Analysis of Singular Stochastic PDE

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1 Introduction

In this master thesis we study an approach to singular Stochastic Partial Differential Equations (singular SPDE or SSPDE for short) based on *mild formulations of PDEs* in the context of the *paracontrolled calculus*.
Mild formulations are integral equations representing certain initial value problems corresponding to evolution equations (see [12] for an introduction). The methods developed here will combine these integral formulations with ideas of the paracontrolled calculus.
The paracontrolled calculus was introduced by Gubinelli, Imkeller and Perkowski in [8] and provides us with a framework for giving a meaning to a certain highly singular SPDEs.
In order to deal with the mild formulations of singular SPDEs, we will consider *random integral operators* and study their regularity in suitable spaces.
Finally, we will apply the developed techniques to study a certain singular SPDE called *Parabolic Anderson Model* or PAM for short.
This be more precise, we will

1. in the first chapter agree upon some conventions and recall basis properties regarding the Fourier transform

2. then introduce the Littlewood-Paley decomposition, Besov spaces and Bony's paraproduct, all concepts and tools essential to our approach

3. in the third chapter study (random) integral operators in Besov spaces and derive regularity results in terms of control of the corresponding integral kernels

4. finally deal with the PAM, derive a sensible concept of solution for this equations in terms of the introduced concept of enhanced noise, show that this formulations admits solutions in a well-posed way and prove that the most important noise for the PAM, called white noise, fits within this framework.

The rest of this introductory chapter is structured as follows:
First we will recall the basic ideas of the paracontrolled approach to singular SPDEs in an intuitive fashion.
After that, we will heuristically introduce the "mild approach" that will be developed rigorously in this master thesis.
Finally, we agree on some notational conventions.

1.1 Paracontrolled Approach

For $T > 0$ we consider the (linear) Parabolic Anderson Model (PAM) which we formally write as the Cauchy problem

$$\partial_t u = \Delta u + u \cdot \xi \text{ on } [0,T] \times \mathbb{T}^2, \quad u(0,\cdot) = u^0(\cdot)$$

where $\mathbb{T}^2$ denotes the two dimensional torus and u is function periodic in space. Here ξ denotes space white noise which heuristically can be thought of independent identically distributed standard Gaussian random variables attached to each point in space and therefore intuitively space white noise should be "a very irregular function". Moreover, u^0 is a suitable initial condition.

In view of the PAM, the first question we ask ourselves is how to interpret the product $u \cdot \xi$ appearing in the formulation of the equation, i.e. we have to understand the analytical properties of the realizations of the white noise. One can prove that white noise can be thought of being a Schwartz distribution on the torus, i.e. $\xi \in \mathscr{S}'$ (Chapter 3.3). Even better, we can prove that for each $\gamma \in (0,1)$ we have that $\xi \in \mathscr{C}^{\gamma-2}$ where $\mathscr{C}^\alpha$ denotes the Hölder-Besov space with regularity α on the torus (Chapter 3.1).

In the latter spaces one can multiply two distributions provided suitable regularity assumption (Chapter 3.2):

$$\mathscr{C}^\alpha \times \mathscr{C}^\beta \to \mathscr{C}^{\min(\alpha,\beta)}, \quad (u,v) \mapsto u \cdot v$$

is a bounded bilinear map if $\alpha + \beta > 0$.

Assume now that u is a solution to the equation on the time interval $[0,T]$. The product $u \cdot \xi$ has to be well-defined and in view of the above theorem it is sensible to assume that $u \cdot \xi \in C([0,T];\mathscr{C}^{\gamma-2})$.

Solving the equation, the Laplacian increases the regularity by two (see the Schauder estimate, Chapter 3.1) and we expect $u \in C([0,T];\mathscr{C}^\gamma)$.

Then, however, the product is ill-defined in terms of the above theorem and consequently the naive approach fails: We cannot give a meaning to the PAM.

As it turns out, in order to get a proper meaning for this equation, we need to *renormalize* the PAM, i.e subtract an "infinite constant" in a suitable way. To achieve this, the paracontrolled approach uses *Bony's paraproduct* (Chapter 3.2), which provides us with a way to decompose the product of two distributions u, v as follows:

$$u \cdot v = u \prec v + u \circ v + u \succ v$$

$u \prec v$ and $u \succ v$ are called paraproducts and $u \circ v$ is called the resonant term. Bony's crucial observation was that, provided $u \in \mathscr{C}^\alpha$ and $v \in \mathscr{C}^\beta$, paraproducts always exist. Furthermore, we can think of $u \prec v$ as a frequency modulation of v and heuristically $u \prec v$ behaves like v at small scales.

However, $u \circ \xi$ is only well-defined if $\alpha + \beta > 0$.

Thus, we localized the difficulty of interpreting the product in the term $u \circ \xi$. The *paracontrolled ansatz* now is as follows: We define *paracontrolled distributions* to be distributions u that can be decomposed in the following way:

$$u = u^\sharp + u^X \prec X$$

Here $u^\sharp \in C([0,T];\mathscr{C}^{2\gamma})$ and $u^X, X \in C([0,T];\mathscr{C}^{\gamma})$ where X is constructed from the white noise and u^X can be thought of an derivative.
We can heuristically think of paracontrolled distributions as distributions that admit a first order noise approximation (since $u^X \prec X$ behaves like X at small scales and $u^\sharp$ has better regularity than the first term). Since the white noise is the only source of irregularity in the PAM it is thus sensible to assume that a hypothetical solution of the PAM is paracontrolled.
Assuming this, we can write the troubling part of the product as

$$u \circ \xi = u^\sharp \circ \xi + (u^X \prec X) \circ \xi$$

Provided that $\gamma \in (2/3, 1)$ the first term is well-defined. The second one can be dealt with by using a purely analytical commutator lemma [7, Chapter 5, 5.2, Lemma 14]:

$$(u^X \prec X) \circ \xi = u^X (X \circ \xi) + \text{ better remainder}$$

Since the remainder has better regularity, we only need to handle $X \circ \xi$.
We make an educated guess and define

$$X(t) = \int_0^t P_t \xi \mathrm{d}r.$$

where P_t is the action of the heat semigroup (Chapter 3.1). Consequently, X satisfies the equation

$$\partial_t X = \Delta X, \quad X(0) = 0.$$

and setting formally

$$X(t) \circ \xi = \int_0^t P_r \xi \mathrm{d}r \circ \xi = \int_0^t P_r \xi \circ \xi \mathrm{d}r$$

one can calculate

$$g_t := \int_0^t \mathbb{E}\left[P_r \xi \circ \xi\right] \mathrm{d}r = \infty$$

Hence the term $X \circ \xi$ admits singular behavior. However, we can prove that

$$X \diamond \xi(t) = \int_0^t P_r \xi \circ \xi - g_r \mathrm{d}r$$

is in fact well-defined and has right regularity. Plugging this into the above equation, we can derive a *renormalized equation* which read as

$$\partial_t u = \Delta u + u \diamond \xi, \quad u(0, \cdot) = u^0(\cdot)$$

and using standard methods we can prove the well-posedness of this equation in a suitable space.

Bibliographical remarks: Paracontrolled distributions were introduced in [8]. In this paper (besides other equations) the following more general version of the PAM is treated:

$$\partial_t u = \Delta u + F(u) \cdot \xi \text{ on } [0,T] \times T^2, \quad u(0,\cdot) = u^0 \tag{1.1.1}$$

Here $F \in C^{2+\epsilon}$ for a suitable $\epsilon > 0$.
See [7] for a gentle introduction to the topic. Moreover see the review in [6]. A different approach to certain singular SPDEs was provided by Hairer in [10].

1.2 Mild Formulations and Paracontrolled Calculus

In the approach put forward in this thesis we consider the mild equation of the PAM which reads as

$$u(t) = \int_0^t P_{t-r} u(r) \cdot \xi \mathrm{d}r + P_t u^0.$$

Using Bony's paraproduct and the resonant term we formally write this as

$$u(t) = \underbrace{\int_0^t P_{t-r}(u(r) \prec \xi)\mathrm{d}r}_{=B_\prec(u,\xi)(t)} + \underbrace{\int_0^t P_{t-r}(u(r) \circ \xi)\mathrm{d}r}_{=B_\circ(u,\xi)(t)} + \underbrace{\int_0^t P_{t-r}(u(r) \succ \xi)\mathrm{d}r}_{=B_\succ(u,\xi)(t)} + P_t u^0.$$

Motivated by the paracontrolled ansatz we hope that

$$u^\sharp(t) := u(t) - B_\prec(u,\xi)(t) \in C([0,T]; \mathscr{C}^{2\gamma})$$

again has better better regularity as it resembles the first order noise approximation in terms of the integral formulation.
For $u^\sharp$ we derive the equation

$$u^\sharp = B_\circ(B_\prec(u,\xi),\xi) + B_\succ(B_\prec(u,\xi),\xi) + B_\succeq(u^\sharp,\xi) + P_t u^0.$$

Using estimates for the paraproduct and the resonant term (Chapter 3.2), we conclude that the operator $u \mapsto B_\circ(B_\prec(u,\xi),\xi)$ is the only one not well-defined.
We formally set

$$\begin{aligned} B_\circ(B_\prec(u,\xi),\xi) &= \int_0^t P_{t-r} \left(\int_0^r P_{r-s}(u(s) \prec \xi)\mathrm{d}s \right) \circ \xi \mathrm{d}r \\ &= \int_0^t P_{t-r} \left(\int_0^r P_{r-s}(u(s) \prec \xi) \circ \xi \mathrm{d}s \right) \mathrm{d}r \end{aligned}$$

and alike above, we can prove that this operator admits singular behavior. Consequently, we need a renormalization to obtain something well-defined. In order to do so, we write

$$\int_0^t P_{r-s}(u(s) \prec \xi) \circ \xi \mathrm{d}s = \int_0^t \int_{\mathbb{T}^2} v^\xi(t,x;r,z) u(r,z) \mathrm{d}z \mathrm{d}r,$$

i.e. in terms of an integral kernel. One now can prove that

$$u \mapsto \int_0^t \int_{\mathbb{T}^2} \left(v^\xi(t,x;r,z) - \mathbb{E}\left[v^\xi(t,x;r,z) \right] \right) u(r,z) \mathrm{d}z \mathrm{d}r$$

is a well-defined bounded operator and thus we can give a sound meaning to the renormalized equation of $u^\sharp$ which we can solve via fixed point iteration arguments in a well-posed way.
Finally, we can solve the equation

$$u = B_{\prec}(u,\xi) + u^\sharp$$

again by a fixed point argument in a well-posed way and thus obtain a notion of solution to the renormalized PAM.

1.3 Notations

In writing $a \lesssim b$ for positive real numbers a, b we mean that there is a constant $C > 0$ independent of a and b such that $a \leq Cb$. Similarly, we define $a \gtrsim b$ by $b \lesssim a$ and write $a \cong b$ provided $a \lesssim b$ and $b \lesssim a$. If we want to denote dependence on some variable explicitly we write $a(t) \lesssim_t b(t)$ or use similar notation.
For a complex number $z \in \mathbb{C}$ we denote its complex conjugate by z^*.
For a multiindex $\mu \in \mathbb{N}^d$ we write $|\mu| = \mu_1 + ... + \mu_d$ and $\partial^\mu = \partial_1^{\mu_1} ... \partial_d^{\mu_d}$.
We denote the torus by $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}$ and write $\mathbb{T}^d$ for the d-dimensional torus. If not stated differently, all functions spaces will have domain $\mathbb{T}^d$ and codomain $\mathbb{C}$, e.g. we write L^p instead of $L^p(\mathbb{T}^d;\mathbb{C})$.
For $\alpha \in \mathbb{R}$ we write C^α for the space of $\lfloor \alpha \rfloor$-times differentiable functions for which the derivatives of order $|\lfloor \alpha \rfloor|$ are $(\alpha - \lfloor \alpha \rfloor)$-Hölder-continuous.
For a Banach space X we denote by $C_T X$ the space of continuous maps from $[0,T]$ to X and write for $u \in C_T X$ the norm of this space as follows: $\|u\|_{C_T X} = \sup_{0 \leq t \leq T} \|u(t)\|_X$. In writing CX we mean the space of continuous maps from $[0,\infty)$ to X.
Moreover, for $\alpha \in (0,1)$ we denote by $C_T^\alpha X$ the subspace of $C_T X$ such that the functions are α-Hölder-continuous. For $u \in C_T^\alpha X$ we use the following notation

$$\|u\|_{C_T^\alpha X} = \sup_{0 \leq s < t \leq T} \frac{\|u(t) - u(s)\|_X}{|t-s|^\alpha}.$$

and write $C^\alpha X$ for the space of functions that are locally in time α-Hölder-continuous in X.
Finally, for Banach spaces X, Y we denote by $L(X,Y)$ the space of bounded operators from X to Y.

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2 Preliminaries

In this chapter we will briefly recall the very basic tools needed for our analysis of singular SPDE and settle on some conventions.
We introduce Schwartz distributions and the Fourier transform both on the torus and the euclidean space. The first provide us with a general framework for handling "irregular functions". The second is an indispensable tool for the techniques presented below.
We follow the conventions used in [7]. See [4, Chapter 3] for a general introduction to the theory.

Definition 2.0.1. *(Schwartz function on torus)The space of Schwartz functions on the d-dimensional torus $\mathbb{T}^d$, denoted by $\mathscr{S} := \mathscr{S}(\mathbb{T}^d)$ is defined as*

$$\mathscr{S} := C^\infty(\mathbb{T}^d; \mathbb{C})$$

The space of continuous linear functionals on the Schwartz space can be characterized as follows:

Definition 2.0.2. *(Schwartz distribution on torus)A linear functional $f : \mathscr{S} \to \mathbb{C}$ is a Schwartz distribution if there exists a constant $C > 0$ and a natural number $k \in \mathbb{N}$ such that for any $\varphi \in \mathscr{S}$*

$$|f(\varphi)| := |\langle \varphi, f \rangle| \leq C \max_{\nu \in \mathbb{N}^d :\, |\nu| \leq k} \sup_{x \in \mathbb{T}^d} |\partial^\nu f|.$$

We denote the space of Schwartz distributions by $\mathscr{S}'$.

Example 2.0.3. *Let $g \in L^1(\mathbb{T}^d)$ be an integrable function. This functions induces a Schwartz distribution $T_g \in \mathscr{S}'$ defined by*

$$T_g(\varphi) := \langle \varphi, g \rangle = \int_{\mathbb{T}^d} \varphi g \mathrm{d}x$$

for any $\varphi \in \mathscr{S}$. In the following, we often will identify T_g with g

Next, we turn to the fundamental objects of Harmonic Analysis on the torus: The Fourier coefficients and the Fourier series.
In the following we use the notation $e_k(\cdot) = e^{i\langle \cdot, k \rangle}/(2\pi)^{d/2}$ and denote its complex conjugation by $e_k^*(\cdot)$.

Definition 2.0.4. *(Fourier transform)Let $f \in \mathscr{S}'$ be a Schwartz distribution and $k \in \mathbb{Z}^d$. We define the k-th Fourier coefficient of f by*

$$\hat{f}(k) := \langle e_k, f \rangle.$$

Remark 2.0.5. *In view of the identification of T_g with g for $g \in L^1$ we write for $k \in \mathbb{Z}^d$ $\hat{g}(k)$ instead of $\widehat{T_g}(k)$.*

Recall that the Fourier transform of a Schwartz distribution $f \in \mathscr{S}'$ is at most of polynomial growth, i.e. there is a natural number $N \in \mathbb{N}$ such that

$$\sup_{k \in \mathbb{Z}^d} |\hat{f}(k)|(1+|k|)^{-N} < \infty.$$

Definition 2.0.6. *(Fourier inversion)Let $(a_k)_{k\in\mathbb{Z}^d}$ be a sequence of complex numbers of at most polynomial growth. We then define the inverse Fourier transform with respect to this sequence as*

$$\mathscr{F}^{-1}(a_k) := \sum_{k \in \mathbb{Z}^d} a_k e_k^*.$$

If $f : \mathbb{R}^d \to \mathbb{C}$ is of at most polynomial growth we define

$$\mathscr{F}^{-1} f := \mathscr{F}^{-1}(f|_{\mathbb{Z}^d}).$$

Moreover, for $f \in \mathscr{S}'$ a Schwartz distribution we use the notation

$$\check{\varphi} := \mathscr{F}^{-1} \hat{f} \tag{2.0.7}$$

We have the following basic results:

Proposition 2.0.8. ([5, Chapter 3, 3.39 and 4, Chapter 12, 12.5.3]) *Let $\varphi \in \mathscr{S}$ and $f \in \mathscr{S}'$*

1. *The Fourier coefficients $(\hat{\varphi}(k))_{k\in\mathbb{Z}^d}$ are of rapid decay, i.e for any $N \in \mathbb{N}$ we have*

   $$\sup_{k \in \mathbb{Z}^d} \hat{\varphi}(k)(1+|k|)^N < \infty$$

   and

   $$\mathscr{F}^{-1}(\hat{\varphi}) = \varphi \text{ in } \mathscr{S}.$$

   Moreover, for any sequence $(a_k)_{k\in\mathbb{Z}^d}$ which is of rapid decay, $\mathscr{F}^{-1}((a_k)_{k\in\mathbb{Z}^d}$ converges to a Schwartz function ψ in $\mathscr{S}$ and $\hat{\psi}(k) = a_k$ for all $k \in \mathbb{Z}^2$.

2. *The sequence $(\hat{f}(k))_{k\in\mathbb{Z}^d}$ is at most of polynomial growth and*

   $$\mathscr{F}^{-1}(\hat{f}) = f \text{ in } \mathscr{S}'$$

   Moreover, for any sequence $(a_k)_{k\in\mathbb{Z}^d}$ which is of at most polynomial growth $\mathscr{F}^{-1}(a_k)$ converges to a Schwartz distribution g in $\mathscr{S}'$ and $\hat{g}(k) = a_k$ for all $k \in \mathbb{Z}^d$.

The above proposition implies that the inverse Fourier transform in fact yields a well-defined object.
We moreover need to notion of convolution.

Definition 2.0.9. *Let $\varphi \in \mathscr{S}$ be a Schwartz function and $f, g \in \mathscr{S}'$ Schwartz distributions. We define*

1. $f * \varphi(z) := \langle \check{\varphi}(\cdot - z), f \rangle$
2. $f * g(u) := \langle g \check{*} \check{u}, f \rangle$

Proposition 2.0.10. ([4, Chapter 12, 12.6.4 and 12.6.5]) *Let $f, g \in \mathscr{S}'$ be Schwartz distributions and $\varphi \in \mathscr{S}$ a Schwartz function.*

1. *$f * \varphi \in \mathscr{S}$ is a Schwartz function.*
2. *$f * g \in \mathscr{S}'$ is a Schwartz distribution and satisfies*

$$\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k).$$

Moreover, the convolution of distributions is commutative and associative.

In the course of this thesis we will also need the corresponding notions on the euclidean space:

Definition 2.0.11. *(Schwartz functions on euclidean space)A smooth function $\varphi \in C^\infty(\mathbb{R}^d)$ is a Schwartz functions on $\mathbb{R}^d$ provided that for any $N \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$*

$$\sup_{x \in \mathbb{R}^d} (1 + |x|)^N |\partial^\alpha \varphi(x)| < \infty.$$

The space of Schwartz functions on $\mathbb{R}^d$ will be denoted by $\mathscr{S}(\mathbb{R}^d)$.

Remark 2.0.12. *We say that a function $g \colon \mathbb{R}^d \to \mathbb{C}$ is of rapid decay if the values decay faster than any polynomial at infinity, i.e. for any natural number $N \in \mathbb{N}$ we have*

$$\sup_{x \in \mathbb{R}^d} g(x)(1 + |x|)^N < \infty.$$

Using this mode of speaking, a smooth functions $\varphi \in C^\infty(\mathbb{R}^d)$ is a Schwartz function if and only if any derivative of φ is of rapid decay.

Definition 2.0.13. *(Fourier transform on euclidean space)Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. The Fourier transform of φ is defined by*

$$\mathscr{F}_{\mathbb{R}^d}\varphi(z) := \hat{\varphi}(z) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle x, z \rangle} \varphi(x) \mathrm{d}x.$$

Moreover, the inverse Fourier transform is defined by

$$\mathscr{F}_{\mathbb{R}^d}^{-1}\varphi(x) = \mathscr{F}_{\mathbb{R}^d}\varphi(-x)$$

Recall that the Fourier transform on the euclidean space of a Schwartz function is again a Schwartz function.

3 Littlewood-Paley Decomposition and Bony's Paraproduct

In this chapter we recall the basics of Littlewood-Paley theory as well as Besov spaces. Moreover, we will introduce Bony's paraproduct and provide fundamental statements regarding this notion.
For a general introduction to the topic see [2, Chapter 2]. Besov spaces - as well as their natural counterparts: Triebel-Lizorkin spaces - on the torus are extensively treated in[12, Chapter 3].
Finally we introduce Bony's paraproduct and provide foundational results for this notion.
In this account, we again loosely follow [7, Chapter 3 and Chapter 5, 5.1]. See also [8, Appendix A.1].

3.1 Dyadic Blocks and Besov Spaces

Generally speaking, Littlewood-Paley theory provides us with a technique to decompose distributions into smooth functions, the so called *dyadic blocks*. The main feature of this decomposition is that the obtained functions are spectrally supported in either a ball or an annulus.
We first need to introduce a family of functions with respect to which we want to define this decomposition:

Definition 3.1.1. *(Dyadic partition of unity) Let $\chi, \rho \in C^\infty(\mathbb{R}^d, [0,1])$ be two smooth, compactly supported radial functions for which the support* $\mathrm{supp}\chi = \mathscr{B} = \{x \colon |x| \leq a\}$ *for a suitable $a > 0$ is a centered ball and* $\mathrm{supp}\rho = \mathscr{A} = \{x \colon b \leq |x| \leq c\}$ *for suitable $c > b > 0$ is a centered annulus such that*

1. *for any $x \in \mathbb{R}^d$, $\chi(x) + \sum_{j\geq 0} \rho(2^{-i}x) = 1$*
2. $\mathrm{supp}\chi \cap \mathrm{supp}\rho(2^{-j}\cdot) = \emptyset$ *if $j \geq 1$ and* $\mathrm{supp}\rho(2^{-j}\cdot) \cap \mathrm{supp}\rho(2^{-i}\cdot) = \emptyset$ *for $i, j \geq 0$ whenever $|i - j| > 1$*

Writing

$$\rho_{-1} \coloneqq \chi, \qquad \rho_i(\cdot) \coloneqq \rho(2^{-i}\cdot) \text{ for } i \geq 0$$

we call the family $(\rho_j)_{j\geq -1}$ of functions a dyadic partition of unity.

First note:

Proposition 3.1.2. ([2, Chapter 2, 2.10]) *Dyadic partitions of unity exist.*

In the following we fix an arbitrary dyadic partition of unity $(\rho_j)_{j\geq -1}$.

Definition 3.1.3. *(Dyadic blocks)Let $f \in \mathscr{S}$ be a Schwartz distribution and $j \geq -1$. The j-th dyadic block of f is defined by*

$$\Delta_j f \coloneqq \mathscr{F}^{-1}(\rho_j \hat{f}) = \sum_{k\in\mathbb{Z}^d} \rho_j(k)\hat{f}(k)e_k^*.$$

We moreover set

$$S_i f := \sum_{j\geq -1}^{i-1} \Delta_j f$$

Remark 3.1.4. *Setting* $K_j = (2\pi)^{-d/2}\mathscr{F}^{-1}\rho_j$ *a straightforward computation yields*

$$K_j * f = \Delta_j f$$

for $f \in \mathscr{S}$.

It is an immediate consequence of the definition that for a Schwartz distribution $f \in \mathscr{S}'$ and natural numbers $i, j \geq -1$ both $\Delta_j f$ and $S_i f$ are Schwartz functions and the Fourier transform of both $\Delta_j f$ and $S_i f$ is compactly supported.
Furthermore, we have:

Proposition 3.1.5. 1. *Let* $f \in \mathscr{S}'$. *Then* $S_i f \to f$ in $\mathscr{S}'$.

2. *There exists a centered annulus* $\mathscr{A}'$ *such that for all Schwartz distributions* $f, g \in \mathscr{S}'$ *and numbers* $i, j \in \mathbb{N}$ *such that* $i \leq j-2$ *we have*

$$\mathrm{supp}\mathscr{F}\Big(\Delta_i g \Delta_j f\Big) \subset 2^j \mathscr{A}'.$$

Proof. The first assertion follows from the facts that

$$\langle \varphi, \Delta_j f\rangle = \langle \Delta_j \varphi, f\rangle$$

and $S_i\varphi \to \varphi$ as $i \to \infty$ in $\mathscr{S}$ for all $\varphi \in \mathscr{S}$.
The second assertion can be proven by noting

$$\mathscr{F}\left(\Delta_i g \Delta_j f\right)(k) = \sum_{l\in\mathbb{Z}^2} \rho_i(l)\rho_j(k-l)\hat{g}(l)\hat{f}(k-l)$$

and the properties of the dyadic partition of unity. □

After this preparation, we are able to define *Besov spaces*. These spaces consist of Schwartz distributions whose dyadic blocks enjoy certain regularity properties:

Definition 3.1.6. *(Besov spaces)Let* $\alpha \in \mathbb{R}$ *and* $p, q \in [1, \infty]$. *We define the Besov space* $B^\alpha_{p,q}$ *by*

$$B^\alpha_{p,q} := B^\alpha_{p,q}(\mathbb{T}^d) := \left\{ f \in \mathscr{S}' \colon \|f\|_{B^\alpha_{p,q}} := \left(\sum_{j\geq -1} (2^{j\alpha} \|\Delta_j f\|_{L^p})^q\right)^{\frac{1}{q}} < \infty \right\}$$

with the usual modification if $q = \infty$. *Moreover, we set*

$$\mathscr{C}^\alpha := B^\alpha_{\infty,\infty}, \quad \|\cdot\|_{\mathscr{C}^\alpha} := \|\cdot\|_{B^\alpha_{\infty,\infty}}.$$

The latter spaces we call Hölder-Besov spaces.

Our definition of Besov spaces rely on the dyadic blocks and thus might be dependent on the chosen dyadic partition of unity. This is issue is dealt with in the following proposition:

Proposition 3.1.7. ([13, Chapter 3, 3.5.1]) *For all $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$ the space $B^{\alpha}_{p,q}$ is a Banach spaces. The norm $\|\cdot\|_{B^{\alpha}_{p,q}}$ is dependent on the choice of a specific partition, but the space $B^{\alpha}_{p,q}$ is not and different norms induced by different dyadic partitions of unity are equivalent.*

We also may interpret the Hölder-Besov spaces in a more elementary way as already suggested by its name:

Proposition 3.1.8. ([13, Chapter 3, 3.5.4]) *Let $\alpha \in \mathbb{R} \setminus \mathbb{N}$. Then $\mathscr{C}^{\alpha} = C^{\alpha}$.*

In our analysis of singular SPDE we will mostly work in the context of Hölder-Besov spaces. The following fundamental inequalities will be of used heavily throughout the text:

Proposition 3.1.9. ([7, Chapter 3, 3.10]) *We have*

1. $\|\cdot\|_{\mathscr{C}^{\alpha}} \leq \|\cdot\|_{\mathscr{C}^{\beta}}$ *provided* $\alpha \leq \beta$
2. $\|\cdot\|_{L^{\infty}} \lesssim_{\alpha} \|\cdot\|_{\mathscr{C}^{\alpha}}$ *provided* $\alpha > 0$
3. $\|\cdot\|_{\mathscr{C}^{\alpha}} \lesssim \|\cdot\|_{L^{\infty}}$ *provided* $\alpha \leq 0$
4. $\|S_i \cdot\|_{L^{\infty}} \lesssim 2^{i\alpha} \|\cdot\|_{\mathscr{C}^{\alpha}}$ *provided* $\alpha < 0$

We moreover need to understand how Besov spaces with different parameters are related. This is dealt with in the following theorem.

Theorem 3.1.10. ([12, Chapter 3, 3.5.5]) *(Besov embedding)Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$ as well as $\alpha \in \mathbb{R}$. Then $B^{\alpha}_{p_1,q_1} \subset B^{\alpha - d(1/p_1 - 1/p_2)}_{p_2,q_2}$ is a continuous embedding.*

Furthermore, we will need conditions under which we may conclude that a certain function is in fact contained in a suitable Hölder-Besov space:

Lemma 3.1.11. ([7, Chapter 3, 3.10])

1. *Let $\mathscr{A}'$ be a centered annulus and $\alpha \in \mathbb{R}$. Assume that $(u_j)_{j \geq -1}$ is a family of Schwartz functions such that $\operatorname{supp} \mathscr{F} u_j \subset 2^j \mathscr{A}'$ and $\|u_j\|_{L^{\infty}} \lesssim 2^{-j\alpha}$. Then*

$$u := \sum_{j \geq -1} u_j \in \mathscr{C}^{\alpha} \text{ and } \|u\|_{\mathscr{C}^{\alpha}} \lesssim \left\| 2^{\alpha j} \|u_j\|_{L^{\infty}} \right\|_{\ell^{\infty}}.$$

2. *Let $\mathscr{B}'$ be centered ball and $\alpha > 0$. Assume that $(u_j)_{j\geq -1}$ is a family of Schwartz functions such that* $\operatorname{supp}\mathscr{F}u_j \subset 2^j\mathscr{B}'$ *and* $\|u_j\|_{L^\infty} \lesssim 2^{-j\alpha}$. *Then*
$$u := \sum_{j\geq -1} u_j \in \mathscr{C}^\alpha \text{ and } \|u\|_{\mathscr{C}^\alpha} \lesssim \left\|2^{\alpha j}\|u_j\|_{L^\infty}\right\|_{\ell^\infty}.$$

When dealing with the Parabolic Anderson Model we will also need to gain temporal scaling factors in order to set up fixed point iterations. This is accomplished by working in the following *parabolic spaces*.

Definition 3.1.12. *Let $T > 0$ and $\alpha \in (0,2)$. We define the parabolic space*

$$\mathscr{L}_T^\alpha := C_T\mathscr{C}^\alpha \cap C_T^{\frac{\alpha}{2}}L^\infty$$

and endow this space with the norm $\|\cdot\|_{\mathscr{L}_T^\alpha} := \max(\|\cdot\|_{C_T\mathscr{C}^\alpha}, \|\cdot\|_{C_T^{\frac{\alpha}{2}}L^\infty})$.

We have

Proposition 3.1.13. *For $T > 0$ and $\alpha \in (0,2)$ the space $\mathscr{L}_T^\alpha$ is a Banach space.*

Proof. Noting that by proposition 3.1.9 we have $\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_{\mathscr{C}^\alpha}$, the claim follows. □

In the parabolic spaces we can in fact gain scaling factors by passing to a larger space:

Proposition 3.1.14. ([9, Chapter 2, 2.3, 2.11])*Let $T > 0$ and $\alpha \in (0,2)$, $\delta \in (0,\alpha)$ as well as $u \in \mathscr{L}_T^\alpha$ Then*

$$\|u\|_{\mathscr{L}_T^\delta} \lesssim T^{\frac{\alpha-\delta}{2}}\|u\|_{\mathscr{L}_T^\alpha} + \|u(0)\|_{\mathscr{C}^\delta}$$

Next we deal with the issue of gluing functions in the parabolic spaces. In the following we denote the time shift of a function f by $\tau^T f(t) := f(T+t)$.

Proposition 3.1.15. *Let $T > 0$ and $T_1 \in (0,T)$. Moreover, let $\alpha > 0$ and assume that $u \in \mathscr{L}_T^\alpha$. Then*

$$\|u\|_{\mathscr{L}_T^\alpha} \leq \|u\|_{\mathscr{L}_{T_1}^\alpha} + \left\|\tau^{T_1}u\right\|_{\mathscr{L}_{T-T_1}^\alpha}.$$

Conversely, if $u \in C_T\mathscr{C}^\alpha$ such that $u \in \mathscr{L}_{T_1}^\alpha$ and $\tau^{T_1}u \in \mathscr{L}_{T-T_1}^\alpha$, then $u \in \mathscr{L}_T^\alpha$

Proof. Clearly

$$\|u\|_{C_T\mathscr{C}^\alpha} \leq \|u\|_{C_{T_1}\mathscr{C}^\alpha} + \left\|\tau^{T_1}u\right\|_{C_{T-T_1}\mathscr{C}^\alpha}. \tag{3.1.16}$$

Thus, it suffices to show

$$\|u\|_{C_T^{\frac{\alpha}{2}} L^\infty} \leq \|u\|_{C_{T_1}^{\frac{\alpha}{2}} L^\infty} + \left\|\tau^{T_1} u\right\|_{C_{T-T_1}^{\frac{\alpha}{2}} L^\infty}$$

Let $s, t \in [0, T]$. The only cases we need to consider is $s < T_1 < t$ and $t < T_1 < s$. Without loss of generality, we may assume $s < T_1 < t$. We calculate

$$\begin{aligned}
\|u(t) - u(s)\|_{L^\infty} &\leq \|u(t) - u(T_1)\|_{L^\infty} + \|u(T_1) - u(s)\|_{L^\infty} \\
&\leq \left\|\tau^{T_1} u\right\|_{\mathscr{L}^\alpha_{T-T_1}} (t - T_1)^{\alpha/2} + \|u\|_{\mathscr{L}^\alpha_{T_1}} (T_1 - s)^{\alpha/2} \\
&\leq \left(\left\|\tau^{T_1} u\right\|_{\mathscr{L}^\alpha_{T-T_1}} + \|u\|_{\mathscr{L}^\alpha_{T_1}} \right) (t - s)^{\alpha/2}
\end{aligned}$$

which implies the first claim.
The second claim follows immediately from the last inequality. □

Next we are concerned with the *Schauder estimate* for the heat semigroup.
The action of the heat semigroup on a distribution $f \in \mathscr{S}'$ is given by

$$\mathscr{F}(P_t f)(k) = e^{-t|k|^2} \hat{f}(k)$$

We state two lemmata first:

Lemma 3.1.17. ([8, Appendix A.1, A.7])*Let $T > 0$ and assume $t \in (0, T]$. Let $u \in \mathscr{S}'$ and $\delta \geq 0$.*

$$\|P_t u\|_{\mathscr{C}^{\alpha+\delta}} \lesssim_T t^{-\delta/2} \|u\|_{\mathscr{C}^\alpha} \quad \text{and} \quad \|P_t u\|_{\mathscr{C}^\delta} \lesssim_T t^{-\delta/2} \|u\|_{L^\infty}$$

If $\mathscr{F}u$ is supported outside of a ball centered at 0 the estimates are uniform in $t > 0$.

Lemma 3.1.18. ([8, Appendix 1, A.8]) *Let $\alpha \in (0, 1)$ and $\beta \in \mathbb{R}$. Assume that $u \in \mathscr{C}^\beta$. Then for all $t \geq 0$.*

$$\|(P_t - Id)u\|_{L^\infty} \lesssim t^{\alpha/2} \|u\|_{\mathscr{C}^\alpha}.$$

These two lemmata can be used the proof the Schauder estimate:

Theorem 3.1.19. ([7, Chapter 3, Lemma 11], see also [8, Appendix A.1 A.9]) *(Schauder estimate) Let $\alpha \in (0, 2)$. For $f \in C\mathscr{C}^{\alpha-2}$ we define $If(t) := \int_0^t P_{t-r} f \mathrm{d}r$. We then have*

$$\|If\|_{\mathscr{L}^\alpha_T} \lesssim (1 + T) \|f\|_{C_T \mathscr{C}^{\alpha-2}}$$

for all $T > 0$ and for $g \in \mathscr{C}^\alpha$

$$\|P_\cdot g\|_{\mathscr{L}^\alpha_T} \lesssim \|u\|_{\mathscr{C}^\alpha}.$$

In the course of this master thesis we also need a variation of the Schauder estimate which deals with functions that are singular at 0.

Theorem 3.1.20. *Let* $\alpha \in (0,2)$. *Assume that for a sufficiently small* $0 < \epsilon < 1$ *we have that* $r^\epsilon f(r) \in C_T\mathscr{C}^{\alpha-2}$. *Then*

$$\|If\|_{\mathscr{L}_T^{\alpha-2\epsilon}} \lesssim \|r^\epsilon f(r)\|_{C_T\mathscr{C}^{\alpha-2}}$$

Proof. First note that there exists $0 < \beta < 2$ such that $\alpha - 2 + \beta > 0$. Using lemma 3.1.18 we estimate

$$\begin{aligned}\int_0^t \|P_{t-r}(f(r))\|_{L^\infty} \,\mathrm{d}r &\lesssim \int_0^t \|P_{t-r}f(r)\|_{\mathscr{C}^{\alpha-2+\beta}} \,\mathrm{d}r \\ &\lesssim \int_0^t r^{-\epsilon}(t-r)^{-\beta/2} \|r^\epsilon f(r)\|_{\mathscr{C}^{\alpha-2}} \,\mathrm{d}r \\ &= \|r^\epsilon f(r)\|_{C_T\mathscr{C}^{\alpha-2}} \int_0^t r^{-\epsilon}(t-r)^{-\beta/2}\mathrm{d}r \\ &\lesssim \|r^\epsilon f(r)\|_{C_T\mathscr{C}^{\alpha-2}}.\end{aligned}$$

where we used that the last integral is bounded.
Thus, using the Fubini-Tonelli theorem, we obtain

$$\mathscr{F}_x\left(\int_0^t P_{t-r}(f(r))\mathrm{d}r\right) = \int_0^t \mathscr{F}_x\left(P_{t-r}(f(r))\right)\mathrm{d}r$$

and consequently for $j \geq -1$

$$\Delta_j \int_0^t P_{t-r}(u(r))\mathrm{d}r = \int_0^t \Delta_j P_{t-r}(u(r))\mathrm{d}r = \int_0^t P_{t-r}\left(\Delta_j u(r)\right)\mathrm{d}r.$$

Now let $j \geq 0$ and $\delta \in (0, t/2)$. We consider

$$\int_0^t P_{t-r}\left(\Delta_j f(r)\right)\mathrm{d}r = \int_0^\delta P_r\left(\Delta_j f(t-r)\right)\mathrm{d}r + \int_\delta^t P_r\left(\Delta_j f(t-r)\right)\mathrm{d}r.$$

On the one hand, we obtain

$$\begin{aligned}&\left\|\int_0^\delta P_r\left(\Delta_j f(t-r)\right)\mathrm{d}r\right\|_{L^\infty} \\ &\lesssim \int_0^\delta \|(t-r)^\epsilon \Delta_j f(t-r)\|_{L^\infty} (t-r)^{-\epsilon}\mathrm{d}r \\ &\leq \|r^\epsilon f(r)\|_{C_T\mathscr{C}^{\alpha-2}} 2^{-j(\alpha-2)} \int_0^\delta (t-r)^{-\epsilon}\mathrm{d}r \\ &\leq \|r^\epsilon f(r)\|_{C_T\mathscr{C}^{\alpha-2}} 2^{-j(\alpha-2)}\delta^{1-\epsilon}\end{aligned}$$

where we used the inequality

$$\frac{1}{(t-r)^\epsilon} \le \frac{1}{(\delta-r)^\epsilon} \text{ for } 0 \le r < \delta.$$

On the other hand, we estimate using lemma 3.1.17

$$\begin{aligned}
&\left\| \int_\delta^t P_r\left(\Delta_j u(t-r)\right) \mathrm{d}r \right\|_{L^\infty} \\
&\le \int_\delta^t 2^{-j(\alpha-2-2(1-\epsilon))} r^{-(1-\epsilon)} \left\| f(t-r) \right\|_{\mathscr{C}^\alpha} \mathrm{d}r \\
&\lesssim \left\| r^\epsilon f(r) \right\|_{C_T \mathscr{C}^{\alpha-2}} 2^{-j(\alpha-2+2(1-\epsilon))} \int_0^t \frac{1}{r^{1-\epsilon}(t-r)^\epsilon} \mathrm{d}r \\
&\lesssim \left\| r^\epsilon f(r) \right\|_{C_T \mathscr{C}^{\alpha-2}} 2^{-j(\alpha-2+2(1-\epsilon))} \delta^{-1+\epsilon} \int_0^t \frac{1}{(t-r)^\epsilon} \mathrm{d}r \\
&\lesssim \left\| r^\epsilon f(r) \right\|_{C_T \mathscr{C}^{\alpha-2}} 2^{-j(\alpha-2+2(1-\epsilon))} \delta^{-1+\epsilon} \delta^{1-\epsilon}
\end{aligned}$$

where we moreover used that

$$\frac{1}{s^{1-\epsilon}} \le \frac{1}{\delta^{1-\epsilon}} \text{ for } \delta \le s$$

Now setting $\delta = 2^{-2j}$ we obtain that

$$\sup_{j \ge -1} 2^{-j(\alpha-2+2(1-\epsilon)} \left\| \Delta_j I f \right\|_{L^\infty} \lesssim \left\| r^\epsilon f(r) \right\|_{C_T \mathscr{C}^{\alpha-2}}.$$

and hence $If \in C_T \mathscr{C}^{\alpha-2\epsilon}$.
We now apply lemma 3.1.18 to estimate

$$\begin{aligned}
\left\| If(t) - If(s) \right\|_{L^\infty} &\lesssim \left\| (P_{t-s} - Id) If(s) \right\|_{L^\infty} + \int_s^t \left\| P_{t-r} f(r) \right\|_{L^\infty} \mathrm{d}r \\
&\lesssim |t-s|^{(\alpha-2\epsilon)/2} \left\| If \right\|_{C_T \mathscr{C}^{\alpha-2\epsilon}} + \int_s^t \left\| r^\epsilon f(r) \right\| r^{-\epsilon} \mathrm{d}r \\
&\lesssim \left\| r^\epsilon f(r) \right\|_{C_T \mathscr{C}^{\alpha-2}} \left(|t-r|^{(\alpha-2\epsilon)/2} + t^{1-\epsilon} - s^{1-\epsilon} \right) \\
&\lesssim_T \left\| r^\epsilon f(r) \right\|_{C_T \mathscr{C}^{\alpha-2}} |t-r|^{(\alpha-2\epsilon)/2}
\end{aligned}$$

where we used

$$|t^{1-\epsilon} - r^{1-\epsilon}| \le |t-r|^{1-\epsilon} \text{ for } r, t \ge 0.$$

This proves the assertion. □

3.2 Bony's Paraproduct

In this section, we deal with the paraproduct and the resonant term. These notions were introduced by Bony in [3]. For a modern introduction see [2]. We again loosely follow [7]. See also [8] for a similar, brief introduction.

Definition 3.2.1. *(Paraproduct and resonant term)Let $f, g \in \mathscr{S}'$ be Schwartz distributions. We define - whenever well-defined - the paraproduct*

$$f \prec g \coloneqq \sum_{j\geq -1} S_{j-1} f \Delta_j g = \sum_{j\geq -1} \sum_{i\geq -1}^{j-2} \Delta_i f \Delta_j g$$

and the resonant term

$$f \circ g \coloneqq \sum_{i,j\geq -1\colon\, |i-j|\leq 1} \Delta_i f \Delta_j g$$

Remark 3.2.2. *The idea of the paraproduct is to split a "product" of distributions f, g as follows:*

$$f \cdot g = f \prec g + f \circ g + f \succ g \tag{3.2.3}$$

where $f \prec g$ respectively $f \succ g$ can be thought of frequency modulation of g respectively f. On the other hand, $f \circ g$ takes frequencies of similar ranges into account.
The crucial point: $f \prec g$ and $f \succ g$ do always exist as well-defined distributions.
However, $f \circ g$ exists only given suitable regularity assumptions on f, g and thus, if ill-defined, may be heuristically interpreted as resonance.

The above remark is made precise in the following proposition:

Proposition 3.2.4. ([7, Chapter 5, 5.1, Theorem 4])*(Paraproduct estimates) The paraproduct and resonant term enjoy the following bounds:*

1. *Let $\beta \in \mathbb{R}$. Assume that $f \in L^\infty$ and $g \in \mathscr{C}^\beta$. Then*
$$\|f \prec g\|_{\mathscr{C}^\beta} \lesssim_\beta \|f\|_{L^\infty} \|g\|_{\mathscr{C}^\beta}.$$
2. *Let $\alpha < 0$ and $\beta \in \mathbb{R}$. Assume that $f \in \mathscr{C}^\alpha$ and $g \in \mathscr{C}^\beta$. Then*
$$\|f \prec g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim_{\alpha,\beta} \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}.$$
3. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$. Assume that $f \in \mathscr{C}^\alpha$ and $g \in \mathscr{C}^\beta$. Then*
$$\|f \circ g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim_{\alpha,\beta} \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}.$$

Proof. By proposition 3.1.5 we conclude that the Fourier transform of $S_{j-1} f \Delta_j g$ is supported in $2^j \mathscr{A}'$ for a suitable annulus $\mathscr{A}'$. We moreover have

$$\|S_j f \Delta_{j-1} g\|_{L^\infty} \leq \|S_j f\|_{L^\infty} \|\Delta_j g\|_{L^\infty} \lesssim 2^{-j\beta} \|f\|_{L^\infty} \|g\|_{L^\infty}.$$

Now lemma 3.1.11 implies the first claim.
The second claim follows by noting

$$\|S_j f\|_{L^\infty} \lesssim 2^{j\alpha} \|f\|_\alpha$$

provided $\alpha < 0$ (proposition 3.1.9).
The last claim follows again from lemma 3.1.11 noting that for $i, j \geq -1$ such that $|i - j| \leq 1$ the Fourier transform of $\Delta_i f \Delta_j g$ is supported in a suitable ball $\mathscr{B}'$ and

$$\|\Delta_i f \Delta_j g\|_{L^\infty} \leq \|\Delta_i f\|_{L^\infty} \|\Delta_j g\|_{L^\infty} \lesssim 2^{-i\alpha} 2^{-i\beta} \|f\|_\alpha \|g\|_\beta .$$

□

These estimates immediately imply the following theorem:

Corollary 3.2.5. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$ Then*

$$\begin{aligned} &\mathscr{C}^\alpha \times \mathscr{C}^\beta \to \mathscr{C}^{\min(\alpha,\beta)} \\ &(f, g) \mapsto fg \end{aligned}$$

is a bounded bilinear map.

As already mentioned in the introduction, our approach to singular SPDEs will based on integral operators combined with the use of the paraproduct and resonant term in the spirit of the paracontrolled calculus.
To define integral kernels that correspond to these integral operators we need to understand how to interpret the paraproduct and the resonant term as *multipliers* in Fourier space.
This we will deal with next.

Proposition 3.2.6. *Let $u \in \mathscr{C}^\alpha$, $\eta \in \mathscr{C}^\beta$ for $\alpha, \beta \in \mathbb{R}$. For $k \in \mathbb{Z}^d$ we have*

$$\mathscr{F}(u \prec \eta)(k) = \sum_{k_1 + k_2 = k} \hat{u}(k_1) \hat{\eta}(k_2) m_\prec(k_1, k_2)$$

where the multiplier $m_\prec$ is given by

$$m_\prec(k_1, k_2) = \sum_{j \geq -2} \sum_{i=-1}^{j-1} \rho_i(k_1) \rho_j(k_2)$$

Proof. Recall that by the regularity assumption we made
$u \prec \eta := \sum_{j \geq -1} \sum_{i=-1}^{j-2} \Delta_i u \Delta_j \eta \in \mathscr{C}^{\min(\beta, \beta+\alpha)}$ is a well defined distribution on $\mathbb{T}^d$.
Moreover, we recall that

$$\text{supp} \mathscr{F}(\Delta_i u \Delta_j \eta) \subset 2^j \mathscr{A}' \tag{3.2.7}$$

whenever $i \leq j-2$ for a suitable centered annulus $\mathscr{A}'$.
We now calculate

$$\mathscr{F}(u \prec \eta)(k) = (u \prec \eta)(e_k)$$
$$= \left(\sum_{j \geq -1}^{J_k} \sum_{i=-1}^{j-2} \Delta_i u \Delta_j \eta \right) (e_k)$$

where J_k is a finite number such that

$$k \notin \text{supp} \mathscr{F}(\Delta_i u \Delta_j \eta) \text{ for } j > J_k,\ i \leq j-2.$$

To find such an J_k is indeed possible due to (3.2.7).
Noting that

$$\Delta_i u \Delta_j \eta$$
$$= \left(\sum_{k_1 \in \mathbb{Z}^2} e^*_{k_1} \hat{u}(k_1) \rho_i(k_1) \right) \left(\sum_{k_2 \in \mathbb{Z}^2} e^*_{k_2} \hat{\eta}(k_2) \rho_j(k_2) \right)$$
$$= \sum_{k \in \mathbb{Z}^d} e^*_k \sum_{k_1+k_2=k} \hat{u}(k_1) \hat{\eta}(k_2) \rho_i(k_1) \rho_j(k_2)$$

we conclude that the k-th Fourier coefficient is given by

$$\mathscr{F}(u \prec \eta)(k) = \mathscr{F} \left(\sum_{j=-1}^{J_k} \sum_{i=-1}^{j-2} \sum_{k' \in \mathbb{Z}^d} e^*_{k'} \sum_{k_1+k_2=k'} \hat{u}(k_1) \hat{\eta}(k_2) \rho_i(k_1) \rho_j(k_2) \right) (k)$$
$$= \sum_{j=-1}^{J_k} \sum_{i=-1}^{j-2} \sum_{k_1+k_2=k} \hat{u}(k_1) \hat{\eta}(k_2) \rho_i(k_1) \rho_j(k_2)$$
$$= \sum_{k_1+k_2=k} \left(\hat{u}(k_1) \hat{\eta}(k_2) \sum_{j=-1}^{J_k} \sum_{i=-1}^{j-2} \rho_i(k_1) \rho_j(k_2) \right)$$
$$= \sum_{k_1+k_2=k} \left(\hat{u}(k_1) \hat{\eta}(k_2) \sum_{j \geq -1} \sum_{i=-1}^{j-2} \rho_i(k_1) \rho_j(k_2) \right)$$
$$= \sum_{k_1+k_2=k} \hat{u}(k_1) \hat{\eta}(k_2) m_{\prec}(k_1, k_2)$$

where the interchange is justified due to all sums being finite sums. □

Proposition 3.2.8. *Assume that $u \in \mathscr{C}^\alpha$, $\eta \in \mathscr{C}^\beta$ where $\alpha + \beta > 0$. Then for $k \in \mathbb{Z}^d$ we have*

$$\mathscr{F}(u \circ \eta)(k) = \sum_{k_1+k_2=k} \hat{u}(k_1) \hat{\eta}(k_2) m_\circ(k_1, k_2)$$

where the multiplier $m_\circ$ is given by

$$m_\circ(k_1,k_2) = \sum_{|i-j|\le 1} \rho_i(k_1)\rho_j(k_2).$$

Proof. By the assumptions, we have that $u \circ \eta \in \mathscr{C}^{\alpha+\beta}$ is a well defined distribution.
Note first that for arbitrary $i, j \ge -1$ we have

$$\mathscr{F}(\Delta_i u \Delta_j \eta)(k) = \sum_{k_1+k_2=k} \rho_i(k_1)\rho_j(k_2)\hat{u}(k_1)\hat{\eta}(k_2)$$

and that the sum is finite.
Now let $J'_k \ge -1$ such that for all $j \ge J'_k + 1$ we have $k \notin \mathrm{supp}\rho_j$.
Thus we conclude

$$\begin{aligned}
&\mathscr{F}(u \circ \eta)(k) \\
&= \lim_{n\to\infty} \left(\sum_{j\ge -1}^{n} \sum_{i:\ |i-j|\le -1} \Delta_i u \Delta_j \eta \right)(e_k) \\
&= \sum_{j\ge -1}^{J'_k} \sum_{i:\ |i-j|\le -1} (\Delta_i u \Delta_j \eta)(e_k) \\
&= \sum_{j\ge -1}^{J'_k} \sum_{i:\ |i-j|\le -1} \sum_{k_1+k_2=k} \rho_i(k_1)\rho_j(k_2)\hat{u}(k_1)\hat{\eta}(k_2) \\
&= \sum_{k_1+k_2=k} \hat{u}(k_1)\hat{\eta}(k_2) m_\circ(k_1,k_2).
\end{aligned}$$

□

In using these multipliers effectively the following bounds will turn out to be crucial:

Proposition 3.2.9. *(Multiplier estimates)*

1. *There are constants $c_1, C_1 > 0$ such that for all $k_1, k_2 \in \mathbb{Z}^d$ the following estimate holds true:*

$$m_\prec(k_1,k_2) \le \mathbb{1}_{|k_1|<|k_2|}\mathbb{1}_{c_1|k_2|\le|k_1+k_2|\le C_1|k_2|}$$

2. *Moreover, there are constants $c_2, C_2 > 0$ such that for all $k_1, k_2 \in \mathbb{Z}^d$ satisfying $|k_1| \ge 2c$ or $|k_2| \ge 2c$ the following estimate holds true:*

$$m_\circ(k_1,k_2) \le \mathbb{1}_{c_2|k_1|\le|k_2|\le C_2|k_1|}$$

   if $k_1, k_2 \ne 0$. If moreover for numbers $k, k_1, k_2 \in \mathbb{Z}^d$ one has $m_\circ(k_1,k_2) \ne 0$ and $k_1 + k_2 = k$ we may conclude $|k| \lesssim |k_1|, |k_2|$.

Remark 3.2.10. *In the following we will omit the constants c_1, C_1, c_2, C_2 and just write*

$$m_{\prec}(k_1, k_2) \leq \mathbb{1}_{|k_1|<|k_2|} \mathbb{1}_{|k_2| \lesssim |k_1+k_2| \lesssim |k_2|}$$

and

$$m_{\circ}(k_1, k_2) \leq \mathbb{1}_{|k_1| \lesssim |k_2| \lesssim |k_1|}$$

Proof. Let $k_1, k_2 \in \mathbb{Z}^d$ such that $m_{\prec}(k_1, k_2) \neq 0$. Then by definition $|k_1| < |k_2|$. The second claim follows from the fact that

$$\text{supp}\rho_i + \text{supp}\rho_j \subset 2^j \mathscr{A}''$$

provided $j - 2 \geq i$.
Let now $k_1, k_2 \in \mathbb{Z}^d$ such that $m_{\circ}(k_1, k_2) \neq 0$. The assertions now follows from the fact that if $k_1 \in \text{supp}\rho_i$ and $k_2 \in \text{supp}\rho_j$ then $|i - j| \leq 1$ and consequently $c2^i \leq |k_1|, |k_2| \leq C2^i$ if $i \geq 1$. The second claim follows from this fact as well. □

3.3 White Noise

To close this chapter, we briefly introduce the space white noise. Here we choose to use a rather easy definition of white noise lacking the sophistication of more detailed expositions (see for example [11, Chapter 1, Example 1.16]). We will, however, prove some analytical control for white noise which enables us to use the analytical tools developed below.

Definition 3.3.1. *(Space white noise)Spatial white noise on $\mathbb{T}^d$, denoted by ξ, is a centered Gaussian process on a suitable probability space $(\Omega, \mathscr{A}, \mathbb{P})$ indexed by L^2 with covariance*

$$\mathbb{E}\left[\xi(f)\xi(g)\right] = \int_{\mathbb{T}^d} f(z)g(z)\mathrm{d}z$$

for $f, g \in L^2$.

Remark 3.3.2. *One can show that white noise is linear in its index-arguments almost surely.*

Proposition 3.3.3. ([7, Chapter 2, 2.1]) *There exists a random variable $\tilde{\xi}$ on $(\Omega, \mathscr{A}, \mathbb{P})$ such that for all $\omega \in \Omega$ $\tilde{\xi}(\omega) \in \mathscr{S}'$ is a Schwartz distribution and $\mathbb{P}\left[\xi(f) = \tilde{\xi}(f)\right] = 1$ for all $f \in L^2$.*

In the following we will often write $\hat{\xi}(k) = \xi(e_k)$ where $k \in \mathbb{Z}^d$.

Proof. For $0 < \lambda < 1$ consider

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\mathbb{Z}^d} \frac{\exp(\lambda|\hat{\xi}(l)|^2)}{1+|k|^{d+1}}\right] \\
&= \sum_{k\in\mathbb{Z}^d} (1+|k|^{d+1})^{-1}\mathbb{E}\left[\exp(\lambda|\hat{\xi}(k)|^2)\right] \\
&= \sum_{k\in\mathbb{Z}^d} (1+|k|^{d+1})^{-1} \int_{\mathbb{R}^d} \exp(\lambda|x|^2)(2\pi)^{-d/2}\exp(-|x|^2)\mathrm{d}x < \infty.
\end{aligned}$$

where we used that $\hat{\xi}(k)$ is a centered Gaussian random variable with variance 1. Hence

$$\sum_{k\in\mathbb{Z}^d} \frac{\exp(\lambda|\hat{\xi}(k)|^2)}{1+|k|^{d+1}} < \infty \text{ almost surely}$$

and thus

$$\frac{\exp(\lambda|\xi(k)|^2)}{1+|k|^{d+1}} \to 0 \text{ as } |k| \to \infty \text{ almost surely.}$$

Consequently $|\hat{\xi}(k)| \lesssim \ln(|k|)$ almost surely. This implies the claim. □

Proposition 3.3.4. ([7, Chaper 3]) *Let $\epsilon > 0$ arbitrary. Then $\tilde{\xi} \in \mathscr{C}^{-d/2-\epsilon}$ almost surely.*

Proof. Consider for arbitrary $\alpha \in \mathbb{R}$ and $p > 1$

$$\begin{aligned}
\mathbb{E}\left[\left\|\tilde{\xi}\right\|^{2p}_{B^{\alpha}_{2p,2p}}\right] &= \mathbb{E}\left[\sum_{j\geq -1} 2^{j\alpha 2p}\left\|\Delta_j\tilde{\xi}\right\|^{2p}_{L^{2p}}\right] \\
&= \sum_{j\geq -1} 2^{j\alpha 2p}\mathbb{E}\left[\left\|\Delta_j\tilde{\xi}\right\|^{2p}_{L^{2p}}\right] \\
&= \sum_{j\geq -1} 2^{j\alpha 2p}\int_{\mathbb{T}^d} \mathbb{E}\left[|\Delta_j\tilde{\xi}|^{2p}\right] \\
&\lesssim \sum_{j\geq -1} 2^{j\alpha 2p}\int_{\mathbb{T}^d} \mathbb{E}\left[|\Delta_j\tilde{\xi}|^{2}\right]^p
\end{aligned}$$

where the inequality is due to Gaussian hypercontractivity [11, Chapter 3, Theorem 3.50].
We calculate

$$\begin{aligned}
|\mathbb{E}\left[|\Delta_j\tilde{\xi}|^2\right]| &\leq \sum_{k_1,k_2\mathbb{Z}^d} \rho_j(k_1)\rho_j(k_2)\mathbb{E}\left[\hat{\tilde{\xi}}(k_1)\hat{\tilde{\xi}}(k_2)\right] \\
&= \sum_{k_1,k_2\in\mathbb{Z}^d} \rho_j(k_1)\rho_j(k_2)\mathbb{1}_{k_1+k_2=0} \\
&\leq 2^{jd}.
\end{aligned}$$

Using this we obtain

$$\mathbb{E}\left[\left\|\tilde{\xi}\right\|_{B^{\alpha}_{2p,2p}}^{2p}\right] \lesssim \sum_{j\geq -1} 2^{j\alpha 2p}2^{jpd} < \infty$$

provided that $2\alpha + d < 0$, i.e. $\alpha < -d/2$, and hence $\xi \in B^{\alpha}_{2p,2p}$.
Thus, using the Besov embedding theorem, we obtain $\tilde{\xi} \in \mathscr{C}^{\alpha - d/2p}$ for any $p > 1$.
Since $p > 1$ can be chosen arbitrarily large, this implies the claim. □

In the following we will identify ξ with $\tilde{\xi}$.

4 Regularity Results for Random Integral Operator

In this chapter we devote ourselves to the study of (random) operators given by (random) integral kernels. The goal is to prove regularity results in Hölder-Besov spaces for this kind of operators provided the integral kernels satisfies suitable regularity assumptions. As already mentioned, these kind of operators emerge in a natural way in the mild approach to certain SPDEs studied below.
The general set-up we will work in is as follows: For a positive time $T > 0$ and a measurable function $u : [0,T] \times \mathbb{T}^d \to \mathbb{C}$, we consider integral operators of the form

$$Vu(t,x) \coloneqq V_t u(x) \coloneqq \int_0^t \int_{\mathbb{T}^d} v(t,x;r,z)u(r,z)\mathrm{d}z\mathrm{d}r \text{ for } 0 \leq t \leq T.$$

where v is a (random) measurable function such that for any $0 < t \leq T$

$$v(t,\cdot,\cdot,\cdot) : \mathbb{T}^d \times [0,t) \times \mathbb{T}^d \to \mathbb{C}.$$

We say that the kernel v induces or gives rise to the operator V.
Moreover, we will impose impose the following conditions on the kernel:

1. $\|v(t,x;r,z)\|_{L^\infty_x L^\infty_z} < \infty$ for all $0 \leq r < t$
2. the kernel is time-homogeneous, i.e. for any $s > 0$ and $0 \leq r < t$ we have
$$v(t+s,x;r+s,z) = v(t,x;r,z)$$

We call a kernel satisfying these assumptions *admissible*.
As we will see, the class of admissible kernels provides a convenient framework for discussing the regularity of integral operators of the above form.
The rest of this chapter is structured as follows:

First, in a purely analytical step, we will establish the aforementioned regularity results given suitable control of the kernel. Here, probability is not taken into account.
Then we will show how to extend these results to a probabilistic setting and thus break the soil for applications in the study of singular SDPEs. We will especially be concerned with deriving assumptions that not only yield the regularity results but also allow feasible verification.

4.1 Integral Operators in Besov Spaces

An immediate consequence of admissibility is the following:

Lemma 4.1.1. *Let v be an admissible integral kernel and $u \in L^{\infty}$. If $0 \leq r < t$ and $k \in \mathbb{Z}^d$ then*

$$\mathscr{F}_x \left(\int_{\mathbb{T}^d} v(t,x;r,z)u(z)\mathrm{d}z \right)(k) = \int_{\mathbb{T}^d} \mathscr{F}_x \left(v(t,x;r,z) \right)(k) u(z) \mathrm{d}z$$

Proof. Since

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |v(t,x;r,z)||u(z)|\mathrm{d}z\mathrm{d}x \leq \mathrm{vol}(\mathbb{T}^d)^2 \left\| v(t,x,r,z) \right\|_{L^\infty_x L^\infty_z} \left\| u \right\|_{L^\infty} < \infty$$

we conclude that

$$\mathscr{F}_x \left(\int_{\mathbb{T}^d} v(t,x,r,z)u(z)\mathrm{d}z \right)(k) = \int_{\mathbb{T}^d} \mathscr{F}_x(v(t,x;r,z))(k)u(z)\mathrm{d}z$$

by the Fubini-Tonelli theorem. □

Proposition 4.1.2. *Let v be an admissible integral kernel and $u \in L^{\infty}$. Then if $0 \leq r < t$ for any $j \geq -1$ we have*

$$\Delta_j \left(\int_{\mathbb{T}^d} v(t,x;r,z)u(r,z)\mathrm{d}z \right) = \int_{\mathbb{T}^d} \Delta_j v(t,x;r,z)u(r,z)\mathrm{d}z.$$

Proof. The proposition follows immediately from an elementary computation using the last lemma and the fact that the sum appearing in the definition of the dyadic bloc is finite. □

The last proposition essentially provides us with a way to compute dyadic blocks of integral operators given rise to by admissible integral kernels. Thus we can calculate the Besov-norm of functions being defined in terms of these operators.
In order to prove the desired regularity results for integral operators, we also need to understand what role the domain of the operator plays in estimating Besov norms.
First note:

Lemma 4.1.3. *Let $f, g \in L^2$. Then*

$$\int_{\mathbb{T}^d} fg\mathrm{d}z = \sum_{j\geq -1} \int_{\mathbb{T}^d} \Delta_j fg\mathrm{d}z.$$

Proof. We first establish $S_i f \to f$ in L^2 as $i \to \infty$:
Note that

$$\widehat{S_i f}(k) = \sum_{j=-1}^{i-1} \widehat{\Delta_j f}(k) = \sum_{j=-1}^{i-1} \rho_j(k)\hat{f}(k)$$

from which we conclude

$$|\widehat{S_i f}(k)| \leq \sum_{j\geq -1}^{i-1} \rho_j(k)|\hat{f}(k)| \leq |\hat{f}(k)| \text{ and } \lim_{i\to\infty} \widehat{S_i f}(k) = \hat{f}(k).$$

Hence for all $i \in \mathbb{N}$ we obtain using Plancheral's theorem

$$\left\| (\widehat{S_i f}(k))_{\mathbb{Z}^d} \right\|_{\ell^2(\mathbb{Z}^d)} \leq \left\| (\hat{f}(k))_{k\in\mathbb{Z}^d} \right\|_{\ell^2(\mathbb{Z}^d)} = \|f\|_{L^2} < \infty$$

and applying the dominated convergence theorem we conclude

$$(\widehat{S_i f}(k))_{k\in\mathbb{Z}^d} \to (\hat{f}(k))_{\mathbb{Z}^d} \text{ in } \ell^2(\mathbb{Z}^d) \text{ as } i \to \infty.$$

Using Plancheral's theorem again, we obtain

$$\lim_{i\to\infty} \|S_i f - f\|_{L^2} = \lim_{i\to\infty} \left\| (\widehat{S_i f}(k) - \hat{f}(k))_{k\mathbb{Z}^d} \right\|_{\ell^2(\mathbb{Z}^d)} = 0$$

and finally

$$\lim_{i\to\infty} \|(S_i f - f)g\|_{L^1} \leq \lim_{i\to\infty} \|S_i f - f\|_{L^2} \|g\|_{L^2} = 0.$$

This implies

$$\sum_{j\geq -1} \int_{\mathbb{T}^d} \Delta_j fg\mathrm{d}z = \lim_{i\to\infty} \int_{\mathbb{T}^d} S_i fg\mathrm{d}z = \int_{\mathbb{T}^d} fg\mathrm{d}z.$$

□

Using this, we can indeed provide a way of taking the regularity of the domain of the integral operator into account:

Lemma 4.1.4. *Assume that $u \in \mathscr{C}^\alpha$ where $\alpha > 0$ is an arbitrary positive number and let $f \in \mathscr{S}'$ be a distribution. Then*

$$\left| \int_{\mathbb{T}^d} \Delta_j fu\mathrm{d}x \right| \lesssim_\alpha 2^{-j\alpha} \|u\|_\alpha \|\Delta_j f\|_{L^1}$$

Proof. First recall that $\Delta_i f \in \mathscr{S}$.
Lemma 4.1.3 implies that

$$\int_{\mathbb{T}^d} \Delta_j f u \mathrm{d}x = \lim_{n\to\infty} \sum_{i\geq -1}^{n} \int_{\mathbb{T}^d} \Delta_j f \Delta_i u \mathrm{d}x.$$

From

$$\hat{u}(k)^* = \left(\int_{\mathbb{T}^d} u(z) e_k(z) \mathrm{d}z\right)^* = \int_{\mathbb{T}^d} u(z)^* e_k^*(z) \mathrm{d}z = \widehat{u^*}(-k)$$

we conclude

$$(\Delta_j u)^* = \sum_{k\in\mathbb{Z}^d} e_k(x)\rho_j(k)\widehat{u^*}(k) = \Delta_j u^*$$

where we used that ρ_j is real and radially symmetric. Hence

$$(\mathscr{F}((\Delta_j u)^*)(k))^* = (\rho_j(k)\widehat{u^*}(k))^* = \mathscr{F}(\Delta_j u)(-k)$$

and therefore, Parseval's Theorem reads as

$$\begin{aligned}
&\int_{\mathbb{T}^d} \Delta_j f \Delta_i u \mathrm{d}x \\
=&\int_{\mathbb{T}^d} \Delta_j f \left((\Delta_i u)^*\right)^* \mathrm{d}x \\
=&\sum_k \mathscr{F}(\Delta_j f)(k) \mathscr{F}((\Delta_i u)^*)(k)^* \\
=&\sum_k \mathscr{F}(\Delta_j f)(k) \mathscr{F}(\Delta_i u)(-k)
\end{aligned}$$

Furthermore, the dyadic blocks satisfy

$$\operatorname{supp}(\mathscr{F}\Delta_j f) \cap \operatorname{supp}(\mathscr{F}\Delta_i u) = \emptyset \tag{4.1.5}$$

whenever $|i-j| > 1$. We conclude

$$\begin{aligned}
|\int_{\mathbb{T}^d} \Delta_j f u \mathrm{d}x| =& |\lim_{n\to\infty} \sum_{i=-1}^{n} \int_{\mathbb{T}^d} \Delta_j f \Delta_i u \mathrm{d}x| \\
=& |\lim_{n\to\infty} \sum_{i=-1}^{n} \sum_{k\in\mathbb{Z}^d} \mathscr{F}(\Delta_j f)(k) \mathscr{F}(\Delta_i u)(-k)| \\
=& |\sum_{k\in\mathbb{Z}^d} \mathscr{F}(\Delta_j f)(k) \left(\mathscr{F}(\Delta_{j-1} u)(-k) + \mathscr{F}(\Delta_j u)(-k) + \mathscr{F}(\Delta_{j+1} u)(-k)\right)| \\
=& |\int_{\mathbb{T}^d} \Delta_j f \Delta_{j-1} u \mathrm{d}x + \int_{\mathbb{T}^d} \Delta_j f \Delta_j u \mathrm{d}x + \int_{\mathbb{T}^d} \Delta_j f \Delta_{j+1} u \mathrm{d}x| \\
\lesssim& \left(\|\Delta_{j-1} u\|_{L^\infty} + \|\Delta_j u\|_{L^\infty} + \|\Delta_{j+1} u\|_{L^\infty}\right) \|\Delta_j f\|_{L^1}
\end{aligned}$$

where we used Parseval's theorem and (4.1.5) to cancel almost all summands with respect to i.
Using the estimate

$$\|\Delta_j u\|_{L^\infty} \leq 2^{-\alpha j} \|u\|_{\mathscr{C}^\alpha}$$

we obtain

$$\begin{aligned}
&|\int_{\mathbb{T}^d} u\Delta_j f \mathrm{d}x| \\
\leq&(2^{-\alpha(j-1)} \|u\|_{\mathscr{C}^\alpha} + 2^{-\alpha j} \|u\|_{\mathscr{C}^\alpha} + 2^{-\alpha(j+1)} \|u\|_{\mathscr{C}^\alpha}) \|\Delta_i f\|_{L^1} \\
\leq&(2^\alpha + 1 + 2^{-\alpha})2^{-j\alpha} \|u\|_{\mathscr{C}^\alpha} \|\Delta_j f\|_{L^1}.
\end{aligned}$$

□

In order to apply this in our study of regularity of integral operators we introduce *double dyadic blocks.*
In writing

$$\Delta_i v \Delta_j (t, x; r.z).$$

we mean the i-th resp j-th dyadic block with respect to $(\rho_j)_{j\geq -1}$ in the x resp. z-variable of v.
We have:

Lemma 4.1.6. *Let $T > 0$ and $\alpha > 0$, $\beta \in \mathbb{R}$ as well as $u \in C_T\mathscr{C}^\alpha$ and assume that v is an admissible integral kernel. Then for any $p \geq 1$ and $t \in [0, T]$ we have*

$$\|V_t u\|_{B^\beta_{p,p}} \lesssim \int_0^t \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \|u\|_{C_T\mathscr{C}^\alpha} \|\Delta_i v \Delta_j (t, x; r, z)\|_{L^p_x L^1_z} \mathrm{d}r.$$

Proof. Noting that

$$\sum_{j\geq -1} \int_{\mathbb{T}^d} \Delta_i v \Delta_j (t, x; r, z) u(r, z) \mathrm{d}z = \int_{\mathbb{T}^d} \Delta_i v(t, x; r, z) u(r, z) \mathrm{d}z$$

by lemma 4.1.3, we estimate using lemma 4.1.4

$$
\begin{aligned}
&\|V_t u\|_{B^\beta_{p,p}} \\
&= \left\| \int_0^t \int_{\mathbb{T}^d} v(t,x;r,z)u(r,z)\mathrm{d}z\mathrm{d}r \right\|_{B^\beta_{p,p}} \\
&\leq \int_0^t \left\| \int_{\mathbb{T}^d} v(t,x;r,z)u(r,z)\mathrm{d}z \right\|_{B^\beta_{p,p}} \mathrm{d}r \\
&= \int_0^t \left(\sum_{i\geq -1} 2^{ip\beta} \left\| \int_{\mathbb{T}^d} \Delta_i v(t,x,r,z)u(r,z)\mathrm{d}z \right\|^p_{L^p_x} \right)^{\frac{1}{p}} \mathrm{d}r \\
&\leq \int_0^t \sum_{i\geq -1} 2^{i\beta} \left\| \sum_{j\geq -1} \int_{\mathbb{T}^d} \Delta_i v \Delta_j (t,x;r,z)u(r,z)\mathrm{d}z \right\|_{L^p_x} \mathrm{d}r \\
&\lesssim \int_0^t \sum_{i\geq -1} 2^{i\beta} \left(\sum_{j\geq -1} 2^{-j\alpha} \|u\|_{C_T\mathscr{C}^\alpha} \left\| \int_{\mathbb{T}^d} |\Delta_i v \Delta_j (t,x;r,z)|\mathrm{d}z \right\|_{L^p_x} \right) \mathrm{d}r \\
&\leq \|u\|_{C_T\mathscr{C}^\alpha} \int_0^t \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \|\Delta_i v \Delta_j (t,x;r,z)\|_{L^p_x L^1_z} \mathrm{d}r.
\end{aligned}
$$

where to obtain the second inequality we used that $\|\cdot\|_{\ell^p} \leq \|\cdot\|_{\ell^1}$ for any $p \geq 1$. □

Remark 4.1.7. *We could have deduced a slightly stronger result if we would not had used the inequality $\|\cdot\|_{\ell^p} \leq \|\cdot\|_{\ell^1}$ in order to get rid of the exponent p. It seems, however, that this generalization is only of little use in practice.*

Thus we in fact reduced the question of regularity of V to a question about the regularity of the doubly dyadic blocks $\Delta_i v \Delta_j$ of the kernel.
The last lemma motivates the following definitions:

Definition 4.1.8. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$, as well as $1 \leq p \leq \infty$ and $T > 0$. Let v be an admissible kernel.*

1. *For $0 < t \leq T$ we write*

$$
\|v\|_{X^{\alpha,\beta;p}((0,t))} \coloneqq \int_0^t \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \|\Delta_i v \Delta_j (t,x;r,z)\|_{L^p_x L^1_z} \mathrm{d}r \quad (4.1.9)
$$

   and denote by $X^{\alpha,\beta;p}((0,t))$ the space of all admissible kernels v such that $\|v\|_{X^{\alpha,\beta;p}((0,t))}$ is finite.

2. *We write*

$$
\|v\|_{X^{\alpha,\beta;p}(T)} \coloneqq \sup_{0\leq t\leq T} \|v\|_{X^{\alpha,\beta;p}((0,t))}
$$

and denote by $X^{\alpha,\beta;p}(T)$ the space of all admissible kernels v such that $\|v\|_{X^{\alpha,\beta;p}(T)}$ is finite and such that for any $0 \le t \le T$

$$\lim_{s\nearrow t}\int_0^s \sum_{i,j\ge -1} 2^{-j\alpha}2^{i\beta}\left\|\Delta_i v\Delta_j(t,x;r,z)-\Delta_i v\Delta_j(s,x;r,z)\right\|_{L^p_xL^1_z}\,\mathrm{d}r=0$$

and

$$\lim_{t\searrow s}\int_0^s \sum_{i,j\ge -1} 2^{-j\alpha}2^{i\beta}\left\|\Delta_i v\Delta_j(t,x,r,z)-\Delta_i v\Delta_j(s,x;r,z)\right\|\mathrm{d}r=0$$

as well as

$$\lim_{t\searrow s}\int_s^t \sum_{i,j\ge -1} 2^{-j\alpha}2^{i\beta}\left\|\Delta_i v\Delta_j(t,x;r,z)\right\|\mathrm{d}r=0$$

3. *Finally, we write*

$$X^{\alpha,\beta;p}:=\bigcap_{T>0}X^{\alpha,\beta;p}(T).$$

Remark 4.1.10. *The conditions in* 2. *connected to a one-sided limit will be used to obtain continuity in time.*

These spaces enjoy the following basic properties:

Proposition 4.1.11. *Let $\alpha>0$ and $\beta\in\mathbb{R}$ as well as $1\le q\le p\le\infty$ and $T>0$.*

1. $\left(X^{\alpha,\beta;p}(T),\|\cdot\|_{X^{\alpha,\beta;p}(T)}\right)$ *is a normed vector space*
2. *For $v\in X^{\alpha,\beta;p}(T)$ we have*

$$\|v\|_{X^{\alpha,\beta;p}(T)}=\int_0^T \sum_{i,j\ge -1} 2^{-j\alpha}2^{i\beta}\left\|\Delta_i v\Delta_j(T,x;r,z)\right\|_{L^p_xL^1_z}\mathrm{d}r.$$

3. *The inclusion $X^{\alpha,\beta;p}(T)\subset X^{\alpha,\beta;q}(T)$ holds true, to be more precise*

$$\|\cdot\|_{X^{\alpha,\beta;q}(T)}\lesssim_d\|\cdot\|_{X^{\alpha,\beta;p}(T)}.$$

Proof. The first assertion is evident.
By definition we have

$$\int_0^T \sum_{i,j\ge -1} 2^{-j\alpha}2^{i\beta}\left\|\Delta_i v\Delta_j(T,x;r,z)\right\|_{L^p_xL^1_z}\mathrm{d}r\le\|v\|_{X^{\alpha,\beta;p}(T)}.$$

We calculate, using the time homogeneity of the kernel and the transformation $r \mapsto T - t + r$,

$$
\begin{aligned}
&\int_0^t \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j (t,x;r,z) \right\|_{L^p_x L^1_z} \mathrm{d}r \\
&= \int_0^t \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j (T-t+t,x;T-t+r,z) \right\|_{L^p_x L^1_z} \mathrm{d}r \\
&= \int_{T-t}^T \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j (T,x;r,z) \right\|_{L^p_x L^1_z} \mathrm{d}r \\
&\leq \int_0^T \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j (T,x,r,z) \right\|_{L^p_x L^1_z} \mathrm{d}r
\end{aligned}
$$

This proves the second claim.
Finally, noting that the inequality

$$
\left\| \Delta_i v \Delta_j (t,x;r,z) \right\|_{L^q_x L^1_z} \lesssim_d \left\| \Delta_i v \Delta_j (t,x;r,z) \right\|_{L^p_x L^1_z}
$$

holds true, we may conclude the last assertion. □

In the upcoming sections, the following subspaces of $X^{\alpha,\beta;p}(T)$ will be of great importance:

Definition 4.1.12. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$ as well as $\kappa \in [0,1)$, $\delta > 0$, $1 \leq p \leq \infty$ and $T > 0$.*

1. *We write $X^{\alpha,\beta;p}_{\kappa}(T)$ for the space of admissible integral kernels v such that*
$$
\sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j (T,x;r,z) \right\|_{L^p_x L^1_z} \lesssim (T-r)^{-\gamma}
$$
where the constant is independent of r.

2. *We write $X^{\alpha,\beta;p}_{\kappa,\delta}$ for the space of integral kernels $v \in X^{\alpha,\beta;p}_{\kappa}(T)$ such that in addition to the requirements in 1. for all $0 \leq r < s \leq t \leq T$ we have*
$$
\sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j (t,x;r,z) - \Delta_i v \Delta_j (s,x;r,z) \right\|_{L^p_x L^1_z} \lesssim (t-s)^{\delta} (s-r)^{-\kappa}
$$
where the constant is independent of t, s and r.

3. *We define*
$$
X^{\alpha,\beta;p}_{\kappa,\delta} := \bigcap_{T>0} X^{\alpha,\beta;p}_{\kappa,\delta}(T).
$$

These spaces enjoy the following basic properties:

Proposition 4.1.13. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ as well as $\kappa, \kappa' \in [0, 1)$, $p \geq 0$ and $T > 0$.*

1. *Both $X_{\kappa}^{\alpha,\beta;p}(T)$ and $X_{\kappa,\delta}^{\alpha,\beta;p}(T)$ are vector spaces*

2. *Let $v \in X_{\delta}^{\alpha,\beta;p}(T)$. Then for all $0 \leq r < t \leq T$ we have*

$$\sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j(t, x; r, z) \right\|_{L_x^p L_z^1} \lesssim (t - r)^{-\delta}.$$

3. *The inclusion $X_{\kappa,\delta}^{\alpha,\beta;p}(T) \subset X^{\alpha,\beta;p}(T)$ holds true, to be more precise, we have*

$$\|v\|_{X^{\alpha,\beta;p}(T)} \lesssim T^{1-\kappa}.$$

4. *If $\kappa' \leq \kappa$ the inclusion $X_{\kappa}^{\alpha,\beta;p}(T) \subset X_{\kappa'}^{\alpha,\beta;p}(T)$ holds true.*

Proof. The first assertion follows from straightforward calculations.
Let $v \in X_{\delta}^{\alpha,\beta;p}(T)$. Using the time-homogeneity of the kernel we conclude for $0 \leq r < t$

$$\Delta_i v \Delta_j(t, x; r, z) = \Delta_i v \Delta_j(T, x; T - t + r; z).$$

Consequently, we may estimate

$$\begin{aligned}
&\sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j(t, x; r, z) \right\|_{L_x^p L_z^1} \\
=&\sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j(T, x; T - t + r, z) \right\|_{L_x^p L_z^1} \\
\lesssim&(T - T + t - r)^{-\delta} \\
=&(t - r)^{-\delta}.
\end{aligned}$$

This proves the second statement.
Now let $v \in X_{\kappa,\delta}^{\alpha,\beta;p}(T)$. By definition $v \in X_{\kappa}^{\alpha,\beta;p}(T)$. Consequently

$$\begin{aligned}
&\|v\|_{X^{\alpha,\beta;p}(T)} \\
=&\int_0^T \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j(T, x; r, z) \right\|_{L_x^p L_z^1} \mathrm{d}r \\
\lesssim&\int_0^T (T - r)^{-\kappa} \mathrm{d}r \\
\lesssim&T^{1-\kappa}.
\end{aligned}$$

Moreover for $0 < t \leq T$ we have

$$\begin{aligned}&\int_0^s \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j(t,x;r,z) - \Delta_i v \Delta_j(s,x;r,z) \right\|_{L^p_x L^1_z} \mathrm{d}r \\ \lesssim &\int_0^s (t-s)^{\delta}(s-r)^{-\kappa} \mathrm{d}r \\ = &(t-s)^{\delta} s^{1-\kappa} = 0.\end{aligned}$$

and

$$\begin{aligned}&\int_s^t \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i\beta} \left\| \Delta_i v \Delta_j(t,x;r,z) \right\|_{L^p_x L^1_z} \mathrm{d}r \\ \lesssim &\int_s^t (t-r)^{-\kappa} \\ = &(t-s)^{1-\kappa}\end{aligned}$$

Now taking limits $s \nearrow t$ as well as $t \searrow s$ for the first term and the limit $t \searrow s$ for the second term we conclude $v \in X^{\alpha,\beta;p}(T)$.

The final claim follows from the elementary inequality

$$(t-r)^{\kappa-\kappa'} \leq T^{\kappa-\kappa'}$$

provided $0 \leq r < t \leq T$ which implies

$$(t-r)^{-\kappa'} \leq T^{\kappa-\kappa'}(t-r)^{-\kappa}.$$

□

The above introduced spaces of kernels provided a natural framework for the desired regularity results:

Theorem 4.1.14. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, as well as $p \geq 1$ and $T > 0$. Assume that for $0 < t \leq T$ we have $v \in X^{\alpha,\beta+d/p;p}((0,t))$.*

1. *We have*

$$\|V_t\|_{C_T \mathscr{C}^\alpha \to \mathscr{C}^\beta} \lesssim \|v\|_{X^{\alpha,\beta;p}((0,t))}.$$

2. *If instead $v \in X^{\alpha,\beta+d/p;p}(T)$, we have*

$$\|V\|_{C_T \mathscr{C}^\alpha \to C_T \mathscr{C}^\beta} \lesssim \|v\|_{X^{\alpha,\beta;p}(T)}.$$

Proof. Using proposition 4.1.6we estimate

$$\begin{aligned}&\|V_t\|_{C_T \mathscr{C}^\alpha \to B^{\beta+\frac{d}{p}}_{p,p}} \\ \lesssim &\int_0^t \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i(\beta+\frac{d}{p})} \left\| \Delta_i v \Delta_j(t,x;r,z) \right\|_{L^p_x L^1_z} \mathrm{d}r \\ \leq &\|v\|_{X^{\alpha,\beta+d/p;p}((0,t))}\end{aligned}$$

Now the Besov embedding (theorem 3.1.10)yields the first statement.
Next we turn to the second statement of the theorem.
We calculate

$$V_t u(x) - V_s u(x) =$$
$$\int_s^t \int_{\mathbb{T}^d} v(t,x;r,z)u(r,z)\mathrm{d}z\mathrm{d}r + \int_0^s \int_{\mathbb{T}^d} (v(t,x;r,z) - v(s,x;r,z))u(r,z)\mathrm{d}z\mathrm{d}r.$$

Similar arguments to the ones made in the proof of proposition 4.1.6 yield

$$\begin{aligned} &\|V_t u - V_s u\|_{B^{\beta+p/d}_{p,p}} \\ \lesssim &\|u\|_{C_T\mathscr{C}^\alpha} \int_0^s \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i(\beta+d/p)} \|\Delta_i v \Delta_j(t,x;r,z) - \Delta_i v\Delta_j(s,x;t,z)\|_{L^p_x L^1_z} \mathrm{d}r + \\ &\|u\|_{C_T\mathscr{C}^\alpha} \int_s^t \sum_{i,j\geq -1} 2^{-j\alpha} 2^{i(\beta+d/p)} \|\Delta_i v\Delta_j(t,x;r,z)\|_{L^p_x L^1_z} \mathrm{d}r. \end{aligned}$$

The first summand vanishes for each one-sided limit $s \nearrow t$ and $t \searrow s$ by assumption. Using the dominated convergence theorem one concludes that also the second summand vanishes for the limit $s \nearrow t$. For the limit $t \searrow s$ the term vanishes by assumption. Hence

$$\lim_{s\to t} \|V_t - V_s\|_{C_T\mathscr{C}^\alpha \to B^{\beta+\frac{d}{p}}_{p,p}} = 0$$

and, invoking the Besov embedding again, we conclude that

$$\|Vu\|_{C_T\mathscr{C}^\beta} \lesssim \|v\|_{X^{\alpha,\beta;p}(T)} \|u\|_{C_T\mathscr{C}^\alpha}.$$

□

Corollary 4.1.15. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$ as well as $\kappa \in [0,1)$, $\delta > 0$, $p \geq 0$ and $T > 0$.*

1. *If $v \in X^{\alpha,\beta;p}_\kappa(T)$, then for any $0 < t \leq T$*
$$\|V_t\|_{C_T\mathscr{C}^\alpha\to\mathscr{C}^\beta} \lesssim T^{1-\gamma}.$$
2. *If $v \in X^{\alpha,\beta;p}_{\kappa,\delta}(T)$ then*
$$\|V\|_{C_T\mathscr{C}^\alpha\to C_T\mathscr{C}^\beta} \lesssim T^{1-\gamma}.$$

Proof. The result follows from the last theorem and proposition 4.1.13. □

Later on, we will also need to understand in how far convergence of these kind of integral operators in operator norm can be reduced to convergence of their integral kernels. The basic statement in this context is the following:

Theorem 4.1.16. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$ as well as $1 \leq p \leq \infty$ and $T > 0$. Assume the for $n \in \mathbb{N}$ $v^n, v \in X^{\alpha,\beta+d/p;p}(T)$ such that*

$$v^n \to v \text{ in } X^{\alpha,\beta+d/p;p}(T) \text{ as } n \to \infty.$$

Then

$$V^n \to V \text{ in } L\left(C_T\mathscr{C}^\alpha, C_T\mathscr{C}^\beta\right) \text{ as } n \to \infty$$

Proof. Using the above theorem we obtain the bound

$$\|V^n - V\|_{C_T\mathscr{C}^\alpha \to C_T\mathscr{C}^\beta} \lesssim \|v^n - v\|_{X^{\alpha,\beta;p}(T)}$$

which implies the theorem. □

4.2 Random Integral Operators in Besov Spaces

In this section we will extend the above developed results to a probabilistic setting. To be more precise, in the following we will assume the integral kernels under consideration to be random with respect to a probability space $(\Omega, \mathscr{A}, \mathbb{P})$, such that for a random integral kernel v we have that $\mathbb{P}$-almost surely it is admissible. In the rest of this chapter all integral kernels will be assumed to be random and $\mathbb{P}$-almosut surely admissible. For simplicity, we will call these random integral kernels admissible as well.
Most of the results presented here have natural counterparts in the last section. After having proved the basic estimate and having adjusted the definitions, the statements can be proven mutatis mutandis. Therefore we refrain from giving these very apparent proofs.
The investigation starts with a rather technical observation:

Lemma 4.2.1. *Let $1 \leq q \leq p$ and $f\colon \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{C}$ be a random, measurable function. We then have*

$$\mathbb{E}\left[\|f(x,z)\|_{L^p_x L^q_z}\right] \leq \left\|\mathbb{E}\left[|f(x,z)|^p\right]^{\frac{1}{p}}\right\|_{L^p_x L^q_z}.$$

Proof. We have

$$
\begin{aligned}
&\mathbb{E}\left[\|f(x,z)\|_{L^p_x L^q_z}\right] \\
=&\mathbb{E}\left[\left(\int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d}|f(x,z)|^q \mathrm{d}z\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}\right] \\
\leq&\mathbb{E}\left[\int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d}|f(x,z)|^q \mathrm{d}z\right)^{\frac{p}{q}}\mathrm{d}x\right]^{\frac{1}{p}} \\
=&\left(\int_{\mathbb{T}^d}\left(\mathbb{E}\left[\left(\int_{\mathbb{T}^d}|f(x,z)|^q \mathrm{d}z\right)^{\frac{p}{q}}\right]\right)^{\frac{q}{p}\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}} \\
\leq&\left(\int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d}\mathbb{E}\left[|f(x,z)|^{q\frac{p}{q}}\right]^{\frac{q}{p}}\mathrm{d}z\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}} \\
=&\left\|\mathbb{E}\left[|f(x,z)|^p\right]^{\frac{1}{p}}\right\|_{L^p_x L^q_z}
\end{aligned}
$$

where in the first inequality, we used the Jensen inequality for concave functions and the second inequality is a Minkowski-type inequality for multiple integrals which is applicable since $p/q \geq 1$ (see [1, Chapter X, Theorem 6.21]). □

This leads to the following proposition:

Proposition 4.2.2. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$ as well as $p \geq 1$ and $T > 0$. Assume that v is an admissible integral kernel.*

1. *Let $0 < t \leq T$. Then*

$$
\mathbb{E}\left[\|v\|_{X^{\alpha,\beta;p}((0,t))}\right] \leq \int_0^t \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\mathbb{E}\left[|\Delta_i v \Delta_j(t,x,r,z)|^p\right]^{\frac{1}{p}}\right\|_{L^p_x L^1_z}\mathrm{d}r
$$

2. *We moreover have*

$$
\begin{aligned}
\mathbb{E}\left[\|v\|_{X^{\alpha,\beta;p}(T)}\right] &\leq \sup_{0\leq t\leq T}\int_0^t \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\mathbb{E}\left[|\Delta_i v \Delta_j(t,x;r,z)|^p\right]^{\frac{1}{p}}\right\|\mathrm{d}r \\
&= \int_0^T \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\mathbb{E}\left[|\Delta_i v \Delta_j(T,x;r,z)|^p\right]^{\frac{1}{p}}\right\|_{L^p_x L^1_z}\mathrm{d}r
\end{aligned}
$$

Proof. We have that

$$\begin{aligned}
&\mathbb{E}\left[\|v\|_{X^{\alpha,\beta;p}((0,t))}\right]\\
=&\mathbb{E}\left[\int_0^t \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\Delta_i v\Delta_j(t,x,r,z)\right\|_{L^p_x L^1_z}\,\mathrm{d}r\right]\\
=&\int_0^t \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\mathbb{E}\left[\left\|\Delta_i v\Delta_j(t,x;r,z)\right\|_{L^p_x L^1_z}\right]\mathrm{d}r\\
\leq&\int_0^t \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\mathbb{E}\left[|\Delta_i v\Delta_j(t,x;r,z)|^p\right]^{\frac{1}{p}}\right\|_{L^p_x L^1_z}\mathrm{d}r
\end{aligned}$$

where we used Fubini's theorem and lemma 4.2.1 for the inequality.
Recalling that

$$\|v\|_{X^{\alpha,\beta;p}(T)} = \int_0^T \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\Delta_i v\Delta_j(T,x;r,z)\right\|_{L^p_x L^1_z}\mathrm{d}r.$$

we deduce, following the first part of the proposition,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\Delta_i v\Delta_j(T,x;r,z)\right\|_{L^p_x L^1_z}\mathrm{d}r\right]\\
\leq&\int_0^T \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\mathbb{E}\left[|\Delta_i v\Delta_j(T,x;r,z)|^p\right]^{\frac{1}{p}}\right\|_{L^p_x L^1_z}\mathrm{d}r.
\end{aligned}$$

Noting that for any $0 < t \leq T$ we have

$$\begin{aligned}
&\int_0^t \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\mathbb{E}\left[|\Delta_i v\Delta_j(t,x;r,z)|^p\right]^{\frac{1}{p}}\right\|_{L^p_x L^1_z}\mathrm{d}r\\
\leq&\int_0^T \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\mathbb{E}\left[|\Delta_i v\Delta_j(T,x;r,z)|^p\right]^{\frac{1}{p}}\right\|_{L^p_x L^1_z}\mathrm{d}r
\end{aligned}$$

we conclude. □

These basic results motivate the following definitions:

Definition 4.2.3. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, as well as $p \geq 1$. Let v be an admissible integral kernel.*

1. *For $0 < t \leq T$ we write*

$$\|v\|_{\mathbb{X}^{\alpha,\beta;p}((0,t))} := \int_0^t \sum_{i,j\geq -1} 2^{-j\alpha}2^{i\beta}\left\|\mathbb{E}\left[|\Delta_i v\Delta_j(t,x;r,z)|^p\right]^{\frac{1}{p}}\right\|\mathrm{d}r \tag{4.2.4}$$

   and we denote by $\mathbb{X}^{\alpha,\beta;p}((0,t))$ the space of all admissible kernels v such that $\|v\|_{\mathbb{X}^{\alpha,\beta;p}((0,t))}$ is finite.

2. *We write*
$$\|v\|_{\mathbb{X}^{\alpha,\beta;p}(T)} := \sup_{0\le t\le T} \|v\|_{\mathbb{X}^{\alpha,\beta;p}((0,t))}$$
and denote by $\mathbb{X}^{\alpha,\beta;p}(T)$ *the space of all admissible kernels* v *such that* $\|v\|_{\mathbb{X}^{\alpha,\beta;p}(T)}$ *is finite and such that for any* $0 \le t \le T$
$$\lim_{s\nearrow t} \int_0^s \sum_{i,j\ge -1} 2^{-j\alpha} 2^{i\beta} \left\| \mathbb{E}\left[|\Delta_i v \Delta_j(t,x;r,z) - \Delta_i v\Delta_j(s,x;r,z)|^p\right]^{\frac{1}{p}} \right\|_{L^p_x L^1_z} \mathrm{d}r = 0$$
and
$$\lim_{t\searrow s} \int_0^s \sum_{i,j\ge -1} 2^{-j\alpha} 2^{i\beta} \left\| \mathbb{E}\left[|\Delta_i v \Delta_j(t,x,r,z) - \Delta_i v\Delta_j(s,x;r,z)|^p\right]^{\frac{1}{p}} \right\|_{L^p_x L^1_z} \mathrm{d}r = 0$$
as well as
$$\lim_{t\searrow s} \int_s^t \sum_{i,j\ge -1} 2^{-j\alpha} 2^{i\beta} \left\| \mathbb{E}\left[|\Delta_i v \Delta_j(t,x;r,z)|p\right]^{\frac{1}{p}} \right\| \mathrm{d}r = 0$$

We have the following basic properties:

Proposition 4.2.5. *Let* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *as well as* $p \ge 1$ *and* $T > 0$.

1. $(\mathbb{X}^{\alpha,\beta;p}(T), \|\cdot\|_{\mathbb{X}^{\alpha,\beta;p}(T)})$ *is a normed vector space*
2. *If* $v \in \mathbb{X}^{\alpha,\beta;p}(T)$, *then almost surely* $v \in X^{\alpha\beta;p}(T)$.

Proof. The first assertion is evident.
The second claim follows from immediately from proposition 4.2.2. □

Analogously to the non-random case, we introduce the following subspaces of $\mathbb{X}^{\alpha,\beta;p}(T)$:

Definition 4.2.6. *Let* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *as well as* $\kappa \in [0,1)$, $p \ge 1$ *and* $T > 0$.

1. *We write* $\mathbb{X}^{\alpha,\beta;p}_{\kappa}(T)$ *for the space of admissible kernels such that for* $0 \le t \le T$
$$\sum_{i,j\ge -1} 2^{-j\alpha} 2^{i\beta} \left\| \mathbb{E}\left[|\Delta_i v \Delta_j(T,x;r,z)|^p\right]^{\frac{1}{p}} \right\|_{L^p_x L^1_z} \lesssim (T-r)^{-\gamma}$$
where the constant is independent of r.
2. *We write* $\mathbb{X}^{\alpha,\beta;p}_{\kappa,\delta}$ *for the space of integral kernels* $v \in \mathbb{X}^{\alpha,\beta;p}_{\kappa}(T)$ *such that in addition to the requirements in* $0 \le r < s \le t \le T$ *we have*
$$\sum_{i,j\ge -1} 2^{-j\alpha} 2^{i\beta} \left\| \mathbb{E}\left[|\Delta_i v \Delta_j(t,x;r,z) - \Delta_i v\Delta_j(s,x;r,z)|^p\right]^{\frac{1}{p}} \right\|_{L^p_x L^1_z} \lesssim (t-s)^{\delta}(s-r)^{-\kappa}$$
where the constant is independent of t, s *and* r.

3. *Finally, we define*

$$\mathbb{X}^{\alpha,\beta;p}_{\kappa,\delta} := \bigcap_{T>0} \mathbb{X}^{\alpha,\beta;p}_{\kappa,\delta}(T).$$

For these spaces we have analogous statements to the non-random case studied above:

Proposition 4.2.7. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ as well as $\kappa, \kappa' \in [0,1)$, $\delta \geq$, $p \geq 1$ and $T > 0$.*

1. *We have*

$$\mathbb{X}^{\alpha,\beta;p}_{\kappa,\delta}(T) \subset \mathbb{X}^{\alpha,\beta;p}(T)$$

and moreover

$$\mathbb{E}\left[\|v\|_{X^{\alpha,\beta;p}(T)}\right] \lesssim T^{1-\kappa}.$$

2. *If $\kappa' \leq \kappa$, then*

$$\mathbb{X}^{\alpha,\beta;p}_{\kappa}(T) \subset \mathbb{X}^{\alpha,\beta;p}_{\kappa'}(T)$$

Proof. Using proposition 4.2.2 the proof follows from proposition 4.1.13 □

The spaces $\mathbb{X}^{\alpha,\beta;p}(T)$ provided a natural framework for regularity results for random integral operators given by random integral kernels in Besov spaces

Theorem 4.2.8. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, as well as $p \geq 1$ and $T > 0$. Assume that for $0 < t \leq T$ we have $v \in \mathbb{X}^{\alpha,\beta+d/p;p}((0,t))$.*

1. *We have*

$$\mathbb{E}\left[\|V_t\|_{C_T\mathscr{C}^\alpha \to \mathscr{C}^\beta}\right] \lesssim \|v\|_{\mathbb{X}^{\alpha,\beta;p}((0,t))}.$$

2. *If instead $v \in \mathbb{X}^{\alpha,\beta+d/p;p}(T)$, we have*

$$\mathbb{E}\left[\|V\|_{C_T\mathscr{C}^\alpha \to C_T\mathscr{C}^\beta}\right] \lesssim \|v\|_{\mathbb{X}^{\alpha,\beta;p}(T)}.$$

Proof. Recalling theorem 4.1.14 and proposition 4.2.2, the claim immediately follows. □

Corollary 4.2.9. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$ as well as $\kappa \in [0,1)$,$\delta > 0$, $p \geq 0$ and $T > 0$.*

1. *If $v \in \mathbb{X}^{\alpha,\beta;p}_{\kappa}(T)$, then for any $0 < t \leq T$*

$$\mathbb{E}\left[\|V_t\|_{C_T\mathscr{C}^\alpha \to \mathscr{C}^\beta}\right] \lesssim T^{1-\gamma}.$$

2. *If $v \in \mathbb{X}^{\alpha,\beta;p}_{\kappa,\delta}(T)$ then*

$$\mathbb{E}\left[\|V\|_{C_T\mathscr{C}^\alpha \to C_T\mathscr{C}^\beta}\right] \lesssim T^{1-\gamma}.$$

Proof. This follows from the last theorem and proposition 4.2.7 □

Theorem 4.2.10. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$ as well as $p \geq 1$ and $T > 0$. Assume the for $n \in \mathbb{N}$ $v^n, v \in \mathbb{X}^{\alpha,\beta;p}(T)$ such that*

$$v^n \to v \text{ in } \mathbb{X}^{\alpha,\beta;p}(T) \text{ as } n \to \infty.$$

Then

$$V^n \to V \text{ in } L\left(C_T\mathscr{C}^\alpha, C_T\mathscr{C}^\beta\right) \text{ in probability as } n \to \infty$$

Proof. This follows from theorem 4.1.16. □

5 The Parabolic Anderson Model

In this section we treat the (linear) parabolic Anderson model (PAM) in two dimension. We formally write this as the Cauchy problem

$$\partial_t u - \Delta u = u \cdot \xi \text{ on } [0,T] \times \mathbb{T}^2, \quad u(0,\cdot) = u^0. \tag{5.0.1}$$

where $T > 0$, u^0 is an initial datum and ξ is the space white noise on $\mathbb{T}^2$.
In the introduction we already discussed the difficulty of giving a proper meaning to this equation.
A natural approach to the PAM is to understand its solution as the limiting object of solutions to equations with smoothly approximated noise. Due to the heuristic analysis, we expect that in order to obtain a well-defined limit, we in fact need to renormalize the appearing equations in a suitable way, that is subtract large appropriate terms that drift to infinity as the approximations of the driving noise tends to the white noise.
The goal of this chapter is to rigorously derive an intrinsic formulation and show that the in this way obtained *renormalized equation* is globally well-posed. Moreover, we want to show that these solutions are indeed sensible: We will prove that the approximate solutions converge to the "intrinsic solutions" after a suitable renormalization.
This, however, is not only possible for the white noise but for a wider class of noises satisfying certain properties introduced below.
Our ansatz will be based on the mild formulation of this equation combined with a slight variation of the paracontrolled ansatz. In order to deal with the troubling product, we will - like already explained in the introduction - employ Bony's paraproduct.
First, we will be concerned with the simpler case of smooth driving noise.

5.1 Mild Formulation and Random Operators

For $T > 0$ we consider the Cauchy problem

$$\partial_t u - \Delta u = u\eta \text{ on } [0,T] \times \mathbb{T}^2, \quad u(0,\cdot) = u^0$$

where u_0 is an initial datum and η denotes a smooth driving noise not depending on time.
The mild formulation of this equations reads as

$$u(t) = \int_0^t P_{t-r}(u(r)\eta)\mathrm{d}r + P_t u^0. \tag{5.1.1}$$

The approach we will develop settles around this equation.
Using Bony's paraproduct and the resonant term, we may rewrite (5.1.1) as

$$u(t) = \int_0^t P_{t-r}(u(r) \prec \eta)\mathrm{d}r + \int_0^t P_{t-r}(u(r) \succeq \eta)\mathrm{d}r + P_t u^0.$$

This decomposition naturally leads to the following operators:

$$B_\prec(u,\eta)(t) := \int_0^t P_{t-r}(u(r) \prec \eta)\mathrm{d}r$$

as well as

$$B_\circ(u,\eta)(t) := \int_0^t P_{t-r}(u(r) \circ \eta)\mathrm{d}r$$

and

$$B_\succ(u,\eta)(t) := \int_0^t P_{t-r}(u(r) \succ \eta)\mathrm{d}r.$$

We moreover set $B_\succeq := B_\circ + B_\succ$ and

$$B_{\prec\succeq}(u,\eta,\eta)(t) := B_\succeq(B_\prec(u,\eta),\eta)(t).$$

We analogously define the operators $B_{\prec\circ}$ and $B_{\prec\succ}$.

Remark 5.1.2. *For notational convenience , we will in the following often omit the explicit time argument t when writing above the operators bi- and trilinear operators.*

These operators enjoy the following properties:

Proposition 5.1.3. *1. Let $\beta \in (0,2)$. Then*

$$\begin{aligned} B_\prec :& C_T L^\infty \times C_T \mathscr{C}^{\beta-2} \to \mathscr{L}_T^\beta \\ &(u,\eta) \mapsto \int_0^t P_{t-r}(u(r) \prec \eta(r))\mathrm{d}r \end{aligned}$$

is a well-defined and bounded operator.

2. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$. Then

$$\begin{aligned} B_\circ :& C_T \mathscr{C}^\alpha \times C_T \mathscr{C}^\beta \to \mathscr{L}_T^{\alpha+\beta+2} \\ &(u,\eta) \mapsto \int_0^t P_{t-r}(u(r) \circ \eta(r))\mathrm{d}r \end{aligned}$$

is a well-defined and bounded operator.

3. Let $\alpha \in \mathbb{R}, \beta < 0$. Then

$$\begin{aligned} B_{\succ} :& C_T\mathscr{C}^{\alpha} \times C_T\mathscr{C}^{\beta} \to \mathscr{L}_T^{\alpha+\beta+2} \\ &(u,\eta) \mapsto \int_0^t P_{t-r}(u(r) \succ \eta(r))\mathrm{d}r \end{aligned}$$

is a well-defined and bounded operator.

Proof. The proof is an application of the Schauder estimate (theorem 3.1.19) combined with the estimates for the paraproduct and the resonant term (proposition 3.2.4). □

Using this operator, we rewrite the considered equation as

$$u = B_{\prec}(u,\eta) + B_{\succeq}(u,\eta) + P_t u^0$$

and motivated by the paracontrolled ansatz, we define

$$u^{\sharp} := u - B_{\prec}(u,\eta).$$

Combining these two equation, we obtain

$$u^{\sharp} = B_{\prec\succeq}(u,\eta,\eta) + B_{\succeq}(u^{\sharp},\eta) + P_t u^0. \tag{5.1.4}$$

The following observation is a crucial motivation for our approach:

Proposition 5.1.5. *Let $T > 0$ as well as $\alpha > 0$ and assume that $u^0 \in \mathscr{C}^{\alpha}$. Then a function $u \in C_T\mathscr{C}^{\alpha}$ is a solution to the equation* (5.1.1) *if and only if $u^{\sharp}$ satisfies equation* (5.1.4)

Proof. Since η is smooth, all appearing paraproducts and resonat terms are well-defined.
Thus, if u is a solution to equation (5.1.1), then $u^{\sharp}$ clearly satisfies equation (5.1.4).
For the second implication, we consider

$$\begin{aligned} u - B_{\prec}(u,\eta) =& u^{\sharp} \\ =& B_{\prec\succeq}(u,\eta,\eta) + B_{\succeq}(u^{\sharp},\eta) + P_t u^0 \\ =& B_{\prec\succeq}(u,\eta,\eta) + B_{\succeq}(u,\eta) - B_{\succeq}(B_{\prec}(u,\eta),\eta) + P_t u^0 \\ =& B_{\succeq}(u,\eta) + P_t u^0 \end{aligned}$$

from which may conclude

$$u(t) = B_{\prec}(u,\eta)(t) + B_{\succeq}(u,\eta)(t) + P_t u^0 = \int_0^t P_{t-r}(u(r)\eta)\mathrm{d}r + P_t u^0.$$

□

Before we start paying attention to the aforementioned renormalization in the next section, we more carefully investigate the operator $B_{\prec\circ}$: We want to derive a representation of this operator as an integral operator. This will turn out to be useful later and will serve as an easy benchmark for our kernel based approach.
We start with the following observation:

Lemma 5.1.6. *Let $T > 0$. Assume that $u \in C_T L^\infty$. Then for any $0 \leq t \leq T$ and $k \in \mathbb{Z}^2$ we have*

$$\mathscr{F}\left(\int_0^t P_{t-r}(u(r) \prec \eta)\mathrm{d}r\right)(k) = \int_0^t \mathscr{F}(P_{t-r}(u(r) \prec \eta)(k)\mathrm{d}r$$

Proof. Since η is smooth, the paraproduct estimates (proposition 3.2.4) imply that $u \prec \eta \in C_T L^\infty$. Now

$$\int_{\mathbb{T}^2}\int_0^t |P_{t-r}(u(r) \prec \eta)(x)|\mathrm{d}r\mathrm{d}x \leq \mathrm{vol}(\mathbb{T}^2)T\, \|u \prec \eta\|_{C_T L^\infty} < \infty$$

and the claim follows from the Fubini-Tonelli theorem. □

Proposition 5.1.7. *Let $T > 0$ and assume $u \in C_T L^\infty$. Then for any $0 \leq t \leq T$ we have*

$$\int_0^t P_{t-r}(u(r) \prec \eta)\mathrm{d}r \circ \eta = \int_0^t P_{t-r}(u(r) \prec \eta) \circ \eta \mathrm{d}r.$$

Proof. Lemma 5.1.6 implies that for any $i \geq -1$

$$\Delta_i\left(\int_0^t P_{t-r}(u(r) \prec \eta)\mathrm{d}r\right) = \int_0^t \Delta_i P_{t-r}(u(r) \prec \eta)\mathrm{d}r.$$

Since for any $0 \leq r \leq t \leq T$ we have that $P_{t-r}(u(r) \prec \eta) \circ \eta$ is smooth (note that η is smooth) and

$$\sum_{i,j\geq -1:\ |i-j|\leq 1}^{n} \Delta_i P_{t-r}(u(r) \prec \eta)\Delta_j \eta \to P_{t-r}(u(r) \prec \eta) \circ \eta \text{ in } \mathscr{S} \text{ as } n \to \infty$$

we conclude

$$\begin{aligned}\int_0^t P_{t-r}(u(r) \prec \eta) \circ \eta \mathrm{d}r &= \int_0^t \lim_{n\to\infty} \sum_{|i-j|\leq 1}^{n} \Delta_i P_{t-r}(u(r) \prec \eta)\Delta_j \eta \mathrm{d}r \\ &= \lim_{n\to\infty} \int_0^t \sum_{|i-j|\leq 1}^{n} \Delta_i P_{t-r}(u(r) \prec \eta)\Delta_j \eta \mathrm{d}r \\ &= \lim_{n\to\infty} \sum_{|i-j|\leq 1}^{n} \Delta_i \int_0^t P_{t-r}(u(r) \prec \xi)\mathrm{d}r \Delta_j \eta \\ &= \int_0^t P_{t-r}(u(r) \prec \eta)\mathrm{d}r \circ \eta\end{aligned}$$

where the interchange of the limit and the integral is justified by dominated convergence. □

To represent the considered operator as an integral operator we define

$$v^\eta(t,x;r,z) \coloneqq \sum_{k_1,k_2,k_3\in\mathbb{Z}^2} e_k^*(x)e_{k_3}(z)m_\prec(k_3,k_2)m_\circ(k_1,k_2+k_3)P_{t-r}(k_2+k_3)\widehat{\eta}(k_1)\widehat{\eta}(k_2).$$

Before proving that this integral kernel indeed gives rise to the considered operator, we need to prove a result concerning the summability of this kernel:

Lemma 5.1.8. *For any $0 \le r \le t$ we have*

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} m_\prec(k_3,k_2)m_\circ(k_1,k_2+k_3)P_{t-r}(k_2+k_3)|\hat{\eta}(k_1)||\hat{\eta}(k_2)| < \infty.$$

Proof. Let $k' \in \mathbb{Z}^2$. Set $k_1+k_2+k_3 = k$. Using proposition 3.2.9 we calculate

$$\begin{aligned}
&\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} \mathbb{1}_{k=k'}m_\prec(k_3,k_2)m_\circ(k_1,k_2+k_3)P_{t-r}(k_2+k_3)\hat{\eta}(k_1)\hat{\eta}_{k_2}\\
\le &\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} \mathbb{1}_{k=k'}\mathbb{1}_{|k_3|\le|k_2|}\mathbb{1}_{|k_2|\lesssim|k_2+k_3|\lesssim|k_2|}\mathbb{1}_{|k_1|\lesssim|k_2+k_3|\lesssim|k_1|}P_{t-r}(k_2+k_3)\hat{\eta}(k_1)\hat{\eta}(k_2).
\end{aligned}$$

Since η is smooth, its Fourier coefficients decay faster than any polynomial. For an arbitrary number $N \in \mathbb{N}$ we may estimate the last sum - up to some constant - by

$$\begin{aligned}
&\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} \mathbb{1}_{k=k'}\mathbb{1}_{|k_3|<|k_2|}\mathbb{1}_{|k_1|\lesssim|k_2+k_3|\lesssim|k_1|}(1+|k_1|)^{-N}(1+|k_2|)^{-2N}\\
\le &\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} \mathbb{1}_{k=k'}\mathbb{1}_{|k_3|<|k_2|}\mathbb{1}_{|k_1|\lesssim|k_2+k_3|\lesssim|k_1|}(1+|k_1|)^{-N}(1+|k_2|)^{-N}(1+|k_3|)^{-N}\\
\le &\sum_{k_1\in\mathbb{Z}^2|k_1|\gtrsim|k'|} (1+|k_1|)^{-N} \sum_{k_2,k_3\in\mathbb{Z}^2} (1+|k_2|)^{-N}(1+|k_3|)^{-N}\\
\lesssim &(1+|k'|)^{-N+3}
\end{aligned}$$

where we used that for a constant $c > 0$ chosen suitably

$$\begin{aligned}
\sum_{k_1\in\mathbb{Z}^2:\,|k_1|\gtrsim|k'|} (1+|k_1|)^{-N} \lesssim &\int_{c|k'|}^\infty x(1+x)^{-N}\mathrm{d}x\\
\lesssim &(1+c|k'|)^{-N+3}\\
\lesssim &(1+|k'|)^{-N+3}
\end{aligned}$$

provided $N \in \mathbb{N}$ is large enough. We then have for any $K_1, K_2, K_3 \in \mathbb{N}$ the uniform bounds

$$\sum_{k_1,k_2,k_3 \in \mathbb{Z}^2 \colon |k_i| \le K_i} m_{\prec}(k_3, k_2) m_{\circ}(k_1, k_2 + k_3) P_{t-r}(k) |\hat{\eta}(k_1)||\hat{\eta}(k_2)|$$
$$\le \sum_{k' \in \mathbb{Z}^2} \sum_{k_1,k_2,k_3 \in \mathbb{Z}^2 \colon |k_i| \le K_i} \mathbb{1}_{k=k'} m_{\prec}(k_3, k_2) m_{\circ}(k_1, k_2 + k_3) P_{t-r}(k_2 + k_3) |\hat{\eta}(k_1)||\hat{\eta}(k_2)|$$
$$\lesssim \sum_{k' \in \mathbb{Z}^2} (1 + |k'|)^{-N+3} < \infty.$$

again, if N is large enough.
From this we conclude the summability claim letting the K_i's tend to infinity.
□

The last lemma implies that v^η is an admissible integral kernel (note that the time-homogenity trivially holds).
Now we are finally able to prove:

Theorem 5.1.9. *Let $T > 0$ and assume that $u \in C_T L^\infty$. Then for any $0 \le t \le T$ we have*

$$\int_0^t P_{t-r}(u(r) \prec \eta) \mathrm{d}r \circ \eta = \int_0^t \int_{\mathbb{T}^2} v^\eta(t, x; r, z) u(r, z) \mathrm{d}z \mathrm{d}r.$$

Proof. From proposition 5.1.7 we know that

$$\int_0^t P_{t-r}(u(r) \prec \eta) \mathrm{d}r \circ \eta = \int_0^t P_{t-r}(u(r) \prec \eta) \circ \eta \mathrm{d}r.$$

For arbitrary $k' \in \mathbb{Z}^2$ we calculate

$$\mathscr{F}\left(P_{t-r}(u(r) \prec \eta) \circ \eta\right)(k')$$
$$= \sum_{k_1,k_2,k_3 \in \mathbb{Z}^2} m_{\prec}(k_3, k_2) m_{\circ}(k_1, k_2 + k_3) P_{t-r}(k_2 + k_3) \hat{\eta}(k_1) \hat{\eta}(k_2) \hat{u}(r, k_3)$$
$$= \mathscr{F}_x \left(\int_{\mathbb{T}^2} v^\eta(t, x; r, z) u(r, z) \mathrm{d}r \right)$$

Since the above Fourier-coefficients are of rapid decay (proposition 2.0.8) we conclude

$$P_{t-r}(u(r) \prec \eta) \circ \eta = \int_{\mathbb{T}^2} v^\eta(t, x, r, z) u(r, z) \mathrm{d}z$$

which yields the assertion. □

Setting

$$V^\eta u(t, x) \coloneqq \int_0^t \int_{\mathbb{T}^2} v^\eta(t, x; r, z) u(r, z) \mathrm{d}z \mathrm{d}r$$

we are now able to write

$$B_{\prec\circ}(u, \eta, \eta) = I V^\eta(u).$$

5.2 The Renormalized PAM

In this section, we want to provide the announced notion of solution to the linear PAM with suitable driving noise, prove that such solutions exist and that solutions to the renormalized approximate equations converge to this solutions.
The formulation of the main result of this section is orientated towards the respective result in [7, Chapter 5, 5.5]. See also [8, Chapter 5].
In a first step, we will introduce a framework of general noises for which we can solve the PAM.
Then, in the next section, we will show that space white noise on the two-dimensional torus $\mathbb{T}^2$ indeed fits within this framework.
Motivated by the heuristic analysis made in the introduction, we assume in the following considerations - in order to be able to later set up a fixed point iteration - that $u \in C_T\mathscr{C}^\gamma$ for some positive time $T > 0$.
In view of proposition 5.1.4 we consider the equation

$$u^\sharp = B_{\prec\succeq}(u, \eta, \eta) + B_{\succeq}(u^\sharp, \eta) + P_t u_0.$$

Assuming a priori that $u^\sharp \in C_T\mathscr{C}^{2\gamma}$ for some $\gamma \in (2/3, 1)$, one easily sees using the estimates for these operator (proposition 5.1.3) that the second term is well-defined. Splitting the first operator into its two summands, we note:

1. $B_{\prec\succ}(u, \eta, \eta) \in \mathscr{L}_T^{3\gamma}$ using the above regularity results for the appearing operators
2. $B_{\prec\circ}(u, \eta, \eta)$ is not well-defined since $B_{\prec}(u, \eta) \in C_T\mathscr{C}^\gamma$ and $2\gamma - 2 \not> 0$

Thus we localized the singular behaviour of the SPDE in the term $B_{\prec\circ}(u, \eta, \eta)$.
We hope that after a suitable renormalization the troubling term yields a well-defined operator.
As we will need to renormalize the singular operator with a sequence of constants c_n, we introduce the operator

$$M_t u := f * u(t)$$

where $f = \sum_{k\in\mathbb{Z}^2} e_k^* \in \mathscr{S}'$ is a Schwartz distribution.
Provided that u is smooth in its spatial variables and $c \in \mathbb{C}$ we have

$$cM_t u(x) = cu(t, x).$$

since $\widehat{M_t u}(k) = \widehat{h * u}(k) = u(\hat{t})(k)$ for any $k \in \mathbb{Z}^2$ and we can apply proposition 2.0.8.
Moreover note that if $\alpha \in \mathbb{R}$ and $u \in C_T\mathscr{C}^\alpha$ then $M_t u(t) \in \mathscr{C}^\alpha$ due to the above Fourier computation, to be more precise, we have

$$\|M_t u\|_{\mathscr{C}^\alpha} = \|u(t)\|_{\mathscr{C}^\alpha}.$$

In the following we make a virtue out of the necessity of renormalization and will start by defining a notion of enhanced noise for which we can solve the PAM.

Definition 5.2.1. *(Mild PAM-enhancement) Let* $\gamma \in (2/3, 1)$.
Let $T > 0$. *We write* $(\eta, V) \in \mathscr{X}^\gamma(T)$ *if*

1. $\eta \in \mathscr{C}^{\gamma-2}$ *and* $(V_t)_{0<t\leq T}$ *is a collection of bounded operators in* $L(\mathscr{L}_t^\gamma; \mathscr{C}^{2\gamma-2})$ *such that* $s \mapsto V_s$ *is continuous in time on the interval* $(0,T]$ *in* $L\left(\mathscr{L}_T^\gamma, \mathscr{C}^{2\gamma}\right)$ *and* $IV \in L(\mathscr{L}_T^\gamma, \mathscr{L}_T^{2\gamma-})$
2. *there are sequences* $(\eta_n)_{n\geq 0}$ *in* C^∞ *and* $(c_n)_{n\geq 0}$ *in* $\mathbb{C}$ *such that*

$$\eta_n \to \eta \text{ in } \mathscr{C}^{\gamma-2} \text{ as } n \to \infty$$

and for all $0 < t \leq T$

$$I(V^{\eta_n} - c_n M) \to IV \text{ in } L(\mathscr{L}_T^\gamma, \mathscr{L}_T^{2\gamma-}) \text{ as } n \to \infty.$$

We moreover endow the space $\mathscr{X}^\gamma(T)$ *with the norm*

$$\|(\eta, V)\|_{\mathscr{X}^\gamma(T)} \coloneqq \|\eta\|_{\mathscr{C}^{\gamma-2}} + \|IV\|_{\mathscr{L}_T^\gamma \to \mathscr{L}_T^{2\gamma-}}$$

The distance induced by this will be denoted by $d_{\mathscr{X}^\gamma(T)}$.
Finally, we write

$$\mathscr{X}^\gamma = \bigcap_{T>0} \mathscr{X}^\gamma(T)$$

Remark 5.2.2. *Note that by assumption* $\left(\eta_n, V_n^\eta - c_n I\right)_{n\geq 0}$ *converges to* (η, V) *with respect to the distance* $d_{\mathscr{X}^\gamma(T)}$.

If $(\eta, V) \in \mathscr{X}^\gamma(T)$ for $T > 0$ we will in the following often use the notation

$$B_{\prec\diamond}(u, \eta, \eta) \coloneqq IVu$$

for $u \in \mathscr{L}_T^\gamma$.
The above notation is chosen to be reminiscent of the renormalized PAM derived in [8, Chapter 5] written as

$$\partial_t u = \Delta u + u \diamond \xi.$$

Here the term $u \diamond \xi$ is the renormalized product around which the approach to the PAM in [8] settles.

Remark 5.2.3. *In order to avoid awkward notation we will in the rest of this section without loss of generality assume that* $IV \in L(\mathscr{L}_T^\gamma, \mathscr{L}_T^{2\gamma})$, *i.e. the codomain of* IV *is* $\mathscr{L}_T^{2\gamma}$ *instead of* $\mathscr{L}_T^{2\gamma-}$.
The crucial point of the regularity issues dealt with in the enhanced noise

is that the regularity of the codomain of IV determines the regularity of $u^\sharp$ and if $\gamma \in (2/3, 1)$

$$B_{\succeq}(u^\sharp, \xi), \quad u \in \mathscr{L}_T^{2\gamma}$$

is well defined since $2\gamma + \gamma - 2 > 0$.
However, if $u^\sharp \in \mathscr{L}_T^{2\gamma-}$ the operator is still well-defined because there exists $\epsilon > 0$ such that $2\gamma - \epsilon + \gamma - 2 > 0$ and $u \in \mathscr{L}_T^{2\gamma-\epsilon}$.
As we do not want to deal with this nuisance, we choose this convenient simplification.

Proposition 5.2.4. *Let $T > 0$ and assume that $(\eta, V) \in \mathscr{X}^\gamma(T)$.*
Then for all $0 < T' \leq T$ we have that $(\eta, V|_{(0,T']}) \in \mathscr{X}^\gamma(T')$ and

$$\left\|(\eta, V|_{(0,T']})\right\|_{\mathscr{X}^\gamma(T')} \leq \|(\eta, V)\|_{\mathscr{X}^\gamma(T)}.$$

Remark 5.2.5. *Abusing the notation we write $(\eta, V) \in \mathscr{X}^\gamma(T')$*

Proof. We first need to check that $IV \in L(\mathscr{L}_{T'}^{2\gamma}, \mathscr{L}_{T'}^{2\gamma})$. This, however, follows immediately from the definition of the operator I.
Now let $u \in \mathscr{L}_{T'}^{\gamma}$. We define a function $\bar{u}$ by

$$\bar{u}(t,x) = u(t,x) \text{ if } t \in [0,T'] \text{ and } u(t,x) = u(T',x) \text{ else.} \tag{5.2.6}$$

where $x \in \mathbb{T}^2$ arbitrary.
One now readily checks that $\bar{u} \in \mathscr{L}_T^{2\gamma}$ and $\|\bar{u}\|_{\mathscr{L}_T^\gamma} = \|u\|_{\mathscr{L}_{T'}^\gamma}$. This shows

$$\|IV\|_{\mathscr{L}_{T'}^\gamma \to \mathscr{L}_{T'}^{2\gamma}} \leq \|IV\|_{\mathscr{L}_T^\gamma \to \mathscr{L}_T^{2\gamma}}$$

which allows us to conclude. □

The above framework now enables us to formulate a notion of solution for the PAM. Motivated by proposition 5.1.5 we define:

Definition 5.2.7. *Let $T > 0$ as well as $\gamma \in (2/3, 1)$. Assume that $(\eta, V) \in \mathscr{X}^\gamma(T)$. We say that $u \in \mathscr{L}_T^\gamma$ is a (local) mild solution to the renormalized PAM with noise (η, V) and initial datum $u_0 \in \mathscr{C}^{2\gamma}$ provided that $u^\sharp = u - B_{\prec}(u, \eta)$ satisfies*

$$u^\sharp = B_{\prec\diamond}(u, \eta, \eta) + B_{\prec\succ}(u, \eta, \eta) + B_{\succeq}(u^\sharp, \eta) + P_t u_0$$

If moreover $(\eta, V) \in \mathscr{X}^\gamma$ we call $u \in \mathscr{L}^\gamma$ a (global) mild solution to the renormalized PAM with noise (η, V) provided that for any $T > 0$ $u|_{[0,T]}$ is a (local) mild solution to the renormalized PAM with noise $(\eta, V) \in \mathscr{X}^\gamma(T)$.
Finally, we call u_n a (local) approximate solution to the renormalized PAM with noise $(\eta, V) \in \mathscr{X}^\gamma(T)$ and initial data $(u_n^0)_{n\geq 0}$ in $\mathscr{C}^{2\gamma}$ if $u_n^\sharp = u_n - B_{\prec}(u_n, \eta_n)$ satisfies

$$u_n^\sharp = B_{\prec\circ}(u_n, \eta_n, \eta_n) + B_{\prec\succ}(u_n, \eta_n, \eta_n) + B_{\succeq}(u_n^\sharp, \eta_n) + P_t u_n^0 - c_n IMu_n.$$

Similarly, we define global approximate solutions to the renormalized PAM.

Assume that $(\eta, V) \in \mathscr{X}^\gamma(T)$ and that u_n is a smooth approximate solution to the renormalized PAM. Then by definition

$$u_n^\sharp = B_{\prec\circ}(u_n, \eta_n, \eta_n) + B_{\prec\succ}(u_n, \eta_n, \eta_n) + B_{\succeq}(u_n^\sharp, \eta_n) + P_t u_n^0 - c_n I u_n$$

Noting that $u_n^\sharp = u_n - B_\prec(u, \eta_n)$ we obtain the equation

$$u_n = B_\prec(u_n, \eta_n) + B_\succeq(u_n, \eta_n) + P_t u_n^0 - c_n I u_n.$$

Since η_n is smooth we have by definition of $B_\prec, B_\succeq$

$$u_n(t) = \int_0^t P_{t-r}(u_n(r)\eta_n - c_n u_n(r))\mathrm{d}r + P_t u_n^0$$

which is the mild formulation of the Cauchy problem

$$\partial_t u_n = \Delta u_n + u_n \eta_n - c_n u_n \text{ on } [0,T] \times \mathbb{T}^2 \quad u_n(0,\cdot) = u_n^0(\cdot).$$

We now are able to formulate the main result of this section:

Theorem 5.2.8. *Let $\gamma \in (2/3, 1)$ and assume that $(\eta, V) \in \mathscr{X}^\gamma$. Let moreover $u^0 \in \mathscr{C}^{2\gamma}$. Then the renormalized PAM with noise η has a unique mild global solution $u \in \mathscr{L}^\gamma$*
Moreover, the mild solutions depends Lipschitz-continuously on the initial datum as well as the driving noise in the following sense:
Let $T > 0$ s well as $M > 0$ be a constant. Let $u_1^0, u_2^0 \in \mathscr{C}^{2\gamma}$ two initial conditions and $(\eta_1, V_1), (\eta_2, V_2) \in \mathscr{X}^\gamma$ two driving noises. Assume that

$$\|u_i^0\|_{\mathscr{C}^{\gamma-2}} + \|(\eta_i, V_i)\|_{\mathscr{X}^\gamma(T)} \leq M$$

for $i \in \{1,2\}$. Then, for the respective mild solution of the renormalized PAM u_1, u_2, the following holds true:

$$\|u_1 - u_2\|_{\mathscr{L}_T^\gamma} \lesssim_{T,M} \|u_0^1 - u_0^2\|_{\mathscr{C}^{\gamma-2}} + d_{\mathscr{X}^\gamma(T)}\Big((\eta_1, V_1), (\eta_2, V_2)\Big). \tag{5.2.9}$$

From this we can derive an immediate corollary:

Corollary 5.2.10. *Let $\gamma \in (2/3, 1)$ and assume that $(\eta, V) \in \mathscr{X}^\gamma$. Let $u^0 \in \mathscr{C}^{2\gamma}$ and assume that a sequence $(u_n^0)_{n\geq 0}$ in $\mathscr{C}^{2\gamma}$ converges to u^0 in $\mathscr{C}^{2\gamma}$ as n tends to infinity.*
Let $(u_n)_{n\geq 0}$ a sequence of approximate solutions with initial conditions $(u_n^0)_{n\geq 0}$. Then

$$u_n \to u \in \mathscr{L}^\gamma \text{ as } n \to \infty.$$

Proof. The proof is a consequence of the Lipschitz-continuity on the initial data of mild solutions to the renormalized PAM on the initial data as well as the assumption on $(u_n^0)_{n\geq 0}$ and on the approximate solutions. $\square$

We break down the proof of theorem 5.2.8 in several steps:

Lemma 5.2.11. *Let $T > 0$ as well as $\gamma \in (2/3, 1)$. Assume that we have $(\eta_1, V_1), (\eta_2, V_2) \in \mathscr{X}^\gamma(T)$ and $\|(\eta_i, V_i)\|_{\mathscr{X}^\gamma(T)} \leq M$ for $i \in \{1, 2\}$. Moreover assume that $u_1, u_2 \in \mathscr{L}_T^\gamma$ such that $\|u_1\|_{C_T\mathscr{C}^\gamma}, \|u_2\|_{C_T\mathscr{C}^\gamma} \leq C$. Then*

$$\|B_{\prec\diamond}(u_1, \eta_1, \eta_1) - B_{\prec\diamond}(u_2, \eta_2, \eta_2)\|_{\mathscr{L}_T^\gamma} + \|B_{\prec\succ}(u_1, \eta_1, \eta_1) - B_{\prec\succ}(u_2, \eta_2, \eta_2)\|_{\mathscr{L}_T^{2\gamma}}$$
$$\lesssim_{C,M} (1+T)^2 \|u_1 - u_2\|_{C_T\mathscr{C}^\gamma} + (1+T)^2 d_{\mathscr{X}^\gamma(T)}\Big((\eta_1, V_1), (\eta_2, V_2)\Big).$$

Proof. We estimate

$$\begin{aligned}
&\|B_{\prec\diamond}(u_1, \eta_1, \eta_1) - B_{\prec\diamond}(u_2, \eta_2, \eta_2)\|_{\mathscr{L}_T^{2\gamma}} \\
=&\|IV_1u_1 - IV_2u_2\|_{\mathscr{L}_T^{2\gamma}} \\
\leq&\|IV_1(u_1 - u_2)\|_{\mathscr{L}_T^{2\gamma}} + \|(IV_1 - IV_2)u_2\|_{\mathscr{L}_T^{2\gamma}} \\
\leq&\|IV_1\|_{\mathscr{L}_T^\gamma \to \mathscr{L}_T^{2\gamma}} \|u_1 - u_2\|_{\mathscr{L}_T^\gamma} + \|IV_1 - IV_2\|_{\mathscr{L}_T^\gamma \to \mathscr{L}_T^{2\gamma}} \|u_2\|_{\mathscr{L}_T^\gamma} \\
\leq&M\|u_1 - u_2\|_{\mathscr{L}_T^\gamma} + Cd_{\mathscr{X}^\gamma(T)}((\eta_1, V_1), (\eta_2, V_2)) \\
\lesssim_{M,C}&\|u_1 - u_2\|_{\mathscr{L}_T^\gamma} + d_{\mathscr{X}^\gamma(T)}((\eta_1, V_1), (\eta_2, V_2))
\end{aligned}$$

Moreover

$$\begin{aligned}
&\|B_{\prec\succ}(u_1, \eta_1, \eta_1) - B_{\prec\succ}(u_2, \eta_2, \eta_2)\|_{\mathscr{L}_T^{2\gamma}} \\
&\leq \|B_{\prec\succ}(u_1 - u_2, \eta_1, \eta_1)\|_{\mathscr{L}_T^{2\gamma}} + \|B_{\prec\succ}(u_2, \eta_1 - \eta_2, \eta_1)\|_{\mathscr{L}_T^{2\gamma}} + \|B_{\prec\succ}(u_2, \eta_2, \eta_1 - \eta_2)\|_{\mathscr{L}_T^{2\gamma}} \\
&\lesssim (1+T)^2\Big(\|u_1 - u_2\|_{C_T\mathscr{C}^\gamma} \|\eta_1\|_{\mathscr{C}^{\gamma-2}}^2 + \|u_2\|_{C_T\mathscr{C}^\gamma} \|\eta_1 - \eta_2\|_{\mathscr{C}^{\gamma-2}} \|\eta_1\|_{\mathscr{C}^{\gamma-2}} + \\
&\|u_2\|_{C_T\mathscr{C}^\gamma} \|\eta_2\|_{\mathscr{C}^{\gamma-2}} \|\eta_1 - \eta_2\|_{\mathscr{C}^{\gamma-2}} \Big) \\
&\lesssim_{M,C} (1+T)^2 \|u_1 - u_2\|_{\mathscr{L}_T^\gamma} + (1+T)^2 d_{\mathscr{X}^\gamma(T)}((\eta_1, V_1), (\eta_2, V_2))
\end{aligned}$$

This implies the claim. □

Lemma 5.2.12. *Let $T > 0$ and $\gamma \in (2/3, 1)$ as well as $(\eta, V) \in \mathscr{X}^\gamma(T)$, $u \in \mathscr{L}_T^\gamma$ and $u^0 \in \mathscr{C}^{2\gamma}$. Assume that $u^\sharp \in \mathscr{L}_T^{2\gamma}$ solves*

$$u^\sharp = B_{\prec\diamond}(u, \eta, \eta) + B_{\prec\succ}(u, \eta, \eta) + B_{\succeq}(u^\sharp, \eta) + P_t u^0$$

Then

$$\left\|u^\sharp\right\|_{\mathscr{L}_T^{2\gamma}} \lesssim_T \|u\|_{\mathscr{L}_T^\gamma} + \left\|u^0\right\|_{\mathscr{C}^{2\gamma}} + \|(\eta, V)\|_{\mathscr{X}^\gamma(T)}.$$

Proof. Let $T' \leq T$. By assumption

$$
\begin{aligned}
&\left\| u^\sharp \right\|_{\mathscr{L}^{2\gamma}_{T'}} \\
&\leq \| B_{\prec\diamond}(u,\eta,\eta)\|_{\mathscr{L}^{2\gamma}_{T'}} + \| B_{\prec\succ}(u,\eta,\eta)\|_{\mathscr{L}^{2\gamma}_{T'}} + \left\| B_{\succeq}(u^\sharp,\eta)\right\|_{\mathscr{L}^{2\gamma}_{T'}} + \left\| P_t u^0 \right\|_{\mathscr{L}^{2\gamma}_{T'}} \\
&\lesssim \| IV \|_{\mathscr{L}^{\gamma}_{T'} \to \mathscr{L}^{2\gamma}_{T'}} \| u \|_{\mathscr{L}^{\gamma}_{T'}} + (1+T')^2 \| \eta \|^2_{\mathscr{C}^{\gamma-2}} \| u \|_{C_T \mathscr{C}^{\gamma}} \\
&\quad + (1+T') \left\| u^\sharp \succeq \eta \right\|_{C_{T'}\mathscr{C}^{2\gamma-2}} + T' \left\| u^0 \right\|_{\mathscr{C}^{2\gamma}}
\end{aligned}
$$

Now let $2\epsilon > 0$ such that $3\gamma - 2 - 2\epsilon > 0$. We estimate

$$
\begin{aligned}
\left\| u^\sharp \succeq \eta \right\|_{C_{T'}\mathscr{C}^{2\gamma-2}} &\leq \left\| u^\sharp \succeq \eta \right\|_{C_{T'}\mathscr{C}^{3\gamma-2-2\epsilon}} \\
&\lesssim \left\| u^\sharp \right\|_{C_{T'}\mathscr{C}^{2\gamma-2\epsilon}} \| \eta \|_{\mathscr{C}^{\gamma-2}} \\
&\lesssim \left\| u^\sharp \right\|_{\mathscr{L}^{2\gamma-2\epsilon}_{T'}} \| \eta \|_{\mathscr{C}^{\gamma-2}} \\
&\lesssim T'^{\epsilon} \left\| u^\sharp \right\|_{\mathscr{L}^{2\gamma}_{T'}} \| \eta \|_{\mathscr{C}^{\gamma-2}} .
\end{aligned}
$$

We conclude

$$
\begin{aligned}
&\left\| u^\sharp \right\|_{\mathscr{L}^{2\gamma}_{T'}} \\
&\lesssim \left(\| (\eta, V) \|_{\mathscr{X}^{\gamma}(T)} + (1+T)^2 \| \eta \|_{\mathscr{C}^{\gamma-2}} \right) \| u \|_{\mathscr{L}^{\gamma}_{T}} + T \left\| u^0 \right\|_{\mathscr{C}^{2\gamma-2}} + (1+T) T'^{\epsilon} \left\| u^\sharp \right\|_{\mathscr{L}^{2\gamma}_{T'}}
\end{aligned}
$$

Now for sufficiently small but fixed T', which we may depending only on uniform constants and T, we derive

$$
\left\| u^\sharp \right\|_{\mathscr{L}^{2\gamma}_{T'}} \lesssim_T C \left(\| (\eta, V) \|_{\mathscr{X}^{\gamma}(T)}, \| u \|_{\mathscr{L}^{\gamma}_{T}}, \left\| u^0 \right\|_{\mathscr{C}^{2\gamma}} \right).
$$

Recall that by τ^S we denote the time-shift of a function.
Considering $\tau^{T'} u^\sharp$ and using $u^\sharp(T')$ as initial datum instead of u^0 we first note that on the time interval $[0, T - T']$ $\tau^{T'} u^\sharp$ solves the equation

$$
\tau^{T'} u^\sharp = B_{\prec\diamond}(\tau^{T'} u, \eta, \eta) + B_{\prec\succ}(\tau^{T'} u, \eta, \eta) + B_{\succeq}(\tau^{T'} u^\sharp, \eta) + P_t u^\sharp(T').
$$

We then obtain a similar estimates for $\left\| \tau^{T'} u^\sharp \right\|_{\mathscr{L}^{\gamma}_{T''}}$ where $0 < T'' \leq T'$. In view of proposition 3.1.15 we can now estimate

$$
\begin{aligned}
\left\| u^\sharp \right\|_{\mathscr{L}^{2\gamma}_{T'+T''}} &\lesssim \left\| u^\sharp \right\|_{\mathscr{L}^{2\gamma}_{T'}} + \left\| \tau^{T'} u^\sharp \right\|_{\mathscr{L}^{2\gamma}_{T''}} \\
&\lesssim_T C' \left(\| (\eta, V) \|_{\mathscr{X}^{\gamma}_{T}}, \| u \|_{\mathscr{L}^{\gamma}_{T}}, \left\| u^0 \right\|_{\mathscr{C}^{2\gamma}} \right).
\end{aligned}
$$

Iterating this argument up to time T and again using proposition 3.1.15 - note that the iteration terminates since T'' only needs to be smaller than T' if $T' + T'' > T$- the claim follows. □

Remark 5.2.13. *Many of the subsequent results will be established by employing similar iterative arguments. To avoid cumbersome and rather lengthy proofs, we will often refer to this technique by using the keywords "by iteration" and indicate how to establish the results on the "macroscopic" interval from the "microscopic" intervals.*

Next we will be concerned with the well-posedness of the equation of $u^\sharp$:

Lemma 5.2.14. *Let $\gamma \in (2/3, 1)$ and $T > 0$. Assume that $(\eta, V) \in \mathscr{X}^\gamma(T)$. Then for initial data $u \in C_T\mathscr{C}^\gamma$ and $u^0 \in \mathscr{C}^\gamma$ the equation*

$$u^\sharp = B_{\prec\diamond}(u, \eta, \eta) + B_{\prec\succ}(u, \eta, \eta) + B_{\succeq}(u^\sharp, \eta) + P_t u_0$$

has a unique solution in $\mathscr{L}_T^{2\gamma}$. Moreover, the solution depends Lipschitz-continuously on the initial data in the following sense: Given $M > 0$ as well as $u_0^1, u_0^2 \in \mathscr{C}^{2\gamma}$, $u_1, u_2 \in \mathscr{L}_T^\gamma$ and $(\eta_1, V_1), (\eta_2, V_2) \in \mathscr{X}^\gamma(T)$ such that

$$\|u_i\|_{\mathscr{L}_T^\gamma} + \|(\eta_i, V_i)\|_{\mathscr{X}^\gamma(T)} + \left\|u_i^0\right\|_{\mathscr{C}^{2\gamma}} \le M$$

for $i \in \{1, 2\}$, then

$$\left\|u_1^\sharp - u_2^\sharp\right\|_{\mathscr{L}_T^{2\gamma}} \lesssim_{T,M} \|u_1 - u_2\|_{\mathscr{L}_T^\gamma} + \left\|u_1^0 - u_2^0\right\|_{\mathscr{C}^{2\gamma}} + d_{\mathscr{X}^\gamma(T)}\Big((\eta_1, V_1), (\eta_2, V_2)\Big)$$

Proof. Let $0 < T' \le T$. Define the space $\mathscr{L}_{T'}^{2\gamma}(u^0) := \{v \in \mathscr{L}_{T'}^{2\gamma} \colon v(0) = u^0\}$. Since $\mathscr{L}_{T'}^{2\gamma}(u^0) \subset \mathscr{L}_{T'}^{2\gamma}$ is closed we conclude that - with the induced norm - $\mathscr{L}_{T'}^{2\gamma}(u^0)$ is a Banach space. Next consider the map

$$\begin{aligned}\Phi \colon & \mathscr{L}_{T'}^{2\gamma}(u^0) \to \mathscr{L}_{T'}^{2\gamma}(u^0), \\ & v \mapsto B_{\prec\diamond}(v, \eta, \eta) + B_{\prec\succ}(v, \eta, \eta) + B_{\succeq}(v, \eta) + P_t u^0.\end{aligned}$$

Since by assumption $3\gamma - 2 > 0$ for $v \in \mathscr{L}_T^{2\gamma}$ we conclude that $B_{\succeq}(v, \eta) \in \mathscr{L}_T^{3\gamma}$ is well defined and hence so is Φ.
We estimate

$$\begin{aligned}& \|\Phi(v_1) - \Phi(v_2)\|_{\mathscr{L}_{T'}^{2\gamma}} \\ = & \|B_{\succeq}(v_1 - v_2, \eta)\|_{\mathscr{L}_{T'}^{2\gamma}} \\ \lesssim & (1 + T') \|(v_1 - v_2) \succeq \eta\|_{C_{T'}\mathscr{C}^{2\gamma-2}}\end{aligned}$$

using the Schauder estimate.
Let $\epsilon > 0$ such that $3\gamma - 2 - 2\epsilon > 0$. Then

$$\begin{aligned}& \|\Phi(v_1) - \Phi(v_2)\|_{\mathscr{L}_{T'}^{2\gamma}} \\ \lesssim & (1 + T') \|(v_1 - v_2) \succeq \eta\|_{C_{T'}\mathscr{C}^{2\gamma-2}} \\ \le & (1 + T') \|(v_1 - v_2) \succeq \eta\|_{C_{T'}\mathscr{C}^{3\gamma-2-2\epsilon}} \\ \lesssim & (1 + T') \|v_1 - v_2\|_{\mathscr{L}_{T'}^{2\gamma-2\epsilon}} \|\eta\|_{\mathscr{C}^{\gamma-2}} \\ \lesssim & (1 + T')T'^\epsilon \|\eta\|_{\mathscr{C}^{\gamma-2}} \|v_1 - v_2\|_{\mathscr{L}_{T'}^{2\gamma}}.\end{aligned}$$

For small enough $T' > 0$, this map becomes a contraction and we obtain a unique solution to the equation on the time interval $[0, T']$ by Banach's fixed point theorem.
Iterating the process after adjusting the initial data -choosing $\tau^{T'} u^\sharp$ instead of u^0 as well as $\tau^{T'} u$ instead of u in a first step, then accordingly- and thus also the space we obtain a function $u^\sharp \in \mathscr{L}_T^{2\gamma}$ after gluing the piecewise solutions according to proposition 3.1.15.
By construction, this function is a solution to the considered equation.
Assume now that $u_1^\sharp, u_2^\sharp \in \mathscr{L}_T^{2\gamma}$ are two solutions corresponding to the initial data u_1, u_1^0 and u_2, u_2^0 as well as driving terms $(\eta_1, V_1), (\eta_2, V_2)$ which satisfy the above stated bound.
By Lemma we know that $\left\| u_2^\sharp \right\|_{\mathscr{L}_T^{2\gamma}} \leq C(T, M)$ for a constant $C(T, M) > 0$.
We have

$$\begin{aligned}
&\left\| u_1^\sharp - u_2^\sharp \right\|_{\mathscr{L}_{T'}^{2\gamma}} \\
\leq &\left\| B_{\prec\diamond}(u_1, \eta_1, \eta_1) - B_{\prec\diamond}(u_2, \eta_2, \eta_2) \right\|_{\mathscr{L}_{T'}^{2\gamma}} + \left\| B_{\prec\succ}(u_1, \eta_1, \eta_2) - B_{\prec\succ}(u_2, \eta_2, \eta_2) \right\|_{\mathscr{L}_{T'}^{2\gamma}} \\
&+ \left\| B_{\succeq}(u_1^\sharp, \eta_1) - B_{\succeq}(u_2^\sharp, \eta_2) \right\|_{\mathscr{L}_{T'}^{2\gamma}} + \left\| P_t u_1^0 - P_t u_2^0 \right\|_{\mathscr{L}_{T'}^{2\gamma}} \\
\lesssim_T &\left\| B_{\prec\diamond}(u_1, \eta_1, \eta_1) - B_{\prec\diamond}(u_2, \eta_2, \eta_2) \right\|_{\mathscr{L}_{T}^{2\gamma}} + \left\| B_{\prec\succ}(u_1, \eta_1, \eta_2) - B_{\prec\succ}(u_2, \eta_2, \eta_2) \right\|_{\mathscr{L}_{T}^{2\gamma}} \\
&+ \left\| B_{\succeq}(u_1^\sharp, \eta_1) - B_{\succeq}(u_2^\sharp, \eta_2) \right\|_{\mathscr{L}_{T'}^{2\gamma}} + \left\| P_t u_1^0 - P_t u_2^0 \right\|_{\mathscr{L}_{T}^{2\gamma}}.
\end{aligned}$$

The first two term may estimate according to the last lemma, the last term can be estimated using the Schauder estimate. Moreover, we may estimate for a suitable $\epsilon > 0$

$$\begin{aligned}
&\left\| B_{\succeq}(u_1^\sharp, \eta_1) - B_{\succeq}(u_2^\sharp, \eta_2) \right\|_{\mathscr{L}_{T'}^{2\gamma}} \\
\leq &\left\| B_{\succeq}(u_1^\sharp - u_2^\sharp, \eta_1) \right\|_{\mathscr{L}_{T'}^{2\gamma}} + \left\| B_{\succeq}(u_2^\sharp, \eta_2 - \eta_1) \right\|_{\mathscr{L}_{T'}^{2\gamma}} \\
\lesssim &(1 + T') T'^{\epsilon} \left\| u_1^\sharp - u_2^\sharp \right\|_{\mathscr{L}_{T'}^{\gamma}} \|\eta_1\|_{\mathscr{C}^{\gamma-2}} + \left\| u_2^\sharp \right\|_{\mathscr{L}_{T'}^{2\gamma}} \|\eta_1 - \eta_2\|_{\mathscr{C}^{\gamma-2}} \\
\lesssim &(1 + T) T'^{\epsilon} M \left\| u_1^\sharp - u_2^\sharp \right\|_{\mathscr{L}_{T'}^{2\gamma}} + C(T, M) \|\eta_1 - \eta_2\|_{\mathscr{C}^{\gamma-2}}.
\end{aligned}$$

For small enough $T' \leq T$ we obtain

$$\left\| u_1^\sharp - u_2^\sharp \right\|_{\mathscr{L}_{T'}^{2\gamma}} \lesssim_{T,M} \|u_1 - u_2\|_{\mathscr{L}_T^{\gamma}} + \left\| u_1^0 - u_2^0 \right\|_{\mathscr{C}^{2\gamma}} + d_{\mathscr{X}^\gamma(T)}\Big((\eta_1, V_1), (\eta_2, V_2) \Big).$$

Using proposition 3.1.15, which allows us to estimate $\|\cdot\|_{\mathscr{L}_T^{2\gamma}}$ in terms of $\|\cdot\|_{\mathscr{L}_S^{2\gamma}}$ and $\left\| \tau^S \cdot \right\|_{\mathscr{L}_{T-S}^{2\gamma}}$ for $0 < S \leq T$, we obtain the desired result by an iteration argument. □

Having gathered these results we are now finally able to prove the theorem:

Proof. (of Theorem) Let $T > 0$ arbitrary. By the above Lemma for any $u \in \mathscr{L}_T^\gamma$, $u^0 \in \mathscr{C}^{2\gamma}$ there exists a function $\Gamma(u, u^0, \eta) \in \mathscr{L}_T^{2\gamma}$ for which we have

$$\Gamma(u, u^0) = B_{\prec\diamond}(u, \eta, \eta) + B_{\prec\succ}(u, \eta, \eta) + B_{\succeq}(\Gamma(u, u^0), \eta) + P_t u^0. \quad (5.2.15)$$

Let $0 < T' \leq T$. Consider the linear map

$$\begin{aligned}\Psi :&\mathscr{L}_{T'}^\gamma \to \mathscr{L}_{T'}^\gamma \\ &v \mapsto B_\prec(v, \xi) + \Gamma(v, u^0).\end{aligned}$$

This is a well-defined, bounded operator since $B_\prec(v, \xi) \in \mathscr{L}_T^\gamma$ by prop and $\mathscr{L}_T^{2\gamma} \subset \mathscr{L}_{T'}^{2\gamma} \subset \mathscr{L}_{T'}^\gamma$ are continuous embeddings. We estimate for a suitable $\epsilon > 0$

$$\begin{aligned}&\|\Psi(v_1) - \Psi(v_2)\|_{\mathscr{L}_{T'}^\gamma} \\ \leq &\|B_\prec(v_1 - v_2, \eta)\|_{\mathscr{L}_{T'}^\gamma} + \left\|\Gamma(v_1, u^0) - \Gamma(v_2, u^0)\right\|_{\mathscr{L}_{T'}^\gamma} \\ \lesssim &(1 + T') \|(v_1 - v_2) \prec \eta)\|_{C_{T'}\mathscr{C}^{\gamma-2}} + T'^{\gamma/2} \left\|\Gamma_{T'}(v_1, u^0) - \Gamma_{T'}(v_2, u^0)\right\|_{\mathscr{L}_{T'}^{2\gamma}} \\ \lesssim &(1 + T') \|v_1 - v_2\|_{C_{T'}L^\infty} \|\eta\|_{\mathscr{C}^{\gamma-2}} + T'^{\gamma/2} \|v_1 - v_2\|_{\mathscr{L}_{T'}^\gamma} \\ \lesssim &(1 + T') \|\eta\|_{\mathscr{C}^{\gamma-2}} T'^{\epsilon} \|v_1 - v_2\|_{\mathscr{L}_{T'}^\gamma} + T'^{\gamma/2} \|v_1 - v_2\|_{\mathscr{L}_{T'}^\gamma}.\end{aligned}$$

For small enough T' we have that Ψ is a contraction and hence Banach's fixed point theorem yields a unique fixed point.
An iteration argument yields existence up to time T and hence there is a function $u \in \mathscr{L}_T^\gamma$ such that $u^\sharp := u - B_\prec(u, \eta) = \Gamma(u, u^0)$, i.e.

$$u^\sharp = B_{\prec\diamond}(u, \eta, \eta) + B_{\prec\succ}(u, \eta, \eta) + B_{\succeq}(u^\sharp, \eta) + P_t u^0.$$

Consequently, u is a mild solution of the renormalized PAM with noise (η, V) and since T was arbitrary (and hence can be chosen arbitrarily throughout the above calculation) we conclude that there exists a mild solution $u \in \mathscr{L}^\gamma$ to the PAM with noise (η, V).
Next, let $T > 0$ and $u_1^0, u_2^0 \in \mathscr{C}^{2\gamma}$ and $(\eta_1, V_1), (\eta_2, V_2) \in \mathscr{X}^\gamma$ such that

$$\left\|u_i^0\right\|_{\mathscr{C}^{2\gamma}} + \|(\eta_i, V_i)\|_{\mathscr{X}^\gamma(T)} \leq M. \quad (5.2.16)$$

for $i \in \{1, 2\}$.Let $u_1, u_2 \in \mathscr{L}_T^\gamma$ the solutions according to these initial data and driving noises.
Setting $u_i^\sharp = u_i - B_\prec(u_i, \eta_i)$ for $i \in \{1, 2\}$ we know by Lemma that

$$\left\|u_1^\sharp - u_2^\sharp\right\|_{\mathscr{L}_T^{2\gamma}} \lesssim_{T,M} \|u_1 - u_2\|_{\mathscr{L}_T^\gamma} + \left\|u_1^0 - u_2^0\right\|_{\mathscr{C}^{2\gamma}} + d_{\mathscr{X}^\gamma(T)}\left((\eta_1, V_1), (\eta_2, V_2)\right).$$

Moreover, we can estimate

$$\begin{aligned}
&\|B_{\prec}(u_1,\eta_1) - B_{\prec}(u_2,\eta_2)\|_{\mathscr{L}_T^\gamma} \\
&\le \|B_{\prec}(u_1-u_2,\eta_1)\|_{\mathscr{L}_T^\gamma} + \|B_{\prec}(u_2,\eta_1-\eta_2\|_{\mathscr{L}_T^{2\gamma}} \\
&\lesssim (1+T)\|u_1-u_2\|_{L^\infty}\|\eta_1\|_{\mathscr{C}^{\gamma-2}} + (1+T)\|u_2\|_{\mathscr{L}_T^\gamma}\|\eta_1-\eta_2\|_{\mathscr{C}^{\gamma-2}} \\
&\lesssim_M (1+T)T^\epsilon\|u_1-u_2\|_{\mathscr{L}_T^\gamma} + d_{\mathscr{X}^\gamma(T)}((\eta_1,V_1),(\eta_2,V_2)).
\end{aligned}$$

Putting both estimates together we obtain

$$\begin{aligned}
&\|u_1-u_2\|_{\mathscr{L}_T^\gamma} \le \|B_{\prec}(u_1,\eta_1) - B_{\prec}(u_2,\eta_2)\| + \left\|u_1^\sharp - u_2^\sharp\right\|_{\mathscr{L}_T^{2\gamma}} \\
&\lesssim_{T,M} (T^\epsilon + T^{\gamma/2})\|u_1-u_2\|_{\mathscr{L}_T^\gamma} + \left\|u_1^0 - u_2^0\right\|_{\mathscr{C}^{2\gamma}} + d_{\mathscr{X}^\eta(T)}((\eta_1,V_1),(\eta_2,V_2))
\end{aligned}$$

from which the desired result follows again from an iteration argument. □

5.3 Enhanced White Noise

Finally, in order to be able to solve the renormalized PAM with white noise as driving term, we need to show that indeed white noise fits within the framework of enhanced noise defined above.
Motivated by the results of the penultimate section, for $r > 0$ we want study the operator

$$V_r^\xi u := \int_0^r P_{r-s}(u(s) \prec \xi) \circ \xi \mathrm{d}r.$$

We hope that an anlysis of this operator provides us with a way to construct operators $(V_t)_{t>0}$ such that $V_t \in L\left(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma-2}\right)$ such that $(\eta, V) \in \mathscr{X}^\gamma(T)$ almost surely for any $T > 0$.
First, we want to find a representation of V^ξ using an integral kernel. Since in this case the noise is not smooth, we cannot directly apply theorem 5.1.9 but rather have to slightly alter the above made arguments in the following. Like above, setting $k = k_1 + k_2 + k_3$ we define

$$\begin{aligned}
&v^\xi(t,x;r,z) \\
:= &\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} e_k^*(x)e_{k_3}(z)m_\prec(k_3,k_2)m_\circ(k_1,k_2+k_3)P_{t-r}(k_2+k_3)\hat{\xi}(k_1)\hat{\xi}(k_2).
\end{aligned}$$

Lemma 5.3.1. *For any $0 \le r < t$ we have*

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} m_\prec(k_3,k_2)m_\circ(k_1,k_2+k_3)P_{t-r}(k_2+k_3)|\hat{\xi}(k_1)||\hat{\xi}(k_2)| < \infty.$$

Especially, v^ξ is an admissible integral kernel.

Proof. Recall that the Fourier transform of ξ is of at most polynomial growth, hence there is a natural number $N \in \mathbb{N}$ such that for any $k \in \mathbb{Z}^2$ we have $|\hat{\xi}(k)| \lesssim (1+|k|)^N$. Furthermore note that P_{t-r} is of rapid decay provided that $0 \le r < t$.
Set $k_1 + k_2 + k_3 = k$ and let $k' \in \mathbb{Z}^2$. For an arbitrary $M \in \mathbb{N}$ we obtain

$$\begin{aligned}
&\sum_{k_1,k_2,k_3 \in \mathbb{Z}^2} \mathbb{1}_{k=k'} m_\prec(k_3,k_2) m_\circ(k_1,k_2+k_3) P_{t-r}(k_2+k_3)|\hat{\xi}(k_1)||\hat{\xi}(k_2)| \\
\lesssim &\sum_{k_1,k_2,k_3 \in \mathbb{Z}^2} \mathbb{1}_{k=k'} \mathbb{1}_{|k_3|<|k_2|} \mathbb{1}_{|k_2|\lesssim|k_2+k_3|\lesssim|k_2|} \mathbb{1}_{|k_1|\lesssim|k_2+k_3|\lesssim|k_1|} \times \\
&(1+|k_2+k_3|)^{-3M}(1+|k_1|)^N(1+|k_2|)^N \\
\lesssim &\sum_{k_1 : \, |k'|\lesssim|k_1|} \sum_{k_2,k_3 \in \mathbb{Z}^2} (1+|k_1|)^{-M}(1+|k_2|)^{-M}(1+|k_3|)^{-M}(1+|k_1|)^N(1+|k_2|)^N \\
\lesssim &(1+|k'|)^{N+2-M}
\end{aligned}$$

for sufficiently large M where we used that

$$(1+|k_2+k_3|)^{-M} \le (1+c|k_2|)^{-M} \lesssim (1+|k_2|)^{-M} \le (1+|k_3|)^{-M}$$

and analogous estimates for the other terms. Now for large enough $M \in \mathbb{N}$ we have for any $K_1, K_2, K_3 \in \mathbb{N}$ a uniform bound

$$\begin{aligned}
&\sum_{k_1,k_2,k_3 \in \mathbb{Z}^2 : \, |k_i| \le K_i} m_\prec(k_3,k_2) m_\circ(k_1,k_2+k_3) P_{t-r}(k)|\hat{\xi}(k_1)||\hat{\xi}(k_2)| \\
\le &\sum_{k' \in \mathbb{Z}^2} \sum_{k_1,k_2,k_3 \in \mathbb{Z}^2 : \, |k_i| \le K_i} \mathbb{1}_{k=k'} m_\prec(k_3,k_2) m_\circ(k_1,k_2+k_3) P_{t-r}(k_2+k_3)|\hat{\xi}(k_1)||\hat{\xi}(k_2)| \\
\lesssim &\sum_{k' \in \mathbb{Z}^2} (1+|k'|)^{-M} < \infty.
\end{aligned}$$

This allows us to conclude.
Since v^ξ clearly is homogeneous in time the admissibility follows immediately. □

We now can prove:

Proposition 5.3.2. *Let $0 < \gamma < 1$ as well as $T > 0$ and $u \in C_T \mathscr{C}^\gamma$. Let $0 \le r < t \le T$ we have*

$$P_{t-r}(u(r) \prec \xi) \circ \xi = \int_{\mathbb{T}^2} v^\xi(t,x;r,z) u(r,z) \mathrm{d}z.$$

Proof. For any $0 \le r < t$ we know that $P_{t-r}(u(r) \prec \xi) \circ \xi$ is a smooth function .

Provided $0 \leq r < t$ we may conclude that for any $k' \in \mathbb{Z}^2$

$$
\begin{aligned}
&\mathscr{F}\Big(P_{t-r}(\xi \prec u(r)) \circ \xi\Big)(k') \\
=& \sum_{k_1,k_2,k_3 \in \mathbb{Z}^2} \mathbb{1}_{k_1+k_2+k_3=k'} m_{\prec}(k_3,k_2) m_{\circ}(k_1,k_2+k_3) P_{t-r}(k_2+k_3)\hat{\xi}(k_1)\hat{\xi}(k_2)\hat{u}(r,k_3) \\
=& \mathscr{F}\left(\int_{\mathbb{T}^2} v^{\xi}(t,x;r,z)u(r,z)\mathrm{d}z\right)(k')
\end{aligned}
$$

Since the Fourier-coefficients with respect to x are of rapid decay for $0 \leq r < t \leq T$ and thus equality of the Fourier coefficients implies equality of the functions and we may conclude

$$
\left(P_{t-r}(u(r) \prec \xi) \circ \xi\right)(x) = \int_{\mathbb{T}^2} v^{\xi}(t,x,r,z)u(r,z)\mathrm{d}z
$$

□

Thus we derived an expression for the operator $B_{\prec\circ}(\cdot,\xi,\xi)$ that can be -at least in principle - treated with the methods we obtained in Chapter 4. However, we have the following result:

Proposition 5.3.3. *The kernel $\mathbb{E}\left[v(t,x;r,z)\right]$ admits singular behaviour, to be more precise, for any $t > 0$*

$$
\int_0^t \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi}(t,x;r,z)\right]\mathrm{d}z\mathrm{d}r = \infty
$$

Proof. Fort $0 \leq r < t$ we calculate

$$
\begin{aligned}
&\mathbb{E}\left[v^{\xi}(t,x;r,z)\right] \\
=& \mathbb{E}\left[\sum_{k_1,k_2,k_3 \in \mathbb{Z}^2} e_k^*(x)e_{k_3}(z) m_{\prec}(k_3,k_2) m_{\circ}(k_1,k_2+k_3) P_{r-s}(k_2+k_3)\hat{\xi}(k_1)\hat{\xi}(k_2)\right] \\
=& \sum_{k_1,k_2,k_3 \in \mathbb{Z}^2} e_k^*(x)e_{k_3}(z) m_{\prec}(k_3,k_2) m_{\circ}(k_1,k_2+k_3) P_{r-s}(k_2+k_3)\mathbb{E}\left[\hat{\xi}(k_1)\hat{\xi}(k_2)\right] \\
=& \sum_{k_1,k_2,k_3 \in \mathbb{Z}^2} e_k^*(x)e_{k_3}(z) m_{\prec}(k_3,k_2) m_{\circ}(k_1,k_2+k_3) P_{r-s}(k_2+k_3)\mathbb{1}_{k_1+k_2=0} \\
=& \sum_{k,k' \in \mathbb{Z}^2} e_{k'}^*(x)e_{k'}(z) m_{\prec}(k',k) m_{\circ}(k,k+k') P_{r-s}(k+k').
\end{aligned}
$$

where the interchange of the expectation and the sums is justified by the

Fubini-Tonelli theorem since

$$\sum_{k_1,k_2,k_3} \mathbb{E}\left[m_{\prec}(k_3,k_2)m_{\circ}(k_1,k_2+k_3)P_{t-r}(k_2+k_3)|\hat{\xi}(k_1)\hat{\xi}(k_2)|\right]$$
$$\leq \sum_{k_1,k_2,k_3\in\mathbb{Z}^2} m_{\prec}(k_3,k_2)m_{\circ}(k_1,k_2+k_3)P_{t-r}(k_2+k_3)\mathbb{E}\left[|\hat{\xi}(k_1)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\hat{\xi}(k_2)|^2\right]^{\frac{1}{2}}$$
$$\leq \sum_{k_1,k_2,k_3\in\mathbb{Z}^2} m_{\prec}(k_3,k_2)m_{\circ}(k_1,k_2+k_3)P_{t-r}(k_2+k_3) < \infty$$

due to a slight modification of the proof of lemma 5.3.1.
We consider

$$\begin{aligned}
&\int_{\mathbb{T}^2} \mathbb{E}\left[v(t,x;r,z)\right]\mathrm{d}z \\
&= \int_{\mathbb{T}^2} \sum_{k,k'\in\mathbb{Z}^2} e^*_{k'}(x)e_{k'}(z)m_{\prec}(k',k)m_{\circ}(k,k+k')P_{t-r}(k+k')\mathrm{d}z \\
&= \sum_{k,k'\in\mathbb{Z}^2} e^*_{k'}(x)m_{\prec}(k',k)m_{\circ}(k,k+k')P_{t-r}(k+k')\int_{\mathbb{T}^2} e_{k'}(z)\mathrm{d}z \\
&= \sum_{k\in\mathbb{Z}^2} m_{\prec}(0,k)m_{\circ}(k,k)P_{t-r}(k) \\
&= \sum_{k\in\mathbb{Z}^2:\,1\lesssim|k|} e^{-(t-r)|k|^2}.
\end{aligned}$$

Here, the interchange of the sum and integral is justified by the Fubini-Tonelli theorem and the above estimate. We conclude

$$\begin{aligned}
&\int_0^t \sum_{k\in\mathbb{Z}^2:\,1\lesssim|k|} e^{-(t-r)|k|^2}\mathrm{d}r \\
&= \sum_{k\in\mathbb{Z}^2:\,1\lesssim|k|} \int_0^t e^{-(t-r)|k|^2}\mathrm{d}r \\
&= \sum_{k\in\mathbb{Z}^2:\,1\lesssim|k|} \frac{1-e^{t|k|^2}}{|k|^2} = \infty.
\end{aligned}$$

which proves the assertion. □

The last proposition especially implies that the operator V^{ξ} is singular. The strategy to show that we can enhance white noise is as follows:
First we will prove that after a suitable renormalization the singular operator in fact is well-behaved.+
Then we will show that smooth approximations of the noise give rise to integral kernels that - after a renormalization - converge to the renormalized

operator in a convenient topology.
Since this renormalization will not consist of sequence of constant but rather a sequence of (deterministic) functions, we will in a next step show that these deterministic functions can be renormalized with suitable sequence of constants thus proving the desired result.
The diverging integral motivates the definition of the following *renormalized kernel*:

$$r^{\xi}(t,x;r,z) := v^{\xi}(t,x;r,z) - \mathbb{E}\left[v^{\xi}(t,x;r,z)\right]$$

The random integral operator defined by this kernel will be denoted by R^{ξ}. In order to being able to handle the kernel more easily in the upcoming computations, we set

$$\begin{aligned} &b(t,x;r;k_3) \\ &:= \sum_{k_1,k_2\in\mathbb{Z}^2} e^*_{k_1+k_2+k_3}(x) m_{\prec}(k_3,k_2) m_{\circ}(k_1,k_2+k_3) P_{t-r}(k_2+k_3)\hat{\xi}(k_1)\hat{\xi}(k_2). \end{aligned}$$

In the proof of proposition 5.3.1 we provided the estimate

$$\sum_{k_1,k_2\in\mathbb{Z}^2} m_{\prec}(k_3,k_2) m_{\circ}(k_1,k_2+k_3) P_{t-r}(k_2+k_3)|\hat{\xi}(k_1)||\hat{\xi}(k_2)| \lesssim_N \frac{1}{(1+|k_3|)^N}$$

for any $k_3 \in \mathbb{Z}^2$, $0 \le r < t \le T$ and $N \in \mathbb{N}$.
Using this, we may rewrite the kernel as

$$r^{\xi}(t,x,r,z) = \sum_{k_3\in\mathbb{Z}^2} e^*_{k_3}(z) b(t,x,r;k_3)$$

due to an application of the Fubini Tonelli theorem.
We moreover use the abbreviation

$$\tilde{b}(t,x;r;k_3) := b(t,x;r;k_3) - \mathbb{E}\left[b(t,x;r;k_3)\right].$$

Lemma 5.3.4. *Let $0 \le r < t \le T$. For the double dyadic blocks of the renomarlized kernel the following holds true:*

$$\begin{aligned} &\Delta_i r^{\xi} \Delta_j(t,x;r,z) \\ &= \sum_{k_3\in\mathbb{Z}^2} e_{k_3}(z)\rho_j(k_3)\Delta_i \tilde{b}(t,x;r;k_3) \\ &= \sum_{k_3\in\mathbb{Z}^2} e_{k_3}(z)\rho_j(k_3)\left(\Delta_i b(t,x;r;k_3) - \mathbb{E}\left[\Delta_i b(t,x;r;k_3)\right]\right) \end{aligned}$$

Proof. To obtain the desired results one needs perform multiple applications of the Fubini-Tonelli theorem. These are justified by the bounds we derived for the appearing sums in the above. Moreover note that for $j \ge -1$ and $k_3 \in \mathbb{Z}^2$

$$\Delta_j \mathbb{E}\left[b(t,x;r,k_3)\right] = \mathbb{E}\left[\Delta_j b(t,x;r,k_3)\right]$$

which also follows from these arguments. □

Lemma 5.3.5. *1. Let $\gamma \in (\frac{1}{2}, 1)$. Then there is $\kappa \in [0,1)$ such that for sufficiently large $p > 2$ we have*

$$r^{\xi} \in \mathbb{X}_{\kappa}^{\gamma, 2\gamma-2+d/p;p}.$$

2. Moreover, there are numbers $0 < \kappa < \kappa'$ numbers $\delta > 0$ such that

$$r^{\xi} \in \mathbb{X}_{\kappa',\delta}^{\gamma, 2\gamma-2+d/p;p}.$$

Proof. Let $T > 0$. For arbitrary $2 < p < \infty$ we first note that, due to Gaussian hypercontractivity [11, Chapter 3, Theorem 3.50], we have

$$\mathbb{E}\left[|\Delta_i r^{\xi} \Delta_j(t,x;r,z)|^p\right]^{\frac{1}{p}} \lesssim_p \mathbb{E}\left[|\Delta_i r^{\xi} \Delta_j(t,x;r,z)|^2\right]^{\frac{1}{2}}.$$

for $0 \le r < t \le T$. Moreover, note that

$$\left\|\mathbb{E}\left[|\Delta_i r^{\xi} \Delta_j(t,x;r,z)|^2\right]^{\frac{1}{2}}\right\|_{L^1_z} \lesssim \left\|\mathbb{E}\left[|\Delta_i r^{\xi} \Delta_j(t,x;r,z)|^2\right]^{\frac{1}{2}}\right\|_{L^2_z}$$

which follows from Jensen's inequality.
In the following we set $k := k_1 + k_2 + k_3$, $k' := k_1' + k_2' + k_3$. Plancheral's theorem yields

$$\begin{aligned}
&\left\|\mathbb{E}\left[|\Delta_i r^{\xi} \Delta_j(t,x;r,z)|^2|\right]^{\frac{1}{2}}\right\|^2_{L^2_z}\\
=&\mathbb{E}\left[\int_{\mathbb{T}^2}\left|\sum_{k_3\in\mathbb{Z}^2} e^*_{k_3}(z)\rho_j(k)\Delta_i\tilde{b}(t,x;r;k_3)\right|^2 \mathrm{d}z\right]\\
=&\sum_{k_3\in\mathbb{Z}^2}\rho_j(k_3)^2\mathbb{E}\left[|\Delta_i\tilde{b}(t,x;r,z)|^2\right]\\
=&\sum_{k_3\in\mathbb{Z}^2}\rho_j(k_3)^2\mathrm{Var}\left[\Delta_i b(t,x;r;k_3)\right]\\
=&\sum_{k_3\in\mathbb{Z}^2}\rho_j(k_3)^2\sum_{k_1,k_1',k_2,k_2'\in\mathbb{Z}^2}\rho_i(k)\rho_i(k')e^*_k(x)e^*_{k'}(x)m_{\prec}(k_2,k_3)m_{\prec}(k_2',k_3)\times\\
&m_{\circ}(k_1,k_2+k_3)m_{\circ}(k_1',k_2'+k_3)P_{t-r}(k_2+k_3)P_{t-r}(k_2'+k_3)\mathrm{Cov}\left[\hat{\xi}(k_1)\hat{\xi}(k_2),\hat{\xi}(k_1')\hat{\xi}(k_2')\right].
\end{aligned}$$

Now Wick's theorem [11, Chapter 3, Theorem 1.28] implies

$$\mathrm{Cov}\left[\hat{\xi}(k_1)\hat{\xi}(k_2),\hat{\xi}(k_1')\hat{\xi}(k_2')\right] = \mathbb{1}_{k_1+k_1'=0}\mathbb{1}_{k_2+k_2'=0} + \mathbb{1}_{k_1+k_2'=0}\mathbb{1}_{k_1'+k_2=0}. \tag{5.3.6}$$

We split the following calculations in two parts corresponding to the two summands in the equation (5.3.6).In the following, the constant c may

change from line to line.
Using proposition 3.2.9, the first sum can be bounded as follows:

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} \rho_j(k_3)^2\rho_i(k)\rho_i(k')m_{\prec}(k_3,k_2)^2 m_{\circ}(k_1,k_2+k_3)m_{\circ}(k_1,k_2-k_3)\times$$

$$P_{t-r}(k_2+k_3)P_{t-r}(k_2-k_3)$$

$$\leq \sum_{k_1,k_2,k_3\in\mathbb{Z}^2} \rho_j(k_3)^2\rho_i(k)\rho_i(k')\mathbb{1}_{|k_3|<|k_2|}\mathbb{1}_{|k_2|\lesssim|k_2+k_3|\lesssim|k_2|}\mathbb{1}_{|k_1|\lesssim|k_2+k_3|\lesssim|k_1|}\times$$

$$\mathbb{1}_{|k_1|\lesssim|k_2-k_3|\lesssim|k_1|}e^{-(t-r)|k_2+k_3|^2}e^{-(t-r)|k_2-k_3|^2}$$

$$\lesssim \sum_{k_1:\,|k_1|\gtrsim\max(2^i,2^j)}\ \sum_{k_2,k_3\in\mathbb{Z}^2} \rho_j(k_3)\rho_i(k)\rho_i(k')e^{-2(t-r)c|k_1|^2}$$

$$\lesssim 2^{2i}2^{2j} \sum_{k_1:\,|k_1|\gtrsim\max(2^i,2^j)} e^{-(t-r)2c|k_1|^2}$$

$$\lesssim 2^{2i}2^{2j}(t-r)^{-1}e^{-(t-r)c2^{2i}}e^{-(t-r)c2^{2j}}$$

where we used that

$$\begin{aligned}\sum_{k:\,|k|\gtrsim\max(2^i,2^j)} e^{-(t-r)2c|k|^2} &\lesssim \int_{c'\max(2^i,2^j)}^{\infty} xe^{-(t-r)2cx^2}\mathrm{d}x\\ &\lesssim (t-r)^{-1}e^{-(t-r)2c\max(2^i,2^j)^2}\\ &\leq (t-r)^{-1}e^{-(t-r)c2^{2i}}e^{-(t-r)c2^{2j}}.\end{aligned}$$

For the second sum we note:

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}^2} \rho_j(k_3)^2\rho_i(k)\rho_i(k')m_{\prec}(k_3,k_2)m_{\prec}(k_3,k_1)m_{\circ}(k_1,k_2+k_3)\times$$

$$m_{\circ}(k_2,k_1-k_3)P_{t-r}(k_2+k_3)P_{t-r}(k_1-k_3)$$

$$\lesssim \sum_{k_1,k_2,k_3\in\mathbb{Z}^2} \rho_j(k_3)^2\rho_i(k)\rho_i(k')\mathbb{1}_{|k_3|<|k_2|}\mathbb{1}_{|k_2|\lesssim|k_2+k_3|\lesssim|k_2|}\mathbb{1}_{|k_3|<|k_1|}\mathbb{1}_{|k_1|\lesssim|k_1-k_3|\lesssim|k_1|}\times$$

$$\mathbb{1}_{|k_1|\lesssim|k_2+k_3|\lesssim|k_1|}\mathbb{1}_{|k_2|\lesssim|k_1-k_3|\lesssim|k_2|}P_{t-r}(k_2+k_3)P_{t-r}(k_1-k_3)$$

$$\lesssim \sum_{k_1\in\mathbb{Z}^2:\,|k_1|\gtrsim\max(2^i,2^j)}\ \sum_{k_2,k_3\in\mathbb{Z}^2} \rho_j^2(k_3)\rho_i(k)\rho_i(k')e^{-(t-r)2c|k_1|^2}$$

$$\lesssim 2^{2i}2^{2j}(t-r)^{-1}e^{-(t-r)c2^{2i}}e^{-(t-r)c2^{2j}}$$

by the same calculations carried out above.
We hence have for $0\leq r<t\leq T$

$$\left\|\mathbb{E}\left[|\Delta_i r^{\xi}\Delta_j(t,x;r,z)|^2\right]\right\|_{L_z^2}^2 \lesssim 2^{2i}2^{2j}(t-r)^{-1}e^{-(t-r)c2^{2i}}e^{-(t-r)c2^{2j}}.$$

Thus we may conclude

$$\left\|\mathbb{E}\left[|\Delta_i r^{\xi}\Delta_j(t,x;r,z)|^2\right]^{\frac{1}{2}}\right\|_{L_z^1L_x^\infty} \lesssim (t-r)^{-\frac{1}{2}}2^i e^{-(t-r)c2^i}2^j e^{-(t-r)c2^{2j}}.$$

Finally we consider

$$\begin{aligned}
&\sum_{i,j\geq -1} 2^{i(2\gamma-2+\frac{2}{p})}2^{-j\gamma}\left\|\mathbb{E}\left[|\Delta_i v\Delta_j(t,x;r,z)|^p\right]^{\frac{1}{p}}\right\|_{L^1_z L^\infty_x}\\
\lesssim &\sum_{i,j\geq -1} 2^{i(2\gamma-2+\frac{2}{p})}2^{-j\gamma}2^i 2^j e^{-(t-r)c2^{2i}}e^{-(t-r)c2^{2j}}(t-r)^{-\frac{1}{2}}\\
\lesssim &(t-r)^{-\frac{1}{2}}\int_{-1}^{\infty} 2^{x(2\gamma-1+\frac{d}{p})}e^{-(t-r)c2^{2x}}\mathrm{d}x\int_{-1}^{\infty} 2^{x(1-\gamma)}e^{-(t-r)c2^{2x}}\mathrm{d}x
\end{aligned}$$

The transformation $x\mapsto (t-r)^{\frac{1}{2}}2^x$ in the both integral leads -up to a constant- to

$$\begin{aligned}
&(t-r)^{-\frac{1}{2}}\int_{0}^{\infty}(t-r)^{-\frac{1}{2}(2\gamma-1+\frac{2}{p})}x^{2\gamma-2+\frac{2}{p}}e^{-cx^2}\mathrm{d}x\int_{0}^{\infty}(t-r)^{-\frac{1}{2}(1-\gamma)}x^{-\gamma+\epsilon}e^{-cx^2}\mathrm{d}x\\
\lesssim &(t-r)^{-\frac{1}{2}(\gamma+1+\frac{2}{p})}
\end{aligned}$$

where the integrals are finite provided that $\gamma>\frac{1}{2}-\frac{1}{p}$. If moreover $\gamma<1-\frac{2}{p}$ we have that $\gamma+1+\frac{2}{p}<2$. Setting $\kappa=(\gamma+1+\frac{2}{p})/2$, we conclude $r^\xi\in\mathbb{X}_\kappa^{\gamma-\epsilon,2\gamma-2;p}(T)$ for any $\gamma\in(1/2,1)$ which can be guaranteed by choosing p sufficiently large.
It remains to prove the second assertion. Again by Gaussian hypercontractivity we have respectively

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_i r^\xi\Delta_j(t,x;r,z)-\Delta_i r^\xi\Delta_j(s,x;r,z)|^p\right]^{\frac{1}{p}}\\
\lesssim_p &\mathbb{E}\left[|\Delta_i r^\xi\Delta_j(t,x;r,z)-\Delta_i r^\xi\Delta_j(s,x;r,z)|^2\right]^{\frac{1}{2}}
\end{aligned}$$

for $0\leq r<s\leq t\leq T$. In order to obtain the desired result we need to estimate

$$\begin{aligned}
&\left\|\mathbb{E}\left[|\Delta_i r^\xi\Delta_j(t,x;r,z)-\Delta_i r^\xi\Delta_j(s,x;r,z)|^2\right]^{\frac{1}{2}}\right\|_{L^2_z}^2\\
=&\left\|\mathbb{E}\left[|\sum_{k_3\in\mathbb{Z}^2}\rho_j(k_3)e^*_{k_3}(z)\left(\Delta_i\tilde{b}(t,x;r;k_3)-\tilde{b}(s,x;r;k_3)\right)|^2\right]^{\frac{1}{2}}\right\|_{L^2_z}^2\\
=&\sum_{k_3\in\mathbb{Z}^2}\rho_j(k_3)^2\mathbb{E}\left[|\tilde{b}(t,x;r;k_3)-\tilde{b}(s,x;r;k_3)|^2\right]\\
=&\sum_{k_3\in\mathbb{Z}^2}\rho_j(k_3)^2\mathrm{Var}\left[b(t,x;r;k_3)-b(s,x;r;k_3)\right]
\end{aligned}$$

for $0\leq r<s\leq t\leq T$.

Since

$$\begin{aligned}
&b(t,x;r;k_3)-b(s,x;r;k_3)\\
=&\sum_{k_1,k_2\in\mathbb{Z}^2} e_k^*(x)m_{\prec}(k_3,k_2)m_\circ(k_1,k_2+k_3)P_{t-r}(k_2+k_3)\hat{\xi}(k_1)\hat{\xi}(k_2)\\
&-\sum_{k_1,k_2\in\mathbb{Z}^2} e_k^*(x)m_{\prec}(k_3,k_2)m_\circ(k_1,k_2+k_3)P_{s-r}(k_2+k_3)\hat{\xi}(k_1)\hat{\xi}(k_2)\\
=&\sum_{k_1,k_2\in\mathbb{Z}^2} e_k^*(x)m_{\prec}(k_3,k_2)m_\circ(k_1,k_2+k_3)\left(P_{t-r}(k_2+k_3)-P_{s-r}(k_2+k_2)\right)\hat{\xi}(k_1)\hat{\xi}(k_2)
\end{aligned}$$

the above expression leads to sums being almost identical to the one we dealt with in the proof of the first part with the sole exception that $P_{t-r}(k_2+k_3)P_{t-r}(k_2'+k_3)$ is replaced by

$$\Big(P_{r-s}(k_2+k_3)-P_{t-r}(k_2+k_3)\Big)\Big(P_{s-r}(k_2'+k_3)-P_{t-r}(k_2'+k_3)\Big). \quad (5.3.7)$$

Noting that for arbitrary $0\le a\le b$ and any $\delta\in(0,1/2)$ the inequality

$$1-e^{-(b-a)x}\le(b-a)^\delta x^\delta$$

holds true for any $x\ge 0$ one concludes that (5.3.7) is -up to a constant - bounded by

$$(t-s)^{2\delta}|k_2+k_3|^\delta|k_2'+k_3|^\delta e^{-(s-r)(|k_2+k_3|^2+|k_2'+k_3|^2)}.$$

Since for each $k_3\in\mathbb{Z}^2$ we have for changing constants c, c'

$$\begin{aligned}
\sum_{k_1\in\mathbb{Z}^2}\mathbb{1}_{|k_1|\lesssim|k_3|}|k_1|^{2\delta}e^{-(s-r)2c|k_1|^2}&\lesssim\int_{c'|k_3|}^\infty x^{1+2\delta}e^{-(s-r)c2x^2}\mathrm{d}x\\
&\lesssim(s-r)^{-(1+\delta)}e^{-(s-r)c|k_3|^2}
\end{aligned}$$

we can put forward essentially the same computation like in proof of the first statement, i.e. invoking Wick's theorem etc. and end up with

$$\begin{aligned}
&\sum_{i,j\ge-1}2^{i(2\gamma-2+\frac{2}{p})}2^{-j\gamma}\left\|\mathbb{E}\left[\left|\Delta_i r^\xi\Delta_j(t,x;r,z)-\Delta_i r^\xi\Delta_j(s,x;r,z)\right|^p\right]^{\frac{1}{p}}\right\|_{L_z^1L_x^\infty}\\
\lesssim&(t-s)^\delta(s-r)^{-\frac{1}{2}(1+\gamma+\delta+\frac{2}{p})}.
\end{aligned}$$

for all $\gamma\in(1/2,1)$ after possibly haven chosen p sufficiently large. Since we may choose $p>1$ arbitrarily large, we can choose for a given $\gamma\in(1/2,1)$ the constant $\delta>0$ such that $1+\gamma+\delta+2/p<2$. Setting $\kappa'=(1+\gamma+\delta+2/p)/2$ the assertion follows: □

Corollary 5.3.8. *For any $T>0$ The operator*

$$R^\xi\colon C_T\mathscr{C}^\gamma\to C_T\mathscr{C}^{2\gamma-2}$$

is well-defined and bounded.

Proof. This follows immediately from the last lemma combined with corollary 4.2.9. □

Next we want to show that the operator R^ξ is the limit of suitably renormalized kernels of smooth approximations of the white noise.
In order to show this, we first need to choose a family of smooth approximations:
Let ϕ a smooth mollifier on $\mathbb{R}^2$, i.e. a symmetric function $\phi \in \mathscr{S}(\mathbb{R}^2)$ such that $\phi \geq 0$ and $\int_{\mathbb{R}^2} \phi(x)\mathrm{d}x = 1$. Set $\phi_n(\cdot) := n^2\phi(n\cdot)$ and define $\xi_n := \phi_n * \xi$.

The smooth approximations enjoy the following properties:

Proposition 5.3.9. *Let $k \in \mathbb{Z}^2$. Then*

$$\widehat{\xi_n}(k) = \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k}{n}\right)\hat{\xi}(k). \tag{5.3.10}$$

Proof. We calculate

$$\begin{aligned}
\widehat{\xi_n}(k) &= \sum_{k' \in \mathbb{Z}^2} \mathscr{F}\left(\langle \xi, n^2\phi(n(x + 2\pi k' - \cdot))\rangle\right)(k) \\
&= \sum_{k' \in \mathbb{Z}^2} \hat{\xi}(k)\mathscr{F}(n^2\phi(n(x + 2\pi k')(k) \\
&= \hat{\xi}(k) \sum_{k' \in \mathbb{Z}^2} \int_{\mathbb{T}^2} e_k(x)n^2\phi(n(x + 2\pi k'))\mathrm{d}x \\
&= \hat{\xi}(k) \sum_{k' \in \mathbb{Z}^2} \int_{\mathbb{T}^2} e_{k+2\pi k'}(x)n^2\phi(n(x + 2\pi k'))\mathrm{d}x \\
&= \hat{\xi}(k) \int_{\mathbb{R}^2} e_k(x)n^2\phi(nx)\mathrm{d}x \\
&= \hat{\xi}(k)\mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k}{n}\right)
\end{aligned}$$

where we used dominated convergence to combine the sum and the integral to an overall integration over $\mathbb{R}^2$. □

Corollary 5.3.11. *For $k_1, k_2 \in \mathbb{Z}^2$ we have*

$$\mathbb{E}\left[\widehat{\xi_n}(k_1)\widehat{\xi}(k_2)\right] = \mathbb{1}_{k_1+k_2=0}\mathscr{F}\phi\left(\frac{k_1}{n}\right)\mathscr{F}\phi\left(\frac{k_2}{n}\right)$$

Proof. This is an immediate consequence of the last proposition. □

Recall that in theorem 5.1.9 we associated an integral kernel $v^\eta$to each smooth noise η. We set $r^{\xi_n} := v^{\xi_n} - \mathbb{E}\left[v^{\xi_n}\right]$.

Lemma 5.3.12. *Let $\gamma \in (1/2, 1)$. Then there is $p > 2$ such that for any $T > 0$*

$$r^{\xi_n} \to r^{\xi} \text{ in } \mathbb{X}^{\gamma, 2\gamma-2+d/p;p}(T). \tag{5.3.13}$$

Proof. By definition

$$r^{\xi} - r^{\xi_n} = v^{\xi}(t,x;r,z) - v^{\xi_n}(t,x,r,z) - \mathbb{E}\left[v^{\xi}(t,x;r,z) - v^{\xi_n(t,x;r,z)}\right]. \tag{5.3.14}$$

Using Gaussian hypercontractivity, we obtain

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_i r^{\xi}\Delta_j(t,x;r,z) - \Delta_i r^{\xi_n}\Delta_j(t,x;r,z)|^p\right]^{\frac{1}{p}} \\
\lesssim_p &\mathbb{E}\left[|\Delta_i r^{\xi}\Delta_j(t,x;r,z) - \Delta_i r^{\xi_n}\Delta_j(t,x;r,z)|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Noting that if we set $k = k_1 + k_2 + k_2$ we have

$$\begin{aligned}
&\Delta_i r \Delta_j^{\xi}(t,x;r,z) - \Delta_i r \Delta_j^{\xi_n}(t,x;r,z) \\
= &\sum_{k_3 \in \mathbb{Z}^2} e_{k_3}(z)\rho_j(k_3) \sum_{k_1,k_2 \in \mathbb{Z}^2} e_k^*(x)\rho_i(k) m_{\prec}(k_3,k_2) m_{\circ}(k_1, k_2+k_3) P_{t-r}(k_2+k_3)\hat{\xi}(k_1)\hat{\xi}(k_2) \\
- &\sum_{k_3 \in \mathbb{Z}^2} e_{k_3}\rho_j(k_3)(z) \sum_{k_1,k_2 \in \mathbb{Z}^2} e_k^*(x)\rho_i(k) m_{\prec}(k_3,k_2) m_{\circ}(k_1, k_2+k_3) P_{t-r}(k_2+k_3)\widehat{\xi_n}(k_1)\widehat{\xi_n}(k_2) \\
= &\sum_{k_3 \in \mathbb{Z}^2} e_{k_3}(z)\rho_i(k_3) \sum_{k_1,k_2 \in \mathbb{Z}^2} \Bigg(e_k^*(x)\rho_i(k) m_{\prec}(k_3,k_2) m_{\circ}(k_1,k_2+k_3) P_{t-r}(k_2+k_3) \times \\
&\left(1 - \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k_1}{n}\right)\right)\left(1 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k_2}{n}\right)\right)\hat{\xi}(k_1)\hat{\xi}(k_2)\Bigg)
\end{aligned}$$

where we used proposition 5.3.9. We calculate

$$\begin{aligned}
&\left\|\mathbb{E}\left[\Delta_i r^{\xi}\Delta_j(t,x;r,z) - \Delta_i r^{\xi_n}\Delta_j(t,x;r,z)\right]\right\|_{L_z^1} \\
\lesssim &\left\|\mathbb{E}\left[\Delta_i r^{\xi}\Delta_j(t,x;r,z) - \Delta_i r^{\xi_n}\Delta_j(t,x;r,z)\right]\right\|_{L_z^2} \\
\lesssim &\sum_{k_3 \in \mathbb{Z}^2} \rho_j(k) \sum_{k_1,k_1',k_2,k_2' \in \mathbb{Z}^2} \rho_i(k)\rho_i(k') e_k^*(x) e_{k'}^*(x) m_{\prec}(k_2,k_3) m_{\prec}(k_2',k_3) \times \\
&m_{\circ}(k_1,k_2+k_3) m_{\circ}(k_1',k_2'+k_3) P_{t-r}(k_2+k_3) P_{t-r}(k_2'+k_3) \\
\times &\left(1 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k_1}{n}\right)\right)\left(1 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k_1'}{n}\right)\right)\left(1 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k_2}{n}\right)\right)\left(1 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k_2'}{n}\right)\right) \times \\
&\operatorname{Cov}\left[\hat{\xi}(k_1)\hat{\xi}(k_2), \hat{\xi}(k_1')\hat{\xi}(k_2')\right].
\end{aligned}$$

Recall that

$$\sup_{x \in \mathbb{R}^2} \mathscr{F}_{\mathbb{R}^2}\phi(x) < \infty, \quad \mathscr{F}_{\mathbb{R}^2}\phi(0) = 1.$$

In view of the proof of lemma 5.3.5 the last sum is - up to a constant - bounded by

$$\frac{1}{(t-r)^{1/2}} 2^i 2^j e^{-(t-r)c2^{2i}} e^{-(t-r)c2^{2j}}$$

and like above we may conclude for sufficiently large p and suitable $\kappa \in [0,1)$

$$\sum_{i,j\geq -1} 2^{-j\gamma} 2^{i(2\gamma-2+d/p))} \left\| \mathbb{E}\left[|\Delta_i r^{\xi} \Delta_j(t,x;r,z) - \Delta_i r^{\xi_n} \Delta_j(t,x;r,z)|^p \right]^{\frac{1}{p}} \right\|_{L^1_z L^\infty_z}$$
$$\lesssim (t-r)^{-\kappa}.$$

Since this is integrable against r on the intervall $[0,t)$ we can apply the dominated convergence theorem to conclude that r^{ξ_n} converges to r^{ξ} in $\mathbb{X}^{\gamma,2\gamma-2+2/p}(T)$ for any $T>0$ as n tends to infinity. □

Corollary 5.3.15. *For any $T>0$ and $\gamma \in (2/3,1)$*

$$R^{\xi_n} \to R^{\xi} \text{ in } L(\mathscr{L}^{\gamma}_T, \mathscr{L}^{2\gamma-2}_T) \text{ as } n \to \infty \text{ in probability.}$$

Proof. This follows from the last lemma as well as theorem 4.2.10. □

As already mentioned above, $\mathbb{E}\left[v^{\xi_n}\right]$ is not a constant and thus not a suitable renormalization in terms of the enhanced notion of noise we introduced above.
The aim of what follows is to correct this flaw and show, that we can also renormalize with a sequence of constants.
The proofs that follow will be structurally very similar. To avoid too lengthy arguments, we will spell out all details only in the first proof and indicate what to use in the later ones.
Before we proceed we need to introduce some notation:
We set

$$c_n := \sum_{k\in\mathbb{Z}^2} m_{\prec}(0,k) \frac{1}{|k|^2} \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k_2}{n}\right)^2$$

and for $u \in \mathscr{L}^{\gamma}_t$ we define

$$W^n_t u := \int_0^t \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right] u(r,z) \mathrm{d}z \mathrm{d}r$$

as well as

$$T^n_{1,t} u := \int_0^t \int_{\mathbb{T}} \mathbb{E}\left[v^{\xi}(t,x,r,z)\right] (u(r,z) - u(t,z)) \mathrm{d}z \mathrm{d}r$$

and

$$T^n_{2,t} u(x) := \int_{\mathbb{T}^2} u(t,z) \int_0^t \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right] \mathrm{d}r \mathrm{d}z - c_n u(t,x).$$

We clearly have

$$T^n_{1,t} + T^n_{2,t} = W^{\xi_n}_t - c_n M_t.$$

The ultimate goal of the following is to prove that $W^{n,t}$ converges in $L(\mathscr{L}^\gamma, \mathscr{C}^{2\gamma-2})$ for any $t > 0$.
Recall that

$$\begin{aligned}
&\mathbb{E}\left[v^\xi(t,x;r,z)\right] \\
&= \sum_{k\in\mathbb{Z}^2} e_k(z)e^*_k(x) \sum_{k'\in\mathbb{Z}^2} m_\prec(k,k')m_\circ(k,k+k')P_{t-r}(k+k')\mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)^2.
\end{aligned}$$

and consequently for $i,j \geq -1$ we have

$$\mathbb{E}\left[\Delta_i v^{\xi_n}\Delta_j(t,x;r,z)\right] \neq 0$$

only if $|i-j| \leq 1$.

Lemma 5.3.16. *For any $t > 0$ there exists an operator $T_{1,t} \in L\left(\mathscr{L}^\gamma_t, \mathscr{C}^{2\gamma-2}\right)$ such that*

$$T^n_{1,t} \to T_{1,t} \text{ in } L\left(\mathscr{L}^\gamma_t, \mathscr{C}^{2\gamma-2}\right) \text{ as } n\to\infty, \quad \|T_{1,t}\|_{\mathscr{L}^\gamma_t \to \mathscr{C}^{2\gamma-2}} \lesssim 1 \quad (5.3.17)$$

Proof. Let $t > 0$ and $u \in \mathscr{L}^\gamma_t$.
First, we deal with $(T^n_{1,t})_{n\geq 0}$. We estimate

$$\begin{aligned}
&\int_0^t \left\| \Delta_i \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right](u(r,z)-u(t,z))\mathrm{d}z \right\|_{L^\infty_x} \mathrm{d}r \\
\leq &\int_0^t \left\| \int_{\mathbb{T}^2} |\Delta_i \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right]||u(r,z)-u(t,z)|\mathrm{d}z \right\|_{L^\infty_x} .\mathrm{d}r
\end{aligned}$$

By assumption $\|u(r)-u(t)\|_{L^\infty} \leq \|u\|_{\mathscr{L}^\gamma_t}(t-r)^{\gamma/2}$. Thus we may estimate

$$\begin{aligned}
&\int_{\mathbb{T}^2} |\mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right]\|u\|_{\mathscr{L}^\gamma_t}(t-r)^{\gamma/2}\mathrm{d}z \\
\lesssim &\|u\|_{\mathscr{L}^\gamma_t}(t-r)^{\gamma/2}\left(\int_{\mathbb{T}^2} |\mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right]|^2\mathrm{d}z\right)^{\frac{1}{2}}
\end{aligned}$$

where we used Jensen's Inequality.
Now Plancheral's theorem implies

$$\begin{aligned}
&\int_{\mathbb{T}^2} |\Delta_i \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right]|^2 \mathrm{d}z \\
\lesssim &\sum_{k\in\mathbb{Z}^2} \rho_i(k)^2 \left(\sum_{k'} m_\prec(k,k')m_\circ(k',k+k')P_{t-r}(k+k')\mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{n}\right)\right)^2.
\end{aligned}$$

Note that for $k \in \mathbb{Z}^2$

$$
\begin{aligned}
&|\sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') P_{t-r}(k+k') \mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)| \\
&\lesssim \sum_{k' \in \mathbb{Z}^2} \mathbb{1}_{|k| \lesssim |k'|} e^{-(t-r)c|k'|^2} \\
&\lesssim \int_{c'|k|}^{\infty} x e^{-(t-r)cx^2} \mathrm{d}x \\
&\lesssim \frac{1}{(t-r)} e^{-(t-r)c''|k|^2}
\end{aligned}
$$

for suitable constants $c, c', c'' > 0$. For $i \geq -1$ we consequently obtain

$$
\begin{aligned}
&\int_0^t \left\| \Delta_i \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi_n}(t,x;r,z) \right] (u(r,z) - u(t,z)) \mathrm{d}z \right\|_{L_x^\infty} \mathrm{d}r \\
&\lesssim \|u\|_{\mathscr{L}_t^\gamma} \int_0^t (t-r)^{\gamma/2} \left(\sum_{k \in \mathbb{Z}^2} \rho_i(k)^2 \frac{1}{(t-r)^2} e^{-(t-r)2c''|k|^2} \right)^{1/2} \mathrm{d}r \\
&\lesssim \|u\|_{\mathscr{L}_t^\gamma} \int_0^t (t-r)^{\gamma/2} \left(2^{2i} \frac{1}{(t-r)^2} e^{-(t-r)c'''2^{2i}} \right)^{1/2} \mathrm{d}r \\
&\lesssim \|u\|_{\mathscr{L}_t^\gamma} 2^i \int_0^t \frac{(t-r)^{\gamma/2}}{(t-r)} e^{-(t-r)2^{2i}c'''/2} \mathrm{d}r \\
&\lesssim \|u\|_{\mathscr{L}_t^\gamma} 2^i 2^{-i\gamma} \int_0^\infty r^{\gamma/2-1} e^{-rc'''/2} \mathrm{d}r \\
&\lesssim \|u\|_{\mathscr{L}_t^\gamma} 2^{i(1-\gamma)}
\end{aligned}
$$

where we used the transformation $r \mapsto t-r$ as well as $r \mapsto 2^{2i} r$ and the fact that the last appearing integral is bounded.
Hence

$$
\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi_n}(t,x;r,z) \right] (u(r,z) - u(t,z)) \mathrm{d}z \mathrm{d}r \right\|_{\mathscr{C}^{2\gamma-2}} \\
&\leq \sup_{i \geq -1} 2^{i(2\gamma-2)} \int_0^t \left\| \Delta_i \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi_n}(t,x;r,z) \right] (u(r,z) - u(t,z)) \mathrm{d}z \right\|_{L_x^\infty} \mathrm{d}r \\
&\lesssim \sup_{i \geq -1} 2^{i(2\gamma-2)} \|u\|_{\mathscr{L}_t^\gamma} 2^{i(1-\gamma)} \\
&\lesssim \|u\|_{\mathscr{L}_t^\gamma}
\end{aligned}
$$

since $\gamma < 1$. Thus we established that for all all $n \geq 0$

$$
\left\| T_{1,t}^n \right\|_{\mathscr{L}_t^\gamma \to \mathscr{C}^{2\gamma-2}} \lesssim 1.
$$

Here the constant is independent of n and thus the sequence $(T^n_{1,t})_{n\geq 0}$ is bounded in $L\left(\mathscr{L}^\gamma_t, \mathscr{C}^{2\gamma-2}\right)$ for any $t > 0$.
Next, for $n, m \in \mathbb{N}$ we estimate

$$\begin{aligned}
&\left\|\left(T^n_{1,t} - T^m_{1,t}\right) u\right\|_{\mathscr{C}^{2\gamma-2}} \\
&= \sup_{i\geq -1} 2^{i(2\gamma-2)} \int_0^t \left\| \int_{\mathbb{T}^2} \left(\Delta_i \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right] - \Delta_i \mathbb{E}\left[v^{\xi_m}(t,x;r,z)\right]\right) |u(r,z) - u(t,z)| \mathrm{d}z \right\|_{L^\infty_x} \mathrm{d}r \\
&\leq \sup_{i\geq -1} 2^{i(2\gamma-2)} \|u\|_{\mathscr{L}^\gamma_t} \int_0^t \Big(\sum_{k\in\mathbb{Z}^2} \rho_i(k)^2 \big(\sum_{k'\in\mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k,k+k') P_{t-r}(k+k') \times \\
&\quad |\mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{n}\right)^2 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{m}\right)^2 | \big) \Big)^{1/2} (t-r)^{\gamma/2} \mathrm{d}r
\end{aligned}$$

Note that the above made arguments imply that

$$\begin{aligned}
&\sup_{0\leq t<\infty} \int_0^t \Big(\sum_{k\in\mathbb{Z}^2} \rho_i(k)^2 \big(\sum_{k'\in\mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k,k+k') P_{t-r}(k+k') \times \\
&|\mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{n}\right)^2 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{m}\right)^2 | \big) \Big)^{1/2} (t-r)^{\gamma/2} \mathrm{d}r \\
&\lesssim 2^{i(1-\gamma)}
\end{aligned}$$

Since for any fixed $L \in \mathbb{N}$ and $\epsilon > 0$ we may choose $K \in \mathbb{N}$ such that for $m, n \geq K$ we have

$$|\mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{n}\right)^2 - \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{m}\right)^2| < \epsilon$$

for $|k'| \leq L$ and noting that the above bound is indeed uniform in t we conclude that $\left(T^n_{1,t}\right)$ is a Cauchy sequence in $L(\mathscr{L}^\gamma_t, \mathscr{C}^{2\gamma-2})$ and consequently the sequence $(T^n_{1,t})_{n\geq 0}$ converges for any $t > 0$ to an operator $T_{1,t} \in L(\mathscr{L}^\gamma_t, \mathscr{C}^{2\gamma-2})$ in this space. This proves the assertion. □

Remark 5.3.18. *Note that the proof also implies that that for $\epsilon > 0$ we can choose $K \in \mathbb{N}$ such that for all $m, n \geq K$ we have*

$$\left\|T^n_{1,t} - T^m_{1,t}\right\|_{\mathscr{L}^\gamma_t, \mathscr{C}^{2\gamma-2}} < \epsilon$$

independent of t. Since this will also occur below we say that the Cauchy-property of $T^n_{1,t}$ is independent of time.

Next, we deal with the operators $(T^n_{2,t})_{n\geq 0}$. Consider

$$\begin{aligned}
&T^n_{2,t}\\
=&\int_{\mathbb{T}^2} u(t,z)\int_0^t \mathbb{E}\left[v^\xi(t,x;r,z)\right]\mathrm{d}r\mathrm{d}z - c_n M_t u\\
=&\int_{\mathbb{T}^2} u(t,z)\int_0^t \sum_{k\in\mathbb{Z}^2} e_k^*(x)e_k(z)\sum_{k'\in\mathbb{Z}^2} m_\prec(k,k')m_\circ(k',k+k')P_{t-r}(k+k')\mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)\mathrm{d}r\mathrm{d}z\\
&- c_n M_t u\\
=&\int_{\mathbb{T}^2} u(t,z)\sum_{k\in\mathbb{Z}^2} e_k^*(x)e_k(z)\sum_{k'\in\mathbb{Z}^2} m_\prec(k,k')m_\circ(k',k+k')\frac{1-e^{-t|k+k'|^2}}{|k+k'|^2}\mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)\mathrm{d}z\\
&- c_n M_t u
\end{aligned}$$

In order to be able to easier handle $T^n_{2,t}$, we define operator

$$S^n_{1,t}u := -\int_{\mathbb{T}^2} u(t,z)\sum_{k\in\mathbb{Z}^2} e_k^*(x)e_k(z)\sum_{k'\in\mathbb{Z}^2} m_\prec(k,k')m_\circ(k',k+k')\frac{e^{-t|k+k'|^2}}{|k+k'|^2}\mathrm{d}z$$

as well as

$$\begin{aligned}
&S^n_{2,t}u\\
:=&\int_{\mathbb{T}^2} u(t,z)\sum_{k\in\mathbb{Z}^2} e_k^*(x)e_k(z)\sum_{k'\in\mathbb{Z}^2} m_\prec(k,k')m_\circ(k',k+k')\frac{1}{|k'|^2}\mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{n}\right)\mathrm{d}z - c_n M_t u
\end{aligned}$$

Note that we have: $T^n_{2,t} = S^n_{1,t} + S^n_{2,t}$.

Lemma 5.3.19. *For any $t>0$ there exists an operator $S_{1,t}\in L\left(\mathscr{L}^\gamma_t,\mathscr{C}^{2\gamma-2}\right)$ such that for each $\epsilon>0$*

$$S^n_{1,t}\to S_{1,t}\in L\left(\mathscr{L}^\gamma_t,\mathscr{C}^{2\gamma-2}\right),\quad \|S_{1,t}\|_{\mathscr{L}^\gamma_t}\lesssim\frac{1}{t^\epsilon}$$

Proof. Note that for each $\epsilon>0$ we have

$$e^{-t|k+k'|^2}\lesssim\frac{1}{t^\epsilon|k+k'|^{2\epsilon}}$$

and thus for $k\in\mathbb{Z}^2$

$$\begin{aligned}
&\sum_{k'\in\mathbb{Z}^2} m_\prec(k,k')m_\circ(k',k+k')\frac{e^{-t|k+k'|^2}}{|k+k'|^2}\\
\lesssim&\sum_{k'\in\mathbb{Z}^2}\frac{1}{t^\epsilon}\mathbb{1}_{|k|<|k'|}\frac{1}{|k'|^{2+2\epsilon}}\\
\lesssim&\frac{1}{t^\epsilon}\frac{1}{|k|^{2\epsilon}}.
\end{aligned}$$

By the Fubini-Tonelli theorem we obtain for $j \geq -1$

$$\Delta_j \left(\int_{\mathbb{T}^d} u(t,z) \sum_{k \in \mathbb{Z}^2} e_k^*(x) e_k \sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') \frac{e^{-t|k+k'|^2}}{|k+k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi \left(\frac{k'}{n}\right)^2 \mathrm{d}z \right)$$

$$= \int_{\mathbb{T}^d} u(t,z) \Delta_j \left(\sum_{k \in \mathbb{Z}^2} e_k^*(x) e_k(z) \sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') \frac{e^{-t|k+k'|^2}}{|k+k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi \left(\frac{k'}{n}\right)^2 \right) \mathrm{d}z.$$

since $\mathscr{F}_{\mathbb{R}^2}\phi$ is of rapid decay.
Using Placheral's theorem, we estimate

$$\int_{\mathbb{T}^2} |\sum_{k \in \mathbb{Z}^2} \rho_j(k) e_k^*(x) e_k(z) \sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') \frac{e^{-t|k+k'|^2}}{|k+k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi \left(\frac{k'}{n}\right)^2 |^2 \mathrm{d}z$$

$$\lesssim \sum_{k \in \mathbb{Z}^2} \rho_j(k)^2 \left(\sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') \frac{e^{-t|k+k'|^2}}{|k+k'|^2} |\mathscr{F}_{\mathbb{R}^2}\phi \left(\frac{k'}{n}\right)|^2 \right)^2$$

$$\lesssim \frac{1}{t^{2\epsilon}} 2^{2j} \frac{1}{2^{4j\epsilon}}.$$

and consequently, using lemma 4.1.4,

$$\left\| \Delta_j \left(\int_{\mathbb{T}^d} u(t,z) \sum_{k \in \mathbb{Z}^2} e_k^*(x) e_k(z) \sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') \frac{e^{-t|k+k'|^2}}{|k+k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi \left(\frac{k'}{n}\right)^2 \mathrm{d}z \right) \right\|_{L^\infty_x}$$

$$= \left\| \int_{\mathbb{T}^d} u(t,z) \Delta_j \left(\sum_{k \in \mathbb{Z}^2} e_k^*(x) e_k(z) \sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') \frac{e^{-t|k+k'|^2}}{|k+k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi \left(\frac{k'}{n}\right)^2 \right) \mathrm{d}z \right\|_{L^\infty_x}$$

$$\lesssim \|u\|_{\mathscr{C}^\gamma} 2^{-j\gamma} \left\| \sum_{k \in \mathbb{Z}^2} \rho_j(k) e_k^*(x) e_k(z) \sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') \frac{e^{-t|k+k'|^2}}{|k+k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi \left(\frac{k'}{n}\right)^2 \right\|_{L^\infty_x L^2_z}$$

$$\lesssim \frac{\|u\|_{\mathscr{C}^\gamma}}{t^\kappa} 2^{j(1-\gamma)} 2^{-j2\kappa}$$

which proves that $S^n_{1,t} u \in \mathscr{C}^{2\gamma-2}$ and

$$\|S^n_{1,t}\|_{\mathscr{L}^\gamma_t, \mathscr{C}^{2\gamma-2}} \lesssim \frac{1}{t^\kappa}$$

for all suitably small $\epsilon > 0$ uniform in n.
A very similar argument to the one used in the proof of lemma 5.3.16 shows that $(S^n_{1,t})_{n \geq 0}$ is a Cauchy sequence in $L(\mathscr{L}^{2\gamma}_t, \mathscr{C}^{2\gamma-2})$.
This implies the claim. □

Here again, the proof implies that the Cauchy-property of $(S^n_{1,t})_{n \geq 0}$ is independent of time.

Lemma 5.3.20. *For any* $t > 0$ *there exists an operator* $S_{2,t} \in L\left(\mathscr{L}_t^{\gamma}, \mathscr{C}^{2\gamma-2}\right)$ *such that*

$$S_{2,t}^n \to S_{2,t} \in L\left(\mathscr{L}_t^{\gamma}, \mathscr{C}^{2\gamma-2}\right), \quad \|S_{2,t}\|_{\mathscr{L}_t^{\gamma} \to \mathscr{C}^{2\gamma-2}} \lesssim 1$$

Proof. Recall that

$$\begin{aligned} &S_{2,t}^n u \\ :=& \int_{\mathbb{T}^2} u(t,z) \sum_{k \in \mathbb{Z}^2} e_k^*(x) e_k(z) \sum_{k' \in \mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k', k+k') \frac{1}{|k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{n}\right) \mathrm{d}z - c_n M_t u \end{aligned}$$

Setting

$$\begin{aligned} S_n = \sum_{k \in \mathbb{Z}^2} e_k^* \sum_{k' \in \mathbb{Z}} &\Big(m_{\prec}(k,k') m_{\circ}(k, k+k') \frac{1}{|k+k'|^2} \mathscr{F}\left(\frac{k'}{n}\right) \\ &- m_{\prec}(0,k) \frac{1}{|k|^2} \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k_2}{n}\right) \Big) \end{aligned}$$

we first conclude that for all $n \geq 0$ we have that $S_n \in \mathscr{S}'$ and

$$S_{2,t}^n u = S_n * u(t) \text{ in } \mathscr{S}'.$$

Now recall that for $j \geq -1$

$$\Delta_j S_n * u(t) = K_j * S_n * u(t)$$

and for $i/geq - 1$ we have $\sum_{i \geq -1} \Delta_i \Delta_j S_n * u(t) = \Delta_j S_n * u(t)$ and

$$\Delta_i \Delta_j S_n * u(t) = K_i * S_m * K_j * u(t) \neq 0 \tag{5.3.21}$$

only if $|i-j| \leq 1$. We estimate using Young's inequality

$$\|K_j * S_n * K_j * u(t)\|_{L^\infty} \leq \|K_j * S_n\|_{L^1} \|K_j * u(t)\|_{L^\infty}. \tag{5.3.22}$$

First note that

$$\|K_j * u(t)\|_{L^\infty} = \|\Delta_j u(t)\|_{L^\infty} \lesssim 2^{-j\gamma} \|u\|_{\mathscr{L}_t^{\gamma}}. \tag{5.3.23}$$

Next we estimate by using Jensen's inequality

$$\|K_j * S_n\|_{\|L\|^1} \lesssim \|\Delta_j S_n\|_{L^2}.$$

We have

$$\begin{aligned} &\Delta_j S_n \\ =& \sum_{k \in \mathbb{Z}^2} \rho_j(k) e_k^* \sum_{k' \in \mathbb{Z}} \Big(m_{\prec}(k,k') m_{\circ}(k, k+k') \frac{1}{|k+k'|^2} \mathscr{F}\left(\frac{k'}{n}\right) \\ &- m_{\prec}(0,k) \frac{1}{|k|^2} \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k_2}{n}\right) \Big) \end{aligned}$$

For $k \in \mathbb{Z}^2$ we define sets $A_k := \{k' \in \mathbb{Z}^2 \colon m_{\prec}(k,k') \neq 1\}$. Plancheral's theorem allows us to estimate

$$\begin{aligned}
&\|\Delta S\|_{L^2}^2 \\
\lesssim= &\sum_{k\in\mathbb{Z}^2} \rho_j(k)^2 \Bigg(\sum_{k'\in\mathbb{Z}} \Big(m_{\prec}(k,k') m_{\circ}(k,k+k') \frac{1}{|k+k'|^2} \mathscr{F}\left(\frac{k'}{n}\right) \\
&- m_{\prec}(0,k) \frac{1}{|k|^2} \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k_2}{n}\right)\Big)\Bigg)^2
\end{aligned}$$

Note that for every $n \in \mathbb{N}$ the term $\mathscr{F}_{\mathbb{R}^2}(k'/n)$ is of rapid decay in k' and hence the double sum if finite.
We split the inner sum into two parts. Consider

$$\sum_{k\in\mathbb{Z}^2} \sum_{k'\notin A_k} \left(m_{\prec}(k,k') - m_{\prec}(0,k')\right) \frac{1}{|k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{n}\right).$$

We have

$$\begin{aligned}
&\sum_{k'\notin A_k} \left(m_{\prec}(k,k') - m_{\prec}(0,k')\right) \frac{1}{|k'|^2} \mathscr{F}_{\mathbb{R}^2}\phi\left(\frac{k'}{n}\right) \\
&\lesssim \sum_{k'\in\mathbb{Z}^2 \colon k'\neq 0} \mathbb{1}_{|k'|\lesssim|k|} \frac{1}{|k'|^2} \\
&\lesssim \ln(|k|)
\end{aligned}$$

Now we need to deal with the sum

$$\sum_{k'\in A_k} m_{\circ}(k,k+k') \frac{1}{|k+k'|^2} \left(\mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)^2 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k+k'}{n}\right)^2 \right).$$

Using that $\mathscr{F}_{\mathbb{R}^2}\phi$ is a Schwartz function on $\mathbb{R}^2$ we conclude:

$$\begin{aligned}
&|\mathscr{F}_{\mathbb{R}^2}\left(\frac{k+k'}{n}\right)^2 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)^2| \\
=&|\mathscr{F}_{\mathbb{R}^2}\left(\frac{k+k'}{n}\right) - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)||\mathscr{F}_{\mathbb{R}^2}\left(\frac{k+k'}{n}\right) + \mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)| \\
\lesssim&\frac{|k|}{n}\frac{1}{(1+|k'/n|)} \\
\leq&\frac{|k|}{(1+|k'|)}.
\end{aligned}$$

This allows us to estimate

$$\begin{aligned}
&\sum_{k'\in A_k} m_\circ(k,k+k')\frac{1}{|k+k'|^2}\left(\mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)^2 - \mathscr{F}_{\mathbb{R}^2}\left(\frac{k+k'}{n}\right)^2\right)\\
&\lesssim \sum_{k'\in\mathbb{Z}^2} \mathbb{1}_{|k|\lesssim|k'|}\frac{1}{|k'|^2}\frac{|k|}{(1+|k'|)}\\
&\lesssim k\sum_{k'\in\mathbb{Z}^2} \mathbb{1}_{|k|\lesssim|k'|}\frac{1}{|k'|^3}\\
&\lesssim 1.
\end{aligned}$$

Hence we conclude that

$$\|\Delta_j S_n\|_{L^2} \lesssim 2^{2j}(ln(2^i)+1)^2$$

and we conclude that

$$\begin{aligned}
\left\|\Delta_j S^n_{2,t}u\right\|_{L^\infty} &\|\Delta_j S_n * u\|\\
&\lesssim \|\Delta_j u\|_{L^\infty}\|\Delta_j S_n\|_{L^2}\\
&\lesssim 2^{j(1-\gamma)} j\,\|u\|_{\mathscr{L}_t^\gamma}
\end{aligned}$$

which implies that $S^n_{2,t}u \in \mathscr{C}^{2\gamma-2}$ with an uniform bound in n.
Using the approach, we can akin to the above also show that $(S^n_{2,t})_{n\geq 0}$ constitutes a Cauchy sequence in $L\left(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma}-2\right)$ and hence, for any $t>0$ there is an operator $S_{2,t} \in L\left(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma-2}\right)$ such that

$$S^n_{2,t} \to S_{2,t} \text{ in } L\left(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma-2}\right) \text{ as } n\to\infty.$$

□

Finally note that also for this operator the Cauchy-property is independent of time.
This enables us to prove the following corollary

Corollary 5.3.24. *For any $t>0$ there exists an operator $T_{2,t}$ in $L(\mathscr{L}_t^\gamma, \mathscr{C}^{\gamma-2})$ such that for each $\kappa>0$*

$$T^n_{2,t} \to T_{2,t} \text{ in } L(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma-2}) \text{ as } n\to\infty, \quad \|T_{2,t}\|_{\mathscr{L}_t^\gamma\to\mathscr{C}^{2\gamma-2}} \lesssim \max(t^{-\kappa},1).$$

Proof. Using the respective results for the operators $S_{1,t}, S_{2,t}$ the claim follows. □

Corollary 5.3.25. *For any $t>0$ there exists an operator $W_t \in L(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma-2})$ such that*

$$W^n_t - c_n M_t \to W_t \text{ in } L(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma-2}) \text{ as } n\to\infty, \quad \|W_t\|_{\mathscr{L}_t^\gamma,\mathscr{C}^{2\gamma-2}} \lesssim \max(t^{-\kappa},1).$$

Moreover,the map $s\mapsto W_t u$ is continuous on $(0,t]$ in $L\left(\mathscr{L}_t^{2\gamma};\mathscr{C}^{2\gamma-2}\right)$.

Proof. For the first claim we just just have to use the respective results for the operators $T_{1,t}, T_{2,t}$.
For the continuity recall first that by definition

$$W_t^n u = \int_0^t \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right] u(r,z)\mathrm{d}z\mathrm{d}r$$

For $0 < s \leq t$ we consequently have

$$\begin{aligned}
&W_t^n u - W_s^n u \\
&= \int_s^t \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right] u(r,z)\mathrm{d}z\mathrm{d}r \\
&\quad + \int_0^s \int_{\mathbb{T}^2} \left(\mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right] - \mathbb{E}\left[v^{\xi}(s,x,r,z)\right]\right) u(r,z)\mathrm{d}z\mathrm{d}r
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^2} \rho_j(k) \sum_{k'\in\mathbb{Z}^2} m_{\prec}(k,k') m_{\circ}(k',k+k') P_{t-r}(k+k') |\mathscr{F}_{\mathbb{R}^2}\left(\frac{k'}{n}\right)|^2 \\
&\lesssim 2^{2i} \frac{e^{-(t-r)c2^{2i}}}{(t-r)}
\end{aligned}$$

We now can prove using the same arguments as above

$$\begin{aligned}
&\left\| \Delta_j \int_s^t \int_{\mathbb{T}^2} \mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right] u(r,z)\mathrm{d}z\mathrm{d}r \right\|_{L_x^\infty} \\
&\lesssim \|u\|_{C_t\mathscr{C}^\gamma} 2^j 2^{-j\gamma} \int_s^t \frac{e^{-(t-r)c2^{2j}}}{(t-r)^{1/2}}\mathrm{d}r \\
&\lesssim \|u\|_{C_t\mathscr{C}^\gamma} 2^{j(1-\gamma)} \int_s^t \frac{1}{(t-r)^{1/2}}\mathrm{d}r.
\end{aligned}$$

The last integral is finite since by assumption $s > 0$ and converges to zero as $s \to t$.
Moreover note that for $\delta \in (0,1)$

$$e^{-(s-r)|k+k'|^2} - e^{-(t-r)|k+k'|} \leq (t-s)^\delta |k+k'|^\delta e^{-(s-r)|k+k'|^2}$$

and using this, we obtain by very similar computation

$$\begin{aligned}
&\left\| \Delta_j \left(\int_0^s \int_{\mathbb{T}^2} \left(\mathbb{E}\left[v^{\xi_n}(t,x;r,z)\right] - \mathbb{E}\left[v^{\xi}(s,x,r,z)\right]\right) u(r,z)\mathrm{d}z\mathrm{d}r \right) \right\|_{L_x^\infty} \\
&\lesssim \|u\|_{C_t\mathscr{C}^\gamma} 2^{j(1-\gamma)} (t-s)^\delta
\end{aligned}$$

On the other hand, one can by the same methods show that

$$\begin{aligned}
&\|c_n M_t u - c_n M_s u\|_{\mathscr{C}^{2\gamma-2}} \\
&\lesssim_n \|u(t) - u(s)\|_{\mathscr{C}^\gamma}
\end{aligned}$$

and consequently, for each $n \in \mathbb{N}$ the continuity property holds. Then

$$\begin{aligned}
&\lim_{s\to t} \|W_t u - W_s u\|_{\mathscr{C}^{2\gamma-2}} \\
\leq &\lim_{s\to t} \|W_t^n - c_n M_t - W_t u\|_{\mathscr{C}^{2\gamma-2}} \lim_{s\to t} \|W^n u_s - c_n M_s u - W_t^n u - c_n M_s u\|_{\mathscr{C}^{2\gamma-2}} \\
&+ \lim_{s\to t} \|W_s^n u - c_n M_s u - W_s u\|_{\mathscr{C}^{2\gamma-2}}
\end{aligned}$$

tends to zero uniform in u if $n \to \infty$ since the Cauchy-properties of the operators that add up to W_t^n is independent of time.
This proves the claim. □

Set $V := R^\xi + W$. We note that for all $t > 0$ by construction

$$R_t^{\xi_n} - c_n M_t \to V \text{ in } L\left(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma-2}\right) \text{ as } n \to \infty$$

since $R_t^{\xi_n} - c_n M_t = R_t^{\xi_n} - W_t^n + W_t^n - c_n M_t$.
Using this, we are finally able to prove:

Theorem 5.3.26. *Let $\gamma \in (2/3, 1)$ then $(\xi, V) \in \mathscr{X}^\gamma$ almost surely.*

Proof. Using the results obtained above we first note that for all $T > 0$ we have $(V_t)_{0<t\leq T}$ is a sequence of operators in $L\left(\mathscr{L}_t^\gamma, \mathscr{C}^{2\gamma-2}\right)$ and satisfies using $(\xi_n)_{n\geq 0}$ as a sequence of smooth approximations for ξ the approximation and continuity properties.
Next note that for each $T > 0$ and $\epsilon > 0$

$$\|V\|_{\mathscr{L}_t^\gamma \to \mathscr{C}^{2\gamma-2}} \lesssim \max(t^{-\epsilon}, 1).$$

Hence

$$t^\epsilon V_t \in L\left(\mathscr{L}_T^\gamma; C_T \mathscr{C}^{2\gamma-2}\right).$$

Now we can apply theorem 3.1.20 to conclude that $IV \in L(\mathscr{L}_T^\gamma; \mathscr{L}_T^{2\gamma-})$ for any $T > 0$.
We conclude $(\eta, V) \in \mathscr{X}^\gamma$. □

We now get the following theorem

Theorem 5.3.27. *Let u^0 be a random variable that almost surely takes values in $\mathscr{C}^{2\gamma}$.*
Then the renormalized PAM with driving noise (η, V) and initial datum u^0 admits a unique global solution $u \in \mathscr{L}^\gamma$ and if we the take smooth approximations of the white noise constructed above (i.e. ξ_n), for the approximate solutions to the renormalized PAM $u_n \in \mathscr{L}^\gamma$ with initial data $(u_n^0)_{n\geq 0}$ in $\mathscr{C}^{2\gamma}$ such that u_n^0 converges to u^0 in probability we have

$$u_n \to u \text{ in } \mathscr{L}^\gamma \text{ as } n \to \infty \text{ in probability.}$$

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