the divergences up to first order. Then we run the linearized flow with initial data given by

$$u_N^{(N)} = -\lambda^N \phi^3 - \lambda^N m_1 \phi = -\lambda^N \phi^3 + 3\lambda^{2N} \rho_N \phi$$

where we set

$$\rho_k = \lambda^{-k}\rho.$$

Using (3.1), we can compute the flow explicitly and get

$$u_n^{(N)} = -\lambda^n \left((\phi + \eta_n^{(N)})^3 - 3\rho_{N-n}(\phi + \eta_n^{(N)})\right).$$

Indeed,

$$\begin{aligned}\mathcal{L}_{n+1} \ldots \mathcal{L}_N(-\lambda^N \phi^3) &= \lambda^{(N-n)(3-5)/2}(-\lambda^N)(\phi + \eta_n^{(N)})^3 \\ &= -\lambda^n (\phi + \eta_n^{(N)})^3, \\ \mathcal{L}_{n+1} \ldots \mathcal{L}_N(-\lambda^N \phi^3) &= 3\lambda^{-2(N-n)} \lambda^{2N} \rho_N (\phi + \eta_n^{(N)}) \\ &= 3\lambda^{2n} \rho_N (\phi + \eta_n^{(N)}) \\ &= \lambda^n 3\lambda^{(n-N)} \rho(\phi + \eta_n^{(N)}) \\ &= \lambda^n 3\rho_{N-n}(\phi + \eta_n^{(N)}).\end{aligned}$$

3.2 Second-order perturbation

Define
$$\mathcal{G}_n(v,\bar{v})(\phi) = (\mathcal{L}_n v)(\phi + \Gamma_{\lambda,1}\bar{v}(\phi)) - (\mathcal{L}_n v)(\phi)$$
so that (2.10) becomes
$$v_{n-1}^{(N)} = \mathcal{L}_n v_n^{(N)} + \mathcal{G}_n(v_n^{(N)}, v_{n-1}^{(N)}).$$
Recall that we decomposed $v_n^{(N)} = u_n^{(N)} + w_n^{(N)}$ where $u_{n-1}^{(N)} = \mathcal{L}_n u_n^{(N)}$ and
$$w_{n-1}^{(N)} = \mathcal{L}_n w_n^{(N)} + \mathcal{G}_n(u_n^{(N)} + w_n^{(N)}, u_{n-1}^{(N)} + w_{n-1}^{(N)}).$$
The initial condition for w is given by
$$w_N(\phi) = -\lambda^{2N}(m_2 \log \lambda^N + m_3)\phi.$$
Since
$$\mathcal{G}_n(u_n^{(N)}, u_{n-1}^{(N)})(\phi) = u_{n-1}^{(N)}(\phi + \Gamma_{\lambda,1} u_{n-1}^{(N)}(\phi)) - u_{n-1}^{(N)}(\phi)$$
by Taylor expansion, it is
$$\mathcal{G}_n(u_n^{(N)}, u_{n-1}^{(N)})(\phi) = Du_{n-1}^{(N)}(\phi)\Gamma_{\lambda,1} u_{n-1}^{(N)}(\phi) + \mathcal{O}(\lambda^n)$$
where it is
$$Du_{n-1}^{(N)} = -3\lambda^{n-1}\left((\phi + \eta_{n-1}^{(N)})^2 - \rho_{N-(n-1)}\right).$$
For a fixed $w_n^{(N)}$, $w_{n-1}^{(N)}$ then satisfies
$$w_{n-1}^{(N)} = \mathcal{L}_n w_{n-1}^{(N)} + Du_{n-1}^{(N)}\Gamma_{\lambda,1} u_{n-1}^{(N)} + \mathcal{F}_n(w_{n-1}^{(N)})$$
where
$$\mathcal{F}_n(w_{n-1}^{(N)}) = \mathcal{G}_n(u_n^{(N)} + w_n^{(N)}, u_{n-1}^{(N)} + w_{n-1}^{(N)}) - Du_{n-1}^{(N)}\Gamma_{\lambda,1} u_{n-1}^{(N)}.$$
It will turn out in section 4 that $\mathcal{F}_n$ will contract in a suitable norm, i.e. it is irrelevant under the RG. We can solve for $w_{n-1}^{(N)}$ up to second order, i.e. solving
$$U_{n-1}^{(N)} = \mathcal{L}_n U_n^{(N)} + Du_{n-1}^{(N)}\Gamma_{\lambda,1} u_{n-1}^{(N)}$$
so that we have the decomposition
$$w_n^{(N)} = U_n^{(N)} + \nu_n^{(N)}$$
and $\nu_{n-1}^{(N)}$ satisfies
$$\nu_{n-1}^{(N)} = \mathcal{L}_n \nu_n^{(N)} + \mathcal{F}_n(U_{n-1}^{(N)} + \nu_{n-1}^{(N)})$$
with initial conditions
$$U_N^{(N)} = -\lambda^{2N}(m_2 \log \lambda^N + m_3)\phi, \quad \nu_N^{(N)} = 0$$

where m_2 and m_3 are determined in the next subsection. We can compute $U_n^{(N)}$ explicitly. It holds

$$U_n^{(N)} = Du_n\Gamma_{\lambda^{N-n},1}u_n - \lambda^{2n}(m_2\log\lambda^N + m_3)(\phi + \eta_n^{(N)}).$$

Indeed, starting from the initial condition, we easily see that

$$\begin{aligned} U_{N-1}^{(N)} &= \mathcal{L}_n U_N^{(N)} + Du_{N-1}^{(N)}\Gamma_{\lambda,1}u_{N-1}^{(N)} \\ &= -\lambda^{2N-2}(m_2\log\lambda^N + m_3)(\phi + \eta_{N-1}^{(N)}) + Du_{N-1}^{(N)}\Gamma_{\lambda,1}u_{N-1}^{(N)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\mathcal{L}_n(Du_n^{(N)}\Gamma_{\lambda^{N-n},1}u_n^{(N)})(\phi) \\ =&\lambda^{-2}s^{-1}\left[Du_n^{(N)}(s(\phi+\Gamma_{\lambda,1}\xi_{n-1}))\Gamma_{\lambda^{(N-n)},1}u_n^{(N)}(s(\phi+\Gamma_{\lambda,1}\xi_{n-1}))\right] \\ =&\lambda^{-2}s^{-1}Du_n^{(N)}(s(\phi+\Gamma_{\lambda,1}\xi_{n-1}))s^{-1}\Gamma_{\lambda^{N-n},1}ss^{-1}u_n^{(N)}(s(\phi+\Gamma_{\lambda,1}\xi_{n-1})) \\ =&\mathcal{L}_nDu_n^{(N)}(\phi)\Gamma_{\lambda^{N-n+1},\lambda}\lambda^{-2}s^{-1}u_n^{(N)}(s(\phi+\Gamma_{\lambda,1}\xi_{n-1})) \\ =&D(\mathcal{L}_nu_n^{(N)})(\phi)\Gamma_{\lambda^{N-n+1},\lambda}\mathcal{L}_nu_n^{(N)}(\phi) \\ =&Du_{n-1}^{(N)}(\phi)\Gamma_{\lambda^{N-n+1},\lambda}u_{n-1}^{(N)}(\phi) \end{aligned}$$

which gives

$$\mathcal{L}_n(Du_n^{(N)}\Gamma_{\lambda^{N-n},1}u_n^{(N)}) + Du_{n-1}^{(N)}\Gamma_{\lambda,1}u_{n-1} = Du_{n-1}^{(N)}\Gamma_{\lambda^{N-(n-1)},1}u_{n-1}^{(N)}$$

and

$$\mathcal{L}_n(\lambda^{2n}(m_2\log\lambda^N + m_3)(\phi+\eta_n^{(N)})) = \lambda^{2n-2}(m_2\log\lambda^N + m_3)(\phi+\eta_{n-1}^{(N)}))$$

3.3 Noise estimates and function spaces

We are going to need estimates of the noise in order to show contractivity. The following terms including the noise appear in the fixed point equation

$$\nu_{n-1}^{(N)} = \mathcal{L}_n\nu_n^{(N)} + \mathcal{F}_n(U_{n-1}^{(N)} + \nu_{n-1}^{(N)})$$

$\mathcal{L}_n$ is dependent of $\Gamma_{\lambda,1}\xi_{n-1}$. Also u_n and U_n involve the noise. To see what terms are relevant or irrelevant under the RG map, let's develop U_n up to second order

$$U_n^{(N)}(\phi) = U_n^{(N)}(0) + DU_n^{(N)}(0)\phi + R_n^{(N)}(\phi).$$

We compute explicitly by using that

$$\begin{aligned} U_n^{(N)}(\phi)(t,x) = 3\lambda^{2n}((\phi(t,x)+\eta_n^{(N)}(t,x))^2 - \rho_{N-n})\int\Gamma_n^{(N)}(t,s,x,y) \\ \cdot\Big((\phi(s,y)+\eta_n^{(N)}(s,y))^3 - 3\rho_{N-n}(\phi(s,y)+\eta_n^{(N)}(s,y))\Big)\,\mathrm{d}s\,\mathrm{d}y \\ -\lambda^{2n}(m_2\log\lambda^N + m_3)(\phi(t,x)+\eta_n(N)(t,x)) \end{aligned}$$

and where we may exchange derivative and integral since $\Gamma_n^{(N)}$ is infinitely smoothing

$$
\begin{aligned}
&U_n^{(N)}(0)\\
&= Du_n(0)\Gamma_{1,\lambda^{(N-n)}}u_n(0) - \lambda^{2n}(m_2 \log\lambda^N + m_3)\eta_n^{(N)}\\
&= -3\lambda^{2n}(\eta_n^{(N)^2} - \rho_{N-n})\Gamma_{1,\lambda^{(N-n)}}(\eta_n^{(N)^3} - 3\rho_{N-n}\eta_n^{(N)}) - \lambda^{2n}(m_2\log\lambda^N + m_3)\eta_n^{(N)}\\
&= \lambda^{2n}\omega_n,\\
&(DU_n^{(N)}(0)\phi)(t,x)\\
&= \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} U_n(0+\varepsilon\phi)(t,x)\\
&= 6\lambda^{2n}\eta_n^{(N)}(t,x)\phi(t,x)\int \Gamma_n^{(N)}(t,s,x,y)\left(\eta_n^{(N)}(s,y)^3 - 3\rho_{N-n}\eta_n^{(N)}(s,y)\right)\,\mathrm{d}s\,\mathrm{d}y\\
&\quad 3\lambda^{2n}(\eta_n^{(N)}(t,x)^2 - \rho_{N-n})\int \Gamma_n^{(N)}(t,s,x,y)\left(3\eta_n^{(N)}(s,y)^2\phi(s,y) - 3\rho_{N-n}\phi(s,y)\right)\,\mathrm{d}s\,\mathrm{d}y\\
&\quad -\lambda^{2n}(m_2\log\lambda^N + m_3)\phi(t,x)\\
&= \lambda^{2n}\mathfrak{z}_n^{(N)}(t,x)\phi(t,x) + \lambda^{2n}\int z_n(t,x,s,y)\phi(s,y)\,\mathrm{d}s\,\mathrm{d}y
\end{aligned}
$$

where we denote

$$
\begin{aligned}
\mathfrak{z}_n^{(N)}(t,x) &= 6\eta_n^{(N)}(t,x)\int \Gamma_n^{(N)}(t,s,x,y)\left(\eta_n^{(N)}(s,y)^3 - 3\rho_{N-n}\eta_n^{(N)}(s,y)\right)\,\mathrm{d}s\,\mathrm{d}y,\\
z_n^{(N)}(t,s,x,y) &= 9(\eta_n^{(N)}(t,x)^2 - \rho_{N-n})\Gamma_n^{(N)}(t,s,x,y)\left(\eta_n^{(N)}(s,y)^2 - \rho_{N-n}\right)\\
&\quad -(m_2\log\lambda^N + m_3)\delta(t-s)\delta(x-y).
\end{aligned}
$$

Remark
It turns out that the covariance of the field $9(\eta_n^{(N)}(t,x)^2 - \rho_{N-n})\Gamma_n^{(N)}(t,s,x,y)\left(\eta_n^{(N)}(s,y)^2 - \rho_{N-n}\right)$ diverges as $N-n\to\infty$. That is the reason for the second renormalization constant. ◇

In the analysis, we need to know the size of the following random fields

$$\eta_n^{(N)},\ (\eta_n^{(N)})^2 - \rho_{N-n},\ (\eta_n^{(N)})^3 - 3\rho_{N-n}\eta_n^{(N)},\ \omega_n^{(N)},\ \mathfrak{z}_n^{(N)} \text{ and } z_n^{(N)}.$$

We also need to constrain $\Gamma_{\lambda,1}\xi_n$. Since in the limit $N\to\infty$, these fields become distributions, we need to measure them in an appropriate negative index Sobolev type norm. This will also take care of the lack of regularity of the field z_n. Define the operator K_1 on $L^2(\mathbb{R})$ by

$$K_1 = (-\partial_t^2 + 1)^{-1}$$

which has the integral kernel

$$K_1(t,s) = \frac{1}{2}e^{-|t-s|}$$

and define K_2 on $L^2(\mathbb{T}_n)$ via

$$K_2 = (-\Delta + 1)^{-2}$$

which has a continuous kernel satisfying $K_2(x,y) = K_2(x-y)$ with

$$K_2(x) \le Ce^{-|x|}.$$

Now we are able to define the function space $\mathcal{V}_n$ as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{T}_n)$ w.r.t. the norm

$$\|v\|_{\mathcal{V}_n} = \sup_{i \in \mathbb{Z}\times(\mathbb{Z}^3 \cap \mathbb{T}_n)} \|Kv\|_{L^2(c_i)}$$

where $K := K_1 K_2$ and c_i is the unit cube centered in $i \in \mathbb{Z} \times (\mathbb{Z}^3 \cap \mathbb{T}_n)$. For the kernel z, we defiine the corresponding norm via

$$\|z\|_{\mathcal{V}_n} = \sup_{i \in \mathbb{Z}\times(\mathbb{Z}^3 \cap \mathbb{T}_n)} \sum_{j \in \mathbb{Z}\times(\mathbb{Z}^3 \cap \mathbb{T}_n)} \|(K \otimes K)z\|_{L^2(c_i \times c_j)}.$$

We can now define the admissible set of noise. Therefore we choose a smooth bump $h \in C^\infty(\mathbb{R})$ such that

$$h(t) = 1 \text{ for } t \le -\lambda^2, \quad h(t) = 0 \text{ for } t \ge -\frac{1}{2}\lambda^2$$

and we set

$$h_k(t) = h(t - \lambda^{-2k}).$$

Thus h_k cuts off times larger than $\lambda^{-2k} - \frac{1}{2}\lambda^2$.

(3.2) Definition

Let $\gamma > 0$ (to be fixed later) and define for $m \in \mathbb{N}$ a measurable set $\mathcal{A}_m \subset \Omega$ via: $\omega \in \mathcal{A}_m$ if the following conditions hold:

(1) For any

$$\zeta_n^{(N)} \in \{\eta_n^{(N)},\ (\eta_n^{(N)})^2 - \rho_{N-n},\ (\eta_n^{(N)})^3 - 3\rho_{N-n}\eta_n^{(N)},\ \omega_n^{(N)},\ \mathfrak{z}_n^{(N)}, z_n^{(N)}\}$$

and for all $m \le n \le N$ it holds

$$\|h_{n-m}\zeta_n^{(N)}\|_{\mathcal{V}_n} \le \lambda^{-\gamma n}.$$

(2) For any two cut-offs χ, χ' with bounded C^1-norm and for any fields

$$\zeta_n^{(N)} \in \{\eta_n^{(N)},\ (\eta_n^{(N)})^2 - \rho_{N-n},\ (\eta_n^{(N)})^3 - 3\rho_{N-n}\eta_n^{(N)},\ \omega_n^{(N)},\ \mathfrak{z}_n^{(N)}, z_n^{(N)}\},$$
$$\zeta_n'^{(N)} \in \{\eta_n'^{(N)},\ (\eta_n'^{(N)})^2 - \rho'_{N-n},\ (\eta_n'^{(N)})^3 - 3\rho'_{N-n}\eta_n'^{(N)},\ \omega_n'^{(N)},\ \mathfrak{z}_n'^{(N)}, z_n'^{(N)}\}$$

and for all $m \le n \le N$ it holds

$$\|h_{n-m}(\zeta_n'^{(N)} - \zeta_n^{(N)}\|_{\mathcal{V}_n} \le \lambda^{\gamma(N-n)}\lambda^{-\gamma n}.$$

(3) For all $n > m$ it holds

$$\|s\Gamma_{\lambda,1}\xi_{n-1}\|_{\Phi_n} \le \lambda^{-\gamma n}.$$

◇

Here we denote by Φ_n the space of functions $\phi\colon [1, \lambda^{-2(n-m)}] \times \mathbb{T}_n \to \mathbb{C}$ which are C^2 in t and C^4 in x and satisfy

$$\partial_t^i \phi(1, x) = 0 \quad \text{for } 0 \leq i \leq 2 \text{ and } x \in \mathbb{T}_n.$$

We denote its norm, the sup-norm, by

$$\|\phi\|_{\Phi_n} = \sum_{i \leq 2, |\alpha| \leq 4} \|\partial_t^i \partial_x^\alpha \phi\|_\infty.$$

We will set up the fixed point equations in spaces of functions on Φ_n.

(3.3) Proposition
There exist renormalization constants m_2 and m_3 such that for some $\gamma > 0$ almost surely there is some $m < \infty$ such that $\mathcal{A}_m$ holds, i.e.

$$\mathbb{P}\left(\bigcup_{m=1}^{\infty} \mathcal{A}_m \right) = 1.$$

◇

The purpose of this thesis was to understand the analysis of the method studied. That is the reason we omit the proof of the proposition here and refer to ??.

3.4 Synopsis

Chapter 4 - The analysis of the fixed point equation

4.1 Spaces and analytic functions on Banach spaces

We now turn to the analysis of the problem. As a convention, we will be indifferent to the numerical change of constants, i.e. our constants denoted by $C(\cdot)$ may change from line to line. We denote the parameter dependence in the argument of the constant.
The main part of the section will be concerned with solving the fixed point problem. Once, we establish existence of the flow of effective potential, it will be comparatively simple to deduce the main theorem by looking at the flow of bookkeeping functions $f_n^{(N)}$.

$$v_{n-1}^{(N)}(\phi) = \lambda^{-2}s^{-1}v_n^{(N)}\left(s(\phi+\Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi)+\xi_{n-1})\right) \tag{4.1}$$

$$= (\mathcal{L}_n^{(N)})(\phi+\Gamma_{\lambda,1}v_{n-1}^{(N)}). \tag{4.2}$$

In the perturbative analysis, we wrote

$$v_{n-1}^{(N)} = \mathcal{L}_n v_n^{(N)} + \mathcal{G}_n(v_n^{(N)}, v_{n-1}^{(N)})$$

and

$$v_n^{(N)} = u_n^{(N)} + w_n^{(N)}$$

where

$$u_n^{(N)}(\phi) = -\lambda^n\left((\phi+\eta_n^{(N)})^3 - 3\rho_{N-n}(\phi+\eta_n^{(N)})\right),$$
$$w_{n-1}^{(N)} = \mathcal{L}_n w_n^{(N)} + \mathcal{G}_n(u_n^{(N)}+w_n^{(N)}, u_{n-1}^{(N)}, w_{n-1}^{(N)}).$$

In second order perturbation theory, it is

$$w_{n-1}^{(N)} = \mathcal{L}_n w_n^{(N)} + Du_{n-1}^{(N)}\Gamma_{\lambda,1}u_{n-1}^{(N)} + \mathcal{F}_n(w_{n-1}^{(N)})$$

and we furthermore decomposed

$$w_n^{(N)} = U_n^{(N)} + \nu_n^{(N)}$$

where

$$U_n^{(N)} = Du_n^{(N)}\Gamma_n^{(N)}u_n^{(N)} - \lambda^{2n}\left(m_2\log\lambda^N + m_3\right)(\phi+\eta_n^{(N)}),$$
$$\nu_{n-1}^{(N)} = \mathcal{L}_n\nu_n^{(N)} + \mathcal{F}_n(U_{n-1}^{(N)}+\nu_{n-1}^{(N)}).$$

Here we start from the initial condition $\nu_N^{(N)} = 0$. We want to solve the fixed point equation for $\nu_n^{(N)}$. Let in the following $\omega \in \mathcal{A}_m$. We drop the ω-dependence of all equations.

(4.1) Definition (Spaces)
We define the following function spaces.

1) Let $K_1 = (-\partial_t^2 + 1)^{-1}$ on $L^2(\mathbb{R})$, $K_2 = (-\Delta + 1)^{-2}$ on $L^2(\mathbb{T}_n)$, $K := K_1 K_2$. We define $\mathcal{V}_n$ as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{T}_n)$ w.r.t. the norm

$$\|v\|_{\mathcal{V}_n} = \sup_{i \in \mathbb{Z} \times (\mathbb{Z}^3 \cap \mathbb{T}_n)} \|Kv\|_{L^2(c_i)}$$

where c_i is the unit cube centered in $i \in \mathbb{Z} \times (\mathbb{Z}^3 \cap \mathbb{T}_n)$.

2) Denote by Φ_n the space of functions $\phi\colon [1, \lambda^{-2(n-m)}] \times \mathbb{T}_n \to \mathbb{C}$ which are C^2 in t and C^4 in x and satisfy

$$\partial_t^i \phi(1, x) = 0 \quad \text{for } 0 \leq i \leq 2 \text{ and } x \in \mathbb{T}_n.$$

We denote its norm, the sup-norm, by

$$\|\phi\|_{\Phi_n} = \sum_{i \leq 2, |\alpha| \leq 4} \|\partial_t^i \partial_x^\alpha \phi\|_\infty.$$

3) Let $B_n \subset \Phi_n$ the open ball centered in the origin with radius $r_n = \lambda^{-2\gamma n}$ and denote

$$\mathcal{W}_n(B_n) = \{f\colon B_n \to \mathcal{V}_n \text{ analytic functions}\}$$

and denote by $\|\cdot\|_{B_n}$ the sup-norm. ◇

Remark

There is some arbitrariness in the choice of the spaces. We need to choose $\mathcal{V}_n$ such that the space-time white noise and the random fields live and converge in $\mathcal{V}_n$. Essentially, -2 time derivatives and -4 space derivatives is just one choice that works and where we have good decay of the kernels. Since we are considering polynomials in $\phi \in \Phi_n$ with coefficients in random fields that live and converge in $\mathcal{V}_n$, if we raise the exponents in K_1 and K_2, we have to control more derivatives of ϕ, so we need more derivatives in the definition of Φ_n which complicates the proofs. ◇

We will summarize some basic properties of analytic functions on Banach spaces that will be important in the construction of fixed points later. We refer to ?? for more facts and the proofs.

In the following, let $E_1, \ldots, E_m, E, F$ Banach spaces. Let $L(E_1, \ldots, E_m; F)$ the space of all continuous multilinear maps $E_1 \times \ldots \times E_m \to F$ and $L^m(E; F) = L(E, \ldots, E; F)$. Denote by $L_s^m(E; F)$ the subset of all symmetric maps, i.e. $A(\phi_1, \ldots, \phi_m) = A(\phi_{\sigma(1)}, \ldots, \phi_{\sigma(m)})$ for all permutations $\sigma \in \mathcal{S}_m$. For $A \in L_s^m(E; F)$ denote $A\psi^m = A(\psi, \ldots, \psi)$.

A power series from E to F about $\xi \in E$ is given by

$$\sum_{m=0}^{\infty} A_m(\phi - \xi)^m$$

where $A_m \in L_s^m(E; F)$ and $L_s^0(E; F) = F$. A finite power series (i.e. $A_m = 0$ for $m \geq m_0$) is called a polynomial. This notion of power series shares a lot of the properties with usual

power series on finite-dimensional vector spaces. E.g. power series are smooth on their radius of convergence and we have the identity theorem, i.e. if two power series about ξ converge to the same function on an open subset, their coefficients are identical.
Let $U \subset E$ open and $f\colon U \to F$. We say f is analytic in $\xi \in U$ if there is a power series about ξ

$$\sum_{m=0}^{\infty} A_m(\phi - \xi)^m$$

which converges uniformly in a ball $B_r(\xi) \subset U$ to f. By the identity theorem, the power series is unique. Also f is smooth in $B_r(\xi)$ whenever it is analytic in ξ and since $D^m f(\xi) = m! A_m$, we can write

$$f(\phi) = \sum_{m=0}^{\infty} A_m(\phi - \xi)^m = \sum_{m=0}^{\infty} \frac{1}{m!} D^m f(\xi)(\phi - \xi)^m \quad \text{on } B_r(\xi).$$

We call f analytic in U when it is analytic in every $\xi \in U$. As usual, analytic functions form a sheaf and composition of analytic functions are analytic whenever they are well-defined.
We have Cauchy estimates for analytic functions. Let E, F complex Banach spaces and $f\colon U \to F$ analytic. (Our spaces are usually real Banach spaces but the results are transferable using the complexification.) Denote by $T_{f,n,\xi}(x)$

$$T_{f,n,\xi}(x) = \sum_{m \le n} \frac{1}{m!} D^m f(\xi)(x - \xi)^m.$$

We have the usual Cauchy formulas, especially given $x \in U$ and assume that $x \in \overline{B_r}(\xi) \subset U$, then it holds

$$\|f(x) - T_{f,m,\xi}(x)\|_F \le \frac{\|x - \xi\|_E^{m+1}}{r^m(r - \|x - \xi\|_E)} \sup\{\|f(t)\|_F : \|t - \xi\|_E = r\}. \tag{4.3}$$

4.2 Analysis of the flow

We start by proving the following lemma.

(4.2) Lemma

We have the following properties.

1) The operators $s\Gamma_{\lambda,1}\colon \mathcal{V}_{n-1} \to \Phi_n$ and $h_{n-1-m}\Gamma_{\lambda,1}\colon \mathcal{V}_{n-1} \to \mathcal{V}_{n-1}$ are bounded operators with operator norms bounded by $C(\lambda)$. Moreover

$$s\Gamma_{\lambda,1} h_{n-1-m} v = s\Gamma_{\lambda,1} v \quad \text{for any } v \in \mathcal{V}_{n-1}$$

as elements of Φ_n.

2) G_1 is a bounded operator $\mathcal{V}_n \to \Phi_n$ and

$$G_1(h_{n-1-m}(\lambda^2 \cdot) v) = G_1 v$$

as elements of Φ_n.

3) $s\colon \Phi_{n-1} \to \Phi_n$ and $s^{-1}\colon \mathcal{V}_n \to \mathcal{V}_{n-1}$ are bounded operators and

$$\|s\|_{\Phi_{n-1}\to\Phi_n} \le \lambda^{\frac{1}{2}}, \quad \|s^{-1}\|_{\mathcal{V}_n\to\mathcal{V}_{n-1}} \le C\lambda^{-\frac{1}{2}}.$$

4) Let $\phi \in C^{2,4}(\mathbb{R}\times\mathbb{T}_n)$ and $v \in \mathcal{V}_n$, then $\phi v \in \mathcal{V}_n$ and

$$\|\phi v\|_{\mathcal{V}_n} \le C\|\phi\|_{C^{2,4}}\|v\|_{\mathcal{V}_n}. \qquad \diamond$$

We will take advantage of 1), 3) and 4) in the proof of existence of the effective potentials. We won't need 2) before the proof of the main theorem.

Proof
For 1), let $v \in C_0^\infty(\mathbb{R}_+\times\mathbb{T}_{n-1})$. Then

$$\begin{aligned} s\Gamma_{\lambda,1}v(t,x) &= \lambda^{\frac{1}{2}}\int_0^{\lambda^2 t}\left(\chi(\lambda^2 t - s) - \chi\left(\frac{\lambda^2 t - s}{\lambda^2}\right)\right)e^{(\lambda^2 t - s)\Delta}v(s,\lambda x)\,\mathrm{d}s \\ &= \int_0^\infty k(\lambda^2 t - s)e^{(\lambda^2 t - s)\Delta}v(s,\lambda x)\,\mathrm{d}s \end{aligned}$$

where $k(\tau) = \lambda^{\frac{1}{2}}(\chi(\tau) - \chi(\tau/\lambda^2))$. Since $k(\tau)$ vanishes for $\tau \le \lambda^2$, we may extend the integral in s. Thus $s\Gamma v \in C^\infty([1,\infty),\mathbb{T}_n)$. To prove that $s\Gamma\colon \mathcal{V}_n \to \Phi_n$ is bounded and well-defined, we need uniform bounds for the L^∞-norms of the derivatives $\partial_t^i\partial_x^\alpha(s\Gamma_{\lambda,1}v)$. The condition

$$\partial_t^i(s\Gamma v)(1,x) = 0, \quad \text{for } 0 \le i \le 2 \text{ and } x \in \mathbb{T}_n$$

follows from the fact that $k(\tau)$ vanishes for $\tau \le \lambda^2$ and thus $s\Gamma_{\lambda,1}v(t,x) = 0$ for $t < 1$. To prove boundedness for the derivatives, we use integration by parts: write

$$v = (-\partial_t^2 + 1)(-\Delta + 1)^2 Kv = (-\partial_t^2 + 1)(-\Delta + 1)^2 w$$

for $w = Kv$. By integrating by parts, it is

$$\begin{aligned} s\Gamma_{\lambda,1}v(t,x) &= \int_{\mathbb{R}} k(\lambda^2 t - s)e^{(\lambda^2 t - s)\Delta}(-\Delta+1)^2(-\partial_s^2+1)w(s,\lambda x)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} k(\lambda^2 t - s)(-\Delta+1)^2 e^{(\lambda^2 t - s)\Delta}(-\partial_s^2+1)w(s,\lambda x)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} \left[(-\partial_s^2+1)k(\lambda^2 t - s)(-\Delta+1)^2 e^{(\lambda^2 t - s)\Delta}\right] w(s,\lambda x)\,\mathrm{d}s \end{aligned}$$

and we get that for $0 \le i \le 2$, $0 \le |\alpha| \le 4$, it is

$$\partial_t^i\partial_x^\alpha s\Gamma_{\lambda,1}v(t,x) = \int_{\mathbb{R}} \left[(-\partial_s^2+1)\partial_t^i k(\lambda^2 t - s)(-\Delta+1)^2\partial_x^\alpha e^{(\lambda^2 t - s)\Delta}\right] w(s,\lambda x)\,\mathrm{d}s.$$

Define the kernel

$$O_{a,\alpha}(t,s,x,y) = \partial_t^a\left(k(\lambda^2 t - s)\partial_x^\alpha e^{(\lambda^2 t - s)\Delta}\right)(x-y).$$

Then $O_{a,\alpha}$ is smooth, exponentially decreasing in $|x-y|$ and supported on $\lambda^2 t - s \in [\lambda^2, 2]$ for all $a \in \mathbb{N}$ and multi-indices α. It will suffice to bound $O_{a\alpha} 1_{c_i} w(t,x)$ for all $|\alpha| \le 8$, $a \le 4$ and $i \in \mathbb{Z} \times (\mathbb{Z}^3 \cap \mathbb{T}_n)$. Hence, for fixed $(t,x) \in \mathbb{R}_+ \times \mathbb{T}_n$, it is

$$
\begin{aligned}
|(O_{a,\alpha} 1_{c_i} w)(t,x)| &= \left| \int O_{a,\alpha}(t,s,x,y) 1_{c_i} w(s,y)\, \mathrm{d}s\, \mathrm{d}y \right| \\
&\le \left(\int_{[\lambda^2,2]\times c_i} O_{a,\alpha}(t,s,x,y)\, \mathrm{d}s\, \mathrm{d}y \right) \|w\|_{L^2(c_i)} \\
&\le C(\lambda) e^{-cd(i,(t,x))} \|w\|_{L^2(c_i)}
\end{aligned}
$$

where we used that $k(\lambda^2 t - s)$ is bounded by $C(\lambda)$ in $C^{4,8}$ and the exponential decay. This proves that

$$
\|\partial_t^i \partial_x^\alpha s\Gamma_{\lambda,1} v\|_\infty \le \sup_i \sum_{a,\alpha} \|O_{a,\alpha} 1_{c_i} w\|_\infty \le C(\lambda) \sup_i \|w\|_{L^2(c_i)} = C(\lambda)\|v\|_{\mathcal{V}_{n-1}}.
$$

Since all derivatives are bounded, it holds

$$
\|s\Gamma_{\lambda,1} v\|_{\Phi_n} \le C(\lambda) \|v\|_{\mathcal{V}_{n-1}}.
$$

Since $C_0^\infty(\mathbb{R}_+ \times \mathcal{T}_{n-1})$ is dense in $\mathcal{V}_{n-1}$ by definition, $s\Gamma_{\lambda,1}$ extends to a bounded linear operator $\mathcal{V}_{n-1} \to \Phi_n$.

The same argument applies to

$$
G_1 v(t,x) = \int_0^t (1 - \chi(t-s))\, e^{(t-s)\Delta} v(s,x)\, \mathrm{d}s = \int_0^t k(t-s) e^{(t-s)\Delta} v(s,x)\, \mathrm{d}s.
$$

Here k is supported on $t - s \in [1, \infty)$. By the same argument as above, since we cut off the singularity of the heat kernel, we get that G_1 is a bounded operator $\mathcal{V}_n \to \Phi_n$.

For the boundedness of $h_{n-1-m}\Gamma_{\lambda,1}$ as an operator $\mathcal{V}_n \to \mathcal{V}_n$, we apply a similar argument: with $k(t-s) = \chi(t-s) - \chi\left(\frac{t-s}{\lambda^2}\right)$ which is supported on $[\lambda^2, 2]$, we get by integration by parts as above for $v \in \mathcal{V}_{n-1}$ and $w = Kv$

$$
\begin{aligned}
\Gamma_{\lambda,1} v(t,x) &= \int_0^t k(t-s) e^{(t-s)\Delta} v(s,x)\, \mathrm{d}s \\
&= \int_0^t \left[(-\partial^2 + 1) k(t-s) (-\Delta+1)^2 e^{(t-s)\Delta} \right] w(s,x)\, \mathrm{d}s \\
&= \int_0^t \int (-\partial^2 + 1) k(t-s) (-\Delta+1)^2 e^{(t-s)\Delta}(x-y) w(s,y)\, \mathrm{d}y\, \mathrm{d}s.
\end{aligned}
$$

Thus for $i \in \mathbb{Z} \times (\mathbb{Z} \cap \mathbb{T}_{n-1})$

$$\begin{aligned}\|Kh_{n-1-m}\Gamma_{\lambda,1}v\|^2_{L^2(c_i)} &\le \int_{c_i} \left| \int K_1(t-t')K_2(x-x')h_{n-1-m}(t')\Gamma_{\lambda,1}v(t',x')\,\mathrm{d}t'\,\mathrm{d}x' \right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\le C \int_{c_i} \mathrm{d}x\,\mathrm{d}t \int \mathrm{d}t'\,\mathrm{d}x' \int \mathrm{d}s\,\mathrm{d}y \Big| h_{n-1-m}(t')e^{-|t-t'|}e^{-|x-x'|} \\ &\qquad (-\partial^2+1)k(t-s)(-\Delta+1)^2 e^{(t-s)\Delta}(x-y)w(s,y)\Big|^2 \\ &\le C(\lambda)\|w\|^2_{L^2(c_i)}\end{aligned}$$

where we get the last inequality from the exponential bounds above and the cut-off h_{n-1-m}. This shows that $h_{n-1-m}\Gamma_{\lambda,1}$ is a bounded operator $\mathcal{V}_{n-1} \to \mathcal{V}_{n-1}$. Left to prove for 1) and 2) are the equalities

$$s\Gamma_{\lambda,1}h_{n-1-m}v = s\Gamma_{\lambda,1}v \quad \text{for any } v \in \mathcal{V}_{n-1} \quad \forall v \in \mathcal{V}_{n-1}$$

and

$$G_1(h_{n-1-m}(\lambda^2\cdot)v) = G_1 v \quad \forall v \in \mathcal{V}_n$$

as elements of Φ_n, meaning for $t \in [1, \lambda^{-2(n-m)}]$.
For 3) recall that

$$s\phi(t,x) = \lambda^{\frac{1}{2}}\phi(\lambda^2 t, \lambda x)$$

for $(t,x) \in [\lambda^{-2}, \lambda^{-2(n-1-m)})]$. Since $\lambda < 1$, the factors λ^j we pick up differentiating, can be thrown away in the bound so that $\|s\phi\|_{\Phi_n} \le \lambda^{\frac{1}{2}}\|\phi\|_{\Phi_{n-1}}$.
For the bound of $s^{-1}\colon \mathcal{V}_n \to \mathcal{V}_{n-1}$ we have to work a little bit. Let $v \in C_0^\infty(\mathbb{R}_+ \times \mathbb{T}_n)$ and $w = Kv$ again. It is

$$Ks^{-1}v = Ks^{-1}(-\partial_t+1)(-\Delta+1)^2 w = K_1(-\lambda^4\partial_t^2+1)K_2(-\lambda^2\Delta+1)^2 s^{-1}w.$$

We easily compute

$$\begin{aligned}K_1(-\lambda^4\partial_t^2+1) &= \lambda^4 + (1-\lambda^4)K_1, \\ K_2(-\lambda^2\Delta+1)^2 &= \lambda^4 + 2\lambda^2(1-\lambda^2)(-\Delta+1)^{-1} + (1-\lambda^2)^2 K_2.\end{aligned}$$

Hence it suffices to show that the operators K_1, K_2, $(-\Delta+1)^{-1}$ and $\lambda^{\frac{1}{2}}s^{-1}$ are uniformly bounded (in λ) in the norm $\sup_i \|\cdot\|_{L^2(c_i)}$. We argue as above using the exponential decay estimates for the kernels K_1, K_2 given above and the estimate

$$(-\Delta+1)^{-1}(x,y) \le Ce^{-c|x-y|}|x-y|^{-1}.$$

We bound on $\lambda^{\frac{1}{2}}s^{-1}$ can be shown like this. Let c_i a cube centered in $i \in \mathbb{Z} \times (\mathbb{Z} \cap \mathbb{T}_n)$ and let $c_{j_1}, \ldots, c_{j_k}$ the collection of cubes centered in $j_l \in \mathbb{Z} \times (\mathbb{Z} \cap \mathbb{T}_{n-1})$ such that $c_i/\lambda \subset \bigcup c_{j_l}$ where $c_i/\lambda = \{(t/\lambda^2, x/\lambda) : (t,x) \in c_i\}$. Remark the number of cubes needed is uniformly bounded. Then, we can bound

$$\|\lambda^{\frac{1}{2}}s^{-1}w\|^2_{L^2(c_i)} = \lambda^5 \int_{c_i/\lambda} |w|^2 \le \lambda^5 \sum_k \|w\|^2_{L^2(c_{j_k})} \le C.$$

For 4), let $\phi \in C^{2,4}(\mathbb{R} \times \mathbb{T}_n)$ and $v \in C_0^\infty(\mathbb{R}_+ \times \mathbb{T}_n)$. Let as above $w = Kv$ so that

$$\phi v = \phi(-\delta_t^2 + 1)(-\Delta + 1)^2 w.$$

Writing $K(\phi v) = \phi K v - [\phi, K]v$ (here ϕ is seen as a multiplication operator) and using that the commutator $[\phi, K]$ is given by

$$[\phi, K](-\delta_t^2 + 1)(-\Delta + 1)^2 w = -\sum_a O_a(\phi_a w)$$

for $O_a \in \{\partial_t^n K_1, \partial_x^\alpha K_2\}$ for $n \leq 1$ and $|\alpha| \leq 3$ and for $a = (m, \beta)$ $\phi_a = c_{(m,\beta)} \partial_t^m \partial_x^\beta \phi$ for $m \leq 2$, $|\beta| \leq 4$. We get to this form by looking at commutation results like

$$\phi \partial_t^2 f = \partial_t^2(\phi f) - 2\partial_t(\partial_t \phi f) + \partial_t^2 \phi f$$

and similar commutation resuls for $-\Delta$. Thus we end up with

$$K(\phi v) = \phi w + \sum_a O_a(\phi_a w).$$

Provided all O_a are uniformly bounded operators, we may bound

$$\begin{aligned} \|K(\phi v)\|_{L^2(c_i)} &\leq \|\phi w\|_{L^2(c_i)} + \sum_a \|O_a(\phi_a w)\|_{L^2(c_i)} \\ &\leq C\|\phi\|_{C^{2,4}} \max_a \sup_i \sum_j \|O_a\|_{L^2(c_j) \to L^2(c_i)} \|v\|_{\mathcal{V}_n}. \end{aligned}$$

The operators K_1 and $\partial_t K_1$ have exponentially decaying kernels. The operators $\partial_x^\alpha K_2$ are bounded operators in L^2, thus they are uniformly bounded as operators $L^2(c_i) \to L^2(c_j)$. Their kernels $\partial_x^\alpha K_2(x - y)$ are smooth outside the diagonal $x - y = 0$ and exponentially decaying. This shows as before

$$\|O_a\|_{L^2(c_i) \to L^2(c_j)} \leq Ce^{-c|i-j|}$$

which satisfies

$$\max_a \sup_i \sum_j \|O_a\|_{L^2(c_j) \to L^2(c_i)} \leq C$$

so that we conclude $\|K(\phi v)\|_{L^2(c_i)} \leq C\|\phi\|_{C^{2,4}} \|v\|_{\mathcal{V}_n}$. By density, the multiplication operator $\phi \colon \mathcal{V}_n \to \mathcal{V}_n$ is bounded by $C\|\phi\|_{C^{2,4}}$ where C does not depend on ϕ. □

The linear part of the fixed point equation obeys the following bounds.

(4.3) Proposition

Given $\lambda < 1$ and $\gamma > 0$, there exists $n(\gamma, \lambda) \in \mathbb{N}$ such that for any $n \geq n(\gamma, \lambda)$ $\mathcal{L}_n \colon \mathcal{W}_n(B_n) \to \mathcal{W}_{n-1}(\lambda^{-\frac{1}{2}} B_{n-1})$ is a well-defined bounded operator with norm

$$\|\mathcal{L}_n\|_{\mathcal{W}_n(B_n) \to \mathcal{W}_{n-1}(\lambda^{-\frac{1}{2}} B_{n-1})} \leq C\lambda^{-\frac{5}{2}}.$$

◇

Proof
Let $v \in \mathcal{W}_n(B_n)$ and $\phi \in \lambda^{-\frac{1}{2}} B_{n-1}$. Recall that

$$(\mathcal{L}_n v)(\phi) = \lambda^{-2} s^{-1} v(s(\phi + \Gamma_{\lambda,1}\xi_{n-1})).$$

Clearly $(\mathcal{L}_n v)$ is analytic wherever it is defined as composition of analytic functions. We need to show that $s(\phi + \Gamma_{\lambda,1}\xi_{n-1}) \in B_n$. Recall that for $\omega \in \mathcal{A}_m$ we imposed the condition (3) in Definition (3.2)

$$\|s\Gamma_{\lambda,1}\xi_{n-1}\|_{\Phi_n} \leq \lambda^{-\gamma n}.$$

Combining this with Lemma (4.2), it holds $s(\phi + \Gamma_{\lambda,1}\xi_{n-1}) \in \Phi_n$ with

$$\|s(\phi+\Gamma_{\lambda,1}\xi_{n-1})\|_{\Phi_n} \leq \|s\|_{\Phi_{n-1}\to\Phi_n}\|\phi\|_{\Phi_{n-1}}+\|s\Gamma_{\lambda,1}\xi_{n-1}\|_{\Phi_n} \leq \lambda^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\lambda^{-2\gamma(n-1)}+\lambda^{-\gamma n} \leq \lambda^{-2\gamma n}$$

for all $n \geq n(\gamma,\lambda)$ where $n(\gamma,\lambda)$ is large enough. Thus $s(\phi + \Gamma_{\lambda,1}\xi_{n-1}) \in B_n$ and $v(s(\phi + \Gamma_{\lambda,1}\xi_{n-1}))$ is analytic in $\phi \in \lambda^{-\frac{1}{2}} B_{n-1}$, so that $\mathcal{L}_n$ indeed maps $\mathcal{W}_n(B_n) \to \mathcal{W}_{n-1}(\lambda^{-\frac{1}{2}} B_{n-1})$. Using that

$$(\mathcal{L}_n v)(\phi) = \lambda^{-2} s^{-1} v(s(\phi + \Gamma_{\lambda,1}\xi_{n-1}))$$

together with Lemma (4.2), it is

$$\|\mathcal{L}_n v\|_{\lambda^{-\frac{1}{2}} B_{n-1}} \leq \lambda^{-2} C \lambda^{-\frac{1}{2}} \|v\|_{B_n}.$$

□

(4.4) Corollary
For $n \geq m$ and $N \geq n$, it holds

$$\|h_{n-m} u_n^{(N)}\|_{RB_n} \leq CR^3 \lambda^{(1-6\gamma)n},$$
$$\left\| h_{n-m}\left(U_n^{(N)}(0) + DU_n^{(N)}(0)\phi\right)\right\|_{RB_n} \leq CR\lambda^{(2-3\gamma)n}$$

for all $R \geq 1$. ◇

Proof
Applying that for $N \geq n \geq m$

$$\|h_{n-m}\zeta_n^{(N)}\|_{\mathcal{V}_n} \leq \lambda^{-\gamma n}$$

for any of the random fields and

$$\|h_{n-m}u_n^{(N)}(\phi)\|_{\mathcal{V}_n} \leq C\|h_{n-m}\|_{C^{2,4}}\|u_n^{(N)}(\phi)\|_{\mathcal{V}_n}.$$

Recall that $u_n^{(N)}(\phi) = -\lambda^n\left((\phi + \eta_n^{(N)})^3 - 3\rho_{N-n}(\phi + \eta_n^{(N)}\right)$. Let $\phi \in B_n$, then

$$\begin{aligned}\|u_n^{(N)}(R\phi)\|_{\mathcal{V}_n} &= \lambda^n \left\|\left((R\phi + \eta_n^{(N)})^3 - 3\rho_{N-n}(R\phi + \eta_n^{(N)}\right)\right\|_{\mathcal{V}_n}\\ &\leq R^3\lambda^n\left(\|\phi^3\|_{\mathcal{V}_n} + \|p(\phi)\|_{\mathcal{V}_n}\right)\end{aligned}$$

where $p(\phi)$ is a quadratic polynomial in ϕ with coefficients given by $\eta_n^{(N)}$, $\eta_n^{(N)2}$, $\eta_n^{(N)}$. We see that $\|\phi^3\|_{\mathcal{V}_n} \leq \lambda^{-6\gamma}$ while $\|p(\phi)\|_{\mathcal{V}_n} \leq C\lambda^{-5\gamma}$. This proves the first claim since h_{n-m} is uniformly bounded in $C^{2,4}$.
For the second inequality, recall that

$$\begin{aligned} U_n^{(N)}(0) + DU_n^{((N)}(0)\phi = &-3\lambda^{2n}(\eta_n^{(N)^2} - \rho_{N-n})\Gamma_{1,\lambda^{(N-n)}}(\eta_n^{(N)^3} - 3\rho_{N-n}\eta_n^{(N)}) \\ &- \lambda^{2n}(m_2 \log \lambda^N + m_3)\eta_n^{(N)} + \lambda^{2n}\mathfrak{z}_n(t,x)\phi(t,x) \\ &+ \lambda^{2n} \int z_n(t,s,x,y)\phi(s,y)\, \mathrm{d}s\, \mathrm{d}y \end{aligned}$$

The linear dependence on ϕ and that $\|\mathfrak{z}_n(t,x)\phi(t,x)\|_{\mathcal{V}_n} \leq \lambda^{-3\gamma}$ which is the worst bound, concludes the proof of the second inequality. □

We will solve a time-localized version of (4.1) first. Therefore define

$$\tilde{v}_n^{(N)} = h_{n-m} v_n^{(N)}.$$

Using that by Lemma (4.2) it holds $s\Gamma_{\lambda,1} h_{n-1-m} v_{n-1}^{(N)} = s\Gamma_{\lambda,1} v_{n-1}^{(N)}$, applying h_{n-1-m} to (4.1), we see that $\tilde{v}_{n-1}^{(N)}$ solves

$$\begin{aligned} \tilde{v}_{n-1}^{(N)}(\phi) &= h_{n-1-m}(\mathcal{L}_n v_n^{(N)})(\phi + \Gamma_{\lambda,1} v_{n-1}^{(N)}(\phi)) \\ &= h_{n-1-m}(\mathcal{L}_n v_n^{(N)})(\phi + \Gamma_{\lambda,1} \tilde{v}_{n-1}^{(N)}(\phi)) \\ &= h_{n-1-m}(\mathcal{L}_n \tilde{v}_n^{(N)})(\phi + \Gamma_{\lambda,1} \tilde{v}_{n-1}^{(N)}(\phi)) \end{aligned}$$

where the last step follows from the identity

$$h_{n-m-1}(\lambda^2 t) = h_{n-m-1}(\lambda^2 t) h_{n-m}(t).$$

This equality can easily be seen by comparing the supports: $h_{n-1-m(\lambda^2 \cdot)}$ is supported on $[0, \lambda^{-2(n-m)} - \frac{1}{2}]$ and $h_{n-m} = 1$ on $[0, \lambda^{-2(n-m)} - \lambda^2]$. We choose λ small enough so that the identity holds true.
Applying the perturbative analysis again to the localized problem, the fixed point problem for $\tilde{\nu}_n^{(N)} = h_{n-m}\nu_n^{(N)}$ is

$$\tilde{\nu}_{n-1}^{(N)} = h_{n-1-m} \left(\mathcal{L}_n \tilde{\nu}_n^{(N)} + \tilde{\mathcal{F}}_n \left(\tilde{U}_{n-1}^{(N)} + \tilde{\nu}_{n-1}^{(N)} \right) \right), \quad \tilde{\nu}_N^{(N)} = 0. \tag{4.4}$$

Here it is

$$\tilde{\mathcal{F}}_n(w) = \mathcal{G}_n(\tilde{u}_n^{(N)} + \tilde{w}_n^{(N)}, \tilde{u}_{n-1}^{(N)} + w) - D\tilde{u}_{n-1}^{(N)} \Gamma_{\lambda,1} \tilde{u}_{n-1}^{(N)}.$$

We now solve (4.4). This is the core of the method. We will see that the existence follows by application of Banach fixed point theorem. The required estimates come from Cauchy estimates on certain analytic functions.

(4.5) Proposition
There exist $\lambda_0 > 0$, $\gamma_0 > 0$ so that for every $\lambda < \lambda_0$, $\gamma < \gamma_0$ and $m > m(\gamma, \lambda)$, if $\omega \in \mathcal{A}_m$, then for all $N \geq n-1 \geq m$ the fixed point problem (4.4) has a unique solution $\tilde{\nu}_{n-1}^{(N)} \in \mathcal{W}_n(B_{n-1})$. Furthermore, we have the bound

$$\|\tilde{\nu}_n^{(N)}\|_{B_n} \leq \lambda^{\left(3-\frac{1}{4}\right)n} \tag{4.5}$$

and $\tilde{\nu}_n^{(N)}$ converge as $N \to \infty$ in $\mathcal{W}_n(B_n)$ to a limit $\tilde{\nu}_n \in \mathcal{W}_n(B_n)$. The limit $\tilde{\nu}_n$ is independent of the choice of the cut-off χ. ◇

Proof
We divide the proof in two main steps that will itself be split into multiple parts. First we are solving the fixed point problems inductively and derive the bound (4.5). In the second step we show convergence of the solutions.
Step 1: The fixed point problem.
The strategy of the proof is to apply Banach fixed point theorem to (4.4). We will solve it in the subset analytic functions $\nu\colon B' \to \mathcal{V}_n$ where $B' = \lambda^{-\frac{1}{2}} B_{n-1}$ such that

$$\|\nu\|_{B'} \leq \lambda^{\left(3-\frac{1}{4}\right)(n-1)}.$$

We need to use the larger ball for bounding $\tilde{U}_n^{(N)}$ since we will need some space to apply a Cauchy estimate. Define the map

$$\nu \mapsto h_{n-1-m}\left(\mathcal{L}_n \tilde{\nu}_n^{(N)} + \tilde{\mathcal{F}}_n\left(\tilde{U}_{n-1}^{(N)} + \nu\right)\right).$$

We solve the fixed point problem with initial condition $\tilde{\nu}_N^{(N)} = 0$ which satisfies (4.5). So, by induction, assume that (4.5) holds, we then show existence of $\tilde{\nu}_{n-1}^{(N)}$ in the appropriate space.
Step 1.1: We start by proving bounds on $\mathcal{L}_n$ and $\mathcal{F}_n$.
By Proposition (4.3), we have

$$\|\mathcal{L}_n \tilde{\nu}_n^{(N)}\|_{B'} \leq \|\mathcal{L}_n\|_{\mathcal{W}_n(B_n) \to \mathcal{W}_{n-1}(B')} \|\tilde{\nu}_n^{(N)}\|_{B_n} \leq C\lambda^{-\frac{5}{2}} \lambda^{\left(3-\frac{1}{4}\right)n} = C\lambda^{\frac{1}{4}} \lambda^{\left(3-\frac{1}{4}\right)(n-1)}.$$

In order to estimate $\tilde{\mathcal{F}}_n$, we first estimate $\mathcal{G}_n$. Recall that

$$\mathcal{G}_n(v, \bar{v}) = (\mathcal{L}_n v)(\phi + \Gamma_{\lambda,1}\bar{v}) - (\mathcal{L}_n v).$$

So define for $v \in \mathcal{W}_n(B_n)$, $\bar{v} \in \mathcal{W}_{n-1}(B')$

$$f(v, \bar{v})(\phi) = \lambda^{-\frac{5}{2}} s^{-1} v(s(\phi + \Gamma_{\lambda,1}\xi_{n-1} + \Gamma_{\lambda,1}\bar{v}(\phi))).$$

Since by Lemma (4.2) $s\Gamma_{\lambda,1}\colon \mathcal{V}_{n-1} \to \Phi_n$ is a bounded operator and using that since $\omega \in \mathcal{A}_m$, $n > m$, $\|s\Gamma_{\lambda,1}\xi_{n-1}\|_{\Phi_n} \leq \lambda^{-\gamma n}$, we find the bound

$$\|s(\phi + \Gamma_{\lambda,1}\xi_{n-1} + \Gamma_{\lambda,1}\bar{v}(\phi))\|_{\Phi_n} \leq \lambda^{-2\gamma(n-1)} + \lambda^{-\gamma n} + C(\lambda)\|\bar{v}\|_{B'}.$$

Thus there is a constant $c(\lambda) > 0$ such that for $\|\bar{v}\|_{B'} \leq c(\lambda)\lambda^{-2\gamma n}$, $f(v, \bar{v}) \in \mathcal{W}_{n-1}(B')$. We will write

$$\tilde{\mathcal{F}}_n(w) = g_n + \mathcal{G}_n(\tilde{w}_n^{(N)}, \tilde{u}_{n-1}^{(N)} + w) + h_n(w)$$

where we recall that $\tilde{w}_n^{(N)}$ was defined via

$$\tilde{v}_n^{(N)} = \tilde{u}_n^{(N)} + \tilde{w}_n^{(N)}$$

and where

$$g_n = \mathcal{G}_n(\tilde{u}_n^{(N)}, \tilde{u}_{n-1}^{(N)}) - D\tilde{u}_{n-1}^{(N)}\Gamma_{\lambda,1}u_{n-1}^{(N)},$$
$$h_n(w) = \mathcal{G}_n(\tilde{u}_n^{(N)}, \tilde{u}_{n-1}^{(N)} + w) - \mathcal{G}_n(\tilde{u}_n^{(N)}, \tilde{u}_{n-1}^{(N)}).$$

We first esimate g_n and $h_n(w)$: we can write

$$g_n = f(1) - f(0) - f'(0)$$

where $f(z) = f(\tilde{u}_n^{(N)}, z\tilde{u}_{n-1}^{(N)})$. f is analytic whenever $\|z\tilde{u}_{n-1}^{(N)}\|_{B'} \leq c(\lambda)\lambda^{-2\gamma n}$, i.e.

$$|z| < c(\lambda)\lambda^{-2\gamma n}\|\tilde{u}_{n-1}^{(N)}\|_{B'}^{-1}.$$

Applying the Cauchy inequality (4.3) to f, we estimate

$$\begin{aligned}
\|g_n\|_{B'} &\leq C(\lambda)\left(\lambda^{-2\gamma n}\|\tilde{u}_{n-1}^{(N)}\|_{B'}^{-1}\right)^{-2}\|f\|_{B'} \\
&\leq C(\lambda)\lambda^{4\gamma n}\|\tilde{u}_{n-1}^{(N)}\|_{B'}^2\|\tilde{u}_n^{(N)}\|_{B_n} \\
&\leq C(\lambda)\lambda^{4\gamma n}\lambda^{(1-6\gamma)n}\lambda^{2(1-6\gamma)(n-1)} \\
&\leq C(\lambda)\lambda^{(3-14\gamma)n}
\end{aligned}$$

where in the second step we used Lemma (4.2) and in the last steps we used Corollary (4.4). We also write

$$h_n(w) = \tilde{f}(1) - \tilde{f}(0)$$

with $\tilde{f}(z) = f(\tilde{u}_n^{(N)}, \tilde{u}_{n-1}^{(N)} + zw)$ which is analytic provided

$$|z| < c(\lambda)\lambda^{-2\gamma n}\|w\|_{B'}^{-1}.$$

Applying the Cauchy estimate (4.3), we may estimate

$$\begin{aligned}
\|h_n(w)\|_{B'} &\leq C(\lambda)\left(\lambda^{-2\gamma n}\|w\|_{B'}^{-1}\right)^{-1}\|\tilde{f}\|_{B'} \\
&\leq C(\lambda)\lambda^{2\gamma n}\|w\|_{B'}\|\tilde{u}_n^{(N)}\|_{B_n}
\end{aligned}$$

using Lemma (4.2) again. Defining

$$j(z) = (\mathcal{L}_n\tilde{w}_n^{(N)})(\phi + z\Gamma_{\lambda,1}(\tilde{u}_{n-1}^{(N)} + w))$$

it holds

$$\mathcal{G}_n(\tilde{w}_n^{(N)}, \tilde{u}_{n-1}^{(N)}) + w) = j(1) - j(0)$$

so that by the same Cauchy inequality together with Proposition (4.3), we find

$$\|\mathcal{G}_n(\tilde{w}_n^{(N)}, \tilde{u}_{n-1}^{(N)} + w)\|_{B'} \leq C(\lambda)\|\tilde{w}_n^{(N)}\|_{B_n} \left(\|\tilde{u}_{n-1}^{(N)}\|_{B'} + \|w\|_{B'}\right).$$

Step 1.2: Bounds on $\tilde{w}_n^{(N)}$. Recall that

$$\tilde{w}_n^{(N)} = \tilde{U}_n^{(N)} + \tilde{\nu}_n^{(N)}$$

where $\tilde{U}_n^{(N)}$ is iteratively defined via

$$\tilde{U}_{n-1}^{(N)} = \mathcal{L}_n \tilde{U}_n^{(N)} + D\tilde{u}_{n-1}^{(N)} \Gamma_{\lambda,1} \tilde{u}_{n-1}^{(N)} = \mathcal{L}_n \tilde{U}_n^{(N)} + \bar{U}_{n-1}^{(N)}$$

and we can write

$$\bar{U}_{n-1}^{(N)} = D\tilde{u}_{n-1}^{(N)} \Gamma_{\lambda,1} \tilde{u}_{n-1}^{(N)} = f'(0).$$

By the Cauchy estimate (4.3), we get

$$\begin{aligned}
\|\bar{U}_{n-1}^{(N)}\|_{B'} &\leq C(\lambda)\left(\lambda^{-2\gamma n}\|\tilde{u}_{n-1}^{(N)}\|_{B'}^{-1}\right)^{-1}\|f\|_{B'} \\
&\leq C(\lambda)\lambda^{2\gamma n}\|\tilde{u}_{n-1}^{(N)}\|_{B'}\|\tilde{u}_n^{(N)}\|_{B_n} \\
&\leq C(\lambda)\lambda^{2(1-6\gamma)n}\lambda^{2\gamma n}
\end{aligned}$$

Recall that in Section 3 we expanded $\tilde{U}_n^{(N)}$ up to second order and got

$$\tilde{U}_n^{(N)}(\phi) = \tilde{U}_n^{(N)}(0) + D\tilde{U}_n^{(N)}(0)\phi + \tilde{R}_n^{(N)}(\phi)$$

where

$$\tilde{R}_{n-1}^{(N)}(\phi) = (\mathcal{L}_n \tilde{R}_n^{(N)})(\phi) - (\mathcal{L}_n \tilde{R}_n^{(N)})(0) - D\left(\mathcal{L}_n \tilde{R}_n^{(N)}\right)(0)\phi + \bar{R}_{n-1}^{(N)}(\phi)$$

and

$$\bar{R}_{n-1}^{(N)}(\phi) = \bar{U}_{n-1}^{(N)}(\phi) - \bar{U}_{n-1}^{(N)}(0) - D\bar{U}_{n-1}^{(N)}(0)\phi.$$

Thus, it is

$$\|\bar{R}_{n-1}^{(N)}\|_{B'} \leq C(\lambda)\lambda^{(2-10\gamma)n}.$$

Now we are able to estimate $\tilde{R}_{n-1}^{(N)}$. Assume by induction that

$$\|\tilde{R}_n^{(N)}\|_{B_n} \leq \lambda^{(2-11\gamma)n},$$

then we can estimate

$$\|\tilde{R}_{n-1}^{(N)}\|_{B_{n-1}} \leq \|(\mathcal{L}_n \tilde{R}_n^{(N)})(\phi) - (\mathcal{L}_n \tilde{R}_n^{(N)})(0) - D\left(\mathcal{L}_n \tilde{R}_n^{(N)}\right)(0)\phi\|_{B_{n-1}} + \|\bar{R}_{n-1}^{(N)}\|_{B'}.$$

We need to estimate the first term. Again, by analyticity, we use the Cauchy estimate. Here, we must allow for some more room, that is why we go over from B_{n-1} to $B' = \lambda^{-\frac{1}{2}} B_{n-1}$:

$$\begin{aligned}\|(\mathcal{L}_n \tilde{R}_n^{(N)})(\phi) - (\mathcal{L}_n \tilde{R}_n^{(N)})(0) - D\left(\mathcal{L}_n \tilde{R}_n^{(N)}\right)(0)\phi\|_{B_{n-1}} &\leq C\lambda \|\mathcal{L}_n \tilde{R}_n^{(N)}\|_{B'} \\ &\leq C\lambda^{-\frac{3}{2}} \|\tilde{R}_n^{(N)}\|_{B_n}\end{aligned}$$

where we also used Proposition (4.3). So, we find that

$$\|\tilde{R}_{n-1}^{(N)}\|_{B_{n-1}} \leq C\lambda^{-\frac{3}{2}} \|\tilde{R}_n^{(N)}\|_{B_n} + C(\lambda)\lambda^{(2-10\gamma)n} \leq C\lambda^{(2-11\gamma)n-\frac{3}{2}} + C(\lambda)\lambda^{(2-10\gamma)n} \leq \lambda^{(2-11\gamma)(n-1)}$$

provided that γ is small enough and $n \geq n(\lambda)$. By linearity of $U_N^{(N)}$, it is $R_N^{(N)} = 0$ and thus by induction, we have

$$\|\tilde{R}_n^{(N)}\|_{B_n} \leq \lambda^{(2-11\gamma)n} \quad \text{for all } N \geq n \geq n(\lambda).$$

We can now use this to derive a bound for $\tilde{U}_n^{(N)}$. Recall that we start from the initial condition

$$U_N^{(N)}(\phi) = -\lambda^{2N}(m_2 \log \lambda^N + m_3)\phi.$$

We estimate using the decomposition

$$\begin{aligned}\|\tilde{U}_n^{(N)}(\phi)\|_{B_n} &\leq \|h_{n-m}(U_n^{(N)}(0) + DU_n^{(N)}(0)\phi)\|_{B_n} + \|\tilde{R}_n^{(N)}(\phi)\|_{B_n} \\ &\leq C\lambda^{(2-3\gamma)n} + \lambda^{(2-11\gamma)n} \\ &\leq 2\lambda^{(2-11\gamma)n}.\end{aligned}$$

This bound obviously also holds for the initial condition.

Step 1.3: Application of Banach fixed point theorem.
We use Step 1.2 to conclude that whenever $|\nu\|_{B'} \leq \lambda^{\left(3-\frac{1}{4}\right)(n-1)}$, then also

$$\|h_{n-1-m}\left(\mathcal{L}_n \tilde{\nu}_n^{(N)} + \tilde{\mathcal{F}}_n\left(\tilde{U}_{n-1}^{(N)} + \nu\right)\right)\|_{B'} \leq \lambda^{\left(3-\frac{1}{4}\right)(n-1)}.$$

Combining the bounds ??, we find that

$$\|\tilde{\mathcal{F}}_n(\tilde{U}_{n-1}^{(N)} + \nu)\|_{B_{n-1}} \leq \lambda^{\left(3-\frac{1}{4}\right)n} + \lambda^{\frac{1}{2}n}\left(\|\nu\|_{B_{n-1}} + \|\tilde{\nu}_n^{(N)}\|_{B_n}\right) + \|\nu\|_{B_{n-1}}\|\tilde{\nu}_n^{(N)}\|_{B_n}.$$

Thus, we get

$$\begin{aligned}&\|h_{n-1-m}\left(\mathcal{L}_n \tilde{\nu}_n^{(N)} + \tilde{\mathcal{F}}_n\left(\tilde{U}_{n-1}^{(N)} + \nu\right)\right)\|_{B'} \\ \leq& C\|h_{n-1-m}\|_{C^{2,4}}\left(\|\mathcal{L}_n \tilde{\nu}_n^{(N)}\|_{B'} + \|\tilde{\mathcal{F}}_n\left(\tilde{U}_{n-1}^{(N)} + \nu\right)\|_{B'}\right) \\ \leq& C\left(C\lambda^{\frac{1}{4}}\lambda^{\left(3-\frac{1}{4}\right)(n-1)} + \lambda^{\left(3-\frac{1}{4}\right)n} + \lambda^{\frac{1}{2}n}\left(\|\nu\|_{B_{n-1}} + \|\tilde{\nu}_n^{(N)}\|_{B_n}\right) + \|\nu\|_{B_{n-1}}\|\tilde{\nu}_n^{(N)}\|_{B_n}\right) \\ \leq& C\left(C\lambda^{\frac{1}{4}} + \lambda^{\left(3-\frac{1}{4}\right)} + 2\lambda^{\frac{1}{2}n} + \lambda^{\left(3-\frac{1}{4}\right)n}\right)\lambda^{\left(3-\frac{1}{4}\right)(n-1)} \\ \leq& \lambda^{\left(3-\frac{1}{4}\right)(n-1)}\end{aligned}$$

whenever λ is small enough. We use that $\|h_{n-1-m}\|_{C^{2,4}}$ is uniformly bounded by $\|h\|_{C^{2,4}}$ and we can clearly choose λ so small, that $C\left(C\lambda^{\frac{1}{4}}+\lambda^{\left(3-\frac{1}{4}\right)}+2\lambda^{\frac{1}{2}n}+\lambda^{\left(3-\frac{1}{4}\right)n}\right)\leq 1$ for every n. The map is also contractive whenever $n>n(\lambda)$.

Step 2: The convergence as $N\to\infty$.
We need to show that $\tilde{\nu}_n^{(N)}$ converges as $N\to\infty$ and the limit $\tilde{\nu}_n$ is independent of the cut-off χ. Recall that we can do so simultaneously by varying the lower cut-off χ'. Recall that for the choice $\chi'(s)=\chi(\lambda^2 s)$, it is $\Gamma_n^{\prime(N)}=\Gamma_n^{(N+1)}$ so that this choice studies the behaviour as $N\to\infty$.

Step 2.1: Convergence of $\tilde{u}_n^{(N)}$ and of $\tilde{U}_n^{(N)}-\tilde{R}_n^{(N)}$.
Using that since $\omega\in\mathcal{A}_m$ it holds

$$\|h_{n-m}(\zeta_n^{\prime(N)}-\zeta_n^{(N)}\|_{\mathcal{V}_n}\leq\lambda^{\gamma(N-n)}\lambda^{-\gamma n},$$

we get by the same proof as of the Corollary (4.4) that

$$\begin{aligned}\|\tilde{u}_n^{(N)}-\tilde{u}_n^{\prime(N)}\|_{B_n}&\leq C\lambda^{\gamma(N-n)}\lambda^{(1-6\gamma)n},\\ \|(\tilde{U}_n^{(N)}-\tilde{R}_n^{(N)})-(\tilde{U}_n^{\prime(N)}-\tilde{R}_n^{\prime(N)})\|_{B_n}&\leq C\lambda^{\gamma(N-n)}\lambda^{(2-3\gamma)n}\end{aligned}$$

since both are polynomials with coefficients in the random fields $\xi_n^{(N)}$. We immediately conclude convergence of $u_n^{(N)}$ since the right-hand side is summable.

Step 2.2: We need similar summable bounds on $\mathcal{R}_n^{(N)}=\tilde{R}_n^{(N)}-\tilde{R}_n^{\prime(N)}$ and for $\tilde{\nu}_n^{(N)}-\tilde{\nu}_n^{\prime(N)}$. Recall that $\tilde{U}_n^{(N)}=\mathcal{L}_{n+1}\tilde{U}_{n+1}^{(N)}+\bar{U}_n^{(N)}$ and we can wrote $\bar{U}_n^{(N)}=D\tilde{u}_n^{(N)}\Gamma_{\lambda,1}\tilde{u}_n^{(N)}=f'(0)$ with $f(z)=f(\tilde{u}_{n+1}^{(N)},z\tilde{u}_n^{(N)})$. Let $g(z)=f(\tilde{u}_{n+1}^{\prime(N)},z\tilde{u}_n^{\prime(N)})$, then again by applying a Cauchy inequality as before

$$\begin{aligned}\|\bar{U}_n^{(N)}-\bar{U}_n^{\prime(N)}\|_{B'}&=\|f'(0)-g'(0)\|_{B'}\\ &\leq C(\lambda)\lambda^{2\gamma n}\left(\|\tilde{u}_n^{(N)}\|_{B_n}+\|\tilde{u}_n^{\prime(N)}\|_{B_n}\right)\|f-g\|_{B'}\\ &\leq C(\lambda)\lambda^{2\gamma n}\lambda^{(1-6\gamma)n}\|\tilde{u}_{n+1}^{(N)}-\tilde{u}_{n+1}^{\prime(N)}\|_{B_{n+1}}\\ &\leq C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(1-6\gamma)n}\lambda^{(1-4\gamma)n}\\ &\leq C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(2-10\gamma)n}.\end{aligned}$$

The same induction we used already to bound $\tilde{R}_n^{(N)}$ here gives

$$\|\mathcal{R}_{n-1}^{(N)}\|_{B_{n-1}}\leq C\lambda^{-\frac{3}{2}}\|\mathcal{R}_n^{(N)}\|_{B_n}+C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(2-10\gamma)n}$$

and thus

$$\|\mathcal{R}_n^{(N)}\|_{B_n}\leq C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(2-10\gamma)n}$$

for every $N\geq n\geq n(\lambda)$. Combining this with

$$\|\tilde{U}_n^{(N)}-\tilde{U}_n^{\prime(N)}-\mathcal{R}_n^{(N)}\|_{B_n}\leq C\lambda^{\gamma(N-n)}\lambda^{(2-3\gamma)n},$$

we obtain the bound

$$\begin{aligned}\|\tilde{U}_n^{(N)} - \tilde{U}_n^{\prime(N)}\|_{B_n} &\le C\lambda^{\gamma(N-n)}\lambda^{(2-3\gamma)n} + C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(2-10\gamma)n} \\ &\le C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(2-11\gamma)n}.\end{aligned}$$

Now, as before, we get

$$\|\tilde{\mathcal{F}}_n(U_{n-1}^{(N)} + \nu_{n-1}^{(N)}) - \tilde{\mathcal{F}}_n'(U_{n-1}^{\prime(N)} + \nu_{n-1}^{\prime(N)})\|_{B_{n-1}} \le \lambda^{\gamma(N-n)}\lambda^{\left(3-\frac{1}{4}\right)n} + \lambda^{\frac{1}{2}n}\|\nu_{n-1}^{(N)} - \nu_{n-1}^{\prime(N)}\|_{B_{n-1}}.$$

Since $\mathcal{L}_n$ is linear, it holds

$$\|\mathcal{L}_n(\tilde{\nu}_n^{(N)} - \tilde{\nu}_n^{\prime(N)})\|_{B_{n-1}} \le C\lambda^{-\frac{5}{2}}\|\tilde{\nu}_n^{(N)} - \tilde{\nu}_n^{\prime(N)}\|.$$

Now assume inductively that

$$\|\tilde{\nu}_n^{(N)} - \tilde{\nu}_n^{\prime(N)}\|_{B_n} \le C\lambda^{\gamma(N-n)}\lambda^{\left(3-\frac{1}{4}\right)n}.$$

Then using that

$$\tilde{\nu}_{n-1}^{(N)} - \tilde{\nu}_{n-1}^{\prime(N)} = \mathcal{L}_n(\tilde{\nu}_n^{(N)} - \tilde{\nu}_n^{\prime(N)}) + \tilde{\mathcal{F}}_n(U_{n-1}^{(N)} + \nu_{n-1}^{(N)}) - \tilde{\mathcal{F}}_n'(U_{n-1}^{\prime(N)} + \nu_{n-1}^{\prime(N)})$$

we find that

$$\|\tilde{\nu}_{n-1}^{(N)} - \tilde{\nu}_{n-1}^{\prime(N)}\|_{B_n} \le C\lambda^{\gamma(N-n+1)}\lambda^{\left(3-\frac{1}{4}\right)(n-1)}$$

for γ small enough and where C is independent of N. Thus by induction, it holds

$$\|\tilde{\nu}_n^{(N)} - \tilde{\nu}_n^{\prime(N)}\|_{B_n} \le C\lambda^{\gamma(N-n)}\lambda^{\left(3-\frac{1}{4}\right)n}$$

for all $m \le n \le N$. This shows convergence of $\nu_n^{(N)}$ and also that the corresponding limit is independent of the chosen cut-off.
Together with convergence of $\tilde{u}_N^{(N)}$ and $\tilde{U}_n^{(N)}$ to cut-off independent limits, this establishes convergence of $\tilde{v}_n^{(N)}$ to a cut-off independent limit $\tilde{v}_n$. □

4.3 Proof of the main theorem

Now that we showed existence of the flow, we can turn to the proof of the main theorem, i.e. reconstruct the solution to the cut-off equation and show convergence to a non-trivial limit. Recall that we wanted to study the limit of the solutions to the equation

$$\varphi_\varepsilon = G_\varepsilon(-\varphi_\varepsilon^3 - r_\varepsilon\varphi_\varepsilon + \Xi)$$

on the time interval $[0, \frac{1}{2}\lambda^{2m}]$ (where we still assume $\omega \in \mathcal{A}_m$). We will reconstruct φ_{λ^N} from the flow starting at $n = m$ and ϕ_m defined on $[0,1]$ the solution to

$$\phi_m = G_1(v_m^{(N)}(\phi_m) + \xi_m).$$

We study the localized iteration

$$\tilde{f}_{n-1}^{(N)}(\phi) = h_{n-1-m}s^{-1}\tilde{f}_n^{(N)}\left(s(\phi + \Gamma_{\lambda,1}(\tilde{v}_{n-1}^{(N)} + \xi_{n-1}))\right)$$

where $\tilde{f}_n^{(N)} = h_{n-m}f_n^{(N)}$.

(4.6) Proposition
Let $\tilde{\nu}_n^{(N)} \in \mathcal{W}_n(B_n)$, $m \leq n \leq N$ be the fixed point constructed in Proposition (4.5) and let $\tilde{v}_n^{(N)} = \tilde{u}_n^{(N)} + \tilde{U}_n^{(N)} + \tilde{\nu}_n^{(N)}$. Then for $m \leq n \leq N$ it is $\tilde{f}_n^{(N)} \in \mathcal{W}_n(B_n)$ and we have the decomposition

$$f_n^{(N)}(\phi) = \phi + \eta_n^{(N)} + g_n^{(N)}(\phi)$$

with

$$\|\tilde{g}_n^{(N)}\|_{B_n} \leq \lambda^{\frac{3}{4}n}.$$

Furthermore, $\tilde{g}_n^{(N)}$ converges as $N \to \infty$ in $\mathcal{W}_n(B_n)$ to a limit $\tilde{g}_m \in \mathcal{W}_n(B_n)$ that is independent of the choice of the cut-off χ. ◇

Proof
Recall that

$$f_{n-1}^{(N)}(\phi) = s^{-1} f_n^{(N)}(s(\phi + \Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi) + \xi_{n-1})))$$

and $f_N^{(N)}(\phi) = \phi$. By induction, we thus get

$$\tilde{g}_{n-1}^{(N)}(\phi) = h_{n-1-m}\left(\Gamma_{\lambda,1} v \tilde{v}_{n-1}^{(N)}(\phi) + s^{-1}\tilde{g}_n^{(N)}\left(s(\phi + \Gamma_{\lambda,1}(\tilde{v}_{n-1}^{(N)}(\phi) + \xi_{n-1})\right)\right).$$

Since $\tilde{g}_N^{(N)} = 0$, if we assume by induction that $\|\tilde{g}_n^{(N)}\|_{B_n} \leq \lambda^{\frac{3}{4}n}$, we get using Lemma (4.2)

$$\begin{aligned}
&\|\tilde{g}_{n-1}^{(N)}\|_{B_{n-1}} \\
&\leq \|h_{n-1-m}\Gamma_{\lambda,1}\|_{\mathcal{V}_{n-1}\to\mathcal{V}_{n-1}} \|\tilde{v}_{n-1}^{(N)}\|_{B_{n-1}} + \|h_{n-1-m} s^{-1}\tilde{g}_n^{(N)}\left(s(\phi + \Gamma_{\lambda,1}(\tilde{v}_{n-1}^{(N)}(\phi) + \xi_{n-1})\right)\|_{B_{n-1}} \\
&\leq C(\lambda)\lambda^{(1-3\gamma)(n-1)} + C\lambda^{-\frac{1}{2}}\lambda^{\frac{3}{4}n} \\
&\leq \lambda^{\frac{3}{4}(n-1)}.
\end{aligned}$$

The convergence and cut-off independence follows immediately from that of $\tilde{v}_n^{(N)}$. □

Recall the main theorem:

(4.7) Theorem
For every $\varepsilon > 0$ there exists r_ε such that for almost all ω there exists $t(\Xi(\omega)) > 0$ such that (2.2) has a unique smooth solution φ_ε on $t \in [0, t(\Xi(\omega))]$, $x \in \mathbb{T}^3$ and there exists $\varphi \in \mathcal{D}'([0, t(\Xi)] \times \mathbb{T}^3)$ such that $\varphi_\varepsilon \to \varphi$ in distributions. Furthermore, the limit is independent of the chosen cut-off χ. ◇

Now we have collected everything to be able to proof the theorem.

Proof
Step 1: We first reconstruct the solution $\varphi^{(N)} = \varphi_{\lambda^N}$ of

$$\varphi_{\lambda^N} = G_{\lambda^N}(-\varphi_{\lambda^N}^3 - r_{\lambda^N}\varphi + \Xi).$$

Claim: if $\omega \in \mathcal{A}_m$, then $\varphi^{(N)}$ is given on the time interval $[0, \frac{1}{2}\lambda^{-2m}]$ by

$$\varphi^{(N)} = s^{-m}\tilde{f}_m^{(N)}(0).$$

We reconstruct in the following way. Define $\phi_n \in \Phi_n$ iteratively via

$$\begin{aligned} \phi_m &= 0, \\ \phi_n &= s\left(\phi_{n-1} + \Gamma_{\lambda,1}\left(\tilde{v}^{(N)}_{n-1}(\phi_{n-1}) + \xi_{n-1}\right)\right). \end{aligned}$$

To prove the claim, we first prove that the ϕ_n are the solution to the effective equations

$$\phi_n = G_1(\tilde{v}^{(N)}_n(\phi_n) + \xi_n).$$

Recall that

$$G_1 f(t) = \int_0^t (1 - \chi(t-s))\, e^{(t-s)\Delta} f(s)\, \mathrm{d}s$$

where $\chi(t) = 1$ for $t \in [0,1]$. So, whenever $t \in [0,1]$, we have $t - s \in [0,1]$ so that $G_1 f(s)$ vanishes identically on $[0,1]$.
For the induction step assume that $\phi_{n-1} \in B_{n-1}$ satisfies

$$\phi_{n-1} = G_1(\tilde{v}^{(N)}_{n-1}(\phi_{n-1} + \xi_{n-1}).$$

We first show that $\phi_n \in B_n$: here we use Lemma (4.2) and the bound on $s\Gamma_{\lambda,1}\xi_{n-1}$ since $\omega \in \mathcal{A}_m$.

$$\begin{aligned} \|\phi_n\|_{\Phi_n} &= \left\| s\left(\phi_{n-1} + \Gamma_{\lambda,1}\left(\tilde{v}^{(N)}_{n-1}(\phi_{n-1}) + \xi_{n-1}\right)\right)\right\|_{\Phi_n} \| \\ &\le \lambda^{\frac{1}{2}} \|\phi_{n-1}\|_{\Phi_{n-1}} + \|s\left(\Gamma_{\lambda,1}\left(\tilde{v}^{(N)}_{n-1}(\phi_{n-1}) + \xi_{n-1}\right)\right)\|_{\Phi_n} \\ &\le \lambda^{\frac{1}{2}} \|\phi_{n-1}\|_{\Phi_{n-1}} + C(\lambda)\|\tilde{v}^{(N)}_{n-1}(\phi_{n-1})\|_{\mathcal{V}_{n-1}} + \lambda^{-\gamma n} \\ &\le \lambda^{\frac{1}{2}} \|\phi_{n-1}\|_{\Phi_{n-1}} + C(\lambda)\lambda^{-\gamma n} \\ &\le \lambda^{-2\gamma n}. \end{aligned}$$

Thus $\phi_n \in B_n$. Furthermore, we compute

$$\begin{aligned} \phi_n &= s\left(\phi_{n-1} + \Gamma_{\lambda,1}\left(\tilde{v}^{(N)}_{n-1}(\phi_{n-1}) + \xi_{n-1}\right)\right) \\ &= s\left(G_1(\tilde{v}^{(N)}_{n-1}(\phi_{n-1} + \xi_{n-1}) + \Gamma_{\lambda,1}\left(\tilde{v}^{(N)}_{n-1}(\phi_{n-1}) + \xi_{n-1}\right)\right) \\ &= s\left((G_1 + \Gamma_{\lambda,1})\left(\tilde{v}^{(N)}_{n-1}(\phi_{n-1}) + \xi_{n-1}\right)\right) \\ &= s\left(G_\lambda s^{-1} s\left(\tilde{v}^{(N)}_{n-1}(\phi_{n-1}) + \xi_{n-1}\right)\right) \\ &= G_1\left(\lambda^2 s\left(\tilde{v}^{(N)}_{n-1}(\phi_{n-1}) + \xi_{n-1}\right)\right) \\ &= G_1\left(h_{n-1-m}(\lambda^2 \cdot)\tilde{v}^{(N)}_n(\phi_n) + \xi_n\right) \\ &= G_1\left(\tilde{v}^{(N)}_n(\phi_n) + \xi_n\right). \end{aligned}$$

Since $\phi_m = 0$, we get

$$\begin{aligned}\tilde{f}_m^{(N)}(\phi_m) &= h_0 s^{-1} \tilde{f}_{m+1}^{(N)}(\phi_{m+1}) = h_0 s^{-1} h_1 s^{-1} \tilde{f}_{m+2}^{(N)}(\phi_{m+2}) \\ &= h_0 h_1(\cdot/\lambda^2) s^{-2} \tilde{f}_{m+2}^{(N)}(\phi_{m+2}) = h_0 s^{-2} \tilde{f}_{m+2}^{(N)}(\phi_{m+2}).\end{aligned}$$

Iterating this, we see that

$$\tilde{f}_m^{(N)}(\phi_m) = h_0 s^{-(N-m)} \tilde{f}_N^{(N)}(\phi_N) = h_0 h_{N-m}(\cdot/\lambda^{2(N-m)} s^{-(N-m)} \phi_N = h_0 s^{-(N-m)} \phi_N.$$

$\phi_N \in B_N$ solves

$$\phi_N = G_1(\tilde{v}_N^{(N)} + \xi_N)$$

and here $\tilde{v}_N^{(N)} = h_{N-m} v_N^{(N)}$ where

$$v_N^{(N)}(\phi) = -\lambda^N \phi^3 - (\lambda^N m_1 + \lambda^{2N}(m_2 \log \lambda^N + m_3)\phi).$$

Also recall that $h_{N-m}(t) = 1$ for $0 \le t \le \lambda^{-2(N-m)} - \lambda^2$ so that

$$\phi_N = G_1(\tilde{v}_N^{(N)} + \xi_N) = G_1(v_N^{(N)} + \xi_N)$$

on $[0, \lambda^{-2(N-m)} - \lambda^2]$. Thus, applying s^{-N}, we get that $\varphi^{(N)} = s^{-N}\phi_N$ solves

$$\varphi_{\lambda^N} = G_{\lambda^N}(-\varphi_{\lambda^N}^3 - r_{\lambda^N}\varphi + \Xi)$$

on $[0, \frac{1}{2}\lambda^{2m}]$. We get

$$h_0(\lambda^{-2m}\cdot)\varphi_{\lambda^N} = s^{-m}\tilde{f}_m^{(N)}(0).$$

Step 2: We need to show convergence of $\varphi^{(N)}$ in $\mathcal{D}'([0, \frac{1}{2}\lambda^{2m}])$. By Proposition (4.6) $\tilde{f}_m^{(N)}(0)$ converges in $\mathcal{V}_m$ to a limit ψ_m independently of the chosen cut-off function χ. Convergence in $\mathcal{V}_m$ implies convergence in $\mathcal{D}'([0,1] \times \mathbb{T}_m)$. Now $s^{-m} \colon \mathcal{D}'([0,1] \times \mathbb{T}_m) \to \mathcal{D}'([0, \lambda^{2m}] \times \mathbb{T}])$ is continuous, so that

$$h_0(\lambda^{-2m}\cdot)\varphi_{\lambda^N} = s^{-m}\tilde{f}_m^{(N)}(0) \to s^{-m}\psi_m$$

converges in $\mathcal{D}'([0, \lambda^{2m} \times \mathbb{T}])$. □

4.4 Synopsis

Chapter 5 - Regularization via mollification

In the existence proof we regularized

$$\partial_t \varphi = \Delta\varphi - \varphi^3 - r\varphi + \Xi. \tag{5.1}$$

via truncating the heat kernel, i.e. removing its singularity. In ?? the dynamical Φ^4_3 model is regularized by mollification of the space-time white noise. In ?? the limiting process is given by rescaling of a stochastic PDE with smooth noise. The regularization scheme as described in section 2 cannot handle these situations initially. The purpose of this section will be to describe how to adapt the renormalization group and the existence proof given in sections 2 to 4 to this situation. We shall later discuss additional problems for which the scheme discussed in 5 here can be applied to.
Consider a space-time mollifier $\rho = \rho(t,x)$ and consider

$$\rho_\varepsilon(t,x) = \varepsilon^{-5}\rho(\varepsilon^{-2}t, \varepsilon^{-1}x).$$

Define the regularized space-time white noise by

$$\Xi_\varepsilon = \rho_\varepsilon * \Xi.$$

Then Ξ_ε is a smooth function and we can write down the regularized PDE

$$\partial_t \varphi_\varepsilon = \Delta\varphi_\varepsilon - \varphi_\varepsilon^3 - r_\varepsilon\varphi_\varepsilon + \Xi_\varepsilon. \tag{5.2}$$

Remark
Assume that ρ is a radial bump around 0, i.e. $\rho(-t,-x) = \rho(t,x)$. Then we may, informally, compute the covariance of Ξ_ε

$$\begin{aligned}
\mathbb{E}[\Xi_\varepsilon(t,x)\Xi_\varepsilon(s,y)] &= \mathbb{E}\left[\int \mathrm{d}t'\,\mathrm{d}x' \int \mathrm{d}s'\,\mathrm{d}y' \rho_\varepsilon(t-t', x-x')\Xi(t',x')\rho_\varepsilon(s-s', y-y')\Xi(s',y')\right] \\
&= \int \mathrm{d}t'\,\mathrm{d}x' \int \mathrm{d}s'\,\mathrm{d}y' \rho_\varepsilon(t-t', x-x')\rho_\varepsilon(s-s', y-y')\mathbb{E}[\Xi(t',x')\Xi(s',y')] \\
&= \int \mathrm{d}t'\,\mathrm{d}x' \int \mathrm{d}s'\,\mathrm{d}y' \rho_\varepsilon(t-t', x-x')\rho_\varepsilon(s-s', y-y')\delta(t'-s')\delta(y'-x') \\
&= \int \mathrm{d}t'\,\mathrm{d}x' \rho_\varepsilon(t-t', x-x')\rho_\varepsilon(s-t', y-x') \\
&= \int \mathrm{d}\tilde{t}\,\mathrm{d}\tilde{x} \rho_\varepsilon(-\tilde{t}, -\tilde{x})\rho_\varepsilon(s-t-\tilde{t}, y-x-\tilde{x}) \\
&= \int \mathrm{d}\tilde{t}\,\mathrm{d}\tilde{x} \rho_\varepsilon(\tilde{t}, \tilde{x})\rho_\varepsilon(s-t-\tilde{t}, y-x-\tilde{x}) \\
&= (\rho_\varepsilon * \rho_\varepsilon)(s-t, x-y).
\end{aligned}$$

Especially if $|x-y|$ is large, then the covariance vanishes.
Of course, is non-rigorous since there are no point processes for space-time white noise. But this computation can be made rigorous by using the corresponding operators on distributions. We omit it here deciding that the non-rigorous computation is clearer. To simplify the perturbative analysis, we will assume that ρ is radial. ◇

As usual we relate solutions to the equations 5.1 and 5.2 to a fixed point problem via Duhamel's formula. Define the heat operator

$$(Gf)(t) = \int_0^t e^{(t-s)\Delta} f(s)\,\mathrm{d}s.$$

Then a solution to (5.1) is formally given by φ that solves

$$\varphi = G(-\varphi^3 - r\varphi + \Xi).$$

A solution to (5.2) is given by a fixed point solving

$$\varphi_\varepsilon = G(-\varphi_\varepsilon^3 - r_\varepsilon\varphi_\varepsilon + \Xi_\varepsilon).$$

5.1 Effective equation

So consider the regularized problem

$$\varphi_\varepsilon = G(-\varphi_\varepsilon^3 - r_\varepsilon\varphi_\varepsilon + \Xi_\varepsilon) = G(V_\varepsilon(\varphi_\varepsilon) + \Xi^\varepsilon) \tag{5.3}$$

for $\varphi(t,x)$ given on $(t,x) \in [0,\tau] \times \mathbb{T}^3$ and where at scale ε the nonlinearity is given by

$$V_\varepsilon(\varphi)(t,x) = -\varphi^3(t,x) - r_\varepsilon\varphi(t,x)$$

for r_ε to be determined later. We want to study the limit of φ_ε as $\varepsilon \to 0$. We first describe how to obtain the solution at level ε in terms of the scales at $\varepsilon' > \varepsilon$. While this is a simple one-step argument if one regularizes the heat kernel as it is done in ??, we need a two-step argument for the regularization of the noise. The first step consists of splitting the full heat kernel. We introduce the short-scale cut-off χ which is a smooth function from $\mathbb{R}_+ \to [0,1]$ such that $\chi \equiv 1$ on $[0,1]$ and $\chi \equiv 0$ on $[2,\infty)$. Then we decompose

$$G = G_{\varepsilon'} + \tilde{\Gamma}_{\varepsilon'}$$

where

$$G_\mu f(t) = \int_0^t \left(1 - \chi\left(\frac{t-s}{\mu^2}\right)\right) e^{(t-s)\Delta} f(s)\,\mathrm{d}s$$

and

$$\tilde{\Gamma}_\mu f(t) = \int_0^t \chi\left(\frac{t-s}{\mu^2}\right) e^{(t-s)\Delta} f(s)\,\mathrm{d}s.$$

Remark that $\tilde{\Gamma}_\mu$ heuristically takes care of the scales between 0 and μ^2 and G_μ of the scales greater than μ^2. We then decompose the solution φ_ε into

$$\varphi_\varepsilon = \varphi'_\varepsilon + Z(\varphi'_\varepsilon)$$

where $Z = Z(\varphi'_\varepsilon)$ satisfies

$$Z(\varphi'_\varepsilon) = \tilde{\Gamma}_{\varepsilon'}(V(\varphi'_\varepsilon + Z(\varphi'_\varepsilon)) + \Xi_\varepsilon). \tag{5.4}$$

Remark that since the noise is smooth, this is a well-defined function. Also remark that we do not change the regularization parameter of the noise in this decomposition. We will describe the flow only in terms of Ξ_ε. Then φ'_ε necessarily satisfies

$$\varphi'_\varepsilon = G_{\varepsilon'}(V'(\varphi'_\varepsilon) + \Xi_\varepsilon) \tag{5.5}$$

where the new effective potential is defined by

$$V'(\varphi'_\varepsilon) = V(\varphi'_\varepsilon + Z(\varphi'_\varepsilon)).$$

Combining this with (5.4), V' satisfies the fixed-point problem

$$V'(\cdot) = V(\cdot + \tilde{\Gamma}_{\varepsilon'}(V'(\cdot) + \Xi_\varepsilon)). \tag{5.6}$$

To continue the renormalization group we have to take care of (5.5). Now we are in the exact same setting as with the heat kernel regularization so that for $\varepsilon'' > \varepsilon'$ we can use the decomposition

$$G_{\varepsilon'} = G_{\varepsilon''} + \Gamma_{\varepsilon',\varepsilon''}$$

where for $\mu < \eta$

$$\Gamma_{\mu,\eta} f(t) = \int_0^t \left(\chi\left(\frac{t-s}{\eta^2}\right) - \chi\left(\frac{t-s}{\mu^2}\right)\right) e^{(t-s)\Delta} f(s)\,\mathrm{d}s.$$

This term involves all scales between μ^2 and η^2. Then we decompose

$$\varphi'_\varepsilon = \varphi''_\varepsilon + Z_2(\varphi''_\varepsilon)$$

where Z_2 now involves the scales from ε'^2 to ε''^2, i.e.

$$Z_2(\varphi''_\varepsilon) = \tilde{\Gamma}_{\varepsilon',\varepsilon''}(V'(\varphi''_\varepsilon + Z_2(\varphi''_\varepsilon)) + \Xi_\varepsilon). \tag{5.7}$$

Then φ''_ε satisfies the new effective equation

$$\varphi''_\varepsilon = G_{\varepsilon''}(V''(\varphi''_\varepsilon) + \Xi_\varepsilon)$$

where we define the new effective potential via

$$V''(\varphi''_\varepsilon) = V'(\varphi''_\varepsilon + Z_2(\varphi''_\varepsilon)).$$

Combine this with (5.7) to derive the fixed-point problem

$$V''(\cdot) = V'(\cdot + \Gamma_{\varepsilon',\varepsilon''}(V''(\cdot) + \Xi_\varepsilon)).$$

Now this second step can be iterated to get a flow. Therefore fix $\lambda < 1$ and let $\varepsilon = \lambda^N$, $\varepsilon' = \lambda^{N-1}$. We furthermore set $V_N^{(N)} = V_{\lambda^N}$. Then we define $V_{N-1}^{(N)}$ as the effective potential of the first step, i.e.

$$V_{N-1}^{(N)}(\cdot) = V_N^{(N)}(\cdot + \tilde{\Gamma}_{\lambda^{N-1}}(V_{N-1}^{(N)}(\cdot) + \Xi_\varepsilon)).$$

For $n \leq N-1$, we now iterate the second step to get at level λ^{n-1}:

$$V_{n-1}^{(N)}(\cdot) = V_n^{(N)}(\cdot + \Gamma_{\lambda^n,\lambda^{n-1}}(V_{n-1}^{(N)}(\cdot) + \Xi_\varepsilon)).$$

As before, we also define the reconstructing functions $F_n^{(N)}$ iteratively. Also here, we have a two-step scheme:

$$\begin{aligned} F_N^{(N)}(\varphi) &= \varphi, \\ F_{N-1}^{(N)}(\varphi) &= F_N^{(N)}\left(\varphi + \tilde{\Gamma}_{\lambda^{N-1}}(V_{N-1}^{(N)}(\varphi) + \Xi_{\lambda^N})\right), \\ F_{n-1}^{(N)}(\varphi) &= F_n^{(N)}\left(\varphi + \Gamma_{\lambda^n,\lambda^{n-1}}(V_{n-1}^{(N)}(\varphi) + \Xi_{\lambda^N})\right) \end{aligned}$$

for $n \leq N-1$. As before, the purpose of these functions will be that

$$\varphi = F_n^{(N)}(\varphi_n)$$

is the solution of

$$\varphi = G(V_N^{(N)}(\varphi) + \Xi_{\lambda^N})$$

where φ_n solves

$$\varphi_n = G_{\lambda^n}\left(V_n^{(N)}(\varphi_n) + \Xi_{\lambda^n}\right).$$

5.2 Rescaling the flow

For $\mu > 0$ define the parabolic scaling operator

$$s_\mu f(t,x) = \mu^{\frac{1}{2}} f(\mu^2 t, \mu x).$$

(5.1) Lemma
We have the following identities:

$$\begin{aligned} s_\mu \circ G \circ s_\mu^{-1} &= \mu^2 G, & s_\mu \circ G_\varepsilon \circ s_\mu^{-1} &= \mu^2 G_{\frac{\varepsilon}{\mu}}, \\ s_\mu \circ \tilde{\Gamma}_\varepsilon \circ s_\mu^{-1} &= \mu^2 \tilde{\Gamma}_{\frac{\varepsilon}{\mu}}, & s_\mu \circ \Gamma_{\varepsilon,\varepsilon'} \circ s_\mu^{-1} &= \mu^2 \Gamma_{\frac{\varepsilon}{\mu},\frac{\varepsilon'}{\mu}}, \\ s_\mu \Xi &\overset{d}{=} \mu^{-2}\Xi^{(\mu)}, & s_\mu \Xi_\varepsilon &\overset{d}{=} \mu^{-2}\Xi^{(\mu)}_{\frac{\varepsilon}{\mu}} \end{aligned}$$

where by $\Xi^{(\mu)}$ we denote space-time white noise on the $\mathbb{R} \times \mu^{-1}\mathbb{T}^3$. ◇

Proof
Heuristically, assuming Ξ were a coordinate process $\Xi(t,x)$ the last identity follows from the

calculation

$$\begin{aligned}
&s_\mu \int \varepsilon^{-3}\rho\left(\frac{x-y}{\varepsilon}\right)\Xi(t,y)\,\mathrm{d}y\\
=&\mu^{\frac{1}{2}}\int \varepsilon^{-3}\rho\left(\frac{\mu x-y}{\varepsilon}\right)\Xi(\mu^2 t,y)\,\mathrm{d}y\\
=&\mu^{\frac{1}{2}}\int \varepsilon^{-3}\rho\left(\mu\frac{x-y'}{\varepsilon}\right)\Xi(\mu^2 t,\mu y')\mu^3\,\mathrm{d}y'\\
=&\left(\frac{\varepsilon}{\mu}\right)^{-3}\int \rho\left(\frac{x-y}{\frac{\varepsilon}{\mu}}\right)s_\mu\Xi(t,y)\,\mathrm{d}y\\
=&\mu^{-2}\Xi^{(\mu)}_{\frac{\varepsilon}{\mu}}.
\end{aligned}$$

The rigorous proof is very similar using the corresponding operators on distributions and we omit it here. □

Now we define the dimensionless variables. We rescale to $\mathbb{T}_n = \lambda^{-n}\mathbb{T}^3$. Define

$$v_n^{(N)} = \lambda^{2n} s_{\lambda^n} \circ V_n^{(N)} \circ s_{\lambda^n}^{-1}, \tag{5.8}$$

$$f_n^{(N)} = s_{\lambda^n} \circ F_n^{(N)} \circ s_{\lambda^n}^{-1}. \tag{5.9}$$

We furthermore define

$$\phi_n = s_{\lambda^n}\varphi_n$$

where we drop the superscript whenever the scaling is clear from the context. Then we can compute, setting $s = s_\lambda$ and using $s \circ s_{\lambda^n} = s_{\lambda^{n-1}} \circ s = s_{\lambda^n}$

$$\begin{aligned}
v_{N-1}^{(N)}(\phi) &= \lambda^{-2}s^{-1}v_N^{(N)}\left(s(\phi+\tilde{\Gamma}_1(v_{N-1}^{(N)}(\phi)+\rho_\lambda * \xi_{N-1})\right),\\
v_{n-1}^{(N)}(\phi) &= \lambda^{-2}s^{-1}v_n^{(N)}\left(s(\phi+\Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi)+\rho_{\lambda^{N-n+1}} * \xi_{n-1})\right)
\end{aligned}$$

where we set

$$\xi_n = \lambda^{2n}s_{\lambda^n}\Xi.$$

Then $\xi_n = \Xi^{(\lambda^n)}$ is distributed as space-time white noise on $\mathbb{R}\times\mathbb{T}_n$. For example, the second equation follows from the following calculation

$$\begin{aligned}
v_{n-1}^{(N)}(\phi) &= \lambda^{2(n-1)}s_{\lambda^{n-1}} \circ V_{n-1}^{(N)} \circ s_{\lambda^{n-1}}^{-1}(\phi)\\
&= \lambda^{2(n-1)}s_{\lambda^{n-1}} \circ V_n^{(N)}\left(s_{\lambda^{n-1}}^{-1}\phi + \Gamma_{\lambda^n,\lambda^{n-1}}(V_{n-1}^{(N)}(s_{\lambda^{n-1}}^{-1}\phi)+\Xi_{\lambda^N})\right)\\
&= \lambda^{-2}s^{-1}\lambda^{2n}s_{\lambda^n}\circ V_n^{(N)} \circ s_{\lambda^n}^{-1}\left(s\phi + s s_{\lambda^{n-1}}\Gamma_{\lambda^n,\lambda^{n-1}}\left(V_{n-1}^{(N)}(s_{\lambda^{n-1}}^{-1}(\phi))+\Xi_{\lambda^N}\right)\right)\\
&= \lambda^{-2}s^{-1}v_n^{(N)}\left(s\left(\phi+\Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi)+\lambda^{2(n-1)}s_{\lambda^{n-1}}\rho_{\lambda^N} * \Xi)\right)\right)\\
&= \lambda^{-2}s^{-1}v_n^{(N)}\left(s\left(\phi+\Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi)+\rho_{\lambda^{N-n+1}} * \xi_{n-1})\right)\right).
\end{aligned}$$

Remark
Upon iteration the noise becomes rougher and rougher since $\lambda^{N-n} \to 0$ as $N - n \to \infty$. ◇

We also compute that

$$f_{N-1}^{(N)}(\phi) = s^{-1} f_N^{(N)}(s(\phi + \tilde{\Gamma}_1(v_{n-1}^{(N)}(\phi) + \rho_\lambda * \xi_{N-1}))), \tag{5.10}$$

$$f_{n-1}^{(N)}(\phi) = s^{-1} f_n^{(N)}(s(\phi + \Gamma_{\lambda,1}(v_{n-1}^{(N)}(\phi) + \rho_{\lambda^{N-n+1}} * \xi_{n-1}))) \tag{5.11}$$

for $n \leq N-1$. As before, define the RG map $v_n^{(N)} \mapsto v_{n-1}^{(N)}$ to be

$$v_{n-1}^{(N)} = \mathcal{R}_n v_n^{(N)}.$$

Remark
We get the same scheme if instead of a mollified space-time white noise, we approximate the space-time white noise by the scaling limit of smooth fields. I.e. let η a centered Gaussian noise with stationary covariance

$$\mathbb{E}[\eta(t,x)\eta(s,y)] = \tilde{C}^\varepsilon(t-s,x,y)$$

for $\tilde{C}^\varepsilon(t-s,x,y) = \Sigma(t-s,x-y)$ if $\mathrm{dist}(x,y) \leq 1$ for a smooth, positive function $\Sigma\colon [0,1] \times B(0,1) \to \mathbb{R}^+$. Then it is well known that the rescaled fields

$$\eta_\varepsilon(t,x) = \varepsilon^{-\frac{5}{2}} \eta\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$$

converge in distribution to the space-time white noise on $\mathbb{R} \times \mathbb{T}^3$. ◇

5.3 Main theorem

We want to prove that also regularizing the equation by mollification of the space-time white noise leads to the same result as the one proved in sections 2 to 4.

(5.2) Theorem
For every $\varepsilon > 0$ there exists r_ε such that for almost all ω there exists $t(\Xi(\omega)) > 0$ such that (5.2) has a unique smooth solution φ_ε on $t \in [0, t(\Xi(\omega))]$, $x \in \mathbb{T}^3$ and there exists $\varphi \in \mathcal{D}'([0,t(\Xi)] \times \mathbb{T}^3)$ such that $\varphi_\varepsilon \to \varphi$ in distributions. Furthermore, the limit is independent of the chosen cut-off χ and the choice of the mollifier ρ. ◇

Remark
It is well-known, see ??, that the correct choice of the renormalization constant for this problem is given by

$$r_\varepsilon = \frac{m_1}{\varepsilon} + m_2 \log(\varepsilon) + m_3$$

where m_1 and m_3 depend explicitly on the choice of the mollifier ρ and on the choice of the cut-off χ and m_2 is universal. ◇

With the renormalization group scheme introduced in the previous paragraphs, the analytic part of the proof will basically be the same as before. Since mollification is a continuous operator on $\mathcal{V}_n$, the stochastic estimates for the noise fields will, up to constants that we will not care for, remain the same. Setting up the perturbative analysis as done before, we will end up with adapting the proof of Propositions (4.5) and (4.6) to get the result.

5.4 Perturbative analysis

We adapt the perturbative analysis like done in section 3. There are some minor changes since random fields will now involve the non-truncated heat kernel. We will adapt the notation from 3 and drop superscripts (N) in the definition of the operators belonging to the perturbative expansion of the RG map.
Fix $N \in \mathbb{N}$. Define the linear flow map $\mathcal{L}_n = D\mathcal{R}_n(0)$ given by

$$(\mathcal{L}_N v)(\phi) = \lambda^{-2} s^{-1} v(s(\phi + \tilde{\Gamma}_1(\rho_\lambda * \xi_{N-1}))), \tag{5.12}$$

$$(\mathcal{L}_n v)(\phi) = \lambda^{-2} s^{-1} v(s(\phi + \Gamma_{\lambda,1}(\rho_{\lambda^{N-n+1}} * \xi_{n-1}))) \tag{5.13}$$

for $n \leq N-1$. As before, it holds

$$v_{N-1}^{(N)}(\phi) = (\mathcal{L}_N v_N^{(N)})(\phi + \tilde{\Gamma}_1 v_{N-1}^{(N)}(\phi))$$
$$v_{n-1}^{(N)}(\phi) = (\mathcal{L}_n v_n^{(N)})(\phi + \Gamma_{\lambda,1} v_{n-1}^{(N)}(\phi)).$$

It is obvious, that the linear flow with initial data u_N is given by

$$u_n(\phi) = (\mathcal{L}_{n+1} \dots \mathcal{L}_N u_N)(\phi) = \lambda^{-2(N-n)} s^{n-N} u_N(s^{N-n}(\phi + \eta_n^{(N)})).$$

Here we define the stochastic field $\eta_n^{(N)}$ to be

$$\eta_n^{(N)} = \tilde{\Gamma}_1(\rho_{\lambda^{N-n}} * \xi_n) =: \Gamma_n^{(N)} \xi_n.$$

As before, we may write

$$\Gamma_n^{(N)} f(t,x) = \int_0^t \int \chi(t-s)\, H_n(t-s, x-y) \rho_{\lambda^{(N-n)}} * f(s,y)\, \mathrm{d}s\, \mathrm{d}y.$$

Also, our linear flow has the same "eigenfunctions"

$$\mathcal{L}_n(\phi + \eta_n^{(N)})^k = \lambda^{(k-5)/2}(\phi + \eta_{n-1}^{(N)})^k$$

so that if $u_N(\phi) = \phi^k$, it is

$$u_n(\phi) = (\mathcal{L}_{n+1} \dots \mathcal{L}_N) u_N(\phi) = \lambda^{(N-n)(k-5)/2}(\phi + \eta_n^{(N)})^k$$

and we obtain the same behaviour as $N - n \to \infty$. As before, we have divergence of

$$\mathbb{E}[\eta_n^{(N)}(t,x)^2] = \mathbb{E}\left[\left(\int \chi(t-s)\, H_n(t-s, x-y)(\rho_{\lambda^{(N-n)}} * \xi_n)(s,y)\, \mathrm{d}s\, \mathrm{d}y\right)^2\right]$$
$$= \int\int \mathrm{d}s\, \mathrm{d}y\, \mathrm{d}s'\, \mathrm{d}y' \chi(t-s)\, \chi(t-s')\, H_n(t-s, x-y) H_n(t-s', x-y') \rho_{\lambda^{(N-n)}} * \rho_{\lambda^{(N-n)}}(s-s', y-y')$$

The limiting behaviour of this object is as in section (3) since $\rho_{\lambda^{(N-n)}} * \rho_{\lambda^{(N-n)}}(s-s', y-y') \to \delta(s-s')\delta(y-y')$ and we get a first renormalization constant θ (that was called ρ before)

dependent on χ and ρ and Lemma (3.1) also holds in this case.
We run the linearized flow with initial data given by

$$u_N^{(N)} = -\lambda^N \phi^3 - \lambda^N m_1 \phi = -\lambda^N \phi^3 + 3\lambda^{2N} \rho_N \phi$$

where we set

$$\rho_k = \lambda^{-k} \rho.$$

As before, it is

$$u_n^{(N)} = -\lambda^n \left((\phi + \eta_n^{(N)})^3 - 3\rho_{N-n}(\phi + \eta_n^{(N)} \right)$$

and we decompose

$$v_n^{(N)} = u_n^{(N)} + w_n^{(N)}.$$

Also in the perturbation theory of second order, the only objects that change will be the coefficients of the analytic functions that decompose the effective potential. We review the notation here since the operators for the first renormalization group step $n = N$ need to be adapted. It is straight-forward that the coefficients have the same limiting behaviour as before, especially we get the same renormalization behaviour.
Define

$$\mathcal{G}_N(v, \bar{v})(\phi) = (\mathcal{L}_N v)(\phi + \tilde{\Gamma}_1 \bar{v}(\phi)) - (\mathcal{L}_N v)(\phi),$$
$$\mathcal{G}_n(v, \bar{v})(\phi) = (\mathcal{L}_n v)(\phi + \Gamma_{\lambda,1} \bar{v}(\phi)) - (\mathcal{L}_n v)(\phi)$$

for $n < N$ so that

$$v_{n-1}^{(N)} = \mathcal{L}_n v_n^{(N)} + \mathcal{G}_n(v_n^{(N)}, v_{n-1}^{(N)}).$$

Recall that we decomposed $v_n^{(N)} = u_n^{(N)} + w_n^{(N)}$ where $u_{n-1}^{(N)} = \mathcal{L}_n u_n^{(N)}$ and

$$w_{n-1}^{(N)} = \mathcal{L}_n w_n^{(N)} + \mathcal{G}_n(u_n^{(N)} + w_n^{(N)}, u_{n-1}^{(N)} + w_{n-1}^{(N)}).$$

The initial condition for w is given by

$$w_N(\phi) = -\lambda^{2N}(m_2 \log \lambda^N + m_3)\phi.$$

Since

$$\mathcal{G}_N(u_N^{(N)}, u_{N-1}^{(N)})(\phi) = u_{N-1}^{(N)}(\phi + \tilde{\Gamma}_1(u_{N-1}^{(N)}(\phi))) - u_{N-1}^{(N)}(\phi),$$
$$\mathcal{G}_n(u_n^{(N)}, u_{n-1}^{(N)})(\phi) = u_{n-1}^{(N)}(\phi + \Gamma_{\lambda,1} u_{n-1}^{(N)}(\phi)) - u_{n-1}^{(N)}(\phi)$$

for $n < N$, by Taylor expansion, it is

$$\mathcal{G}_N(u_N^{(N)}, u_{N-1}^{(N)})(\phi) = Du_{N-1}^{(N)}(\phi)\tilde{\Gamma}_1 u_{N-1}^{(N)}(\phi) + \mathcal{O}(\lambda^N),$$
$$\mathcal{G}_n(u_n^{(N)}, u_{n-1}^{(N)})(\phi) = Du_{n-1}^{(N)}(\phi)\Gamma_{\lambda,1} u_{n-1}^{(N)} + \mathcal{O}(\lambda^n)$$

for $n < N$ where it is

$$Du_{n-1}^{(N)} = -3\lambda^{n-1}\left((\phi + \eta_{n-1}^{(N)})^2 - \rho_{N-(n-1)}\right).$$

For a fixed $w_n^{(N)}$, $w_{n-1}^{(N)}$ then satisfies

$$\begin{aligned} w_{N-1}^{(N)} &= \mathcal{L}_N w_{N-1}^{(N)} + Du_{N-1}^{(N)}\tilde{\Gamma}_1 u_{N-1}^{(N)} + \mathcal{F}_N(w_{N-1}^{(N)}), \\ w_{n-1}^{(N)} &= \mathcal{L}_n w_{n-1}^{(N)} + Du_{n-1}^{(N)}\Gamma_{\lambda,1} u_{n-1}^{(N)} + \mathcal{F}_n(w_{n-1}^{(N)}) \end{aligned}$$

for $n < N$ where

$$\begin{aligned} \mathcal{F}_N(w_{N-1}^{(N)}) &= \mathcal{G}_N(u_N^{(N)} + w_N^{(N)}, u_{N-1}^{(N)} + w_{N-1}^{(N)}) - Du_{N-1}^{(N)}\tilde{\Gamma}_1 u_{N-1}^{(N)}, \\ \mathcal{F}_n(w_{n-1}^{(N)}) &= \mathcal{G}_n(u_n^{(N)} + w_n^{(N)}, u_{n-1}^{(N)} + w_{n-1}^{(N)}) - Du_{n-1}^{(N)}\Gamma_{\lambda,1} u_{n-1}^{(N)}. \end{aligned}$$

We can also decompose

$$w_n^{(N)} = U_n^{(N)} + \nu_n^{(N)}$$

solving up to second order where we define

$$\begin{aligned} U_{N-1}^{(N)} &= \mathcal{L}_N U_N^{(N)} + Du_{N-1}^{(N)}\tilde{\Gamma}_1 u_{N-1}^{(N)}, \\ U_{n-1}^{(N)} &= \mathcal{L}_n U_n^{(N)} + Du_{n-1}^{(N)}\Gamma_{\lambda,1} u_{n-1}^{(N)} \end{aligned}$$

and $\nu_n^{(N)}$ solves

$$\nu_{n-1}^{(N)} = \mathcal{L}_n \nu_n^{(N)} + \mathcal{F}_n(U_{n-1}^{(N)} + \nu_{n-1}^{(N)})$$

with initial conditions

$$U_N^{(N)} = -\lambda^{2N}(m_2 \log\lambda^N + m_3)\phi, \quad \nu_N^{(N)} = 0.$$

By continuity of dependence on the mollifier, m_2 and m_3 can be determined by the stochastic analysis as before. Especially, up to constants, the noise estimates will hold for the mollified noise. We compute that

$$U_n^{(N)} = Du_n\tilde{\Gamma}_1 u_n - \lambda^{2n}(m_2 \log\lambda^N + m_3)(\phi + \eta_n^{(N)}).$$

5.5 Proof of the main theorem

We use the stochastic bounds given in Definition (3.2) and the spaces introduced in Definition (4.1). Since we have the same bounds, the only change in the analysis lies in the first RG step. It is obviously sufficient to show that Proposition (4.5) does hold for this new scheme. If we can construct $\tilde{\nu}_{N-1}^{(N)} \in \mathcal{W}_N(B_{N-1})$ and it satisfies the bound

$$\|\tilde{\nu}_{N-1}^{(N)}\|_{B_n} \le \lambda^{\left(3-\frac{1}{4}\right)(N-1)}$$

we will immediately have proven the proposition. The rest of the analysis then is identical to the proof given in section 4 and we refer to it. Recall that we defined $\tilde{v}_n^{(N)} = h_{n-m} v_n^{(N)}$ where $m \in \mathbb{N}$ is fixed such that $\omega \in \mathcal{A}_m$. The localized flow for $\tilde{\nu}_n^{(N)}$ is given by

$$\tilde{\nu}_{n-1}^{(N)} = h_{n-1-m}\left(\mathcal{L}_n \tilde{\nu}_n^{(N)} + \tilde{\mathcal{F}}_n\left(\tilde{U}_{n-1}^{(N)} + \tilde{\nu}_{n-1}^{(N)}\right)\right), \quad \tilde{\nu}_N^{(N)} = 0$$

with

$$\tilde{\mathcal{F}}_n(w) = \mathcal{G}_n(\tilde{u}_n^{(N)} + \tilde{w}_n^{(N)}, \tilde{u}_{n-1}^{(N)} + w) - D\tilde{u}_{n-1}^{(N)} \Gamma_{\lambda,1} \tilde{u}_{n-1}^{(N)}.$$

The main difference to the proof before is that the fixed point problem involves the operators $\tilde{\Gamma}_1$ that is not infinitely smoothing.
Recall from the proof of Proposition (4.5), that one needs to study

$$\mathcal{G}_N(v, \bar{v}) = (\mathcal{L}_N v)(\phi + \tilde{\Gamma}_1 \bar{v}) - (\mathcal{L}_N v).$$

The bounds on the linear equation only depend on those obtained for $\tilde{\Gamma}_1(\rho_\lambda * \xi_{N-1})$ that we get by smoothness of the noise. We then studied

$$f(v, \bar{v}) = \lambda^{-\frac{5}{2}} s^{-1} v(s(\phi + \tilde{\Gamma}_1(\rho_\lambda * \xi_{N-1}) + \tilde{\Gamma}_1 \bar{v}(\phi))).$$

Essentially, the bounds we obtained in section 4, followed from the fact that $s\Gamma_{\lambda,1}\bar{v} \in B_N$ which in turn yielded that f is an analytic function. It will thus be sufficient, to ensure that $s\tilde{\Gamma}_1 \bar{v} \in B_N$ whenever $\bar{v} \in \mathcal{W}_{N-1}(B')$. This is clear though since at this scale, $s\tilde{\Gamma}_1 \bar{v}(\phi)$ will be at least as smooth as ϕ is using the usual properties of the heat kernel and $\phi \in \Phi_N$. The bound now can be obtained as in the proof of Lemma (4.2).
Now, redoing the proof of Propositions (4.5) and 4.6, we end up being in the same situation as before and we can construct the solution as it is done in the proof of the main theorem.

5.6 Synopsis

Chapter 6 - Index of symbols

Funktionenräume

$\mathbb{T}^n$	n-dimensional torus
$\Xi^{(\mu)}$	space-time white noise on $\mathbb{R} \times \mu^{-1}\mathbb{T}^3$
Ξ_ε	mollified space-time white noise $\rho_\varepsilon * \Xi$
χ	smooth bump, $\chi \geq 0$, $\chi(t) = 1$ for $t \in [0,1]$, $\chi(t) = 0$ for $t \in [2\infty)$
G	heat kernel $Gf(t) = \int_0^t e^{(t-s)\Delta} f(s)\,\mathrm{d}s$
G_ε	truncated heat kernel $G_\varepsilon f(t) = \int_0^t \left(1 - \chi\left(\frac{t-s}{\varepsilon^2}\right)\right) e^{(t-s)\Delta} f(s)\,\mathrm{d}s$
$\Gamma_{\mu,\eta}$	small-scale part of heat kernel $\Gamma_{\mu,\eta} f(t) = \int_0^t \left(\chi\left(\frac{t-s}{\eta^2}\right) - \chi\left(\frac{t-s}{\mu^2}\right)\right) e^{(t-s)\Delta} f(s)\,\mathrm{d}s$
$\tilde{\Gamma}_\mu$	$\tilde{\Gamma}_\mu f(t) = \int_0^t \chi\left(\frac{t-s}{\mu^2}\right) e^{(t-s)\Delta} f(s)\,\mathrm{d}s$
s_λ	scaling operator
s	$s = s_\lambda$ for fixed λ
$V_n^{(N)}$	flow of effective potentials
$v_n^{(N)}$	rescaled flow of effective potentials, $v_n^{(N)} = \lambda^{2n} s_{\lambda^n} \circ V_n^{(N)} \circ s_{\lambda^n}^{-1}$
$F_n^{(N)}$	reconstructing functions for flow of effective potentials
$f_n^{(N)}$	rescaled reconstructing functions $f_n^{(N)} = s_{\lambda^n} \circ F_n^{(N)} \circ s_{\lambda^n}^{-1}$
$\mathcal{R}_n$	renormalization group map, $\mathcal{R}_n v_n^{(N)} = v_{n-1}^{(N)}$
$\mathcal{L}_n$	first order term of $\mathcal{R}_n$, see ??
$\mathcal{G}_n$	second order term of $\mathcal{R}_n$, see ??
$\mathcal{F}_n$	third order term of $\mathcal{R}_n$, see ??
$u_n^{(N)}$	first order part of $v_n^{(N)}$, see ??
$w_n^{(N)}$	remainder: $v_n^{(N)} = u_n^{(N)} + w_n^{(N)}$
$U_n^{(N)}$	second order part of $v_n^{(N)}$, see ??
$\nu_n^{(N)}$	remainder: $w_n^{(N)} = U_n^{(N)} + \nu_n^{(N)}$, see ??
K_1	$K_1 = (-\partial_t^2 + 1)^{-1}$ on $L^2(\mathbb{R})$
K_2	$K_2 = (-\Delta + 1)^{-2}$ on $L^2(\mathbb{T}_n)$
K	$K = K_1 K_2$
$\mathcal{V}_n$	function space, see Definition (4.1)
Φ_n	function space, see Definition (4.1)
B_n	open ball centered in origin with radius $r_n = \lambda^{-2\gamma n}$ in Φ_n
$\mathcal{W}_n(B_n)$	function space of analytic functions $B_n \to \mathcal{V}_n$, see Definition (4.1)

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