Topics on Mean Field Game Theory

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1 Introduction

Mean field game theory is the analysis of symmetric games with a very large number of players that interact with each other, change positions and endure costs or gain profit depending on their positions throughout and in the end of the game. The players follow a strategy with respect to which they decide how to move when they find themselves in a specific position. This strategy is chosen such that it minimizes their cost or respectively maximizes their profit. During the analysis, we take the limit case where the number of players is infinite, and thus each player receives statistically the same mean impact from every opponent in the game and where the impact is felt through the empirical distribution of the dynamics of the positions of all the players. The name mean field has been inspired by the mean field theory in physics which is used to study the movement of a particle that lives and interacts in a space with a very large number of other particles.

This kind of games have, in our knowledge, first been studied by Jean-Michel Lasry and Pierre-Louis Lions. They approached these games with the theory of PDEs and they formulated the limiting problem as a coupled system of the Hamilton-Jacobi-Bellman and the backward Kolmogorov equation. The first one is a forward equation that ensures the optimization problem is solved, whereas the backward equation takes care of the time consistency of the statistical distribution of the positions of the players.

In this thesis, we are following the probabilistic approach of René Carmona and Franois Delarue in [5] and the results presented here, unless otherwise stated, are results from this paper. In this approach, we do not work with PDEs but instead we create a coupled system of a forward and a backward SDE (FBSDE). To do this, they reformulated the problem as a fixed point problem in a space of flows of probability measures and with the use of the fixed point theory prove the existence of such a point. The resulting FBSDE system can then be solved with the use of the stochastic maximum principle, which results in a strategy which, if followed by all the players, no one player would want to change as this would result in a disadvantage. In other words, the resulting strategy is proved to be an approximate Nash equilibrium.

This thesis is split in three chapters. In the first chapter we give basic definitions and results that will be used throughout the analysis and to which we will refer repeatedly. Furthermore, we denote the spaces in which we will be working in, describe the setting of the game and give an outline of the steps that we will follow to solve the mean field game. At the end of the section we write the most important assumptions of the game and which will come in force gradually.

In the following chapter we introduce the theory of mean field games (MFG) and seek the strategies that optimize the costs of the players. During the MFG the distribution of the dynamics of the positions of the players is the empirical distribution created by their private states. As a first step, we assume some fixed arbitrary distribution for the private states of the players. The reduces the problem to a standard stochastic control problem. For this reason, we introduce the notions of the value function and the stochastic maximum principle, which play a key role for the solution of the MFG. In order to apply the stochastic maximum principle, we define the Hamiltonian and prove that it has a unique minimizer. Afterwards, we create a FBSDE system for which we prove that it is solvable and we find the value function. From this point on, we stop working with the fixed distribution and we change the general FBSDE by adding another restriction to our problem, this of finding the correct distribution. We then define what can be considered as a solution to the MFG and prove the existence of such a solution.

In the last chapter, we move back to our initial game with the large, yet finite, number of players and compare it to our theoretical approach of the FBSDE solution. Then we prove that the strategy that was found during the theoretical model, can indeed be used for the large game as it is an approximate Nash equilibrium. We end the analysis by proving an even stronger result which has to do with the fact that the cost of the player that decides to rebel will definitely worsen, depending also on the kind of strategy chosen, whereas the cost of (at least one) of his opponents will still remain close to the optimal cost.

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1.1 Preliminary Definitions

In the following, we state some definitions that are used throught this thesis as a reminder. The definition and the intuiton for the admissible strategy have been taken from [2]. The definition for the probability measures of finite momentum can be found in the article [5], as well as the definition of the Wasserstein distance. In the book of Pham [16] one finds the stochastic control problem, the optimal control and the value function. Gronwall's inequality appears in [17], whereas Prof. Bill Jackson in his lectures in [13] analyzes the definition of a Hausdorff space. In [18] one can find the implicit function theorem for Lipschitz functions.

- **Admissible strategy:**

Definition 1.1. *The progressively measurable stochastic process* $\beta = (\beta_t)_{0\le t\le T} \in \mathbb{R}^k$ *is said to satisfy the admissibility condition if*

$$E\left[\int_0^T |\beta_t|^2 dt\right] < \infty. \tag{1}$$

- **Nash equilibrium:**
In a game with N players, a strategy $\beta = (\beta_1, \dots, \beta_N)$ constitutes a Nash equilibrium if no player has the incentive (i.e. is better off) to change their strategy while every other player retains theirs. Mathematically, this is formulated as follows

Definition 1.2. *If* J *is a function denoting the outcome of the game given the* N *players follow the strategies* $\beta^i, i \in \{1, 2, \dots, N\}$*, then the set of admissible strategies* β^i *is said to form a Nash equilibrium for the game if*

$$J^i(\beta^1, \beta^2, \dots, \beta^N) \ge J^i(\beta^1, \beta^2, \dots, \beta^{i-1}, \alpha^i, \beta^{i+1}, \dots, \beta^N)$$

for all $i \in \{1, 2, \dots, N\}$*, where* α^i *is any strategy that differs from* β^i*.*

- **Probability measures with finite momentum:**

Definition 1.3. *For* E *a separable Banach space and* p *an integer greater than* 1*, a probability measure* μ *is said to have a moment of order* p *if*

$$M_{p,E}(\mu) = \left(\int_E ||x||_E^p d\mu(x)\right)^{1/p} < +\infty.$$

We write M_p for $M_{p,\mathbb{R}^d}$.

- **Wasserstein distance:**

Definition 1.4. *For a separable Banach space E, for all* $p \ge 1$, $\mu, \mu' \in \mathcal{P}_p(E)$*, the Wasserstein distance* $W_p(\mu, \mu')$ *is defined by*

$$W_p(\mu, \mu') = \inf\left\{ \left[\int_{E\times E} |x-y|_E^p \pi(dx, dy)\right]^{1/p} : \right.$$
$$\left. \pi \in \mathcal{P}(E \times E) \text{ with marginals } \mu \text{ and } \mu' \right\}.$$

- **Stochastic control problem:** For a measurable space $(A, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a d-dimensional Brownian motion $W_t = (W_t^1, \cdots, W_t^d)$ with respect to $\mathcal{F}$, assume two given functions $b(t, x, \alpha)\colon [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ and $\sigma(t, x, \alpha)\colon [0, T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d\times k}$, as well as stochastic process X_t, which solves the following stochastic differential equation (SDE)

$$dX_t = b(t, x, \alpha_t)dt + \sigma(t, x, \alpha_t)dW_t.$$

Definition 1.5. *A stochastic control problem is given by*

- *a dynamical system whose state is characterized by the evolution of the stochastic variable* X_t*;*
- *the process* α_t *is an admissible control, i.e. a process satisfying the admissibility condition;*
- *the goal of the problem is to find such an admissible control* α_t *that optimizes a given cost function* $J(t, x, \alpha_t)$ *defined on* $[0, T] \times \mathbb{R}^d \times A$ *and going to* $\mathbb{R}^d$*.*

- **Optimal control:**

Definition 1.6. *Given a stochastic control problem as the one defined above and for all the possible initial points* $(t, x) \in [0, T] \times \mathbb{R}^d$ *of the problem, an admissible control* α_t *is called optimal if the functional* J *obtains its optimum value when* α_t *is applied to it.*

- **Value function:**

Definition 1.7. *For any point* $(t, x) \in [0, T] \times \mathbb{R}^n$ *let a function* $\upsilon \colon [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *denote the optimal value of the functional* J*, that is the value of* J *under an optimal control. This function* υ *is called the value (or utility) function. More specifically this means that*

- *for a maximization problem the value function is defined as* $\upsilon(t, x) = \sup_{\alpha_t \in A} J(t, x, \alpha_t)$*,*
- *for a minimization problem the value function is defined as* $\upsilon(t, x) = \inf_{\alpha_t \in A} J(t, x, \alpha_t)$*.*

- **Gronwall's inequality:**

Theorem 1.8. *Let* α *be a function from* $\mathbb{R}_+$ *to itself, and suppose*

$$\alpha(s) \leq c + k \int_0^t \alpha(r) dr < +\infty$$

for $0 \leq s \leq t$*. Then* $\alpha(t) \leq ce^{kt}$*. Moreover, if* $c = 0$ *then* α *vanishes identically.*

- **Hausdorff space:**

Definition 1.9. *A topological space* X *is Hausdorff if for any* $x, y \in X$ *with* $x \neq y$ *there exist open sets* U *containing* x *and* V *containing* y *such that* $U \cap V = \emptyset$*.*

Remark: Every metric space is Hausdorff, in particular $\mathbb{R}^d$.

- **A useful estimate:**
  The following is a useful estimate of the 2-Wasserstein distance which will be used during the proof of the approximate Nash equilibrium.

  Theorem 1.10. *We define $\mathcal{P}_p(\mathbb{R}^d)$ to be the subspace of the space of probability measures of order p, i.e. haveing a finite moment of order p as defined before. Given $\mu \in \mathcal{P}_{d+1}(\mathbb{R}^d)$, there exists a constant c depending only upon d and $\mathcal{M}_{d+5}(\mu)$, such that*

  $$\mathbb{E}\left[W_2^2(\bar{\mu}^N, \mu)\right] \leq CN^{-2/d+4},$$

  where $\bar{\mu}^N$ denotes the empirical measure of any sample of size N.

- **The implicit function theorem for Lipschitz functions:**

  Theorem 1.11. *Let $U_m \subset \mathbb{R}^d$ and $U_n \subset \mathbb{R}^d$ open. Next fix $a \in U_m$ and $b \in U_n$ where $U = U_m \times U_n$. Consider $F\colon U \to R_n$ a Lipschitz function such that $F(a, b) = 0$ and with the property that there exists a constant $K > 0$ for which $|F(x, y_1) - F(x, y_2)| \geq K|y_1 - y_2|$ for all $(x, y_j) \in U$ where $j = 1, 2$. Then there exists $V_m \subset R_m$ open, such that $a \in V_m$ and a Lipschitz function $\phi\colon V_m \to U_n$ such that $\Phi(a) = b$, and $(x, y) \in V_m \times U_n \colon F(x, y) = 0 = (x, \Phi(x))$. In particular, $F(x, \Phi(x)) = 0$ for all $x \in V_m$.*

1.2 The setting

First, we define the spaces in which we will be working

- the space of the possible positions of the players $A \subset \mathbb{R}^k$, for some $k \in \mathbb{N}$,
- the space of strategies $\mathbb{A} = \mathbb{H}^{2,k}$ of progressively measureable A-valued stochastic processes $\beta = (\beta_t)_{0\leq t\leq T}$ satisfying the admissibility condition,
- $\mathbb{A}^N$ is the space of the strategies $\beta = (\beta^1, \ldots, \beta^N)$ of all the players and it denotes the product of N copies of $\mathbb{A}$,
- for a topological space E equipped with a Borel σ- field, we define $\mathcal{P}(E)$ to be the space of its probability measures. $\mathcal{P}(E)$ is also endowed with the Borel σ -field generated by the topology of weak convergence of measures,
- $\mathcal{P}_p(E)$ stands for the subspace of $\mathcal{P}(E)$ with probability measures of order p, (see also preliminary definitions),
- bounded subsets of $\mathcal{P}_p(E)$ are defined as sets of probability measures with uniformly bounded moments of order p,
- all regularity properties with respect to the measure μ during the analysis are understood in the sense of the 2-Wasserstein distance W_2.

Throughout the thesis, we avoid defining the initial starting point for each player, as this has no significant influence on the modelling and the solution of the game, with the exception of the dynamic programming principle and the proofs of Lemma (2.8) and Lemma (2.9).

1.3 The game

A mean field game consists of N players, where we let $N \in \mathbb{N}$ be very large. We are conducting the analysis from the viewpoint of one of the players. This player $i \in \{1, 2, ..., N\}$ begins their journey from the starting point x_0 and in each step chooses the next step β_t^i of their strategy in such a way that minimizes their final cost (or in a similar setting, that maximizes their final pay off). Remember that a strategy β_t is in the space $\mathbb{A}$ if it satisfies the admissibility condition. It is assumed that the dynamic of the position $U_t^i \in \mathbb{R}^d$ of the player i at time $t \in [0, T]$ is given by the following Ito stochastic differential equation

$$dU_t^i = b(t, U_t^i, \mu_t^N, \beta_t^i)dt + \sigma(t, U_t^i, \mu_t^N, \beta_t^i)dW_t^i, \tag{2}$$

where

- $0 \leq t \leq T$,
- $i \in \{1, \dots, N\}$,
- $\mu_t^N \in \mathcal{P}(\mathbb{R}^d)$ is the empirical distribution of the positions U_t^i of all players in the game at time t, defined as

$$\bar{\mu}_t^N(dx') = \frac{1}{N}\sum_{i=1}^{N} \delta_{U_t^i}(dx'),$$

- $[b^i, \sigma^i] \colon [0,T] \times \mathbb{R}^d \times P(\mathbb{R}^d) \times \mathbb{A} \hookrightarrow \mathbb{R}^d \times \mathbb{R}^{m\times d}$ are deterministic measurable functions,
- W_t^i is the m-dimensional Wiener Process driving the SDE with independent coefficients.

The final outcome of the player i depends on a measurable function $f^i \colon [0,T] \times \mathbb{R}^d \times P(\mathbb{R}^d) \times \mathbb{A} \hookrightarrow \mathbb{R}$, which evaluates the cost of each position that the player has occupied throughout the game, as well as a terminal function $g^i \colon \mathbb{R}^d \times P(\mathbb{R}^d) \hookrightarrow \mathbb{R}$ capturing the cost of the final position of the player. This means that the final outcome of player i is given by

$$J^i(\beta, \mu) = \mathbb{E}\left[\int_0^T f^i(t, U_t^i, \mu_t^N, \beta_t^i)dt + g^i(U_T^i, \mu_T^N)\right] \tag{3}$$

for a strategy $\beta_t \in \mathbb{A}^N$.

The goal of each player in this game is to follow such a strategy that minimizes the overall cost that they need to pay during the game. That is to find the optimal strategy α_t that gives the minimum value for $J^i(\beta, \mu^N)$, that is such that

$$J^i(\alpha, \mu^N) = \inf_{\beta^i \in \mathbb{A}} \mathbb{E}\left[\int_0^T f^i(t, U_t^i, \mu_t^N, \beta_t^i)dt + g^i(U_T^i, \mu_T^N)\right]. \tag{4}$$

Note that the final cost depends upon the whole vector of β, i.e. the strategies of all players and not only on the strategy of player i. This is because the strategy of player i depends on the empirical distribution μ, which is a result of the positions of all players.

Since the number of players in the game is very large, it is assumed that each opponent affects the outcome of our player in a statistically identical way. This means that the functions $\beta^i, \sigma^i, f^i, g^i$ actually do not depend on player i, but are the same for each player and thus they will be denoted as β, σ, g and f respectively. Obviously, this is not true for the cost function J, as it depends on the position of the individual player i and thus cannot be the same for all players. Moreover, note that in this case, we will let the volatility be an uncontrolled constant matrix $\sigma \in \mathbb{R}^{d\times m}$, in order to avoid many technicalities.

1.4 Main steps of the process

This model resembles a stochastic control problem. What makes it different from a typical stochastic problem is the fact that the players' distribution is not fixed, but rather depends on the strategy that each player chooses and the resulting position they have at each time $t \in [0, T]$. Therefore, the way to solve it, is by following the next steps

1. Assume that μ_t is a fixed deterministic distribution throughout the game;

2. Try to solve the stochastic control problem that results, i.e.

$$\begin{aligned}&\inf_{\beta \in A} E\left[\int_0^T f(t, U_t, \mu_t, \beta_t)dt + g(U_T, \mu_T)\right],\\ &dX_t = b(t, U_t, \mu_t, \beta_t)dt + \sigma dW_t, \qquad U_0 = u_0\end{aligned}$$

   using the usual methods of stochastic control theory;

3. Use the theory of propagation of chaos in order to find a function

$$\begin{cases} [0, T] \hookrightarrow \mathcal{P}(\mathbb{R}^d) \\ \qquad t \mapsto \mu_t \end{cases}$$

   s.t. for all t it holds $\mathbb{P}_{U_t} = \mu_t$.

The last step is the most important as this is what gives the mean field solution, i.e. the solution for an undefined distribution μ. More importantly, this solution "orders" the strategy that each player would be best off following, given that all her co-players "obey" to the optimal strategy, i.e. forms an approximate Nash equilibrium, as it will be defined later on.

1.5 Assumptions of the model

As is the case in most models, there are certain assumptions that need to hold in order to prove and find the solution for the game. These assumptions will be layed out here, as they have been written in [5] and they will come in force gradually during this thesis

(A.1) The drift b is an affine function of α in the sense that it is of the form

$$b(t, x, \mu, \alpha) = B_1(t, x, \mu) + b_2(t)\alpha,$$

where

- the mapping

$$\begin{cases} [0, T] \hookrightarrow \mathbb{R}^{d \times k} \\ \qquad t \mapsto b_2(t), \end{cases}$$

  is measurable and bounded, and

- the mapping

$$\begin{cases} [0,T] \hookrightarrow \mathbb{R}^d \\ \qquad t \mapsto B_1(t,x,\mu) \end{cases}$$

is measurable and bounded on bounded subsets of $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

(A.2) There exist two positive constants λ and c_L such that for all $t \in [0,T]$ and $\mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ such that

- the function

$$\begin{cases} \mathbb{R}^d \times \mathbb{A} \hookrightarrow \mathbb{R} \\ \quad (x,\alpha) \mapsto f(t,x,\mu,\alpha) \end{cases}$$

is once continuously differentiable with Lipschitz continuous derivatives, i.e. $f(t,\cdot,\mu,\cdot) \in \mathcal{C}^{1,1}$,

- the Lipschitz constant in x and α is bounded by c_L (so that it is uniform in t and μ),
- the Lipschitz constant in x and α satisfies the convexity assumption

$$\begin{aligned} &f(t,x',\mu,\alpha') - f(t,x,\mu,\alpha) \\ &\qquad - \langle (x'-x, \alpha'-\alpha), \partial_{(x,\alpha)} f(t,x,\mu,\alpha) \rangle \geq \lambda |\alpha' - \alpha|^2, \end{aligned} \quad (5)$$

where $\partial_{(x,\alpha)}$ stands for the gradient in the joint variables (x,α).

Furthermore, f, $\partial_x f$ and $\partial_\alpha f$ are locally bounded over $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{A}$.

(A.3) The function

$$\begin{cases} [0,T] \hookrightarrow \mathbb{R}^d \\ t \mapsto B_1(t,x,\mu) \end{cases}$$

is affine in x, i.e. it has the form
$B_1(t,x,\mu) = b_0(t,\mu) + b_1(t)x$, where $b_0 \in \mathbb{R}^d, b_1 \in \mathbb{R}^{d\times k}$ are bounded on bounded subsets of their respective domains.

(A.4) The function

$$\begin{cases} \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \hookrightarrow \mathbb{R} \\ (x,\mu) \mapsto g(x,\mu) \end{cases}$$

is locally bounded. Moreover, for any $\mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ the function

$$\begin{cases} \mathbb{R}^d \hookrightarrow \mathbb{R} \\ x \mapsto g(x,\mu) \end{cases}$$

is once continuously differentiable and convex and has a c_L-Lipschitz-continuous first order derivative.

(A.5) – The functions

$$\begin{cases}[0,T] \hookrightarrow \mathbb{R}^d \\ t \mapsto f(t,0,\delta_0,0)\end{cases}, \begin{cases}[0,T] \hookrightarrow \mathbb{R}^d \\ t \mapsto \partial_x f(t,0,\delta_0,0)\end{cases} \quad \text{and} \quad \begin{cases}[0,T] \hookrightarrow \mathbb{R}^d \\ t \mapsto \partial_\alpha f(t,0,\delta_0,0)\end{cases}$$

are bounded by c_L,

– for all $t \in [0,T]$, $x, x' \in \mathbb{R}^d$, $\alpha, \alpha' \in \mathbb{A}$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, it holds

$$\begin{aligned}|(f,g)(t,x',\mu',\alpha') - (f,g)(t,x,\mu,\alpha)| \leq& \\ c_L \left[1 + |(x',\alpha')| + |(x,\alpha)| + M_2(\mu) + M_2(\mu')\right]& \\ \left[|(x',\alpha') - (x,\alpha)| + W_2(\mu',\mu)\right],&\end{aligned}$$

– b_0, b_1 and b_2 as defined in (A.3) are bounded by c_L,

– for any $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, b_0 satisfies the following inequality

$$|b_0(t,\mu') - b_0(t,\mu)| \leq c_L W_2(\mu,\mu').$$

(A.6) For all $t \in [0,T], x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ it holds $|\partial_\alpha f(t,x,\mu,0)| \leq c_L$.

(A.7) For all $(t,x) \in [0,T] \times \mathbb{R}^d$

$$\langle x, \partial_x f(t,0,\delta_x,0)\rangle \geq -c_L(1+|x|),$$

$$\langle x, \partial_x g(0,\delta_x)\rangle \geq -c_L(1+|x|).$$

Note that when assumptions (A.1) and (A.3) are valid, function b reads

$$b(t,x,\mu,\alpha) = b_0(t,\mu) + b_1(t)\cdot x + b_2(t)\cdot \alpha. \tag{6}$$

2 Solving the MFG

2.1 Freezing the distribution

As we have already mentioned, we cannot directly solve the MFG, since there are too many variables. More specifically, we need to find the solution with respect to the distribution of the private states of the players, while at the same time detecting the optimal distribution. For this exact reason as a first step we choose to fix the distribution and consider it to be known, therefore we can use the usual tools for solving a standard stochastic control problem.

2.2 Solving the Standard SCP

After fixing the distribution of the private states, to be some μ, which is assumed to be known, our initial problem reduces to solving the stochastic control problem that arises.

For our setting we are going to use two important aspects of the solution of a stochastic control problem. The first one is the notion of the value function, which can be best understood through the dynamical programming principle(DPP), and the second is Pontryagin's maximum principle, also called the stochastic maximum principle. In this section we will explain the value function and present the stochastic maximum principle and define the Hamiltonian. The information about the dynamical programming principle, as they are presented next, appear in [16] and have been adjusted to our setting.

During the analysis of DPP, the initial point where the stochastic process starts its path is of great importance, and thus we will add the initial point into our notation for this section. Furthermore, the theory illustrated here is valid for a game that happens within a certain period of time $[0, T]$ and not for arbitrarily long time, thus it is still valid in the mean field game that we are studying throughout the thesis.

2.2.1 The dynamic programming principle

Mathematically, it is easy to define the value function. Let us consider a process X_t that starts at a point $x_0 \in \mathbb{R}^d$ at time $t_0 \in [0, T]$, denoted by $X_t^{t_0,x_0}$. Furthermore, let J stand for the cost function of the game, as it was defined in (3) and let μ^N be the fixed distribution in the game. Then the value function is defined to be

$$v^{\mu}(t_0, x_0) = \inf_{\beta \in \mathbb{A}} J(\beta, \mu^N).$$

Before we proceed, we will also define the set $\mathcal{T}_{t_0,T}$ to be the set of all stopping times in the set $[t_0, T]$. We quote from Pham "the intepretation of the DPP is that the optimization problem can be split in two parts an optimal control on the whole time interval $[t_0, T]$ may be obtained by first searching for an optimal control from time θ given the state value $X_\theta^{t_0,x_0}$, i.e. compute $\upsilon^\mu(\theta, X_\theta^{t_0,x_0})$, and then maximizing over controls on $[t, \theta]$ the quantity

$$\mathbb{E}\left[\int_t^\theta f(s, X_s^{t_0,x_0}, \beta_s)ds + \upsilon^\mu(\theta, X_\theta^{t_0,x_0})\right].\text{''}$$

After stating the principle formally we will illustrate it with an example to make the intuition clear. The proof of the theorem can be found in [16].

Theorem 2.1 (Dynamic programming principle)**.** *Let $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ be the initial starting point of the process X_t. Then we have*

$$\upsilon^\mu(t_0, x_0) = \inf_{\beta \in \mathbb{A}} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_{t_0}^\theta f(s, X_s^{t_0,x_0}, \beta_s)ds + \upsilon^\mu(\theta, X_\theta^{t_0,x_0})\right].$$

Remark: Pham in [16] notes that the there is an even stronger version than the usual version of the DPP, which is the following

$$\upsilon(t_0, x_0) = \inf_{\beta \in \mathbb{A}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t_0,x_0}, \beta_s)ds + \upsilon^\mu(\theta, X_\theta^{t_0,x_0})\right]$$

for any stopping time $\theta \in \mathcal{T}_{t_0,T}$.

We will try to illustrate this a bit better in the following example, which has been inspired by the examples of [19]

Example 2.2. *Assume that you live in the city o S and that you want to go on vacation in one of the islands I1, I2 or I3. You have no personal preference between the islands, other than minimizing the total cost of the vacation, i.e. to minimize the sum of going and staying there. Therefore, you have to choose the trip (i.e. the island and the way to reach it) such that you have to spent the least money.*

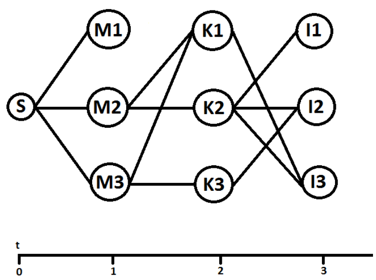

Figure 1: Trying to move from city S to one of the islands I1, I2 or I3 by minimizing the total cost.

The cost of staying in each island Ik is depicted by a function $g\colon \{I1, I2, I3\} \to \mathbb{R}$ that takes some positive values

$$g(I1) = c_1, \qquad g(I2) = c_2, \qquad g(I3) = c_3,$$

that depict the cost of staying in the respective island. The arrows between the cities on the picture denote that there is a road connecting these cities and for each road there is a toll that needs to be payed. This cost is depicted by a function $f\colon \{S, M1, M2, M3, I1, I2, I3\}^2 \to \mathbb{R}$.

From the picture it is clear that the only way to reach the island $I1$ is through city K2. Thus the minimum cost J to visit island $I1$ is

$$J = \min\{\textit{start at city } S \textit{ and reach city } K2\} + f(K2, I1) + g(I1).$$

If we try to write it in the same way as it was expressed in the DPP, then, with a slide abuse of the notation, we would write

$$v(0, S) = \inf_{\beta \in \mathbb{A}} \mathbb{E}\left[\int_0^2 f(m, X_m^{0,S}, \beta_m) dm + v^{\mu}(2, X_2^{0,S})\right].$$

2.2.2 Pontryagin's maximum principle

Pontryagin's maximum principle, also known as stochastic maximum principle, was introduced in 1956 by Lev Pontryagin. In order to define the stochastic maximum principle, they introduced the Hamiltonian, which is a tool necessary to find the optimum value. For reasons of consistency, we follow the definition of the Hamiltonian that can be found in [5].

As we said, the volatility is an uncontrolled constant matrix in our setting and thus we can use the following version for the Hamiltonian

$$H(t, x, \mu, y, \alpha) = \langle b(t, x, \mu, \alpha), y \rangle + f(t, x, \mu, \alpha)$$

for $t \in [0, T]$, $x, y \in \mathbb{R}^d, \alpha \in \mathbb{R}^k$ and $\mu \in \mathcal{P}(\mathbb{R}^d)$.

With this principle, Pontryagin proves that in order to find the optimum value of a stochastic control problem, it is enough to find the optimum value of the Hamiltonian. Then, we know that the stochastic control problem obtains its optimum value at the same point, where the Hamiltonian obtains its minimum.

For each $\alpha \in \mathbb{A}$, we consider a backward SDE (BSDE), called the adjoint equation, which is as follows

$$-dY_t = \partial_x H(t, X_t, \mu_t, Y_t, \alpha) dt + Z_t dW_t.$$

Theorem 2.3 (Pontryagin's maximum principle). *Let $\hat{\alpha} \in \mathbb{A}$ be a control and X be the associated controlled diffusion of the system. Suppose that the BSDE has a solution $(\hat{Y}_t, \hat{Z}_t)$ such that*

$$H(t, X_t, \mu_t, \hat{Y}_t, \hat{\alpha}) = \max_{\alpha \in \mathbb{A}} H(t, X_t, \mu_t, Y_t, \alpha),$$

and also such that

$$(x, \alpha) \to H(t, x, \mu, y, \alpha)$$

is a convex function for all $t \in [0, T]$. *Then* $\hat{\alpha}$ *is an optimal control, i.e. it holds*

$$J(\hat{\alpha}) = \inf_{\alpha \in \mathbb{A}} J(\alpha).$$

Proof. The proof can be found in [16] and is similar to the proof of Theorem (2.6) and will, thus, be skipped. □

2.2.3 Minimizing the Hamiltonian

As we have mentioned, we will use the Pontryagin maximum principle to find the solution of the standard stochastic problem.

The Pontryagin maximum principle proves that the minimizer of the Hamiltonian is also the minimizer of the SDE. Thus, our next step will be to find if there exists (at least one) a minimizer of the Hamiltonian.

Lemma 2.4. *If we let the assumptions (A.1) and (A.2) be in force, then for all* $(t, x, \mu, y) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d$, *there exists a unique minimizer* $\hat{\alpha}(t, x, \mu, y)$ *of H.*

Proof. Letting the assumption (A.1) be in force, the Hamiltonian has the following form

$$H(t, x, \mu, y, \alpha) = \langle b_1(t, x, \mu), y \rangle + \langle b_2(t)\alpha, y \rangle + f(t, x, \mu, \alpha).$$

The function $\alpha \hookrightarrow H(t, x, \mu, y, \alpha)$ is strictly convex and continuously differentiable for any (t, x, μ, y). Thus we know that this function has a unique minimum, which will be achieved at the same point where the gradient hits the origin, i.e. where $\partial_\alpha H(t, x, \mu, y, \alpha) = 0$. □

In fact, we even know a bit more about this minimizer

Lemma 2.5. *Assuming the assumptions (A.1) and (A.2) are in force, the function*

$$\begin{cases} [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \hookrightarrow \mathbb{R}^k \\ \qquad\qquad (t, x, \mu, y) \mapsto \hat{\alpha}(t, x, \mu, y), \end{cases}$$

which is the minimizer function of H, is measurable, locally bounded and Lipschitz-continuous with respect to (x, y), *uniformly in* $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, *where the Lipschitz constant depends only upon* λ *(from (A.2)), the supremum norm of* b_2 *and the Lipschitz constant of* $\partial_\alpha f$ *in* x.

Proof. The strict convexity property and the gradient descent algorithm result in the measurability of the minimizer. The local boundedness of

$\hat{\alpha}(t,x,\mu,y)$ also follows from strict convexity. Firstly, since $\hat{\alpha}(t,x,\mu,y)$ is the minimizer, it obviously holds that

$$H(t,x,\mu,y,0) \geq H(t,x,\mu,y,\hat{\alpha}(t,x,\mu,y)).$$

Furthermore, we will use assumption (A.2) to obtain another inequality for the Hamiltonian which we gain by applying inequality (5) for $x'=x$, $\alpha=0$ and $\alpha'=\hat{\alpha}$. This then yields

$$\begin{aligned}&f(t,x,\mu,\hat{\alpha}(t,x,\mu,y)) - f(t,x,\mu,0)\\&\qquad - \Big\langle \big(0,\hat{\alpha}(t,x,\mu,y)\big), \partial_{(x,\alpha)} f(t,x,\mu,0)\Big\rangle \geq \lambda|\hat{\alpha}(t,x,\mu,y)|^2.\end{aligned}$$

Adding and subtracting the inner product norm of the Hamiltonian we obtain

$$\begin{aligned}&H(t,x,\mu,y,\hat{\alpha}(t,x,\mu,y)) - H(t,x,\mu,y,0) - \hat{\alpha}^T(t,x,\mu,y)\partial_\alpha H(t,x,\mu,y,0)\\&\qquad - b^T(t,x,\mu,\hat{\alpha}(t,x,\mu,y))y - b(t,x,\mu,0)y\\&\qquad\qquad + \hat{\alpha}(t,x,\mu,y)\partial_\alpha\left(b^T(t,x,\mu,0)y\right) \geq \lambda\left|\hat{\alpha}(t,x,\mu,y)\right|^2,\end{aligned}$$

where the matrix A^T stands for the transpose of A. Taking advantage of the affinity of function b, from assumption (A.1), we obtain

$$\begin{aligned}&H(t,x,\mu,y,\hat{\alpha}(t,x,\mu,y)) - H(t,x,\mu,y,0) - \hat{\alpha}^T(t,x,\mu,y)\partial_\alpha H(t,x,\mu,y,0)\\&\qquad - b_2^T(t)\hat{\alpha}(t,x,\mu,y)y + \hat{\alpha}^T(t,x,\mu,y)b_2^T(t)y \geq \lambda\left|\hat{\alpha}(t,x,\mu,y)\right|^2.\end{aligned}$$

We reorganize the above, therefore that we obtain

$$\begin{aligned}H(t,x,\mu,y,\hat{\alpha}(t,x,\mu,y)) &\geq H(t,x,\mu,y,0)\\&\quad + \hat{\alpha}^T(t,x,\mu,y)\partial_\alpha H(t,x,\mu,y,0) + \lambda|\hat{\alpha}(t,x,\mu,y)|^2.\end{aligned}$$

Combining this with the first inequality that we proved, we obtain

$$\begin{aligned}H(t,x,\mu,y,0) &\geq H(t,x,\mu,y,0)\\&\quad + \hat{\alpha}^T(t,x,\mu,y)\partial_\alpha H(t,x,\mu,y,0) + \lambda|\hat{\alpha}(t,x,\mu,y)|^2.\end{aligned}$$

This then leads us to

$$\begin{aligned}|\hat{\alpha}(t,x,\mu,y)|^2 &\leq \lambda^{-1}\hat{\alpha}^T(t,x,\mu,y)\partial_\alpha H(t,x,\mu,y,0)\\&\leq \lambda^{-1}\hat{\alpha}^T(t,x,\mu,y)\left(\partial_\alpha f(t,x,\mu,0) + b_2^T(t)y\right)\\&\leq \lambda^{-1}|\hat{\alpha}(t,x,\mu,0)|\left|\partial_\alpha f(t,x,\mu,0) + b_2^T(t)y\right|,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. Using the triangle inequality we finally obtain

$$|\hat{\alpha}(t,x,\mu,y)| \leq \lambda^{-1}(|\partial_\alpha f(t,x,\mu,0)| + |b_2(t)||y|), \tag{7}$$

which proves the local boundedness of $\hat{\alpha}(t,x,\mu,y)$. The Lipschitz continuity of $\hat{\alpha}(t,x,\mu,y)$ with respect to (x,y) comes by applying the implicit function theorem on $\partial_\alpha H$. The Lipschitz constant depends on the uniform bound on b_2 and on the Lipschitz-constant of $\partial_{(x,\alpha)} f$. $\square$

2.3 The general FBSDE

In order to be able to use the known theory from the forward backward stochastic differential equations theory, we need to "create" such a system. To do this, we need to construct a stochastic process $(Y_t)_{0\leq t\leq T}$ that moves backwards and which will be coupled with X_t.

We define the filtration $(F_t)_{0\leq t\leq T}$ generated by the Wiener Process W_t and we determine the initial condition of the constructed process to be $Y_T = \partial_x g(X_T, \mu_T)$. Then we start the "construction" of the Backward process by iteration with the following first step

$$Y_t^0 = \mathbb{E}[Y_T|F_t].$$

Since we let Y_t be a square integrable process measurable with respect to F_t, we then know by the martingale representation theorem that there exists a predictable process Z_t adapted to F_t such that

$$dY_s^0 = Z_s^0 dW_s, \qquad \text{when } Y_T^0 = Y_T.$$

Then, assuming that we have constructed the n-first iteration, the next one is defined to be

$$\begin{aligned} Y_t^{n+1} =& \mathbb{E}\left[Y_T - \int_0^T \left(-\partial_x H(s, X_s, \mu_s, Y_s^n, \alpha_s)\right) dt \Big| F_t\right] \\ &+ \int_0^t \left(-\partial_x H(s, X_s, \mu_s, Y_s^n, \alpha_s)\right) ds. \end{aligned}$$

Again by the martingale representation theorem, the conditional expectation can be represented as an Ito integral

$$\mathbb{E}\left[Y_T - \int_0^T \left(-\partial_x H(s, X_s, \mu_s, Y_s^n, \alpha_s)\right) dt \Big| F_t\right] = \int_0^t Z_s^{n+1} dW_s,$$

so that the differential version of the process Y_t can be given by

$$dY_t^{n+1} = -\partial_x H(t, X_t, \mu_T, Y_T^n, \alpha_T)dt + Z_t^{n+1} dW_t.$$

From the above and by letting the number of iterations become very large we finally obtain the Backward SDE

$$dY_t = -\partial_x H(t, X_t, \mu_T, Y_T, \alpha_T)dt + Z_t dW_t,$$

with the initial condition $Y_T = \partial_x g(X_T, \mu_T)$. This BSDE is obviously coupled with the SDE (2) that gives the dynamics of the positions of the players. Thus, we have created a system of a FBSDE system, which is the following

$$\begin{aligned} dX_t &= b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + \sigma dW_t \\ dY_t &= -\partial_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + Z_t dW_t, \end{aligned} \tag{8}$$

where $X_0 = x_0$, $Y_T = \partial_x g(X_T, \mu_T)$ are the border conditions. Of course, the stochastic processes Y_t and Z_t are required to be adapted as well.

At first glance, it might seem that the above FBSDE has come up completely arbitrarily and it is not obvious that there is any connection between (8) and our problem. In the next theorem, which is also the key theorem of this thesis, we will prove just that, i.e. that the optimum solution of the FBSDE, if it exists, is an almost minimizer for the cost function.

Theorem 2.6. *Under the assumptions (A.1)-(A.4), let the mapping*

$$\begin{aligned} [0,T] &\hookrightarrow \mathcal{P}_2\left(\mathcal{C}\left([0,T], \mathbb{R}^{d\times k}\right)\right) \\ t &\mapsto \mu_t \end{aligned}$$

be measurable and bounded, the FBSDE (8) have a solution (X_t, Y_t, Z_t) such that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left(|X_t|^2 + |Y_t|^2\right) + \int_0^T |Z_t|^2 dt\right] < \infty, \tag{9}$$

defining the cost functional $J(\beta, \mu)$ as in (3), where $U = (U_t)_{0\le t\le T}$ is the corresponding controlled diffusion process which solves (2) for $x_0 \in \mathbb{R}^d$ and setting $\hat{\alpha}_t = \hat{\alpha}(t, X_t, \mu, Y_t)$, where the function $\hat{\alpha}$ is the minimizer of the Hamiltonian, then for any admissible control $\beta = (\beta_t)_{0\le t\le T}$, it holds

$$J(\hat{\alpha}, \mu) + \lambda \mathbb{E}\int_0^T |\beta_t - \hat{\alpha}_t|^2 dt \le J(\beta, \mu).$$

Proof. This proof can be done in a similar way as the standard proof of the stochastic maximum principle (see Theorem 6.4.6 [16]). We split the proof into four steps.

Step 1:
First, we need to prove that the solution provided by the FBSDE (8) can be considered to be a solution of the mean field game. In other words, we need to show that it is admissible. This can be shown by using the inequality (7), assumption (A.1) and (A.2) on the functions b_2 and f, and by using the assumption (9) on the solution of the FBSDE.

Step 2:
By the definition of J we know that for some strategy β and the Hamiltonian-minimizing strategy $\hat{\alpha}_t = \hat{\alpha}(t, X_t, \mu_t, Y_t)$ one can write

$$\begin{aligned} J(\hat{\alpha}_t, \mu_t) - J(\beta_t, \mu_t) = \mathbb{E}\bigg[\int_0^T \Big(f(t, X_t, \mu_t, \hat{\alpha}_t) - f(t, U_t, \mu_t, \beta_t)\Big) dt \\ - \Big(g(X_T, \mu_T) - g(U_T, \mu_T)\Big)\bigg]. \end{aligned}$$

In the above equation, the right hand side can be split into the $''f-part''$ and the $''g-part''$ and thus that it is possible to work on them independently.

For the $''f-part''$:
We use the definition of the Hamiltonian and thus we obtain

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^T \Big(f(t,X_t,\mu_t,\hat{\alpha}_t) - f(t,U_t,\mu_t,\beta_t)\Big)dt\Big] = \\
\mathbb{E}\Big[\int_0^T \Big(H(t,X_t,\mu_t,Y_t,\hat{\alpha}_t) - \langle b(t,X_t,\mu_t,\hat{\alpha}_t),Y_t\rangle \\
- H(t,U_t,\mu_t,\widetilde{Y}_t,\beta_t) + \langle b(t,U_t,\mu_t,\beta_t),\widetilde{Y}_t\rangle\Big)dt\Big],
\end{aligned}
$$

where $\widetilde{Y}_t$ is the solution of the backward equation that one can associate with the Forward SDE of U_t.

For the $''g-part''$:
By assumption (A.4) we know that the function g is convex with respect to the variable x, i.e. that for any two points $x,y\in\mathbb{R}^d$ it holds that $g(x,\mu)\geq g(y,\mu)-\partial_x g(y,\mu)(y-x)$. Thus we obtain

$$
\begin{aligned}
\mathbb{E}\Big[g(X_T,\mu_T)-g(U_T,\mu_T)\Big] &\leq \mathbb{E}\left[(U_T-X_T)^T\partial_x g(X_T,\mu_T)\right] \\
&= \mathbb{E}\left[(U_T-X_T)^T Y_T\right],
\end{aligned}
$$

where we have taken advantage of the initial condition of the Backward SDE of (14). Now we need to apply Ito's formula on the stochastic function $h(U_T,X_T,Y_T)=\big[U_T-X_T\big]Y_T$, from which it follows that

$$
\begin{aligned}
h(U_T,X_T,Y_T) = h(U_0,X_0,Y_0) + \int_0^T Y_t^T dU_t - \int_0^T Y_t^T dX_t + \\
\int_0^T (U_t-X_t)^T dY_t + \frac{1}{2}\int_0^T \left(d\langle U,Y\rangle_t - d\langle X,Y\rangle_t\right) \\
= h(U_0,X_0,Y_0) + \int_0^T Y_t^T\left(dU_t-dX_t\right) + \int_0^T (U_t-X_t)^T dY_t.
\end{aligned}
$$

Note that since U and X satisfy SDEs with a (same) Brownian Motion and the same diffusion term σ, we know that they have the same quadratic variation, i.e. $d[U]_t=d[X]_t$ and therefore their quadratic covariations with Y cancel each other out, so that $d\langle U,Y\rangle_t - d\langle X,Y\rangle_t = 0$. Furthermore, U and X are assumed to have the same initial position, and thus $h(U_0,X_0,Y_0,)=0$.

By replacing dX_t, dU_t and dY_t with the equivalent of their respective

SDE, we obtain

$$\begin{aligned}h(U_T, X_T, Y_T) = &\int_0^T Y_t^T \left(b(t, U_t, \mu_t, \beta_t) - b(t, X_t, \mu_t, \hat{\alpha}_t\right) dt \\ &+ \int_0^T (U_t - X_t)^T \partial_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t) dt \\ &+ \int_0^T (U_t - X_t)^T Z_t dW_t.\end{aligned}$$

This results in the following inequality for the $''g - part''$

$$\begin{aligned}&\mathbb{E}\big[g(X_T, \mu_T) - g(U_T, \mu_T)\big] \leq \\ &\qquad \mathbb{E}\Big[\int_0^T Y_t^T \left(b(t, U_t, \mu_t, \beta_t) - b(t, X_t, \mu_t, \hat{\alpha}_t)\right) dt \\ &\qquad - \int_0^T (U_t - X_t)^T \partial_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t) dt \\ &\qquad + \int_0^T (U_t - X_t)^T Z_t dW_t\Big].\end{aligned}$$

Combining the results for the $''f - part''$ and $''g - part''$ we obtain

$$\begin{aligned}J(\hat{\alpha}_t, \mu_t) - J(\beta_t, \mu_t) \leq \mathbb{E}\Big[&\int_0^T \Big(H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t) - b^T(t, X_t, \mu_t, \hat{\alpha}_t) Y_t \\ &- H(t, U_t, \mu_t, \widetilde{Y}_t, \beta_t) + b^T(t, U_t, \mu_t, \beta_t)\widetilde{Y}_t\Big) dt \\ &- \int_0^T Y_t^T \left(b(t, U_t, \mu_t, \beta_t) - b(t, X_t, \mu_t, \hat{\alpha}_t)\right) dt \\ &+ \int_0^T (U_t - X_t)^T \partial_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t) dt \\ &- \int_0^T (U_t - X_t)^T Z_t dW_t\Big].\end{aligned}$$

We note the following points

- $-b^T(t, X_t, \mu_t, \hat{\alpha}_t) Y_t + Y_t^T b(t, X_t, \mu_t, \hat{\alpha}_t) = 0$,
- $\mathbb{E}\Big[\int_0^T (U_t - X_t)^T Z_t dW_t\Big] = 0$, since it is the expectation of a martingale and the processes U_t and X_t have the same initial condition,
- $\partial_\alpha H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t) = 0$ by the definition of the strategy $\hat{\alpha}_t$, and thus $(\beta_t - \hat{\alpha}_t)^T \partial_\alpha H(t, X_t, \mu_t, Y_t, \hat{\alpha}_t) = 0$,
- without loss of generality we can assume that $\widetilde{Y}_t$ and Y_t coincide, and thus we conclude that

$$b^T(t, U_t, \mu_t, \beta_t)\widetilde{Y}_t = Y_t^T b(t, U_t, \mu_t, \beta_t).$$

Therefore, we have proved, that the following inequality holds for H

$$\begin{aligned} J(\beta,\mu) \geq J(\hat{\alpha},\mu) + \mathbb{E}\int_0^T &[H(t,U_t,\mu_t,Y_t,\beta_t) \\ - H(t,X_t,\mu_t,Y_t,\hat{\alpha}_t) &- (U_t - X_t)^T \partial_x H(t,X_t,\mu_t,Y_t,\hat{\alpha}_t) \\ &- (\beta_t - \hat{\alpha}_t)^T \partial_\alpha H(t,X_t,\mu_t,Y_t,\hat{\alpha}_t)]dt. \end{aligned} \quad (10)$$

Step 3:

In this step we are going to prove that if assumption (A.2) is in force, the Hamiltonian satisfies the same convexity assumption. We start by the convexity assumption of f

$$\begin{aligned} &f(t,x',\mu,\alpha') - f(t,x,\mu,\alpha) \\ &\qquad - \langle (x'-x,\alpha'-\alpha), \partial_{(x,\alpha)} f(t,x,\mu,\alpha)\rangle \geq \lambda|\alpha'-\alpha|^2. \end{aligned}$$

We will try to adjust it, to obtain the respective convexity assumption for the Hamiltonian in it.

$$\begin{aligned} &H(t,x',\mu,y',\alpha') - H(t,x,\mu,y,\alpha) \\ &- \langle (x'-x,\alpha'-\alpha), \partial_{(x,\alpha)} H(t,x,\mu,y,\alpha)\rangle - b^T(t,x',\mu,\alpha')y + b^T(t,x,\mu,\alpha)y \\ &\qquad + (x'-x)^T b_1^T(t)y + (\alpha'-\alpha)^T b_2^T(t)y \geq \lambda|\alpha'-\alpha|^2. \end{aligned}$$

Note that we will again denote by y the variable y' without loss of generality. By simple calculations one can see that

$$\begin{aligned} &- b^T(t,x',\mu,\alpha')y + b^T(t,x,\mu,\alpha)y \\ &\qquad + (x'-x)^T b_1^T(t)y + (\alpha'-\alpha)^T b_2^T(t)y = 0, \end{aligned}$$

since by (6) it holds that

$$\begin{aligned} -b^T(t,x',\mu,\alpha')y &+ b^T(t,x,\mu,\alpha)y \\ &= -\left(b_0^T(t,\mu) + b_1^T(t)x'^T + b_2^T(t)\alpha'^T\right) y \\ &\qquad + \left(b_0^T(t,\mu) + b_1^T(t)x^T + b_2^T(t)\alpha^T\right) y \\ &= -b_1^T(t)(x'-x)^T y - b_2^T(t)(\alpha'-\alpha)^T y. \end{aligned}$$

Thus it was proved that the convexity assumption holds for the Hamiltonian, i.e. that

$$\begin{aligned} &H(t,x',\mu,y',\alpha') - H(t,x,\mu,y,\alpha) \\ &\qquad - \langle (x'-x,\alpha'-\alpha), \partial_{(x,\alpha)} H(t,x,\mu,y,\alpha)\rangle \geq \lambda|\alpha'-\alpha|^2. \end{aligned}$$

Step 4:
Using the convexity assumption of the Hamiltonian and applying it in (10), it yields

$$J(\beta,\mu) \geq J(\hat{\alpha},\mu) + \mathbb{E}\int_0^T \lambda|\beta - \hat{\alpha}|^2 dt$$

which completes the proof. □

Remark: From the above one can conclude that the optimal strategy, if it exists, it is also unique. This can be seen easily, since for two optimal solution, say $\hat{\alpha}_t^1$ and $\hat{\alpha}_t^2$ it would hold that

$$J(\hat{\alpha}^1,\mu) + \lambda\mathbb{E}\int_0^T |\alpha_t^2 - \hat{\alpha}_t^1|^2 dt \leq J(\alpha_t^2,\mu),$$

and at the same time

$$J(\hat{\alpha}^2,\mu) + \lambda\mathbb{E}\int_0^T |\alpha_t^1 - \hat{\alpha}_t^2|^2 dt \leq J(\alpha_t^1,\mu),$$

which can only be true if the two strategies coincide.
Remark: It can also be seen that the solution (X_t, Y_t, Z_t) will also be unique as well. As we saw, the optimal strategy function $\hat{\alpha}$ comes from the solution of the Hamiltonian. To obtain the optimal solution of the game, we then have to apply the solution of the FBSDE (8) to the $\hat{\alpha}$ function. From this, one can conclude, that if we have two sets of solutions of the FBSDE (X_t, Y_t, Z_t) and (X'_t, Y'_t, Z'_t), then it holds that

$$\hat{\alpha}(t, X_t, \mu_t, Y_t) = \hat{\alpha}(t, X'_t, \mu_t, Y'_t), \qquad d\mathbb{P}\otimes dt \; a.e..$$

Thus, by the Lipschitz property of b and σ, the coefficients of the forward equation, we can easily conclude that the X_t and X'_t will coincide. Similarly, we can conclude that the same is true also for the pairs Y_t and Y'_t, as well as for Z_t and Z'_t.

Proposition 2.7. *Under the same assumptions and notation as in Theorem 2.6 above, if we consider in addition another measurable and bounded mapping*

$$\begin{gathered}[0,T] \hookrightarrow \mathcal{P}_2(\mathbb{R}^d)\\ t \to \mu'_t\end{gathered}$$

and the controlled diffusion $U' = (U'_t)_{0\leq t\leq T}$ defined by

$$U'_t = x'_0 + \int_0^t b(s, U'_s, \mu'_s, \beta_s)ds + \sigma W_t, \qquad t \in [0,T],$$

for an initial condition $x_0' \in \mathbb{R}^d$ possibly different from x_0, then,

$$J(\hat{\alpha}, \mu) + (x_0' - x_0)^T Y_0 + \lambda \mathbb{E} \int_0^T |\beta_t - \hat{\alpha}_t|^2 dt$$
$$\leq J([\beta, \mu'], \mu) + \mathbb{E}\bigg[\int_0^T \big(b_0(t, \mu_t') - b_0(t, \mu_t)\big)^T Y_t dt \bigg], \tag{11}$$

where

$$J([\beta, \mu'], \mu) = \mathbb{E}\bigg[g(U_T', \mu_T) + \int_0^T f(t, U_t', \mu_t, \beta_t) dt \bigg].$$

The parameter $[\beta, \mu']$ in the cost $J([\beta, \mu'], \mu)$ indicates that the flow of measures in the drift of U' is $(\mu_t')_{0 \leq t \leq T}$ whereas the flow of measures in the cost functions is $(\mu_t)_{0 \leq t \leq T}$. In fact, we should also indicate that the initial condition x_0' might be different from x_0, but we prefer not to specify this, since there is no risk of confusion in the sequel. Also, when $x_0' = X_0$ and $\mu_t' = \mu_t$ for any $t \in [0, T]$, $J([\beta, \mu'], \mu) = J(\beta, \mu)$.

Proof. The proof of the theorem follows the same idea as in (2.6), where the Ito formula is applied on

$$\bigg((U_t' - X_t)^T Y_t + \int_0^T [f(s, U_s', \mu_s', \beta_s) - f(s, X_s, \mu_s, \hat{\alpha}_s)] ds \bigg)_{0 \leq t \leq T}.$$

Since the initial points of the processes U_t and X_t are two possibly different points x_0 and x_0', the function $h(U_0, X_0, Y_0)$ will not cancel out, as in the proof of (2.6) and will thus yield the additional term $(x_0 - x_0')Y_t$ on the left hand side of the statement. The second additional term $\int_0^T \big(b_0(t, \mu_t') - b_0(t, \mu_t)\big)^T Y_t \, dt$ comes from the fact that the drifts of the two processes U_t and X_t follow the different probability distributions μ_t and μ_t'. □

In the following sections, we will see that the above mentioned FBSDE, as well as its mean field version (which will come up later), are key to solving the stochastic control problem when the distribution μ_t is not fixed.

2.3.1 The general FBSDE is uniquely solvable

Theorem (2.6) proves that the solutions of the FBSDE and the simple SDE provide us with our optimal strategy, which is a very important result. However, this theorem also has many conditions that need to be true.

We have already seen that the Hamiltonian H has a minimizer. We also work on the same setting i.e. having the exact FBSDE, the minimizing function and the corresponding SDE, thus these condition are also true. It remains to prove that the FBSDE also has a solution (X_t, Y_t, Z_t). This is exactly what we will do in this section, starting with the next theorem. The following theorem can be found in [5].

Lemma 2.8. *Given* $\mu \in \mathcal{P}_2\left(\mathcal{C}\left([0,T],\mathbb{R}^d\right)\right)$ *with marginal distribution* $(\mu_t)_{0\le t\le T}$ *the FBSDE (2) is uniquely solvable.*

Proof. Since the assumption (A.3) is true, we can take advantage of the simplified form of the function b. Thus we can simplify the Hamiltonian and by simple calculations we can see that

$$\partial_x H(t,x,\mu,y,\alpha) = b_1^T(t)y + \partial_x f(t,x,\mu,\alpha).$$

Note that in the FBSDE (2) the minimizer $\hat{\alpha}$ of H has already been induced. Then, by Lemma (2.5) we conclude that the function $(t,x,\mu,y) \hookrightarrow \hat{\alpha}(t,x,\mu,y)$ is Lipschitz continuous with respect to (x,y) and uniformly in (t,μ). Considering also the form of $\partial_x H$ and the above properties of $\hat{\alpha}$ we know that $\partial_x H$ is also Lipschitz continuous with respect to (x,y) and uniform in t.

Then, by standard results in FBSDE theory (see for example Theorem (5.1) in [15]) we know that there exists a $T_0 > 0$ s.t. for all $t_0 \in (0,T_0]$ and any $x \in \mathbb{R}^d$, the FBSDE (2) has a unique solution. On the other hand, when T is arbitrary and not necessary small, there exists $\delta > 0$ depending on the Lipschitz constant of the coefficients in the variable x and y such that the FBSDE is uniquely solvable on $[T-\delta,T]$ given that the starting point $x_0 \in \mathbb{R}^d$ of X_t occurs at some time $t_0 \in [T-\delta,T]$.

Obviously, as in the case of the DPP, the initial time and point (t_0,x_0) are important. Thus, we define our above solution in each set $[T-\delta,T]$ to be $(X_t^{t_0,x_0}, Y_t^{t_0,x_0}, Z_t^{t_0,x_0})_{t_0\le t\le T}$, where $t_0 = T-\delta$.

Using Theorem (2.6) from ([10]) we conclude that the existence and the uniqueness of the solution holds for the whole $[0,T]$, provided that

$$\forall x_0, x_0' \in \mathbb{R}^d \qquad |Y_{t_0}^{t_0,x_0} - Y_{t_0}^{t_0,x_0'}|^2 \le c|x_0 - x_0'|^2,$$

for some constant c, independent of t_0 and δ.

To finish the proof, we will prove that even under only the assumption (A.1)-(A.4) the necessary condition holds. By Blumenthal's Zero-One-Law one can see that the random variables $Y_{t_0}^{t_0,x_0}$ and $Y_{t_0}^{t_0,x_0'}$ are deterministic. Since assumptions (A.1)-(A.4) are in force, we can use the conclusion of Proposition (2.7) and apply our solution $(X_{t_0}^{t_0,x_0}, Y_{t_0}^{t_0,x_0}, Z_{t_0}^{t_0,x_0})$ two times to the inequality (11), where the second time we interchange x_0 and x_0' which gives us

$$\hat{J}^{t_0,x_0} + \langle x_0' - x_0, Y_{t_0}^{t_0,x_0}\rangle + \lambda\mathbb{E}\int_{t_0}^T |\hat{\alpha}_t^{t_0,x_0} - \hat{\alpha}_t^{t_0,x_0'}|^2 dt \le \hat{J}^{t_0,x_0'},$$

$$\hat{J}^{t_0,x_0'} + \langle x_0 - x_0', Y_{t_0}^{t_0,x_0'}\rangle + \lambda\mathbb{E}\int_{t_0}^T |\hat{\alpha}_t^{t_0,x_0'} - \hat{\alpha}_t^{t_0,x_0}|^2 dt \le \hat{J}^{t_0,x_0},$$

where $\hat{J}^{t_0,x_0} = J\big((\hat{\alpha}_t^{t_0,x_0})_{t_0\le t\le T}\big)$ and $\hat{\alpha}_t^{t_0,x_0} = \hat{\alpha}(t, X_{t_0}^{t_0,x_0}, \mu_t, Y_{t_0}^{t_0,x_0})$. $\hat{J}^{t_0,x_0'}$ and $\hat{\alpha}_t^{t_0,x_0'}$ have been defined accordingly. By summing up the last two inequalities we obtain

$$2\lambda\mathbb{E}\int_{t_0}^{T} |\hat{\alpha}_t^{t_0,x_0'} - \hat{\alpha}_t^{t_0,x_0}|^2 dt \le \langle x_0' - x_0, Y_{t_0}^{t_0,x_0'} - Y_{t_0}^{t_0,x_0}\rangle.$$

Finally, by standard BSDE estimates, which will be proven at the end of the proof, there exists a constant c, independent of t_0 and δ, such that

$$\mathbb{E}[\sup_{t_0\le t\le T} |Y_t^{t_0,x_0} - Y_t^{t_0,x_0'}|^2] \le c\mathbb{E}\int_{t_0}^{T} |\hat{\alpha}_t^{t_0,x_0'} - \hat{\alpha}_t^{t_0,x_0}|^2 dt. \tag{12}$$

Combining the last two inequalities, together with the fact that $Y_{t_0}^{t_0,x_0}$ is deterministic, proves that the necessary condition holds. Thus, we conclude the existence and uniqueness of the solution of the FBSDE (2) in the whole $[0,T]$.

The last step of the proof, will be to prove the estimate in (12). By the construction of the Backward SDE we know that

$$Y_t = \mathbb{E}\left[Y_T + \int_{t_0}^{T} \partial_x H(t, X_s^{t_0,x_0}, \mu, Y_y^{t_0,x_0}, \hat{\alpha}_s^{t_0,x_0}) ds - \int_{t_0}^{T} Z_s dW_s | \mathcal{F}_s\right].$$

Thus, for the difference that we want to estimate, we deduce

$$\begin{aligned}
&|Y_t^{t_0,x_0} - Y_t^{t_0,x_0'}|^2 = \mathbb{E}\Bigg[\Bigg|Y_T^{t_0,x_0} - Y_T^{t_0,x_0'} \\
&+ \int_{t_0}^{T} \Big(\partial_x H(t, X_s^{t_0,x_0}, \mu, Y_y^{t_0,x_0}, \hat{\alpha}_s^{t_0,x_0}) - \partial_x H(t, X_s^{t_0,x_0'}, \mu, Y_y^{t_0,x_0'}, \hat{\alpha}_s^{t_0,x_0'})\Big) ds\Bigg|^2 | \mathcal{F}_s\Bigg],
\end{aligned}$$

where the Ito integral vanishes, since both SDEs have the same diffusion for both processes $Y_t^{t_0,x_0}$ and $Y_t^{t_0,x_0'}$. This now yields

$$\begin{aligned}
\mathbb{E}\sup_{0\le t\le T} |Y_t^{t_0,x_0} - Y_t^{t_0,x_0'}|^2 = \mathbb{E}\Bigg[\Bigg|&Y_T^{t_0,x_0} - Y_T^{t_0,x_0'}| \\
&+ \sup_{0\le t\le T}\int_{t_0}^{T} \Big(\partial_x H(t, X_s^{t_0,x_0}, \mu, Y_y^{t_0,x_0}, \hat{\alpha}_s^{t_0,x_0}) \\
&- \partial_x H(t, X_s^{t_0,x_0'}, \mu, Y_y^{t_0,x_0'}, \hat{\alpha}_s^{t_0,x_0'})\Big) ds\Bigg|^2\Bigg],
\end{aligned}$$

where the conditional expectation becomes normal expectation by the tower property. Thus, we can deduce

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{t_0,x_0}-Y_t^{t_0,x_0'}|^2\right]\le c\mathbb{E}\left[|Y_T^{t_0,x_0}-Y_T^{t_0,x_0'}|^2\right]$$
$$+c(T-t)\int_t^T\mathbb{E}\left[\sup_{0\le t\le T}|Y_s^{t_0,x_0}-Y_s^{t_0,x_0'}|^2\right]$$
$$+c\int_0^T\mathbb{E}|\hat{\alpha}_s^{t_0,x_0}-\hat{\alpha}_s^{t_0,x_0'}|^2ds,$$

which completes the proof of (12). □

2.3.2 The value function

Recall, that our primary goal it so solve the mean field game, i.e. equation (4). According to our main theorem, to find the solution, we need to use the minimizer of the Hamiltonian and then apply the solution (X_t, Y_t, Z_t) to it, to find the optimal strategy of the SDE. More precisely, we need to apply only the processes X_t and Y_t to $\hat{\alpha}$.

One could ask whether one needs to move to the FBSDE every time they have an MFG problem at hand and go into the trouble of solving the FBSDE, in order to find the optimum of the initial problem. Fortunately, the answer is no.

In the next theorem we prove that there exists a function u, that gives us the value of the process Y_t when we apply the process X_t to it. This means, that it is enough to know the minimizer of the Hamiltonian, the value function u and the solution X_t of the forward SDE, in order to find the optimal strategy for the game.

Lemma 2.9. *Assume that $(X_t^{x_0,\mu_t}, Y_t^{x_0,\mu_t}, Z_t^{x_0,\mu_t})_{0\le t\le T}$ is the solution of the FBSDE (8). Then there exists a constant $c>0$, only depending upon the parameters of (A.1-7), and a locally bounded measurable function $u^\mu\colon[0,T]\times\mathbb{R}^d\hookrightarrow\mathbb{R}^d$ such that*

$$\forall x,x'\in\mathbb{R}^d, |u^\mu(t,x')-u^\mu(t,x)|\le c|x-x'|, \tag{13}$$

and $\mathbb{P}$-a.s., for all $t\in[0,T]$, $Y_t^{x_0,\mu_t}=u^\mu(t,X_t^{x_0,\mu_t})$.

Proof. In Lemma (2.8) we proved that since assumption (A.1)-(A.4) hold, it is true that

$$\forall x_0,x_0'\in\mathbb{R}^d\qquad |Y_{t_0}^{t_0,x_0}-Y_{t_0}^{t_0,x_0'}|^2\le c|x_0-x_0'|^2.$$

The above bound triggers the idea that we might be looking for a function u^μ that resembles the process Y_t. Thus, we define

$$u^\mu\colon[0,T]\times\mathbb{R}^d\hookrightarrow\mathbb{R}^d$$
$$u^\mu(t,x)=Y_t^{t,x}.$$

Then from the above inequality, it easily follows that

$$|u^{\mu}(t_0, x_0) - u^{\mu}(t_0, x_0')|^2 \leq c|x_0 - x_0'|^2, \qquad \forall x_0, x_0' \in \mathbb{R}^d,$$

where the constant c is the square root of the constant in the first inequality and which is again independent of t_0 and x_0.

The representation property of Y in terms of X directly follows from the fact that the process X_t is Markov. This can be seen intuitively, since the decision on the new move at time t depends upon the position of the player and not how the player gets there. Mathematically, this can be seen by the fact that the process X_t is the solution of the forward SDE in (8), which according to our assumption has Lipschitz coefficient with respect to x (b is Lipschitz by the assumptions (A.1) and (A.3) and σ is constant and thus also Lipschitz). Thus, Y_t can be represented as a function of X_t, which means that the required representation property is also satisfied (see also Corollary 1.5 of [10]). Moreover, by the definition of the function u^{μ} we can also conclude that it is measurable.

Finally, by the fact that u^{μ} is Lipschitz continuous with respect to x, together with the inequality

$$\sup_{0 \leq t \leq T} |u^{\mu}(t, 0)| < \infty,$$

it follows that u^{μ} is locally bounded, where the last fact is a consequence of the following inequality

$$\sup_{0 \leq t \leq T} |u^{\mu}(t, 0)| = \sup_{0 \leq t \leq T} \left[\mathbb{E}[|u^{\mu}(t, X_t^{0,0}) - u^{\mu}(t, 0)|] + \mathbb{E}[|Y_t^{0,0}|] \right].$$

□

2.4 The mean field FBSDE

Now, that the solution to the standard problem has been found, we can define the mean field problem, where we look for the optimal distribution within the mean field game setting. In order to do this, we move on to define the mean field FBSDE, i.e. the FBSDE where we replace the fixed-"known" distribution μ_t by the family of distributions $\mathbb{P}_{X_t}$. This means that the FBSDE becomes

$$\begin{aligned} dX_t &= b(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t))dt + \sigma dW_t, \\ dY_t &= -\partial_x H(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t))dt + Z_t dW_t, \end{aligned} \tag{14}$$

with $X_0 = x_0 \in \mathbb{R}^d$ and $Y_T = \partial_X g(X_T, \mathbb{P}_{X_t})$.

The goal of this section is to find a distribution (μ_t) such that there exist three stochastic process (X_t, Y_t, Z_t) that satisfy the above system when we induce the distribution (μ_t) to it.

2.4.1 What is a solution?

Before we solve the FBSDE, we need to define what we mean by "solution". In [5] Carmona and Delarue work with the following notion of a solution

Definition 2.10. *Assume that we have a measure* $(\mu_t) \in \mathcal{P}_2(\mathcal{C}([0,T]), \mathbb{R})$ *which generates the stochastic process* $X = X^{x_0,\mu}$ *when applied to the FB-SDE[12], assuming that the initial point is* x_0. *The law of this stochastic process is denoted by* $\mathbb{P}_{X^{x_0,\mu}}$ *and assume the mapping*

$$\begin{aligned}\Phi\colon \mathcal{P}_2(\mathcal{C}([0,T]), \mathbb{R}^d) &\hookrightarrow \mathcal{P}_2(\mathcal{C}([0,T]), \mathbb{R}^d)\\ \mu &\hookrightarrow \mathbb{P}_X^{x_0,\mu}.\end{aligned}$$

Then we say that the measure μ_t *is a solution to (14), if it is a fixed point of* Φ.

In other words, we say that a distribution μ_t is accepted for the mean field FBSDE system, if it coincides with the empirical distribution generated by the dynamics of the positions of the players.

2.4.2 Existence of solution

The above notion of a solution would not be useful at all, would there be no such fixed point of Φ. Thus, we will prove the existence of such a point, by using Schauder's fixed point theorem, which we state next, as it appears in [3], where one can also find the proof.

Theorem 2.11. *[Schauder - Tychonoff] Let* K *be a non-empty compact convex subset of a locally convex Hausdorff linear topological set* E, *and let* T *be a continuous mapping of* K *into itself. Then* T *has a fixed point in* K.

The key to proving the existence of a solution is to take advantage of the convexity of the coefficients, since this will also lead us to compactness. We will first prove that such a fixed point exists for functions f and g that have additionally their partial derivatives ∂f and ∂g be uniformly bounded. The problem is that such functions are unfortunately limited in number. The good thing is that one can take a sequence of such nice functions and use them as approximations of the function f and g that define the mean field FBSDE. Then, this approximation system does indeed have a solution.

We will proceed our analysis by proving that if f and g have their first derivatives bounded, then the mean field FBSDE systems is solvable.

Proposition 2.12. *The mean field FBSDE is solvable, if in addition to the assumption (A.1) to (A.7), we also assume that* $\partial_x f$ *and* $\partial_x g$ *are uniformly bounded, i.e. that for some constant* $c_B > 0$ *it holds that*

$$|\partial_x g(x, \mu)| \leq c_B \text{ and } |\partial_x f(t, x, \mu, \alpha)| \leq c_B,$$

for all $t \in [0,T], x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Proof. In order to prove the existence of the solution we are going to use Schauder's fixed point theorem. To apply it, we are going to work on the space $\mathcal{M}_1(\mathcal{C}([0,T];\mathbb{R}^d))$ of finite signed measure ν of order 1 on $\mathcal{C}([0,T];\mathbb{R}^d)$, which is endowed with the Kantorovich-Rubinstein norm:

$$\|\nu\|_{\mathrm{KR}} = \sup\Big\{ \Big| \int_{\mathcal{C}([0,T];\mathbb{R}^d)} F(w)d\nu(w) \Big| \ ; \ F \in \mathrm{Lip}_1\big(\mathcal{C}([0,T];\mathbb{R}^d)\big) \Big\},$$

for $\nu \in \mathcal{M}_1(\mathcal{C}([0,T];\mathbb{R}^d))$. The reason why we can use this metric for our system, is because it is known to coincide with the Wasserstein distance W_1 on $\mathcal{P}_1(\mathcal{C}([0,T];\mathbb{R}^d))$. For the proof we will work on the space $\mathcal{M}_1(\mathcal{C}([0,T];\mathbb{R}^d))$ and we will prove that there exists a closed convex subset $\mathcal{E} \subset \mathcal{P}_2(\mathcal{C}([0,T];\mathbb{R}^d)) \subset \mathcal{M}_1(\mathcal{C}([0,T];\mathbb{R}^d))$ which is stable for Φ, with a relatively compact range, Φ being continuous on $\mathcal{E}$.

Step 1:

In this step we will prove some a priori estimates for the solution (X_t, Y_t, Z_t) for the solution of the general FBSDE. By the conditions of the theorem we know that the coefficients $\partial_x f$ and $\partial_x g$ are bounded and that the terminal condition in the general FBSDE is bounded. Moreover, the growth of the driver is bounded as follows

$$|\partial_x H\big(t, x, \mu_t, y, \hat{\alpha}(t, x, \mu_t, y)\big)| \leq c_B + c_L|y|.$$

We use standard BSDE and Gronwall's inequality to conclude that there exists a constant c, only depending upon c_B, c_L and T, such that, for any $\mu \in \mathcal{P}_2(\mathcal{C}([0,T];\mathbb{R}^d))$,

$$\forall t \in [0,T], \quad |Y_t^{x_0;\mu}| \leq c$$

holds $\mathbb{P}$-almost surely. Thus, by the estimate for the minimizer $\hat{\alpha}$ found in (7) and the bound in assumption (A.6), we obtain

$$\forall t \in [0,T], \quad \hat{\alpha}\big(t, X_t^{x_0;\mu}, \mu_t, Y_t^{x_0;\mu}\big) \leq c.$$

We can use the above bound for the forward part of the general FBSDE and then we know by standard L^p estimates for SDEs that there exists a constant c', only depending upon c_B, c_L and T, such that also the solution X is bounded by c', i.e.

$$\mathbb{E}\big[\sup_{0\leq t\leq T} |X_t^{x_0;\mu}|^4\big] \leq c'. \tag{15}$$

We consider the restriction of Φ to the subset $\mathcal{E}$ of probability measures of order 4 whose fourth moment is not greater than c', i.e.

$$\mathcal{E} = \big\{\mu \in \mathcal{P}_4\big(\mathcal{C}([0,T],\mathbb{R}^d)\big) : M_{4,\mathcal{C}([0,T],\mathbb{R}^d)}(\mu) \leq c'\big\}.$$

We know that $\mathcal{E}$ is convex and closed for the 1-Wasserstein distance and Φ maps $\mathcal{E}$ into itself.

Step 2:

The two bounds that we found for the drift $\hat{\alpha}$ and the solution X conclude that the family of processes $((X_t^{x_0;\mu})_{0\leq t\leq T})_{\mu\in\mathcal{E}}$ is tight in $\mathcal{C}([0,T];\mathbb{R}^d)$. Tightness then proves that $\Phi(\mathcal{E})$ is relatively compact for the topology of weak convergence of measures. By the bound of X we know that every any weakly convergent sequence $(\mathbb{P}_{X^{x_0;\mu_n}})_{n\geq 1}$, with $\mu_n \in \mathcal{E}$ for any $n \geq 1$, is convergent for the 1-Wasserstein distance. From this we deduce that $\Phi(\mathcal{E})$ is actually relatively compact for the 1-Wasserstein distance on $\mathcal{C}([0,T];\mathbb{R}^d)$.

Step 3:

In this final step we are going to prove that Φ is continuous on $\mathcal{E}$. Using the cost inequality for different measures, proved in Proposition (2.7) and for some measure $\mu' \in \mathcal{E}$ we obtain

$$\begin{aligned} J(\hat{\alpha},\mu)+\lambda\mathbb{E}\int_0^T |\hat{\alpha}_t' - \hat{\alpha}_t|^2 dt& \\ \leq J([\hat{\alpha}_t',\mu'],\mu) + \mathbb{E}\Big[\int_0^T &\big(b_0(t,\mu_t') - b_0(t,\mu_t)\big)^T Y_t dt\Big], \end{aligned} \tag{16}$$

where $\hat{\alpha}_t = \hat{\alpha}(t, X_t^{x_0,\mu}, \mu_t, Y_t^{x_0,\mu})$, for $t \in [0,T]$, and where $\hat{\alpha}_t'$ is defined similarly by replacing μ by μ'. Note that in this case $J(\hat{\alpha},\mu)$ is the cost associated with the flow of measures $(\mu_t')_{0\leq t\leq T}$ and the controlled diffusion process U satisfying

$$dU_t = \big[b_0(t,\mu_t') + b_1(t)U_t + b_2(t)\hat{\alpha}_t\big]dt + \sigma dW_t, \quad t\in[0,T], \quad U_0 = x_0,$$

which is also the reason why in (16) the term of the initial points has vanished. The cost $J(\hat{\alpha},\mu')$ is associated with the flow of measures $(\mu_t)_{0\leq t\leq T}$ and the diffusion process $X^{x_0,\mu}$. The optimality of $\hat{\alpha}'$ for the cost functional $J(\cdot;\mu')$ then results in

$$J\big(\big[\hat{\alpha}',\mu'\big],\mu\big) \leq J\big(\hat{\alpha},\mu'\big) + J\big(\big[\hat{\alpha}',\mu'\big],\mu\big) - J\big(\hat{\alpha}',\mu'\big).$$

Applying this into (16) we deduce

$$\begin{aligned} \lambda\mathbb{E}\int_0^T |\hat{\alpha}_t' - \hat{\alpha}_t|^2 dt \leq J\big(\hat{\alpha},\mu'\big) - J(\hat{\alpha},\mu) + J\big(\big[\hat{\alpha}',\mu'\big],\mu\big)& \\ - J\big(\hat{\alpha}',\mu'\big) + \mathbb{E}\Big[\int_0^T \big(b_0(t,\mu_t') - b_0(t,\mu_t)\big)^T Y_t dt\Big]&, \end{aligned} \tag{17}$$

By Gronwall's lemma, there exists a constant c such that

$$\mathbb{E}\big[\sup_{0\leq t\leq T} |X_t^{x_0,\mu} - U_t|^2\big] \leq c\int_0^T W_2^2(\mu_t,\mu_t')dt.$$

Using the above we can now compare $J(\hat{\alpha}, \mu')$ with $J(\hat{\alpha}, \mu)$ (and respectively $J(\hat{\alpha}', \mu')$ with $J([\hat{\alpha}', \mu'], \mu)$). Since μ and μ' are both in $\mathcal{E}$, we can use the bound for the minimizer $\hat{\alpha}$ and the bound (15), as well as the last part from assumption (A.5) to obtain

$$J\big(\hat{\alpha}; \mu'\big) - J\big(\hat{\alpha}; \mu\big) \leq c\bigg(\int_0^T W_2^2(\mu_t, \mu_t')dt\bigg)^{1/2}.$$

A similar bound can be proved in the same way for $J([\hat{\alpha}', \mu']; \mu) - J(\hat{\alpha}'; \mu')$, where the argument is even simpler as the costs are driven by the same processes. Applying this on (17), using the a priori estimate for $Y_t^{x_0,\mu}$and applying Gronwall's lemma allows us to go back to the controlled SDEs and it yields

$$\mathbb{E}\int_0^T |\hat{\alpha}_t' - \hat{\alpha}_t|^2 dt + \mathbb{E}\big[\sup_{0\leq t\leq T} |X_t^{x_0;\mu} - X_t^{x_0;\mu'}|^2\big] \leq c\bigg(\int_0^T W_2^2(\mu_t, \mu_t')dt\bigg)^{1/2}.$$

Remember that the following holds

$$W_1(\Phi(\mu), \Phi(\mu')) \leq \mathbb{E}\left[\sup_{0\leq t\leq T} |X_t^{x_0;\mu} - X_t^{x_0;\mu'}|\right].$$

As a result and since probability measures in $\mathcal{E}$ have bounded moments of order 4, the Cauchy-Schwartz inequality yields

$$W_1(\Phi(\mu), \Phi(\mu')) \leq c\bigg(\int_0^T W_2^2(\mu_t, \mu_t')dt\bigg)^{1/4} \leq c\bigg(\int_0^T W_1^{1/2}(\mu_t, \mu_t')dt\bigg)^{1/4},$$

which shows that Φ is continuous on $\mathcal{E}$ with respect to the 1-Wasserstein distance W_1 on $\mathcal{P}_1(\mathcal{C}([0,T]; \mathbb{R}^d))$. □

Thus, we have to find a way to use the above theorem for our setting. The way to do this, is by using function f^n and g^n that satisfy the assumptions (A.1)-(A.7) and have furthermore uniformly bounded first derivatives $\partial_x f^n$ and $\partial_x g^n$. In the next theorem, Carmona and Delarue show that if we have such functions, they do indeed solve the mean field FBSDE (14).

Lemma 2.13. *If there exist two sequences $(f^n)_{n\geq 1}$ and $(g^n)_{n\geq 1}$ such that*

(i) there exist two parameters λ' and c_L' such that, for any $n \geq 1$, f^n and g^n satisfy (A.1)-(A.7) with respect to λ' and c_L',

(ii) f^n (resp. g^n) converges towards f (resp. g) uniformly on any bounded subset of $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$ (resp. $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$),

(iii) for any $n \geq 1$, equation (14), with $(\partial_x f, \partial_x g)$ replaced by $(\partial_x f^n, \partial_x g^n)$, has a solution which we denote by (X^n, Y^n, Z^n).

Then, equation (14) is solvable.

Proof. Since this theorem has to do with the approximations f^n and g^n, we will also work with approximations of the other tools as well. For any $n \geq 1$, we define the approximated Hamiltonian

$$H^n(t, x, \mu, y, \alpha) = b^T(t, x, \mu, \alpha)y + f^n(t, x, \mu, \alpha),$$

and its minimizer $(\hat{\alpha}^n(t, x, \mu, y))_{0 \leq t \leq T}$. We define the candidate for approximated optimal strategy as $\hat{\alpha}_t^n = \hat{\alpha}^n(t, X_t^n, \mathbb{P}_{X_t^n}, Y_t^n)$ for any $t \in [0, T]$, where $(X_t^n, Y_t^n, Z_t^n)_{0 \leq t \leq T}$ is the solution of the approximated FBSDE. Since X_t^n gives the dynamics of the states of the player when the strategy $\hat{\alpha}^n$ is followed, we can use theorem (2.6) to compare the resulting optimal cost to the cost generated when some strategy $(\beta_t^n)_{0 \leq t \leq T}$ is being followed, so that we obtain

$$J(\hat{\alpha}_t^n, \mathbb{P}_{X_t^n}) + \lambda \mathbb{E} \int_0^T |\beta_t^n - \hat{\alpha}_t^n|^2 dt \leq J(\beta_t^n, \mathbb{P}_{X_t^n}). \tag{18}$$

We define by $(U_t^n)_{0 \leq t \leq T}$ the outcome process when the player follows strategy β_t^n. The proof can be split in the three following steps. Our first two steps consist in proving that

$$\sup_{m \geq 1} \mathbb{E} \left[\int_0^T |\hat{\alpha}_s^n|^2 ds \right] < +\infty.$$

which can used to prove that the processes $(X^n)_{n \geq 1})$ are tight.

Step 1:
For this step we consider the strategy β^n to be such that $\beta_s^n = \mathbb{E}(\hat{\alpha}_t^n)$ for $0 \leq t \leq T$. In this case, by the affinity of b in (6), for $t \in [0, T]$ we obtain

$$U_t^n = x_0 + \int_0^T [b_0(s, \mathbb{P}_{X_s^n}) + b_1(s)U_s^n + b_2(s)\mathbb{E}(\hat{\alpha}_s^n)]ds + \sigma dW_t.$$

The last equation shows that both U_t^n and X_t^n have the same expectation for $t \in [0, T]$. As a result

$$[U_t^n - \mathbb{E}(U_t^n)] = \int_0^T b_1(s)[U_s^n - \mathbb{E}(U_s^n)]ds + \sigma dW_t.$$

We proceed to the cost function, and for this reason we will work with the approximation functions g^n and f^n, with the same distribution $\mu_t = \mathbb{P}_{X_t^n}$. Now, we will replace the cost functions in (18) and thus we obtain

$$\begin{aligned} \mathbb{E}\left[g^n(X_T^n, \mathbb{P}_{X_T^n})\right] + & \mathbb{E}\left[\int_0^T \left(\lambda|\beta_s^n - \hat{\alpha}_s^n|^2 + f^n(s, X_s^n, \mathbb{P}_{X_s^n}, \hat{\alpha}_s^n\right) ds\right] \\ & \leq \mathbb{E}\left[g^n(U_T^n, \mathbb{P}_{X_T^n}) + \int_0^T f^n(s, U_s^n, \mathbb{P}_{X_s^n}, \beta_s^n) ds\right]. \end{aligned}$$

Using the fact that $\beta_s^n = \mathbb{E}(\hat{\alpha}_t^n)$ and assumption (A.2) which states that g is convex, yields

$$g^n(\mathbb{E}\left[X_T^n\right], \mathbb{P}_{X_T^n}) + \mathbb{E}\left[\int_0^T \left(\lambda|\mathbb{E}(\hat{\alpha}_s^n) - \hat{\alpha}_s^n|^2 + f^n(s, X_s^n, \mathbb{P}_{X_s^n}, \hat{\alpha}_s^n\right) ds\right]$$
$$\leq g^n(\mathbb{E}\left[U_T^n\right], \mathbb{P}_{X_T^n}) + \mathbb{E}\left[\int_0^T f^n(s, U_s^n, \mathbb{P}_{X_s^n}, \mathbb{E}(\hat{\alpha}_s^n))ds\right].$$

Using also the convexity assumption for f as it was stated in (A.4) we obtain

$$g^n(\mathbb{E}\left[X_T^n\right], \mathbb{P}_{X_T^n}) + \left[\int_0^T \left(\lambda \operatorname{Var}(\hat{\alpha}_s^n) + f^n(s, \mathbb{E}(X_s^n), \mathbb{P}_{X_s^n}, \mathbb{E}(\hat{\alpha}_s^n)\right) ds\right]$$
$$\leq g^n(\mathbb{E}\left[U_T^n\right], \mathbb{P}_{X_T^n}) + \left[\int_0^T f^n(s, \mathbb{E}(U_s^n), \mathbb{P}_{X_s^n}, \mathbb{E}(\hat{\alpha}_s^n))ds\right],$$

since $\mathbb{E}(|\mathbb{E}(\hat{\alpha}_s^n) - \hat{\alpha}_s^n|^2) = \operatorname{Var}(\hat{\alpha}_s^n)$. Then by assumption (A.5) we conclude that there exists some constant c, possibly different from line to line and which depends only on λ, c_L, x_0 and T so that

$$\int_0^T \operatorname{Var}(\hat{\alpha}_s^n)ds \leq c\left(1 + \mathbb{E}\left[|U_T^n|^2\right]^{1/2} + \mathbb{E}\left[|X_T^n|^2\right]^{1/2}\right)\mathbb{E}\left[|U_T^n - \mathbb{E}(X_T^n)|^2\right]^{1/2}$$
$$+ \int_0^T \left(1 + \mathbb{E}\left[|U_s^n|^2\right]^{1/2} + \mathbb{E}\left[|X_s^n|^2\right]^{1/2} + \mathbb{E}\left[|\hat{\alpha}_s^n|^2\right]^{1/2}\right)$$
$$\times \mathbb{E}\left[|U_s^n - \mathbb{E}(X_s^n)|^2\right]^{1/2} ds.$$

By definition $\mathbb{E}(X_t^n) = \mathbb{E}(U_t^n)$ for $0 \leq t \leq T$, therefore since the variance of $(U_t^n)_{0\leq t\leq T})$ is uniformly bounded, we obtain

$$\int_0^T \operatorname{Var}(\hat{\alpha}_s^n)dt \leq c\left[1 + \sup_{0\leq s\leq T} \mathbb{E}\left[|X_s^n|^2\right]^{1/2} + \left(\mathbb{E}\int_0^T |\hat{\alpha}_s^n|^2 dt\right)^{1/2}\right].$$

We also know that

$$\sup_{0\leq s\leq T} \mathbb{E}\left[|X_s^n|^2\right] \leq c\left[1 + \mathbb{E}\int_0^T |\hat{\alpha}_s^n|^2 ds\right], \tag{19}$$

so that we can use it, in combination with the linearity of the dynamics of X^n and Gronwall's inequality, to obtain

$$\sup_{0\leq s\leq T} \operatorname{Var}(X_s^n) \leq c\left[1 + \left(\mathbb{E}\int_0^T |\hat{\alpha}_s^n|^2 ds\right)^{1/2}\right],$$

The last bound is used to control the Wasserstein distance between distribution of X_s^n and the Dirac mas at the point $\mathbb{E}(X_s^n)$ for all $\in [0, T]$.

Step 2:
In this step we will use the process U_t^n controlled by the strategy $\beta_t^n = 0$, so the process becomes

$$\tilde{U}_t^n = x_0 + \int_0^T [b_0(s, \mathbb{P}_{X_s^n}) + b_1(s)\tilde{U}_s^n]ds + \sigma dW_t,$$

for all $t \in [0, T]$. We use the boundedness of b_0 defined in assumption (A.5), so that we have

$$\sup_{n\geq 1} \mathbb{E}[\sup_{0\leq t\leq T} |\tilde{U}_t^n|^2] < +\infty.$$

We use the comparison on the costs from theorem (2.6) and the convexity of g^n and f^n, as in the first step, which yields

$$\begin{aligned} g^n(\mathbb{E}\left[X_T^n\right], \mathbb{P}_{X_T^n}) + &\int_0^T \left(\lambda\mathbb{E}(|\hat{\alpha}_s^n|^2) + f^n(s, \mathbb{E}(X_s^n), \mathbb{P}_{X_s^n}, \mathbb{E}(\hat{\alpha}_s^n)\right) ds \\ &\leq \mathbb{E}\left[g^n(\tilde{U}_T^n, \mathbb{P}_{X_T^n}) + \int_0^T f^n(s, \tilde{U}_s^n, \mathbb{P}_{X_s^n}, 0)ds\right]. \end{aligned}$$

Assumption (A.2) provides us with the convexity of f^n with respect to the strategy and so we can combine it with the assumption (A.6) for $f^n(s, U_s^n, \mathbb{P}_{X_s^n}, 0)$ which yields

$$\begin{aligned} g^n(\mathbb{E}\left[X_T^n\right], \delta_{\mathbb{E}(X_T^n)}) + &\int_0^T \left(\lambda\mathbb{E}(|\hat{\alpha}_s^n|^2) + f^n(s, \mathbb{E}(X_s^n), \mathbb{P}_{X_s^n}, 0\right) ds \\ &\leq \mathbb{E}\left[g^n(\tilde{U}_T^n, \mathbb{P}_{X_T^n}) + \int_0^T f^n(s, \tilde{U}_s^n, \mathbb{P}_{X_s^n}, 0)ds\right] + c\mathbb{E}\int_0^T |\hat{\alpha}_s^n|, \end{aligned}$$

for some constant c, with a value that might change from line to line, independent of n. By assumption (A.5) and letting the constant c change it value from line to line, we deduce

$$\begin{aligned} g^n(\mathbb{E}\left[X_T^n\right], \delta_{\mathbb{E}(X_T^n)}) + &\int_0^T \left(\lambda\mathbb{E}(|\hat{\alpha}_s^n|^2) + f^n(s, \mathbb{E}(X_s^n), \delta_{\mathbb{E}(X_T^n)}, 0\right) ds \\ \leq g^n(0, \delta_{\mathbb{E}(X_T^n)}) + &\int_0^T f^n(s, 0, \delta_{\mathbb{E}(X_T^n)}, 0)ds + c\mathbb{E}\int_0^T |\hat{\alpha}_s^n|ds \\ + c&\left(1 + \sup_{0\leq s\leq T}\left[\mathbb{E}\left[|X_s^n|^2\right]^{1/2}\right]\right)\left(1 + \sup_{0\leq s\leq T}[\mathrm{Var}(X_s^n)]^{1/2}\right). \end{aligned}$$

Thus, we use the inequality (19) and apply Young's inequality so that we obtain

$$\begin{aligned} g^n(\mathbb{E}\left[X_T^n\right], \delta_{\mathbb{E}(X_T^n)}) + &\int_0^T \left(\frac{\lambda}{2}\mathbb{E}(|\hat{\alpha}_s^n|^2) + f^n(s, \mathbb{E}(X_s^n), \delta_{\mathbb{E}(X_T^n)}, 0\right) ds \\ \leq g^n(0, \delta_{\mathbb{E}(X_T^n)}) + &\int_0^T f^n(s, 0, \delta_{\mathbb{E}(X_T^n)}, 0)ds + c\left(1 + \sup_{0\leq s\leq T}[\mathrm{Var}(X_s^n)]\right). \end{aligned}$$

Next, we use the result from the first step, to bound the last term by the integral of the expectation of the strategy and so we conclude

$$\begin{aligned}
&g^n(\mathbb{E}\left[X_T^n\right], \delta_{\mathbb{E}(X_T^n)}) + \int_0^T \left(\frac{\lambda}{2}\mathbb{E}(|\hat{\alpha}_s^n|^2) + f^n(s, \mathbb{E}(X_s^n), \delta_{\mathbb{E}(X_T^n)}, 0\right) ds \\
\leq &g^n(0, \delta_{\mathbb{E}(X_T^n)}) + \int_0^T f^n(s, 0, \delta_{\mathbb{E}(X_T^n)}, 0) ds + c\left(1 + \left[\int_0^T \mathbb{E}\left(|\hat{\alpha}_s^n|^2\right) ds\right]^{1/2}\right).
\end{aligned}$$

By assumption (A.2) we know that g^n and f^n are convex and applying also Young's inequality we obtain

$$\begin{aligned}
&(\mathbb{E}(X_T^n))^T \partial_x g^n(0, \delta_{\mathbb{E}(X_T^n)}) \\
&\qquad + \int_0^T \left[\frac{\lambda}{4}\mathbb{E}\left(|\hat{\alpha}_s^n|^2\right) + \left(\mathbb{E}(X_s^n)\right)^T \partial_x f^n(s, 0, \delta_{\mathbb{E}(X_s^n)}, 0)\right] ds \leq c.
\end{aligned}$$

The last of our basic assumptions in the paper, gives us

$$\mathbb{E}\left(|\hat{\alpha}_s^n|^2\right) \leq c\left(1 + \sup_{0\leq s\leq T} \mathbb{E}\left[|X_s^n|^2\right]^{1/2}\right),$$

so that the desired bound (2.4.2) results from (19). This bound then results in

$$\mathbb{E}\left[\sup_{0\leq s\leq T} |X_s^n|^2\right] \leq c. \tag{20}$$

Thus using the last two results of (20) and (2.4.2) we can conclude that the processes $(X^n)_{n\geq 1}$ are tight and thus a convergent subsequence can be found.

Step 3:
Assume that the empirical distribution of the convergent subsequence $(X^{n_p})_{p\geq 1}$ is $(\mathbb{P}_{X^{n_p}})$, and define its limit as μ. Since the supremum of the process X_s^n has be found in (20) to be bounded, we get

$$M_{2,\mathcal{C}([0,T],\mathbb{R}^d)}(\mu) < +\infty.$$

Thus we can use Lemma (2.8) and we now that the general FBSDE has a unique solution, which we define as $(X_t, Y_t, Z_t)_{0\leq t\leq T}$. By Lemma (2.9) we know that the value function $u : [0,T] \times \mathbb{R}^d \hookrightarrow \mathbb{R}^d$ exists and is moreover c-Lipschitz with respect to the variable x for the same constant as in thte statement of the theorem. Therefore, as we proved, it holds that for all $t \in [0,T]$ $Y_t = u(t, X_t)$. More specifically, it holds that

$$\sup_{0\leq t\leq T} |u(t,0)| \leq \sup_{0\leq t\leq T} \left[\mathbb{E}\left[|u(t,X_t) - u(t,0)|\right] + \mathbb{E}[|Y_t|]\right] < +\infty. \tag{21}$$

From the above we conclude that there even exists a constant c' so that

$$|u(t,x)| \le c'(1+|x|), \qquad 0 \le t \le T,\ x \in \mathbb{R}^d. \tag{22}$$

Last but not least, the bound (7) for the optimal strategy, together with the assumption (A.6) result in the fact that the same bound holds for the optimal strategy, i.e. that

$$|\hat{\alpha}(t,x,\mu_t,u(t,x))| \le c'(1+|x|),$$

where the constant c' could have a different value than before. Using this bound in the forward SDE from general FBSDE system, we obtain that

$$\forall \mathcal{L} \ge 1, \qquad \mathbb{E}\left[\sup_{0\le t\le T} |X_t|^{\mathcal{L}}\right] < +\infty. \tag{23}$$

This result then ensures that the optimal strategy remains an admissible strategy even when we plug the solution (X_t, Y_t, Z_t) in it, that is

$$\mathbb{E}\int_0^T |\hat{\alpha}(t,X_t,\mu_t,Y_t)|^2 dt < +\infty, \qquad t \in [0,T].$$

The same argument can be used to prove the same for the approximation solution $(X_t^n)_{0\le t\le T}$. We now use the fact that

$$\sup_{n\ge 1}\mathbb{E}\left[\sup_{0\le t\le T} |X_t^n|^{\mathcal{L}}\right] < +\infty, \tag{24}$$

for all $\mathcal{L} \ge 1$ which will be proved in the final step. Following the same train of thought as in the proof of (17), we obtain

$$\begin{aligned}\lambda\mathbb{E}\int_0^T |\hat{\alpha}_t^n - \hat{\alpha}_t|^2 dt \le & J^n(\hat{\alpha},\mu^n) - J(\hat{\alpha},\mu) + J([\hat{\alpha}^n,\mu^n],\mu) - J^n(\hat{\alpha}^n,\mu^n) \\ & + \mathbb{E}\int_0^T \left(b_0(t,\mu_t^n) - b_0(t,\mu_t)\right)^T Y_t dt. \end{aligned}\tag{25}$$

For the notation difference of $J([\hat{\alpha}^n,\mu^n],\mu)$ and $J(\hat{\alpha}^n,\mu^n)$ can be seen clearly in Proposition (2.7) and $J(\hat{\alpha}^n,\mu^n)$ has as similar definition as $J(\hat{\alpha},\mu)$ except that the approximation function f^n and g^n are used. We define the process $(U_t^n)_{0\le t\le T}$ for $n \ge 1$ as follows

$$dU_t^n = b(t,U_T^n,\mu_t^n,\hat{\alpha}_t)dt + \sigma dW_t,$$

with the initial condition $U_0^n = x_0$ and where the drift b is again affine and has the form defined in (6). From the above we conclude

$$\begin{aligned}J^n(\hat{\alpha},\mu^n) - J(\hat{\alpha},\mu) = & \mathbb{E}\left[g^n(U_T^n,\mu_T^n) - g(X_T,\mu_T)\right] \\ & + \mathbb{E}\int_0^T \left[f^n(t,U_t^n,\mu_t^n,\hat{\alpha}_t) - f(t,X_t,\mu_t,\hat{\alpha}_t)\right] dt.\end{aligned}$$

We use Gronwall' inequality and the convergence of the μ^{n_p} to μ for the 2-Wasserstein distance, we claim that

$$U^{n_p} \xrightarrow{p\to+\infty} X,$$

for the norm $\mathbb{E}\left[\sup_{0\le s\le t}|\cdot_s|^2\right]^{1/2}$. By

- the uniform convergence of g^n and f^n towards f and g on bounded subsets of their respective domains, as stated in the theorem,
- the convergence of μ^{n_p} towards μ,
- the bound in (23), and
- the admissibility of the strategy $\hat{\alpha}(t, X_t, \mu_t, Y_t)$,

we conclude that

$$J^{n_p}(\hat{\alpha}, \mu^{n_p}) \xrightarrow{p\to+\infty} J(\hat{\alpha}, \mu).$$

Similarly, we use

- the finiteness of $\mathbb{E}\left[\sup_{0\le t\le T}|X_t|^{\mathcal{L}}\right]$,
- the finiteness proved in (24), and
- the finiteness of $\sup_{m\ge 1}\mathbb{E}\left[\int_0^T|\hat{\alpha}_s^n|^2 ds\right]$

to conclude that

$$\begin{aligned} b_0(t,\mu_t^n) - b_0(t,\mu_t) &\xrightarrow{p\to+\infty} 0, \\ J([\hat{\alpha}^n,\mu^n],\mu) - J^n(\hat{\alpha}^n,\mu^n) &\xrightarrow{p\to+\infty} 0. \end{aligned}$$

By (25), we thus deduce

$$\hat{\alpha}^{n_p} \xrightarrow{p\to+\infty} \hat{\alpha},$$

in $L^2([0,T]\times\Omega, dt\otimes d\mathbb{P})$. This proves that X is the limit of the subsequence $(X^{n_p})_{g\ge 1}$ for the norm $\mathbb{E}\left[\sup_{0\le s\le t}|\cdot_s|^2\right]^{1/2}$. This finally yields the conclusion that $\mu = \mathbb{P}_{X_t}$ and therefore the mean field FBSDE (14) is solvable.

Step 4:
This step consists of the proof of

$$\sup_{n\ge 1}\mathbb{E}\left[\sup_{0\le t\le T}|X_t^n|^m\right] < +\infty,$$

for all $m \ge 1$. Notice that the constant c in Theorem (13) is independent of n. Furthermore, as can be seen in (20), the second moments of $\sup_{0\le t\le T}|X_t^n|$ are uniformly bounded for $n \ge 1$. By assumption (A.5) we know that the driver in the backward equations of the general FBSDE system has a driver that is at most of linear growth in (x, y, α), so that the finiteness of $\sup_{n\geq 1} \mathbb{E}\int_0^T |\hat{\alpha}_s^n|^2 ds$ and by standard L^2-estimates for backward SDEs, we conclude that also the second moments of $\sup_{0\leq t\leq T} |Y_t^n|$ are uniformly bounded for $n \geq 1$. Repeating the calculations used to prove (23), we conclude the result. $\square$

So we have found that there exists a solution of the FBSDE and that can be approximated by two nice enough sequences. The next step is to see how one can choose these approximating sequences.

Lemma 2.14. *Assume that, in addition to (A.1)-(A.7), there exists a constant $\gamma > 0$ such that the functions f and g satisfy*

$$\begin{aligned} f(t, x', \mu, \alpha') - f(t, x, \mu, \alpha) - \langle (x' - x, \alpha' - \alpha), \partial_{(x,\alpha)} f(t, x, \mu, \alpha)\rangle \\ \geq \gamma |x' - x|^2 + \lambda |\alpha' - \alpha|^2, \quad (26) \\ g(x', \mu) - g(x, \mu) - \langle x' - x, \partial_x g(x, \mu)\rangle \geq \gamma |x' - x|^2. \end{aligned}$$

Then, there exist two positive constants λ' and c'_L, depending only upon λ,c_L and γ, and two sequences of functions $(f^n)_{n\geq 1}$ and $(g^n)_{n\geq 1}$ such that

(i) for any $n \geq 1$, f^n and g^n satisfy (A.1)-(A.7) with respect to the parameters λ' and c'_L and $\partial_x f^n$ and $\partial_x g^n$ are bounded,

(ii) for any bounded subsets of $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^k$, there exists an integer n_0, such that, for any $n \geq n_0$, f^n and g^n coincide with f and g respectively.

Proof. The proof of this theorem is very technical and without particular interest, thus we will not be presented here. The interested reader is referred to [5]. $\square$

One should compare the above convexity assumptions for f and g with (5) from the assumption (A.2) that we have. For the above approximating sequences $(f^n)_{n\geq 1}$ and $(g^n)_{n\geq 1}$, Lemma (2.13) can be applied and thus one obtains a solution of (14). Next, we state and prove the second main result in this thesis.

Theorem 2.15. *Under assumptions (A.1) to (A.7), the forward-backward system (14) has a solution. Moreover, for any solution $(X_t, Y_t, Z_t)_{0\leq t\leq T}$ to (14), there exists a function $u\colon [0, T] \times \mathbb{R}^d \hookrightarrow \mathbb{R}^k$ (i.e. a value function), satisfying the growth and Lipschitz properties*

$$\begin{aligned} |u(t, x)| &\leq c(1 + |x|), \\ |u(t, x) - u(t, x')| &\leq c|x - x'|, \end{aligned}$$

for all $t \in [0,T]$, for all $x, x' \in \mathbb{R}^d$ for some constant $c \geq 0$, and such that, $\mathbb{P}$-a.s., for all $t \in [0,T]$, $Y_t = u(t, X_t)$. In particular, for any $m \geq 1$, $\mathbb{E}[\sup_{0\leq t\leq T} |X_t|^m] < +\infty$.

Proof. In theorem (2.14) we proved that the mean field FBSDE is uniquely solvable if the assumptions (A.1)-(A.7) hold, and furthermore f and g satisfy the additional convexity assumption (26). Then, we know that there exist two approximating function f^n and g^n that converge to f and g and having bounded partial derivatives. Then by (2.14) we know that the FBSDE system (14) has a solution.

If on the other hand, only the assumption (A.1)-(A.7) are satisfied, without the additional convexity assumption, then one defines the following approximating functions

$$f_n(t, x, \mu, \alpha) = f(t, x, \mu, \alpha) + \frac{1}{n}|x|^2, \qquad g_n(x, \mu) = g(x, \mu) + \frac{1}{n}|x|^2,$$

for $(t, x, \mu, \alpha) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)^2$ and for $n \geq 1$. Then, we can see that these approximating functions satisfy the assumptions of Lemma (2.13) and thus, the mean field FBSDE is uniquely solvable. Then, for an arbitrary solution of the mean field FBSDE, one can conclude the existence of the value function by Lemma (2.9) and the bound (21). Last but not least, the bound (23) proves the boundedness of the moments of X_t. $\square$

3 Solving the large game

3.1 The large game in connection to the FBSDE

In this section we will try to show the connection between the solution that we got from the FBSDE and the large game that we wanted to solve initially. But first, let us summarize and make the different notations at hand clear.

- **The FBSDE**
  In this thesis we have defined the mean field FBSDE
  $$dX_t = b(t, X_t, \mathbb{P}_{X_t}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t))dt + \sigma dW_t,$$
  $$dY_t = -\partial_x H(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t))dt + Z_t dW_t,$$
  We proved that the above FBSDE system has a unique solution $(X_t, Y_t, Z_t)_{0\leq t\leq T}$ and furthermore that the associate value function $u(t, x)$ exists. We have defined the flow of marginal probability measures $(\mu_t)_{0\leq t\leq T}$ that are considered in this case, which satisfy the condition $\mu_t = \mathbb{P}_{X_t}$. During the analysis we have found a strategy $\hat{\alpha}(t, X_t, \mu_t, u(t, X_t))$ which is the minimizer of the Hamiltonian of the FBSDE. We denote by J the optimal cost
  $$J = \mathbb{E}\left[g(X_T, \mu_T) + \int_0^T f(t, X_t, \mu_t, \hat{\alpha}_(t, X_t, \mu_t, Y_t)dt\right].$$
  Recall that in this setting there are infinitely many players, i.e. we let $N \to +\infty$.

  To sum up, we have proved that this limiting problem provides us with both the distribution μ_t, as well as the strategy $\hat{\alpha}_t$ that minimize the cost function $J(\beta, \mu_t)$, whose minimum we will denote as J.

- **The large game:**
  In the large game setting, we assume that we have N different players, where N is again large, yet finite. This time we have a system of N stochastic differential equations
  $$dU_t^i = b(t, U_t^i, \bar{\nu}_t^N, \beta_t)dt + \sigma dW_t^i, \qquad \bar{\nu}_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{U_t^j}, \tag{27}$$
  with $t \in [0, T]$ and $U_0^i = u_0$ and where β_t is an admissible strategy. This is the notation that we use when the players follow an arbitrary admissible strategy β_t.

  We want to investigate what happens when the N players of the large game, follow the strategies that have been found in the limiting case. Thus, we define the strategies
  $$\bar{\alpha}_t^{N,i} = \hat{\alpha}(t, X_t^i, \mu_t, u(t, X_t^i)), \tag{28}$$

for $0 \leq t \leq T$ and $i \in \{1, 2, \dots, N\}$.

What is important to understand, is that the strategy $\hat{\alpha}$ only depends on the time t, the position X_t of the player at this time and the positions of their opponents through the empirical distribution, but not on the strategies that the rest of the players follow. Thus, all players would essentially follow the same strategy, should they start at the same initial point and play the game under the exact same circumstances (i.e. without the influence of the random factor W_t).

In this setting, and with the assumption that the player i follows strategy $\bar{\alpha}_t^{N,i}$, the dynamics of the their private state will be given by

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, \bar{\alpha}_t^{N,i})dt + \sigma dW_t^i.$$

Note that the above equation is well defined, since as we saw in Lemma (2.5) the minimizer $\hat{\alpha}(t, x, \mu_t, y)$ of the Hamiltonian is Lipschitz continuous and at most of linear growth with respect to the variables x and y, uniformly in $t \in [0, T]$. Moreover, the function u satisfies the properties as defined in (2.15). As in the limiting case, the cost function for the player i when every player follows the strategy $\bar{\alpha}$, is given by

$$J^{i,\mu}(\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_N) = \mathbb{E}\left[g(X_T, \mu_T) + \int_0^T f(t, X_t, \mu_t, \bar{\alpha}_t)dt\right]. \quad (29)$$

3.2 An approximate Nash equilibrium

In this section, we will prove that the solution provided by the FBSDE, forms an approximate Nash equilibrium for the Stochastic Problem (2) at hand. This proof follows the approache of Bensoussan,Sung, Yam and Yung in "Linear quadratic mean field games" (2011) and Cardaliaguet in "Notes on mean field games" (2010).

Theorem 3.1. *Letting all assumptions in section (1.5) hold, the strategies* $(\bar{\alpha}_t^{N,i})_{0 \leq t \leq T,\, i \in \{1,\dots,N\}}$ *defined in (28) form an approximate Nash equilibrium of the N-player game at hand. More precisely, there exists a constant $c > 0$ and a sequence of positive number $(\epsilon_N)_{N \geq 1}$ such that, for each $N \geq 1$,*

i) $\epsilon_N \leq cN^{-1/(d+4)}$,

ii) for any player $i \in \{1, 2, \dots, N\}$ and any progressively measurable strategy $\beta^i = (\beta_t^i)_{0 \leq t \leq T}$ that is admissible, one has

$$\bar{J}^{N,i}(\bar{\alpha}^{N,1}, .., \bar{\alpha}^{N,(i-1)}, \beta^i, \bar{\alpha}^{N,i+1}, .., \bar{\alpha}^{N,N}) \geq \bar{J}^{N,i}(\bar{\alpha}^{N,1}, .., \bar{\alpha}^{N,N}) - \epsilon_N. \quad (30)$$

Proof. Throughout this proof we will use the three processes U_t^i, X_t^i and $\bar{\bar{X}}_t^i$ and their respective empirical distributions. It is important that their definitions are clear and thus we write them here for clarification.

$$
\begin{aligned}
dU_t^i &= b(t, U_t^i, \bar{\nu}_t^N, \beta_t)dt + \sigma dW_t^i, & \bar{\nu}_t^N &= \frac{1}{N}\sum_{j=1}^N \delta_{U_t^j};\\
dX_t^i &= b(t, X_t^i, \bar{\mu}_t^N, \hat{\alpha}_t(t, X_t^i, \bar{\mu}_t^N, u(t, X_t^i)))dt + \sigma dW_t^i, & \bar{\mu}_t^N &= \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j};\\
d\bar{\bar{X}}_t^i &= b(t, \bar{\bar{X}}_t^i, \mu_t^N, \hat{\alpha}_t(t, \bar{\bar{X}}_t^i, \mu_t^N, u(t, \bar{\bar{X}}_t^i)))dt + \sigma dW_t^i, & \bar{\bar{\mu}}_t^N &= \frac{1}{N}\sum_{j=1}^N \delta_{\bar{\bar{X}}_t^j},
\end{aligned}
$$

where the distribution $\bar{\bar{\mu}}_t^N$ is defined to coincide with the empirical distribution μ_t of the solution X_t of the FBSDE.

Step 1: Bounding the random dynamics U_t^i by the random strategy of the first player.

Since the function $\bar{J}$ is symmetric, we only need to prove the theorem for $i = 1$. Thus we assume that the first player follows a progressively measurable strategy β^1 such that $\mathbb{E}\int_0^T |\beta^1|^2 ds < +\infty$ and that the rest of the players follow the admissible strategies $\bar{\alpha}_t^{N,i}$ as they were defined in (28). We apply the above strategies to the quantities U_t^i and $\bar{J}^{N,i}$ as they were introduced in (27) and (29). Then, by boundedness of b_0, b_1 and b_2, as well as by Gronwalls's lemma, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T} |U_t^1|^2\right] &\le c\left(1 + \mathbb{E}\int_0^T |\beta_t^1|^2 dt\right), \\
\mathbb{E}\left[\sup_{0\le t\le T} |U_t^i|^2\right] &\le c\left(1 + \mathbb{E}\int_0^T |\bar{\alpha}_t^{i,N}|^2 dt\right),
\end{aligned}
\tag{31}
$$

where $i \in \{2, \ldots, N\}$. Using the fact that the strategies $(\bar{\alpha}_t^{N,i})$ satisfy the square integrability condition of admissibility, the last inequality becomes

$$
\mathbb{E}\left[\sup_{0\le t\le T} |U_t^i|^2\right] \le c.
$$

Summing up the bounds for all $\mathbb{E}\left[\sup_{0\le t\le T} |U_t^i|^2\right]$ for $i \in \{1, \ldots, N\}$ yields

$$
\sum_{i=1}^N \mathbb{E}\left[\sup_{0\le t\le T} |U_t^i|^2\right] \le c\left(N + \mathbb{E}\int_0^T |\beta_t^1|^2 dt\right),
$$

or equivalently,

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\sup_{0\le t\le T}|U_t^i|^2\right]\le c\left(1+\frac{1}{N}\mathbb{E}\int_0^T|\beta_t^1|^2dt\right), \tag{32}$$

Step 2: The distance of the distributions μ_t and μ_t^N can be estimated.

As the description of this step reveals, we will be working with a copy of X_t. Thus we introduce for all $i\in\{1,\dots,N\}$ a new stochastic process $\bar{\bar{X}}_t$ that occurs when the players follow strategy

$$\bar{\bar{\alpha}}_t^{N,i}=\hat{\alpha}(t,\bar{\bar{X}}_t^i,\mu_t,u(t,\bar{\bar{X}}_t^i)),$$

and which then results to the following system of decoupled independent and identically distributed states

$$d\bar{\bar{X}}_t^i=b(t,\bar{\bar{X}}_t^i,\bar{\bar{\mu}}_t^N,\bar{\bar{\alpha}}_t^{N,i})dt+\sigma dW_t^i,$$

for $0\le t\le T$. Since the processes $\bar{\bar{X}}_t^i$ are by construction independent copies of X^i (note that these processes are a copy of the process from the mean field game, not the approximation X_t^i) for all $i\in\{1,\dots,N\}$, it also holds that $\bar{\bar{\mu}}_t=\mathbb{P}_{\bar{\bar{X}}_t^i}=\mu_t$ for any $t\in[0,T]$. Using the properties of the value function and the uniform boundedness of $(M_{d+5}(\mu_t))_{0\le t\le T}$ proved in Theorem (2.15), together with the useful estimate that can be found in the introduction, one can apply Theorem (3.2) so that we obtain

$$\sup_{1\le i\le N}\mathbb{E}\left(\sup_{0\le t\le T}|X_t^i-\bar{\bar{X}}_t^i|^2\right)\le cN^{\frac{-2}{d+4}},$$

where in our case becomes even stronger so that

$$\max_{1\le i\le N}\mathbb{E}\left(\sup_{0\le t\le T}|X_t^i-\bar{\bar{X}}_t^i|^2\right)\le cN^{\frac{-2}{d+4}}. \tag{33}$$

Recall that X_t^i solves

$$dX_t^i=b(t,X_t^i,\bar{\mu}_t^N,\bar{\alpha}_t^{N,i})dt+\sigma dW_t^i.$$

Moreover, for each $t\in[0,T]$

$$W_2^2(\bar{\mu}_t^N,\mu_t)\le\frac{2}{N}\sum_{i=1}^{N}|X_t^i-\bar{\bar{X}}_t^i|^2+2W_2^2(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i},\mu_t).$$

We take expectations in both sides and use the bound for the maximum in (33) together with the useful estimate presented in the introduction. The above equation then becomes

$$\sup_{0\le t\le T}\mathbb{E}\left[W_2^2(\bar{\mu}_t^N,\mu_t)\right]\le cN^{\frac{-2}{d+4}}. \tag{34}$$

Step 3: The cost $\bar{J}^{N,i}(\bar{\alpha}^{N,1},\dots,\bar{\alpha}^{N,N})$ approximates the optimal cost J.

To prove the next step we will use the local Lipschitz regularity of the coefficients f and g as it was defined in the assumption (A.5) and the Cauchy Schwarz inequality. Thus, for all $i \in \{1,\dots,N\}$ and for a constant $c > 0$, that might change from line to line, we obtain

$$
\begin{aligned}
&|J - \bar{J}^{N,i}(\bar{\alpha}^{N,1},\dots,\bar{\alpha}^{N,N}| \\
&\quad = \Bigg|\mathbb{E}\bigg[g(\bar{\bar{X}}_T^i,\mu_T) + \int_0^T f(t,\bar{\bar{X}}_t^i,\mu_t,\hat{\alpha}_t^i)dt - g(X_T^i,\bar{\mu}_T^N) \\
&\qquad\qquad - \int_0^T f(t,X_t^i,\bar{\mu}_t^N,\bar{\alpha}_t^{N,i})dt\bigg]\Bigg| \\
&\leq c\mathbb{E}\left[1 + |\bar{\bar{X}}_T^i|^2 + |X_T^i|^2 + \frac{1}{N}\sum_{j=1}^N |X_T^j|^2\right]^{\frac{1}{2}} \mathbb{E}\left[|\bar{\bar{X}}_T^i - X_T^i|^2 + W_2^2(\mu_T,\bar{\mu}_T^N)\right]^{\frac{1}{2}} \\
&\qquad + c\int_0^T \Bigg[\mathbb{E}\left[1 + |\bar{\bar{X}}_t^i|^2 + |X_t^i|^2 + |\hat{\alpha}_t^i|^2 + |\bar{\alpha}_t^{N,i}|^2 + \frac{1}{N}\sum_{j=1}^N |X_T^j|^2\right]^{\frac{1}{2}} \\
&\qquad\quad \times \mathbb{E}\left[|\bar{\bar{X}}_t^i - X_t^i|^2 + |\bar{\alpha}_t^{N,i} - \hat{\alpha}_t^i|^2 + W_2^2(\mu_t,\bar{\mu}_t^N)\right]^{\frac{1}{2}}\Bigg]dt.
\end{aligned}
$$

By the boundedness of b_0, the properties of the value function of the mean field game and by the estimate we found for the minimizer strategy in (7), we obtain

$$
\sup_{N\geq 1}\max_{1\leq i\leq N}\left[\mathbb{E}[\sup_{0\leq t\leq T}|X_t^i|^2] + \mathbb{E}\int_0^T |\bar{\alpha}_t^{N,i}|^2 dt\right] < +\infty. \tag{35}
$$

Applying this to the above inequality for the cost function we deduce

$$
\begin{aligned}
|J - \bar{J}^{N,i}(\bar{\alpha}^{N,1},\dots,\bar{\alpha}^{N,N}| &\leq c\mathbb{E}\left[|\bar{\bar{X}}_T^i - X_T^i|^2 + W_2^2(\mu_T,\bar{\mu}_T^N)\right]^{\frac{1}{2}} \\
&\quad + c\left(\int_0^T \mathbb{E}\left[|\bar{\bar{X}}_t^i - X_t^i|^2 + |\bar{\alpha}_t^{N,i} - \hat{\alpha}_t^i|^2 + W_2^2(\mu_t,\bar{\mu}_t^N)\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

Using the Lipschitz properties of the minimizer strategy $\hat{\alpha}_t$ and the value function u, we obtain

$$
\begin{aligned}
|\hat{\alpha}_t^i - \bar{\alpha}_t^{N,i}| &= |\hat{\alpha}_t^i(t,\bar{\bar{X}}_t^i,\mu_t,u(t,\bar{\bar{X}}_t^i)) - \bar{\alpha}_t^{N,i}(t,X_t^i,\mu_t,u(t,X_t^i))| \\
&\leq c|\bar{\bar{X}}_t^i - X_t^i|.
\end{aligned}
$$

By the inequalities (34) and (33) we finally obtain

$$
\bar{J}^{N,i}(\bar{\alpha}^{N,1},\dots,\bar{\alpha}^{N,N}) = J + O(N^{\frac{-1}{d+4}}). \tag{36}
$$

Step 4: The distance between $\bar{\nu}_t$ and μ_t generated by the optimal distribution can be estimated.

The fact that the cost of each player i is almost the optimal cost, when we assume that all players follow the prescribed strategy, leads to the conclusion that it is enough to prove that

$$\bar{J}^{N,i}(\beta^1, \bar{\alpha}^{N,2}, \dots, \bar{\alpha}^{N,N}) \geq J - \epsilon_N.$$

By similar computations as the one that we used to prove (32), it can be shown that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |U_t^1 - X_t^1|^2\right] \leq \frac{c}{N}\int_0^T \sum_{j=1}^N \mathbb{E}\left[\sup_{0\leq r\leq s} |U_r^j - X_r^j|^2\right] ds$$
$$+ c\mathbb{E}\int_0^T |\beta_t^1 - \bar{\alpha}_t^{N,1}| dt$$

for the rebel player. For the rest of the players, for $i \in \{2, \dots, N\}$ it even holds that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |U_t^i - X_t^i|^2\right] \leq \frac{c}{N}\int_0^T \sum_{j=1}^N \mathbb{E}\left[\sup_{0\leq r\leq s} |U_r^j - X_r^j|^2\right] ds.$$

Thus, we apply Gronwall's inequality and obtain

$$\frac{1}{N}\sum_{j=1}^N \mathbb{E}\left[\sup_{0\leq t\leq T} |U_t^j - X_t^j|^2\right] \leq \frac{c}{N}\mathbb{E}\int_0^T \left[\sup_{0\leq r\leq s} |\beta_t^1 - \bar{\alpha}_t^{N,i}|^2\right] ds.$$

Furthermore, when $i \in \{2, \dots, N\}$, we can even conclude that

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|U_t^j - X_t^j|^2\right] \leq \frac{c}{N}\mathbb{E}\int_0^T \left[|\beta_t^1 - \bar{\alpha}_t^{N,i}|^2\right] dt.$$

This last conclusion, combined with (33) and (35) proves that for any $A > 0$, there exists a constant c_A depending on A, such that if

$$\mathbb{E}\int_0^T |\beta_t^1|^2 dt \leq A \Rightarrow \max_{2\leq i\leq N} \sup_{0\leq t\leq T} \mathbb{E}\left(|U_t^i - \bar{\bar{X}}_t^i|^2\right) \leq c_A N^{\frac{-2}{d+4}}. \tag{37}$$

For the next step of the proof we will fix a number A such that $\mathbb{E}\int_0^T |\beta_t^1|^2 dt \leq A$. In the last part of the proof, we discuss why we don't need to take into account the opposite case, where $\mathbb{E}\int_0^T |\beta_t^1|^2 dt > A$. Thus, let us fix some $A > 0$ such that the above condition is satisfied. By (37) we get that

$$\frac{1}{N-1}\sum_{i=2}^N \mathbb{E}\left(|U_t^i - \bar{\bar{X}}_t^i|^2\right) \leq \frac{1}{N-1}\max_{2\leq i\leq N} \sup_{0\leq t\leq T} (N-1)\mathbb{E}\left(|U_t^i - \bar{\bar{X}}_t^i|^2\right)$$
$$\leq c_A N^{\frac{-2}{d+4}},$$

where the constant c_A might change its value from line to line. Next, we will apply the triangle inequality for the Wasserstein distance, so that we obtain

$$\mathbb{E}\left[W_2^2(\bar{\nu}_t^N,\mu_t)\right] \le c\Bigg\{\mathbb{E}\left[W_2^2\left(\frac{1}{N}\sum_{j=1}^N \delta_{U_t^j}, \frac{1}{N-1}\sum_{j=2}^N \delta_{U_t^j}\right)\right]$$
$$+\frac{1}{N-1}\sum_{j=2}^N \mathbb{E}\left[|U_t^j - \bar{\bar{X}}_t^j|^2\right] + \mathbb{E}\left[W_2^2\left(\frac{1}{N-1}\sum_{j=2}^N \delta_{\bar{\bar{X}}_t^j}, \mu_t\right)\right]\Bigg\}. \tag{38}$$

Using the fact that

$$\mathbb{E}\left[W_2^2\left(\frac{1}{N}\sum_{j=1}^N \delta_{U_t^j}, \frac{1}{N-1}\sum_{j=2}^N \delta_{U_t^j}\right)\right] \le \frac{1}{N(N-1)}\sum_{j=2}^N \mathbb{E}\left[|U_t^1 - U_t^j|^2\right],$$

we can simplify (38), so that we obtain

$$\mathbb{E}\left[W_2^2(\bar{\nu}_t^N,\mu_t)\right] \le c\Bigg\{\frac{1}{N(N-1)}\sum_{j=2}^N \mathbb{E}\left[|U_t^1 - U_t^j|^2\right]$$
$$+\frac{1}{N-1}\sum_{j=2}^N \mathbb{E}\left[|U_t^j - \bar{\bar{X}}_t^j|^2\right] + \mathbb{E}\left[W_2^2\left(\frac{1}{N-1}\sum_{j=2}^N \delta_{\bar{\bar{X}}_t^j}, \mu_t\right)\right]\Bigg\}.$$

Furthermore, we have already seen that the second term can be estimated, and we also have the useful estimate stated in the preliminary definitions. Thus, the above inequality becomes

$$\mathbb{E}\left[W_2^2(\bar{\nu}_t^N,\mu_t)\right] \le c\Bigg\{\frac{1}{N(N-1)}\sum_{j=2}^N \mathbb{E}\left[|U_t^1 - U_t^j|^2\right] + c_A N^{\frac{-2}{d+4}} + c_A N^{\frac{-2}{d+4}}\Bigg\}.$$

Since the first term is $O(N^{-1})$, as can be seen by the first three estimates of this proof, we conclude that

$$\mathbb{E}\left[W_2^2\left(\bar{\nu}_t^N,\mu_t\right)\right] \le c_A N^{\frac{-2}{d+4}}, \tag{39}$$

which is the desired estimate for the distance of the two distributions.

Step 5: Proving (30) for (β_t^1) whose expected square integral is not too large.

For this step of the prove we define the process $(\bar{\bar{U}}_t^1)_{0\le t\le T}$ of the private state of the rebel player, which is defined as the solution of

$$d\bar{\bar{U}}_t^1 = b(t,\bar{\bar{U}}_t^1,\mu_t,\beta_t^1),$$

for $0 \leq t \leq T$ and the initial condition $\bar{\bar{U}}_0^1 = x$. Using also the definition of the process $(U_t^1)_{0\leq t\leq T}$, and the form of the drift as assumed in (6), we obtain

$$U_t^1 - \bar{\bar{U}}_t^1 = \int_0^t \left[b_0(s, \mu_s) - b_0(s, \bar{\nu}_s^N)\right] ds + \int_0^t \left[b_1(s)[U_s^1 - \bar{\bar{U}}_s^1]\right] ds.$$

In the last equality we take advantage of the Lipschitz property of b_0, the boundedness of b_1 and the estimate found in the last step, and by applying Gronwall's inequality, we conclude

$$\sup_{0\leq t\leq T} \mathbb{E}\left[U_t^1 - \bar{\bar{U}}_t^1\right] \leq c_A N^{\frac{-2}{d+4}}.$$

We define the mean-field cost of the rebel player as $J(\beta^1)$, i.e.

$$J(\beta_t^1) = \mathbb{E}\left[g(\bar{\bar{U}}_T^1, \mu_T) + \int_0^T f(t, \bar{\bar{U}}_t^1, \mu_t, \beta_t^1)dt\right].$$

Following the same calculations as in the third step, where we exchange J for $J(\beta^1)$ and $\bar{J}^{N,i}(\bar{\alpha}^{N,1}, \ldots, \bar{\alpha}^{N,N})$ for $\bar{J}^{N,i}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N})$, we obtain

$$\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N}) \geq J - c_A N^{\frac{-1}{d+4}}. \tag{40}$$

To end this step, we combine the above inequality with (36) and obtain easiy the desired approximate Nash equilibrium equation.

Step 6: Explain how the bound A can be chosen.
For this step we are going to use the convexity of g with respect to x in a neighbourhood around 0, as well as the convexity of f in the variables (x, α) again in a neighbourhood around $(0, 0)$. From this, it follows that

$$\begin{aligned}
\bar{J}^{N,1}(\beta^1, \bar{\alpha}^{N,2}, \ldots, \bar{\alpha}^{N,N}) &\geq \mathbb{E}\left[g(0, \bar{\nu}_T^N) + \int_0^t f(t, 0, \bar{\nu}_t^N, 0)dt\right] \\
+ \lambda\mathbb{E}\int_0^T |\beta_t^1|^2 dt &+ \mathbb{E}\bigg[\left((U_T^1)^T \partial_x g(0, \bar{\nu}_T^N)\right) \\
&+ \int_0^t \left[(U_t^1)^T \partial_x f(t, 0, \bar{\nu}_t^N, 0) + \left((\beta_t^1)^T \partial_\alpha f(t, 0, \bar{\nu}_t^N, 0)\right)\right] dt\bigg].
\end{aligned}$$

The assumption over the local-Lipschitz continuity with respect to the Wasserstein distance, as well as the definition of the metric ensure the existence of some constant $c > 0$, such that for all $0 \leq t \leq T$ we obtain

$$\begin{aligned}
\mathbb{E}\left[|f(t, 0, \bar{\nu}_t^N, 0) - f(t, 0, \delta_0, 0)|\right] &\leq c\mathbb{E}\left[1 + M_2^2(\bar{\nu}_t^N)\right] \\
&= c\left[1 + \left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[|U_t^i|^2\right]\right)\right],
\end{aligned}$$

so that

$$\mathbb{E}\left[|f(t,0,\bar{\nu}_t^N,0)\right] \geq f(t,0,\delta_0,0) - c\left[1+\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|U_t^i|^2\right]\right)\right]$$

for f. A similar inequality holds for g. Thus, we deduct that

$$\begin{aligned}
&\bar{J}^{N,1}(\beta^1,\bar{\alpha}^{N,2},\dots,\bar{\alpha}^{N,N})\\
&\geq g(0,\delta_0)+\int_0^t f(t,0,\delta_0,0)dt+\lambda\mathbb{E}\int_0^T|\beta_t^1|^2dt\\
&\quad+\mathbb{E}\Bigg[\left((U_T^1)^T\partial_x g(0,\bar{\nu}_T^N)\right)+\int_0^t (U_t^1)^T\partial_x f(t,0,\bar{\nu}_t^N,0)dt\\
&\quad+\int_0^t\left((\beta_t^1)^T\partial_\alpha f(t,0,\bar{\nu}_t^N,0)\right)dt\Bigg]-c\left[1+\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{0\leq t\leq T}\mathbb{E}\left[|U_t^i|^2\right]\right)\right].
\end{aligned}$$

Since by assumption (A.5) we know that $\partial_x g$, $\partial_x f$ and ∂_α are at most of linear growth for the variable μ_t, we can conclude that for any $\delta > 0$, there exists a constant c_δ, so that we obtain

$$\begin{aligned}
\bar{J}^{N,1}(\beta^1,\bar{\alpha}^{N,2},\dots,\bar{\alpha}^{N,N}) &\geq g(0,\delta_0)+\int_0^t f(t,0,\delta_0,0)dt+\frac{\lambda}{2}\mathbb{E}\int_0^T|\beta_t^1|^2dt\\
&\quad-\delta\sup_{0\leq t\leq T}\mathbb{E}\left[|U_t^1|^2\right]-c_\delta\left[1+\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{0\leq t\leq T}\mathbb{E}\left[|U_t^i|^2\right]\right)\right].
\end{aligned}$$

The estimates for $\sup_{0\leq t\leq T}|U_t^i|^2$, where $i\in\{1,\dots,N\}$, that we found in the beginning of this proof, show that for the appropriate c and for δ small enough we deduce

$$\bar{J}^{N,1}(\beta^1,\bar{\alpha}^{N,2},\dots,\bar{\alpha}^{N,N}) \geq -c+\left(\frac{\lambda}{4}\frac{-c}{N}\right)\mathbb{E}\int_0^T|\beta_t^1|^2dt.$$

Thus, we know that there exists some positive integer N_0 such that for any $N\geq N_0$ and constant $\bar{A}$, we can choose the constant A such that

$$\mathbb{E}\int_0^T|\beta_t^1|^2dt \leq A \;\Rightarrow \bar{J}^{N,1}(\beta^1,\bar{\alpha}^{N,2},\dots,\bar{\alpha}^{N,N}) \geq J+\bar{A}. \tag{41}$$

This means that if the strategy β_1 has an expected square integrable that is very large, then the cost for the rebel player will definitely be more than the optimal value. However, the theorem states that the minimum cost for the rebel player can get very close, or in an extreme case, can even get less than the optimal value. We now know, that this can only happen when the expected square integrable of the strategy does not get too large and so it is enough to prove the statement for this case, as we did in the proof. The statement in (41) give us the guidelines on how to choose the bound A. $\square$

Remark: As the above proof shows, if all the players follow the prescribed strategy $\bar{\alpha}^{N,i}$ for all $i \in \{1, \dots, N\}$ and if there is a large number of players participating in the game, the result is an ϵ-Nash equilibrium, with a precise quantification of the relationship between N and ϵ, even though the quantification is not optimal.

For reasons of completeness we will write the part of Theorem (1.3) from [10], that is used in the above proof. We have adjusted the proof into our setting.

Theorem 3.2. *Assume that x_0 and $(W_t)_{0 \le t \le T}$ are square integrable, and that the diffusion mapping σ is Lipschitz continuous when $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ is endowed with the product of the canonical topology on $\mathbb{R}^d$ and the Wasserstein metric W_p on $\mathcal{P}_2(\mathbb{R}^d)$. If $\mathbb{E}\left(|X_0|^{d+5} + |W_T|^{d+5}\right) < +\infty$, then*

$$\sup_{i \le N} \mathbb{E}\left(\sup_{0 \le t \le T} |X_t^i - \bar{X}_t^i|^2\right) \le CN + \frac{2}{d+4},$$

where the constant C does not depend on N and where $\bar{X}_t^i$ are independent, distributed like X_t^i and the solutions of the same SDE as X_t^i.

3.3 A stronger result

From the last step of the proof of the approximate Nash equilibrium one can conclude that we have an even stronger result. Namely, if we let player 1 diverge from the prescribed strategy $\bar{\alpha}_t^{1,N}$ and let them use some arbitrary strategy β_t^1 instead, whose expected square integral is not too large, then we can see in a similar way as in the proof above, by using the estimates (39), (37) and (31) as in (36), that equation (40) holds for the cost of any player, i.e. that for any $i \in \{2, \dots, N\}$

$$\bar{J}^{N,i}(\beta_1, \bar{\alpha}^{N,2}, \dots, \bar{\alpha}^{N,N}) = J - c_A N^{\frac{-1}{d+4}}. \tag{42}$$

Theorem 3.3. *Under the assumptions of section 1.5, not only does*

$$((\bar{\alpha}_t^{N,i} = \hat{\alpha}(t, X_t^i, \mu_t, u(t, X_t^i)))_{1 \le i \le N})_{0 \le t \le T}$$

form an approximate Nash equilibrium of the N-player game, but even more

(i) there exists an integer N_0 such that, for any $N \ge N_0$ and $\bar{A} > 0$, there exists a constant $A > 0$ such that, for any player $i \in \{1, \dots, N\}$ and any admissible strategy $\beta^i = (\beta_t^i)_{0 \le t \le T}$,

$$\mathbb{E}\int_0^T |\beta_t^i|^2 dt \ge A \Longrightarrow$$
$$\bar{J}^{N,i}(\bar{\alpha}^{1,N}, \dots, \bar{\alpha}^{i-1,N}, \beta^i, \bar{\alpha}^{i+1,N}, \dots, \bar{\alpha}^{N,N}) \ge J + \bar{A}.$$

(ii) Moreover, for any $A > 0$, there exists a sequence of positive real number $(\epsilon_N)_{N\geq 1}$ converging towards 0, such that for any admissible strategy $\beta^j = (\beta^j_t)_{0\leq t\leq T}$, for the j-th player, where $j \in \{1, \ldots, N\}$

$$\mathbb{E}\int_0^T |\beta^j_t|^2 dt \leq A \Longrightarrow$$
$$\min_{1\leq i\leq N} \bar{J}^{N,i}(\bar{\alpha}^{N,1}, .., \bar{\alpha}^{N,j-1}, \beta^j, \bar{\alpha}^{N,j+1}, .., \bar{\alpha}^{N,N}) \geq J - \epsilon_N. \quad (43)$$

Proof. This theorem is a result from the proof of the approximate Nash equilibrium. □

The first case of the theorem says that if there is one player that decides to rebel and that chooses a strategy other than $\bar{\alpha}_t$,which has a very large expected square integral, then his own cost will never be optimal. On the other hand, the second result of the theorem shows what happens with the cost of the other players when all but one follow the almost optimal strategy. In this case, if the expected square integral of the arbitrary strategy β^j_t does not get values that are too large, then for some of the other players the cost might remain almost optimal, depending also on how many players there are in the game.

It is important to notice that in (3.1) we are exploring how the cost of one player is affected by their own unruliness. The same is being explored in the first case of the last theorem. On the other hand in (42) and in (43) we investigate how the cost of any player is affected, when it is a different player that has decided to follow an arbitrary strategy.

4 Outlook

Throughout this thesis, we studied the theory of mean field games from the probabilistic viewpoint. Our goal was to find an optimal control strategy for the cost functional of the players. To do this and in order to take advantage of the theory of propagation of chaos, we assumed that the number of the players tends to infinity and that there is a statistically identical influence from one player to the other. This allowed us to reduce our problem to solving only one SDE (or in our case to solving a FBSDE system), instead of a system of N-coupled SDEs. Moreover, we have assumed that the SDEs that represent the dynamics of the positions of the players, each have individual noise σ which is some arbitrary constant matrix, but there is no common noise for all the players. As for the players, we have assumed that the game is symmetric and also that the decisions of the players are independent from the other players (with the exception of the indirect influence that comes from the empirical distribution of the game). There are several ways that one could generalize and apply the above results.

One way to apply the above theory in the real world are the evacuation scenarios. In this problem, we assume that we have a big crowd of people that are located inside a room and that for some reason and at some time t they all simultaneously decide that they want to leave the room as quickly as possible. In this case we could define the cost functional to be the time needed to exit the room. A relatively simple way to study such a problem, is to take the drift to be only a function of the distance between the initial position of the player and the door. A complete model would include a drift that depends on the distance from the door, the velocity of the player, the overall density of the crowd and the exit time. This problem has been studied by Burger, Di Fransesco, Markowich and Wolfram in [4].

Another interesting application of the theory of mean field games is the case of interbank borrowing, which was studied in [7]. In interbank borrowing we assume that we have N banks whose evolution of the log-monetary reserves are described by a system of differential equations that are coupled with each other. Should the monetary reserves of one bank become low, the bank has to borrow money from a central bank. Should on the other hand the monetary reserves of some bank become high, then the bank must lend to the central bank. In this game, the control process is the rate of borrowing (or respectively of lending) the money from the central bank, which is determined by the bank itself. Each bank depends on the empirical distribution created by the log-monetary reserves from the rest of the banks, in the sense that the “systemic risk is characterized by a large number of banks reaching a default threshold by a given time horizon”, as we read in [7].

One modification of the last application, which one could argue to be closer to the real world, would be to assume that there is one player, whose influence on the other players will never become statistically identical to the rest of the players, no matter how big N becomes. This has been studied in [9] with the title of a mean-field game with major and minor players. This seems to be a more realistic example since in the banking system there are a few systemically important financial institutions, where the impact of their actions never becomes negligible, nor even similar to that of the small banking institution, no matter how many banks there are in total.

In the stochastic differential equations theory there are weak and strong solutions depending on whether the solution is adapted to the given filtration. In the same sense, one could distinguish between weak and strong MFG solutions. In this case, the term weak refers to the fact that the limit of the fixed point distribution may not be adapted to the filtration of the common noise any longer. In this case, one would need to condition the fixed point distribution with respect to the common noise B and should assume that the resulting distribution will be independent from the individual noise that comes from the Wiener processes. In this sense, a weak solution is also strong when the associate measure flow is measurable with respect to the common noise B. More on this can be seen in [6].

Mean field game theory seems to still have much potential to evolve, generalize the existing theory and find more real life applications. We look forward to what is yet to come!

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