Topics in Quantum Probability

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Abstract

In 1984, R.L. Hudson and D. Applebaum introduced the concept of Fermion diffusions with one degree of freedom. Here, we generalize that concept and do some computations in order to find the explicit form of Fermion diffusions with higher degrees of freedom. More precisely, we study the time evolution of operators acting on antisymmetric Fock spaces that satisfy the canonical anticommutation relations and that are odd with respect to some $\mathbb{Z}_2$-grading. Furthermore, we give a general overview of the stochastic calculus developed in symmetric Fock spaces with the sole purpose of illustrating the similarities with Fermion stochastic calculus. The central tools are the commutator and anticommutator operators, the anticommuting tensor product, the Fermion stochastic integrals, the Fermion Ito formula and the Fermion uniqueness theorem. We use the Fermion Ito formula to give conditions that Fermion diffusions need to satisfy. With these conditions we find the zero terms of the even coefficients of Fermion diffusions with three degrees of freedom. The Fermion uniqueness theorem allows us to know how many generators the C*-algebra generated by the operators satisfying the canonical anticommutation relations has.

We observe that the number of terms of the coefficients of Fermion diffusions grows more quickly than the amount of consistency conditions, which leads us to conjecture the impossibility of giving the explicit form of Fermion diffusions with a large number of degrees of freedom. Finally, we find the zero terms of Fermion diffusions with n-degrees of freedom by assuming certain conditions on the even and odd coeficients.

Contents

1 Introduction

1.1 On Bosons and Fermions

Currently, the Standard Model of particle physics considers Fermions and Bosons to be the two most fundamental particles existing in the universe [30]. Fermions are particles with half-integer spin that obey the Pauli exclusion principle: no two identical Fermions can be in the same quantum state. Contrarily, Bosons are particles with integer spin that do not satisfy the Pauli exclusion principle: more than one Boson can occupy the same quantum state.[1] Furthermore, Fermions are particles that are, generally speaking, considered to be matter, such as protons, neutrons, and electrons. On the other hand, intermediate force carrying particles, are known as Bosons. For example, photons that carry the electromagnetic force and the pion, that carries the nuclear force.[2] Mathematically, we represent Fermions and Bosons in the antisymmetric and symmetric Fock spaces. The different nature of these Fock spaces gives rise to two different non-commutative generalizations of stochastic analysis: Boson and Fermion stochastic calculus.

The inspiration, for non-commutative calculus, comes from quantum mechanics, where the expected value of the result of an experiment is given by $\langle u, Au\rangle$, where A is an operator acting on a Hilbert space $\mathcal{H}$ and u is a normalized element of $\mathcal{H}$ [1]. This way of measuring probabilities gave rise to the developement of quantum probability.[3] The latter being a non-commutative generalization of probability theory.[4]

In this master thesis, we are interested in the similarities between Fermion and Boson stochastic analysis, and in the concept of Fermion diffusions, which is a quantum analogue of Ito diffusions.[5]

We will be interested in dealing with operators acting on "initial" Hilbert spaces tensored with Fock spaces, that is:

$$\widetilde{\mathcal{H}}_s = h_0 \otimes \underbrace{\Gamma_s(\mathcal{H})}_{noise}, \quad \widetilde{\mathcal{H}}_a = h_0 \otimes \underbrace{\Gamma_a(\mathcal{H})}_{noise}.$$

Intuitively, we can view h_0 as the Hilbert space for describing the events and observables concerning a system, plus noise, $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$ for describing the same objects concerning a noise process (heat bath). Then $\widetilde{H}_s$ and $\widetilde{H}_a$ can be used to describe events and observables concerning a system plus noise.[6]

1.2 Main results

As previously stated, we will be concerned with two kinds of non-commutative stochastic calculus: Fermionic and Bosonic. In Section 2 we will introduce the concepts of symmetric and antisymmetric Fock spaces (Definition 2.9), which will help us model the "quantum noise" of the Fermion and Boson stochastic differential equations, as Brownian motions do in

[1] See page 144 in [30]. There are 12 elementary particles with half-integer spin and five with integer spin in the Standard model.

[2] See paragaph 4 in Chapter 8 of [30].

[3] See Chapter I in [1].

[4] In this theory we have a quantum central limit theorem, quantum Gaussians, among other generalizations. See [9] and [1] for a detailed threatment of this theory, for the basic setting see Appendix B.

[5] See in [32] the unification of Fermion and Boson Stochastic Calculus.

[6] See paragraph 3 and 4 in [1].

stochastic analysis [26]. More precisely, the creation and annihilation operators (Definitions 2.21 and 2.29) will help us define quantum Brownian motions (Definition 7.8). Furthermore, we will introduce the concepts of Fermion and Boson total sets (Definitions 2.17) and 2.37). In the Boson case, we prove that the coherent vectors are total (Proposition 2.16). For both, Bosons and Fermions, we have Fock exponential properties (Propositions 2.12 and 2.19), which allows us to give quantum analogues of filtered probability spaces.

In Section 3, we define the Weyl operators (Definition 3.4), present some results (e.g. Proposition 3.4) and concepts that are useful to make calculations with the Boson creation, annihilation and conservation processes (Definition 4.15) that are fundamental for Boson stochastic integration in Section 4.

In Sections 4 and 5 we see that as in classical stochastic analysis[26], quantum stochastic differential equations have a "deterministic" and a "non-deterministic" part. The "deterministic" part is given by a Lebesgue measure and the "non-deterministic" part by some "quantum noise". The quantum noise is a quantum analogue of Brownian motions (Definition 7.8).[7]

In order to define Fermion and Boson stochastic differential equations (Remarks 4.24 and 5.40) we are going to define Fermion and Boson Ito integrals (Definitions 4.17 and 5.37). Quantum Ito integrals are operators acting on the symmetric and antisymmetric Fock spaces (Definition 2.9), respectively. We conclude both sections by stating non-commutative Ito formulas (Theorems 4.26 and 5.41).

In Section 6 we are concerned with Fermion diffusions (Definition 6.11), the C^*-algebra generated by Fermions (Proposition 6.1 and Theorem 6.2) and the consistency conditions (Propositions 6.14, 6.15 and 6.16). In Section 7 we give the explicit form of Fermion diffusion with one and two degrees of freedom (Propositions 7.7 and 7.18) by finding, with the consistency conditions, the zero terms of the even and odd coefficients of the Fermion diffusions with one and two degrees of freedom. Additionally, we calculate the zero terms of the even coefficients of a Fermion diffusion with three degrees of freedom (Proposition 7.21). Finally, in Section 8 we find some general properties of Fermion diffusions by imposing some conditions on the even and odd coefficients.

1.3 Notes to the reader

This work can be seen as an in-depth study of [3] and as a general overview of Boson stochastic analysis. In Sections 2 and 5 we give the results and concepts needed for the developement of [3], which were probably left out due to space concerns.[8] Furthermore, Sections 6 and 7 can be understood as an extension or generalization of [3]. Let us highlight some of our own contributions here:
In Section 6 we generalize the concept of Fermion diffusions (Defintion 6.11) which in [3] was only defined for Fermions with one degree of freedom. As a result of this generalization, we obtain consistency conditions for Fermion diffusions with n-degrees of freedom (Propositions 6.14, 6.15 and 6.16) that allows us to find the zero terms of the even and odd coefficients of Fermion diffusions.
In Section 7 we extend the results of [3] by calculating the explicit form of Fermion diffusions with two degrees of freedom (Proposition 7.18) and find the zero terms of the even coefficients of Fermion diffusions with three degrees of freedom (Proposition 7.21) Since the computations

[7] Although, in the case of Boson stochastic calculus there exists a more general setting, where the deterministic part is given by measures of bounded variation, see Appendix E for a short summary of this theory.

[8] For an in-depth study of [3], Subsubsection 2.3.1 can be skipped.

in Sections 6 and 7 are rather lenghty, we often suppress time dependency of the Fermion operators and the tensored algebraic extension, that is, b_i, $i \in \{1, ..., n\}$, should be understood as $b^i(t)\hat{\otimes}I_{L^2[t,\infty)}$.
For an overview of Boson stochastic calculus, we present some of the main results and concepts of this theory in Sections 2, 3 and 4.[9] Furthermore, in order to complement this sections we wrote the Appendices B.2, C and D, where we present the concepts, results and the notation required for the developement of this theory. To conclude, we give a short summary of Boson stochastic integration with measures of bounded variation.
Finally, let us point out that we use some notation without explanation that we consider to be standard. Nevertheless, the meaning of some notation can be consulted in the list of symbols.

1.4 Acknowledgments

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[9] For an overview of Boson stochastic calculus, Subsubsection 2.3.2 can be skipped.

2 Fock spaces

Fock spaces are used in physics to represent quantum states of a known (or unknown) number of identical particles, for instance: an electron gas or photons.[10] In the context of quantum stochastic analysis, Fock spaces play the role of quantum probability spaces.[11] This section summarizes some results and concepts from [17] and [1].

2.1 Tensor Products

We introduce the symmetric and antisymmetric tensor products which we will use to define the Fermion and Boson Fock spaces.

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space and let $n \in \mathbb{N}$. For any $u_1, ..., u_n \in \mathcal{H}$ we define the* ***symmetric tensor product*** *as*

$$u_1 \circ \cdots \circ u_n := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

where $\mathcal{S}_n$ is the group of permutations of $\{1,...,n\}$, and we define the ***antisymmetric tensor product*** *as*

$$u_1 \wedge \cdots \wedge u_n := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \varepsilon_\sigma u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$$

where ε_σ is the signature of the permutation σ.

Remark 2.2. *Let $\mathcal{H}$ be a complex Hilbert space and let $n \in \mathbb{N}$. For any integer $n \geqslant 1$ we denote as*

$$\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ times}}$$

the n-fold ***tensor product*** *of $\mathcal{H}$. We denote as $\mathcal{H}^{\wedge n}$ the closed subspace of $\mathcal{H}^{\otimes n}$ generated by the products $u_1 \wedge \cdots \wedge u_n$ defined in Definition 2.1 and we call it the n-fold* ***antisymmetric tensor product****. Similarily, we denote as $\mathcal{H}^{\circ n}$ the closed subspace of $\mathcal{H}^{\otimes n}$ generated by the products $u_1 \circ \cdots \circ u_n$ defined in Definition 2.1 and we call it the n-fold* ***symmetric tensor product****.*

Remark 2.3. *If $n = 0$ then we put*

$$\mathcal{H}^{\otimes 0} = \mathcal{H}^{\wedge 0} = \mathcal{H}^{\circ 0} = \mathbb{C}.$$

Definition 2.4. *Let $\mathcal{H}$ be a Hilbert space. For u_i, v_j with $i, j \in \{1, ..., n\}$, we define a scalar product in $\mathcal{H}^{\wedge n}$ as*

$$\langle u_1 \wedge \cdots \wedge u_n, v_1 \wedge \cdots \wedge v_n \rangle_\wedge := \mathrm{Det}[(\langle u_i, v_j \rangle)_{ij}].$$

and we denote it as $\langle \cdot, \cdot \rangle_\wedge$. Similarly, we define a scalar product in $\mathcal{H}^{\circ n}$ as

$$\langle u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \rangle_\circ := \mathrm{Per}(\langle u_i, v_j \rangle)_{ij},$$

[10] See Paragraph 2 in [17].
[11] strictly speaking we need to give a state, which we do not do, see Definition B.2.

Remark 2.5. *From the previous definition we get that*

$$\langle u_1 \circ \cdots \circ u_n, u_1 \circ \cdots \circ u_n \rangle_\wedge = n! \langle u_1 \circ \cdots \circ u_n, u_1 \circ \cdots \circ u_n \rangle_\otimes.$$

Remark 2.6. Per *denotes in Definition 2.4 the permanent of a matrix, which is "the determinant with only positive signs".* Det *denotes the determinant of a matrix. For* $u_i, v_j \in \mathcal{H}$ *with* $i, j \in 1, ..., n$ *and* $\mathcal{H}$ *a Hilbert space,* $(\langle u_i, v_j \rangle)_{ij}$ *denotes the matrix with entries* $\langle u_i, v_j \rangle$ *with* $u_i, v_i \in \mathcal{H}$.

Remark 2.7. *The* $\langle \cdot, \cdot \rangle_\wedge$ *scalar product is different from the one induced by* $\mathcal{H}^{\otimes n}$*, indeed:*

$$\begin{aligned}\langle u_1 \wedge \cdots \wedge u_n, v_1 \wedge \cdots \wedge \ v_n \rangle &= \frac{1}{(n!)^2} \sum_{\sigma, \tau \in \mathcal{S}_n} \varepsilon_\sigma \varepsilon_\tau \langle u_{\sigma(1)}, v_{\tau(1)} \rangle \cdots \langle u_{\sigma(n)}, v_{\tau(n)} \rangle \\ &= \frac{1}{n!} \mathrm{Det}[(\langle u_i, v_j \rangle)_{ij}].\end{aligned}$$

2.2 Full, Fermionic and Bosonic Fock spaces

We now introduce the quantum probability spaces in which the Boson and Fermion stochastic integrals are defined.

Remark 2.8. *In the following* $\bigoplus$ *is going to denote the direct sum of Hilbert spaces.*

Definition 2.9. *We call* ***Full Fock Space*** *over* $\mathcal{H}$*, the space*

$$\Gamma_f(\mathcal{H}) := \overline{\bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}}.$$

We call ***symmetric*** *(or* ***Boson****) Fock space over* $\mathcal{H}$*, the space*

$$\Gamma_s(\mathcal{H}) := \overline{\bigoplus_{n=1}^{\infty} \mathcal{H}^{\circ n}}.$$

We call ***antisymmetric*** *(or* ***Fermion****) Fock space over* $\mathcal{H}$*, the space*

$$\Gamma_a(\mathcal{H}) := \overline{\bigoplus_{n=1}^{\infty} \mathcal{H}^{\wedge n}},$$

where the bar denotes the Hilbert space completion with respect to the inner products $\langle \cdot, \cdot \rangle_\otimes, \langle \cdot, \cdot \rangle_\circ$ *and* $\langle \cdot, \cdot \rangle_\wedge$*, respectivley.*

Remark 2.10. *For notational convenience we will drop the bar of the Fock spaces in Definition 2.9.*

Remark 2.11. *For* $\Gamma_f(\mathcal{H}), \Gamma_s(\mathcal{H})$ *and* $\Gamma_a(\mathcal{H})$ *each* $\mathcal{H}^{\otimes n}$*,* $\mathcal{H}^{\wedge n}$ *and* $\mathcal{H}^{\wedge n}$ *have inner products* $\langle \cdot, \cdot \rangle_\otimes, \langle \cdot, \cdot \rangle_\circ$ *and* $\langle \cdot, \cdot \rangle_\wedge$*, respectively. We call* $1 \in \mathbb{C}$ *the* ***vacuum vector*** *and we denote it as* Ω*. Observe, that by taking* $\mathcal{H} = \mathbb{C}$ *we get a simple case of a symmetric Fock space given by* $\Gamma_s(\mathbb{C}) = \ l^2(\mathbb{N})$*. If* $\mathcal{H}$ *has finite dimension* n *then* $\mathcal{H}^{\wedge m} = 0$ *for* $m > n$ *and thus* $\Gamma_a(\mathcal{H})$ *has finite dimension, which can be proven to be* 2^n*. The symmetric Fock space* $\Gamma_s(\mathcal{H})$ *has always infinite dimension.*

In the asymmetric Fock space, we have the following property which we will use repeatedely in Sections 5 and 6. We will also use it to define Fermion adapted processes (Definition 5.21).

Proposition 2.12 (Fermion exponential property)**.** *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Furthermore, $u_i \in \mathcal{H}_1$, $v_j \in \mathcal{H}_2$, $1 \leq i \leq n$. Then, there exists a unique unitary isomorphism defined as*

$$U : \Gamma_a(\mathcal{H}_1 \oplus \mathcal{H}_2) \longrightarrow \Gamma_a(\mathcal{H}_1) \otimes \Gamma_a(\mathcal{H}_2)$$

which satisfies the relations:

$$U[(m+n)!]^{\frac{1}{2}} u_1 \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge v_n = \{(m!)^{\frac{1}{2}} u_1 \wedge \cdots \wedge u_m\} \otimes \{(n!)^{\frac{1}{2}} v_1 \wedge \cdots \wedge v_n\}$$

for $m = 1, 2, ..., n = 1, 2, ...$ and

$$U(\Omega) = \Omega_1 \otimes \Omega_2$$

Ω, Ω_1, Ω_2 being the vacuum vectors in $\Gamma_a(\mathcal{H})$, $\Gamma_a(\mathcal{H}_1)$, $\Gamma_a(\mathcal{H}_2)$, respectively.

Proof. See Proposition 19.7 in [1]. □

The following example give us a sort of Fermion "filtered" probability space

Example 2.13. *Take $\mathcal{H}_1 = L^2([0,t])$, $\mathcal{H}_2 = L^2([t,\infty))$, and $\mathcal{H} = L^2(\mathbb{R}_+)$, then it holds by the Fermion exponential property that*

$$\Gamma_a(L^2(\mathbb{R})) \cong \Gamma_a(L^2([0,t)) \otimes \Gamma_a(L^2([t,\infty])).$$

For the Boson exponential property, we first introduce the exponential vectors.

2.3 Total sets

Our purpose now, is to give dense sets, that will help us define the domains of the Boson and Fermion stochastic integrals in Sections 4 and 5, respectively.

2.3.1 Boson total set

Here, we give a dense subset of the symmetric Fock space which will allows us in Section 4 to define Boson stochastic integrals.

Definition 2.14. *A subset of a Hilbert space $\mathcal{H}$ is* ***total*** *if its span is dense in $\mathcal{H}$.*

Definition 2.15. *Let $\mathcal{H}$ be a Hilbert space and let $u \in \mathcal{H}$, the* ***coherent vector*** *(or exponential vector) associated to u is given by*

$$e(u) := \sum \frac{u^{\otimes n}}{n!}$$

so that it holds that

$$\langle e(u), e(v) \rangle = \exp(\langle u, v \rangle)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\Gamma_s(\mathcal{H})$.

Proposition 2.16. *The set $\{e(u) \mid u \in \mathcal{H}\}$ of all coherent vectors is linearly independent and total in $\Gamma_a(\mathcal{H})$.*

Proof. Let $\{u_j | 1 \leq j \leq n\}$ be a finite subset of $\mathcal{H}$. The sets $E_{jk} = \{u \mid \langle u, u_j\rangle \neq \langle u, u_k\rangle\}$ are open and dense in $\mathcal{H}$ for $j \neq k$, if we take a Cauchy sequence of elements in the set $\{u \mid \langle u, u_i\rangle = \langle u, u_k\rangle\}$ it clearly converges to an element in this set and therefore is this set closed. By taking $\|u\| = 1$ and taking a $\varepsilon > 0$ we get

$$|\varepsilon\langle u, u_i - u_j\rangle| \leq \varepsilon\|u\|\|u_i - u_j\| = \varepsilon\|u_i - u_j\|$$

therefore the sets E_{jk} are dense in $\mathcal{H}$, then there exists a v in $\mathcal{H}$ such that the scalars $\theta_j = \langle v, v_j\rangle$ for $1 \leq i, j \leq n$, where i and j are distinct. Suppose that α_j, $1 \leq j \leq n$ are scalars such that

$$\sum_{j=1}^{n} \alpha_j e(u_j) = 0.$$

Then, we have

$$0 = \left\langle e(zv), \sum_{i=1}^{n} \alpha_i e(v_i) \right\rangle = \sum_{i=1}^{n} \alpha_i \exp(z\theta_i)$$

for all $z \in \mathbb{C}$. Since the functions $z \longrightarrow \exp(z\theta_i)$ are linearly independent it follows that the α_i are all equal to zero, therefore the family of vectors $\{e(u_1), ..., e(u_n)\}$ is linearly independent. For the rest of the proof see Proposition 19.3 in [1]. □

Definition 2.17. *We call the linear span of the exponential vectors the **Boson total set** and we denote it $\mathcal{E}$.*

A straightforward consequence is the following

Corollary 2.18. *If $S \subseteq \mathcal{H}$ is a dense subset, then the space $\mathcal{E}(S)$ generated by the exponential vectors associated to elements of S is dense in $\Gamma_s(\mathcal{H})$.*

Proof. See Corollary 19.5 in [1]. □

The Propositions 2.19 and 2.12 give the theoretical setting for the definition of quantum adapted processes[12]. Further, this will allows us, roughly speaking, to give a non-commutative filtred probability space.

Proposition 2.19 (Boson exponential property)**.** *Let $\mathcal{H}, \mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$,. Then, the mapping from $\Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2)$ to $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$, defined as*

$$U(e(u \oplus v)) = e(u) \otimes e(v)$$

with $u \in \mathcal{H}_1$, $v \in \mathcal{H}_2$, can be extended to a unitary isomorphism.

[12]See Definition 4.9 and 5.21.

Proof. We have, for $u, u_1 \in \mathcal{H}_1$ and $v, v_1 \in \mathcal{H}_2$, that

$$\begin{aligned}
\langle e(u \oplus v), e(u_1 \otimes v_1)\rangle &\overset{2.15}{=} \exp\langle u \oplus v, u_1 \oplus v_1\rangle \\
&= \exp(\langle u, u_1\rangle + \langle v, v_1\rangle) \\
&= \exp\langle u, u_1\rangle \exp\langle v, v_1\rangle \\
&\overset{2.15}{=} \langle e(u), e(u_1)\rangle\langle e(v), e(v_1)\rangle \\
&= \langle e(u) \otimes e(v), e(u_1) \otimes e(v_1)\rangle.
\end{aligned}$$

Therefore, the operator U is isometric. Since the coherent vectors are dense in $\Gamma_a(\mathcal{H})$, we get that the set $\{e(u) \otimes e(v) \mid u \in \mathcal{H}_1,\ v \in \mathcal{H}_2\}$ is total in $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$, we get that U can be extended to a unitary operator. □

Example 2.20. *Take $\mathcal{H}_1 = L^2([0,t]), \mathcal{H}_2 = L^2([t,\infty))$, and $\mathcal{H} = L^2(\mathbb{R}_+)$, then it holds by the Boson exponential property that*

$$\Gamma_s(L^2(\mathbb{R})) \cong \Gamma_s(L^2([0,t)) \otimes \Gamma_s(L^2([t,\infty]))$$

where L^2 denotes the square integrable functions.

We will use the following definition in Section 4 to define Boson creation and annihilation processes 4.15. Against these processes we will define the Boson stochastic integrals.

Definition 2.21. *Let $\mathcal{H}$ be a Hilbert space, then for any $u \in \mathcal{H}$:*

(i) ***The Boson creation operator*** *with domain $\mathcal{H}^{\circ n}$ and codomain $\mathcal{H}^{\circ(n+1)}$ is defined as*

$$a^*(u)(u_1 \circ \cdots \circ u_n) := u \circ u_1 \cdots \circ u_n;$$

(ii) ***The Boson annhiliation operator*** *which goes from $\mathcal{H}^{\circ n}$ to $\mathcal{H}^{\circ(n-1)}$ is defined as*

$$a(u)(u_1 \circ \cdots \circ u_n) := \sum_{i=1}^{n} \langle u, u_i\rangle u_1 \circ \cdots \circ \hat{u}_i \circ \cdots \circ u_n;$$

Remark 2.22. *Let Ω_{Boson} denote the vacuum in $\Gamma_s(\mathcal{H})$. Let u be an element of the Hilbert space $\mathcal{H}$. Then, we observe that the following relations hold:*

$$a^*(u)\Omega_{Boson} = u, \quad a(u)\Omega_{Boson} = 0,$$

2.3.2 Fermion total set

Now, we introduce the domain in which the Fermion stochastic integrals are defined. We also state some results that are important for the developement of the theory of Fermion stochastic integrals, developed in Section 5. This section summarizes some results and concepts of [3], [5] and [27].

Definition 2.23. *The* ***anticommutator*** *of two operators A and B acting on a Hilbert space is defined as*

$$\{A, B\} := AB + BA.$$

Definition 2.24. *A **Fermion system** with one degree of freedom is given by a pair of operators* $(b^0, b^{*(0)})$ *acting on a Hilbert space* $\mathcal{H}_0$ *and satisfying the relations*

$$\{b^{(0)},\ b^{*(0)}\} = I, \quad \{b^{(0)},\ b^{(0)}\} = \{b^{*(0)},\ b^{*(0)}\} = 0,$$

where I *denotes the identity operator acting on* $\mathbb{C}^2$.

Example 2.25. *The operators*

$$b^{(0)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b^{(0)\dagger} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

acting on the Hilbert space $\mathbb{C}^2$*, satisfy the canonical anticommutation relations.*

Definition 2.26. *Let* $\mathcal{H}$ *be a Hilbert space and let* $f \to a(f)$ *be a conjugate map from* $\mathcal{H}$ *to* $\mathcal{B}(\mathcal{H})$*, the bounded operators acting on* $\mathcal{H}$*. Then, we say that the operators* $a(f)$ *satisfy the **canonical anticommutation relations**, if*

$$\{a(f), a(g)\} = 0, \quad \{a(f), a^{\dagger}(g)\} = \langle f, g\rangle I, \tag{2.3.1}$$

for all $f, g \in \mathcal{H}$ *and we call it* CAR.

Example 2.27. *Take* $\mathcal{M}_{2x2}(\mathbb{C})$ *as the* C^**-algebra*[13] *with involution the conjugate transpose and consider*

$$\psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then, it clearly holds that $\psi_1 = b^*\psi_0$ *and* ψ_0 *form a total set in* $\mathbb{C}^2$.

Remark 2.28. *Let* $f = (f_1, ..., f_n)$ *and* $\mathcal{H}$ *be a Hilbert space, such that* $f_i \in \mathcal{H}$ *for all* $i \in \{1, ..., n\}$*. Then,* f^j *denotes the omission of the j-th element of* f.

We introduce the Fermion creation and annihilation operators, which we will use in Section 5 to define the Fermion creation and annihilation processes (Definition 5.19). Against the differentials of these processes we will define the quantum stochastic integrals. We will also see that the Fermion creation and annihilation operators satisfy CAR.

Definition 2.29. *Let* $\mathcal{H}$ *be a Hilbert space, then for any* $u \in \mathcal{H}$*:*

(i) ***The Fermion creation operator*** *with domain* $\mathcal{H}^{\wedge n}$ *and codomain* $\mathcal{H}^{\wedge n+1}$ *is defined as*

$$b^*(u)(u_1 \wedge \cdots \wedge u_n) := u \wedge u_1 \wedge \cdots \wedge u_n;$$

(ii) ***The Fermion annihilation operator*** $b(u)$ *which goes from* $\mathcal{H}^{\wedge n}$ *to* $\mathcal{H}^{\wedge(n-1)}$ *is defined as*

$$b(u)(u_1 \wedge \cdots \cdot \wedge u_n) := \sum_{i=1}^{n} (-1)^i \langle u, u_i\rangle u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots u_n,$$

[13] See in Appendix A.3 the definition of a C*-algebra.

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$ and $\hat{u}$ means that u is to be omitted.

Remark 2.30. *Let $\Omega_{Fermion}$ denote the vacuum in $\Gamma_a(\mathcal{H})$. Let u be an element of the Hilbert space $\mathcal{H}$. Then, we observe that the following relations hold:*

$$b(u)\Omega_{Fermion} = 0, \quad b^*(u)\Omega_{Fermion} = u.$$

Proposition 2.31. *The operator $a^*(u)$ is the adjoint of $a(u)$*

Proof. See Proposition 8.15 in [17] □

Remark 2.32. *The Proposition 2.31 allows us to interchange the symbol * with the adjoint symbol † in Definition 2.29.*

Example 2.33. *If $\mathcal{H} = L^2(\mathbb{R}^v)$, then the Boson and Fermion spaces consists of sequences $\{\psi^{(n)}\}_{n\geq 0}$ of functions of n variables $x_i \in \mathbb{R}^v$ which are totally symmetric ($+sign$) or totally antisymmetric ($-sign$). The action of the annihilation and creation operators is given by*

$$(a_{\pm}(f)\psi)^{(n)}(x_1, ..., x_n) = (n+1)^{1/2} \int \overline{f(x)}\psi^{(n+1)}(x, x_1, ..., x_n)\mathrm{d}x,$$
$$(a^*_{\pm}(f)\psi)^{(n)}(x_1, ..., x_n) = n^{-\frac{1}{2}} \sum_{i=1}^{n} (\pm 1)^{i-1} f(x_i)\psi^{(n-1)}(x_1, ..., \hat{x}_i, ..., x_n),$$

where $\hat{x}_i$ denotes that the i-th variable is to be omitted.[14]

Proposition 2.34. *The Fermion creation and annihilation operators satisfy the canonical anticommutation relations.*

Proof. See Proposition 23.4 in [1]. □

Definition 2.35. *Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \to \mathcal{H}$ be an operator, let V be a vector subspace of $\mathcal{H}$, then we say that V is **invariant** under A if $A(V) \subset V$.*

Definition 2.36. *A set $\mathcal{M}$, of bounded operators, on a Hilbert space $\mathcal{H}$, is defined to be **irreducible**, if the only closed subspaces of $\mathcal{H}$ which are invariant under the action of $\mathcal{M}$, are the trivial subspaces $\{0\}$ and $\mathcal{H}$.*[15]

The following proposition will allows us to define a domain for the Fermion stochastic integrals in Section 5.

Proposition 2.37. *Let $\mathcal{H}$ be a Hilbert space, consider the Fermion creation and annihilation operators and consider a unit vector $\psi_0 \in \mathcal{H}$ such that $a(f)\psi_0 = 0$ for all $f \in \mathcal{H}$, then the set of vectors*

$$\psi_m(f_1, ..., f_m) = a^\dagger(f_m)... \, a^\dagger(f_1)\psi_0,$$

with $m = 0, 1, 2, ..., f_1, ..., f_m \in \mathcal{H}$, is total in $\Gamma_a(\mathcal{H})$.

Proof. See in [27]. □

[14] See in [27] for more details on this example

[15] See 2.3.7 in [28].

Remark 2.38. *For each $m \in \mathbb{N}$, we call the vectors of the previous proposition* ***Fermion total vectors*** *and the set ot these vectors the* ***Fermion total set****.*

Proposition 2.39. *Let $\mathcal{H}$ be a Hilbert space, $\Gamma_a(\mathcal{H})$ the Fermion Fock space, $f, g \in \mathcal{H}$, $a(f)$ and $a^*(g)$ the corresponding annihilation and creation operators on $\Gamma_a(\mathcal{H})$. It follows that*

(i) $\|a(f)\| = \|f\| = \|a^*(f)\|$.

(ii) If $\Omega = (1, 0, 0, ...)$ and $\{f_\alpha\}$ is an orthonormal basis of $\mathcal{H}$, then

$$\psi_m(f_1, ..., f_m) = a^\dagger(f_m) \cdots a^\dagger(f_1)\psi_0$$

is an orthonormal basis of $\Gamma_a(h)$ when $\{f_1, ..., f_m\}$ runs over the finite subsets of $\{f_\alpha\}$.

(iii) The set of bounded operators $\{a(f), a^*(g) : f, g \in h\}$ is irreducible on $\Gamma_a(h)$.

Proof. Since the Fermion creation and annihilation operators satisfy CAR, for $f \in \mathcal{H}$, it follows that

$$a^*(f)a^*(f)a(f)a(f) = 0. \tag{2.3.2}$$

Therefore,

$$\begin{aligned}
(a^*(f)a(f))^2 &= a^*(f)a(f)a^*(f)a(f) \\
&\overset{(2.3.2)}{=} a^*(f)a(f)a^*(f)a(f) + a^*(f)a^*(f)a(f)a(f) \\
&= a^*(f)\left(a(f)a^*(f)a(f) + a^*(f)a(f)a(f)\right) \\
&= a^*(f)(a(f)a^*(f) + a^*(f)a(f))a(f) \\
&\overset{2.23}{=} a^*(f)\{a(f), a^*(f)\}a(f) \\
&\overset{2.26}{=} \|f\|^2 a^*(f)a(f).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|a(f)\|^4 &= \|a(f)\|\|a(f)\|\|a(f)\|\|a(f)\| \\
&= \|a^*(f)\|\|a(f)\|\|a^*(f)\|\|a(f)\| \\
&= \|a^*(f)a(f)\| \, \|a^*(f)a(f)\| \\
&= \left\|(a^*(f)a(f))^2\right\| \\
&\overset{2.26}{=} \|f\|^2\|a^*(f)a(f)\| \\
&= \|f\|^2\|a(f)\|^2.
\end{aligned}$$

Since $a(f) \neq 0$, if and only if $f \neq 0$, it follows that

$$\|a(f)\|^2 = \|f\|^2$$

and then

$$\|a(f)\| = \|f\|.$$

For the rest of the proof see Proposition 5.22 in [27]. □

Remark 2.40. *The first statement of Proposition 2.39 implies that the Fermion creation and annihilation operators are bounded.*

The following result is a formula for the **Fermion total vectors**, which we need for the developement of the theory of Fermion stochastic integration. Specifically, to find the "differential" rules given in Remark 5.44.

Proposition 2.41. *Let $\mathcal{H}$ be a Hilbert space, let $a(f)$ be operators that satisfy* CAR *with $f \in \mathcal{H}$, and let $\psi_0 \in \mathcal{H}$ be a unit vector such that*

$$a(g)\psi_0 \ = \ 0,$$

for all $g \in \mathcal{H}$. Then, for fixed $f_1, ..., f_m \in \mathcal{H}$, it holds that

$$a(g)\psi_m(f) \ = \ \sum_{j=1}^{m}(-1)^{m-j}\langle g, f_j\rangle \psi_{m-1}(f^j),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$, $f \ = \ (f_1, ...f_m)$ and where

$$\psi_m(f_1, ..., f_m) = a^\dagger(f_m) \cdots a^\dagger(f_1)\psi_0, \tag{2.3.3}$$

For $m = 1, 2, ...$

Proof. We prove the equality by induction in m.
For the case $m = 2$, we infer from

$$a(g)a^\dagger(f) \ = \ \langle g, f\rangle I - a^\dagger(f)a(g)$$

that

$$\begin{aligned} a(g)a^\dagger(f_2)a^\dagger(f_1)\psi_0 \ &\overset{2.26}{=} \ (\langle g, f_2\rangle I - a^\dagger(f_2)a(g))a(f_1)\psi_0 \\ &= \ \langle g, f_2\rangle a^\dagger(f_1)\psi_0 - a^\dagger(f_2)a(g)a(f_1)\psi_0 \\ &= \ \langle g, f_2\rangle a^\dagger(f_1)\psi_0 - a^\dagger(f_2)(\langle g, f_1\rangle - a^\dagger(f_1)a(g))\psi_0 \\ &\overset{2.26}{=} \ \langle g, f_2\rangle a^\dagger(f_1)\psi_0 - a^\dagger(f_2)\langle g, f_1\rangle\psi_0 + a^\dagger(f_2)a^\dagger(f_1)a(g)\psi_0 \\ &\overset{2.26}{=} \ \langle g, f_2\rangle a^\dagger(f_1)\psi_0 - a^\dagger(f_2)\langle g, f_1\rangle\psi_0 \\ &= \ \sum_{j=1}^{2}(-1)^{2-j}\langle g, f_j\rangle\psi_1(f^j) \end{aligned}$$

assume that the equality holds for $m = n$, that is

$$a(g)\psi_n(f) = \sum_{j=1}^{n}(-1)^{n-j}\langle g, f_j\rangle\psi_{n-1}(f^j). \tag{2.3.4}$$

By the anticommutation relations we have

$$a(g)a^\dagger(f_{n+1}) \cdots a^\dagger(f_1)\psi_0 = (\langle g, f_{n+1}\rangle I - a^\dagger(f_{n+1})a(g))a^\dagger(f_n) \cdots a^\dagger(f_1)\psi_0$$

which by hypothesis of induction is equal to

$$
\begin{aligned}
& (\langle g, f_{n+1}\rangle) a^\dagger(f_n)\cdots a^\dagger(f_1)\psi_0 - a^\dagger(f_{n+1})\left(\sum_{j=1}^{n}(-1)^{n-j}\langle g, f_j\rangle \psi_{n-1}(f^j)\right) \\
\overset{2.3.4}{=} \;& \langle g, f_{n+1}\rangle a^\dagger(f_n)\cdots a^\dagger(f_1)\psi_0 - \sum_{j=1}^{n}(-1)^{n-j}\langle g, f_j\rangle \psi_n(f^j) \\
= \;& \langle g, f_{n+1}\rangle a^\dagger(f_n)\cdots a^\dagger(f_1)\psi_0 + \sum_{j=1}^{n}(-1)^{n+1-j}\langle g, f_j\rangle \psi_n(f^j) \\
\overset{2.3.3}{=} \;& \langle g, f_{n+1}\rangle \psi_n(f^{n+1}) + \sum_{j=1}^{n}(-1)^{n+1-j}\langle g, f_j\rangle \psi_n(f^j) \\
= \;& \sum_{j=1}^{n+1}(-1)^{n+1-j}\langle g, f_j\rangle \psi_n(f^j)
\end{aligned}
$$

which proves the statement. □

3 Weyl operators

The Weyl operators allow us to define the creation and annihilation operators and the results given here help to simplify computations related to these operators. For a review of basic operator theory and Stone's theorem see Appendices B.2 and C, respectively. This section summarizes some results and concepts of chapter 20 of [1].

Definition 3.1. *Let $\mathcal{H}$ be a Hilbert space and $u, v \in \mathcal{H}$. For a unitary operator U acting $\mathcal{H}$, we define the action of the pair (u, U) as*

$$(u, U)v := Uv + u.$$

Further, we will denote as $\mathcal{U}(\mathcal{H})$ the set of all unitary operators acting on a Hilbert space $\mathcal{H}$.

Remark 3.2. *Since U is a unitary operator the inverse of this pair is well defined and it is given by*

$$(-U^{-1}u, U^{-1}).$$

Remark 3.3. *We observe that for $u_1, u_2, v \in \mathcal{H}$ and $U_1, U_2 \in \mathcal{U}(\mathcal{H})$ it holds*

$$\begin{aligned} (u_1, U_1)(u_2, U_2)v &\overset{3.1}{=} (u_1, U_1)\,(U_2 v + u_2) \\ &= U_1 U_2 v + U_1 u_2 + u_1 \\ &\overset{3.1}{=} ((u_1 + U_1 u_2), U_1 U_2)v \end{aligned}$$

and therefore we have the composition for the pairs $(u_j, U_j), j = 1, 2,$

$$(u_1, U_1)(u_2, U_2) = (u_1 + U_1 u_2, U_1 U_2). \tag{3.0.1}$$

Definition 3.4. *For a pair (u, U), with $u \in \mathcal{H}$ and $U \in \mathcal{U}(\mathcal{H})$, the associated **Weyl operator** is given by*

$$W(u, U)e(v) \;=\; \exp\left(-\frac{1}{2}\|u\|^2 - \langle u, Uv\rangle\right)e(Uv + u), \tag{3.0.2}$$

for all v in $\mathcal{H}$.

Remark 3.5. *The Weyl operators have as **domain** the finite linear combinations of the exponential vectors.*

Proposition 3.6. *Let $\mathcal{H}$ be a Hilbert space, $U \in \mathcal{U}(\mathcal{H})$ and $u, v_1, v_2 \in \mathcal{H}$. Then, the Weyl operators are unitary.*

Proof.

$$\begin{aligned} &\langle W(u, U)e(v_1), W(u, U)e(v_2)\rangle \\ \overset{(3.0.2)}{=}\; &\langle \exp\{-\|u\|/2 - \langle Uv_1, u\rangle\}\, e(Uv_1 + u), \exp\{-\|u\|/2 - \langle u, Uv_2\rangle\}e(Uv_2 + u)\rangle \\ =\; &\exp\{-\|u\| - \langle u, Uv_1\rangle - \langle u, Uv_2\rangle\}\langle e(Uv_1 + u), e(Uv_2 + u)\rangle \\ \overset{2.15}{=}\; &\exp\{-\|u\| - \langle u, Uv_1\rangle - \langle u, Uv_2\rangle\}\exp\langle Uv_1 + u, Uv_2 + u\rangle \\ =\; &\exp\{\langle Uv_1, Uv_2\rangle\} \\ =\; &\exp\{\langle v_1, v_2\rangle\} \\ \overset{2.15}{=}\; &\langle e(v_1), e(v_2)\rangle. \end{aligned}$$

Therefore, since the coherent vectors are total, we can extend the Weyl operators to an isometry. Now, we prove that they are invertible

$$\begin{aligned}
& W(U,u)W(U^*,-U^*u)e(v) \\
\overset{3.0.2}{=} \; & \exp(-\langle -U^*u, -U^*u\rangle/2 - \langle -U^*u, U^*v\rangle) W(U,u) e(U^*v - U^*u) \\
\overset{3.0.2}{=} \; & \exp(-\langle -U^*u, -U^*u\rangle/2 - \langle -U^*u, U^*v\rangle) \exp\big(-\|u\|^2/2 - \langle u, v-u\rangle\big) e(v-u+u) \\
= \; & \exp(-\langle U^*u, -U^*u\rangle/2 + \langle U^*u, U^*v\rangle - \langle u,u\rangle/2 - \langle u,v\rangle) e(v) \\
= \; & \exp(-\langle U^*u, -U^*u\rangle/2 + \langle U^*u, U^*v\rangle - \langle u,u\rangle/2 - \langle u, v-u\rangle) e(v) \\
= \; & \exp(-\langle u,u\rangle/2 + \langle u,v\rangle - \langle u,u\rangle/2 + \langle u,u\rangle - \langle u,v\rangle) e(v) \\
= \; & e(v).
\end{aligned}$$

Thus, the Weyl operators are invertible and hence unitary. □

Proposition 3.7. *Given a Hilbert space $\mathcal{H}$, $u_1, u_2 \in \mathcal{H}$ and two unitary operators U_1, U_2 acting on the Hilbert space $\mathcal{H}$, it follows that*

$$W(u_1,U_1)W(u_2,U_2) = \exp(-i\,\mathrm{Im}\langle u_1, U_1u_2\rangle) W((u_1,U_1)(u_2,U_2)).$$

Proof. We have that

$$\begin{aligned}
W(u_1,U_1)W(u_2,U_2)e(v) \overset{3.0.2}{=} \; & W(u_1,U_1)\exp\left(-\frac{1}{2}\|u_2\|^2 - \langle u_2, U_2v\rangle\right) e(U_2v+u_2) \\
\overset{3.0.2}{=} \; & \exp\left(-\frac{1}{2}\|u_2\|^2 - \langle u_2, U_2v\rangle\right) \\
& \exp\left(-\frac{1}{2}\|u_1\|^2 - \langle u_1, U_1(U_2v+u_2)\rangle\right) \\
& e\left(U_1(U_2v+u_2)+u_1\right).
\end{aligned}$$

On the other side, we have that

$$\begin{aligned}
& \exp\left(-i\,\mathrm{Im}\langle u_1, U_1u_2\rangle\right) W((u_1+U_1u_2, U_1U_2))e(v) \\
\overset{3.0.2}{=} \; & \exp(-i\,\mathrm{Im}\langle u_1, U_1u_2\rangle)\exp(-\langle u_1+U_1u_2, U_1U_2v\rangle) e(U_1U_2v + u_1 + U_1u_2) \\
= \; & \exp(-i\,\mathrm{Im}\langle u_1, U_1u_2\rangle)\exp\left(-\frac{1}{2}(\langle u_1,u_1\rangle)\right)\exp\left(-\frac{1}{2}\langle u_1, U_1u_2\rangle\right) \\
& \exp\left(-\frac{1}{2}\langle U_1u_2, u_1\rangle\right)\exp\left(-\frac{1}{2}\langle U_1u_2, U_1u_2\rangle\right) \\
& \exp(-\langle u_1, U_1U_2v\rangle)\exp(-\langle U_1u_2, U_1U_2v\rangle) \\
& e(U_1(U_2v+u_2)+u_1) \\
= \; & \exp\left(-\frac{1}{2}\|u_1\|^2 - \frac{1}{2}\|u_2\|^2\right) \\
& \exp(-\langle u_1, U_1u_2\rangle)\exp(-\langle u_1, U_1U_2v\rangle - \langle U_1u_2, U_1U_2v) \\
& e(U_1(U_2v+u_2)+u_1)
\end{aligned}$$

and therefore the equality holds. □

Remark 3.8. *From Proposition 2.16 we know that the linear manifold generated by* $\{e(u) \mid u \in \mathcal{H}\}$ *is a dense subset of* $\Gamma_a(\mathcal{H})$ *and we denote this set as* $\mathcal{E}(\mathcal{H}) = \mathcal{E}$.

Definition 3.9. *We call the operator*

$$\Gamma(U) \;=\; W(0, U)$$

with domain $\mathcal{E}$ *and* $U \in \mathcal{U}(\mathcal{H})$ *the* ***second quantization*** *of* U.

Additionally, by denoting

$$W(u) \;:=\; W(u, 1),$$

we get the following proposition

Proposition 3.10. *For* $u \in \mathcal{H}$ *and* $U \in \mathcal{U}(\mathcal{H})$*, it holds that*

(i) $W(u)W(v) \;=\; \exp(-i\,\mathrm{Im}\langle u, v\rangle)W(u+v)$,

(ii) $W(u)W(v) \;=\; W(v)W(u)\{\exp(-2i\,\mathrm{Im}\langle u, v\rangle)\}$,

(iii) $\Gamma(U)\Gamma(V) \;=\; \Gamma(UV)$,

(iv) $\Gamma(U)W(u)\Gamma(U)^{-1} \;=\; W(Uu)$,

(v) $W(su)W(tu) \;=\; W((\overline{s+t})u),\;\; s, t \in \mathbb{R}$.

Proof. It holds that

$$\begin{aligned} W(u, I)W(v, I) &\overset{3.7}{=} \exp(-i\,\mathrm{Im}\langle u_1, u_2\rangle)W((u_1, I)(u_2, I)) \\ &\overset{(3.0.1)}{=} \exp(-i\,\mathrm{Im}\langle u_1, u_2\rangle)W(u_1 + u_2, I) \end{aligned}$$

the rest of the equalities follow similarly. □

Remark 3.11. *The last equality of Proposition 3.10 implies that every* $u \in \mathcal{H}$ *yields a one-parameter unitary group*[16] $\{W(tu) \mid t \in \mathbb{R}\}$[17] *and hence by Stone's theorem*[18] *we have that*

$$W(tu) = \exp(-itp(u)),$$

$t \in \mathbb{R}, u \in \mathcal{H}$*, where* $p(u)$ *is an observable (a self-adjoint operator).*

Definition 3.12. *From the third equality of Proposition 3.10 we infer that for each one-parameter unitary group* $U_t = \exp(-itH)$ *in* $\mathcal{H}$ *corresponds a one parameter unitary group* $\{\Gamma(U_t) \mid t \in \mathbb{R}\}$ *in* $\Gamma_s(\mathcal{H})$*. We denote its Stone generator by* $\lambda(H)$ *so that we have from Stone's theorem that*

$$\Gamma(\exp(-itH)) = \exp(-it\lambda(H)),$$

where $t \in \mathbb{R}$*. We call the self-adjoint operator*[19] $\lambda(H)$ *with domain described by C.0.1* ***the differential second quantization of H****.*

[16] See Definition C.1.
[17] See in [1] for more details on Weyl operators and Stone generators.
[18] See the domain in C.0.1.
[19] See Definitions B.19 and B.15.

The proofs of the following propositions use, as in the previous cases, the definition of coherent vectors, the Weyl operators and Stone's theorem. They give us useful formulas for the creation and annihilation operators that are used for the developement of the Boson stochatic integration.

Proposition 3.13. *The self-adjoint operators $p(u)$ and $\lambda(H)$ obey the following relations:*

(i) $[p(u), p(v)]e(w) = 2i\,\mathrm{Im}\langle u, v\rangle e(w)$ *for all* $u, v, w \in \mathcal{H}$;

(ii) $i[p(u), \lambda(H)]e(v) = -p(iHu)e(v)$ *for all* $u, v \in D(H^2)$;

(iii) For any two bounded observables $H_1, H_2 \in \mathcal{H}$ and $v \in \mathcal{H}$

$$i[\lambda(H_1), \lambda(H_2)]e(v) = \lambda(i[H_1, H_2])e(v).$$

Proof. See Theorem 20.10 in [1]. □

For the next propostion we write

$$\begin{aligned} q(u) &:= -p(iu), \\ a(u) &:= \frac{1}{2}(q(u) + ip(u)), \\ a^\dagger(u) &:= \frac{1}{2}(q(u) - ip(u)). \end{aligned}$$

Proposition 3.14. *For any $\psi_1, \psi_2, \psi_3 \in \mathcal{E}$, the following relations hold:*

(i) $a(u)e(v) = \langle u, v\rangle e(v)$;

(ii) $\langle a^\dagger(u)\psi_1, \psi_2\rangle = \langle \psi_1, a(u)\psi_2\rangle$;

(iii) $\langle \lambda^\dagger(H)\psi_1, \psi_2\rangle = \langle \psi_1, \lambda(H)\psi_2\rangle$;

(iv) The restrictions of $a(u)$ and $a^\dagger(u)$ to $\mathcal{E}$ are respectively antilinear and linear in the variable u. The restriction of $\lambda(H)$ to $\mathcal{E}$ is linear in the variable H;

$$\begin{aligned} [a(u), a(v)]\psi &= [a^\dagger(u), a^\dagger(v)]\psi = 0, \\ [a(u), a^\dagger(v)]\psi &= \langle u, v\rangle\psi, \\ [\lambda(H_1), \lambda(H_2)]\psi &= ([H_1, H_2])\psi, \\ [a(u), \lambda(H)]\psi &= a(H^*)\psi, \\ [a^\dagger, \lambda(H)]\psi &= -a^\dagger(Hu)\psi. \end{aligned}$$

Proof. See Propoosition 20.12 in [1]. □

Proposition 3.15. *The operators $a^\dagger(u)$, $u \in \mathcal{H}$ and $\lambda(H), H \in \mathcal{B}(\mathcal{H})$ obey the following relations*

(i) $\langle e(v), \lambda(H)e(w)\rangle = \langle v, Hw\rangle \exp(\langle v, w\rangle)$;

(ii) $\left\langle a^\dagger(u_1)e(v), a^\dagger(u_2)e(w)\right\rangle = \{\langle u_1, w\rangle\langle v, u_2\rangle + \langle u_1, u_2\rangle\} \exp(\langle v, w\rangle)$;

(iii) $\langle \lambda(H_1)e(v), \lambda(H_2)e(w)\rangle = \{\langle H_1 v, w\rangle\langle v, H_2 w\rangle + \langle H_1 v, H_2 w\rangle\} \exp(\langle v, w\rangle)$,

(iv) $\left\langle a^\dagger(u)e(v), \lambda(H)e(w)\right\rangle = \{\langle u, w\rangle\langle v, Hw\rangle + \langle u, Hw\rangle\} \exp(\langle v, w\rangle)$.

Proof. See Proposition 20.13 in [1]. □

The following proposition is useful when proving the statements of Boson stochastic integration in Chapter 4.

Proposition 3.16. *The operator $a^\dagger(u)$, with $u \in \mathcal{H}$, satisfies*

$$a^\dagger(u)e(v) = \frac{\mathrm{d}}{\mathrm{d}t} e(v + tu)\bigg|_{t=0}.$$

Proof. See Proposition 20.14 in [1]. □

Definition 3.17. *Let $\mathcal{H}$ be a Hilbert space, let f, $g \in \mathcal{H}$, then in the domain $\mathcal{E}$ we define the operators*

$$\exp\left(a(f)\right) e(g) := \exp(\langle f, g\rangle)e(g),$$
$$\exp\left(a^\dagger(f)\right) e(g) := e(g + f).$$

Proposition 3.18. *In the domain $\mathcal{E}$, the Weyl operators $W(u, U)$ admits the factorization*

$$W(u, U) = \exp\left(-\frac{1}{2}\|u\|^2\right) \exp\left(a^\dagger(u)\right)\Gamma(U) \exp\left(-a(U^{-1}u)\right),$$

for all $u \in \mathcal{H}$, $U \in \mathcal{U}(\mathcal{H})$.

Proof. See Proposition 20.15 in [1]. □

And we see that we have the following definition

Definition 3.19. *We call $\lambda(H)$ the conservation operator associated with H.*

Remark 3.20. *$a(u)$, $a^\dagger(u)$ are the creation and annihilation operators associated with u.*

4 Boson stochastic integration

Now we define Boson stochastic integrals with respect to creation, annihilation and conservation processes, which will allow us to define Boson stochastic differential equations. Finally, we will give a Boson Ito formula.

Here, we give a non-commutative generalization of stochastic analysis. The similarities between Fermion and Boson stochastic calculus are quite interesting.[20] We will like to point out that, the similarity between the Bosonic and Fermionic theory is more clear following the presentations of [5] and [8] rather than the more general presentation of Bosons given in [1] (see Appendix E).

Due to the length of the proofs of this theory, we skip most of them. Most of the results, concepts and proofs of this section can be found in [8], as this section is based on [8].

Remark 4.1. *We denote the **algebraic tensor product** of vector spaces $\mathcal{H}_1$, $\mathcal{H}_2$ as $\mathcal{H}_1 \underline{\otimes} \mathcal{H}_2$, if they are Hilbert spaces, $\mathcal{H}_1 \otimes \mathcal{H}_2$ denotes the Hilbert space completion.*

Definition 4.2. *Let T be an operator, $\mathcal{H}$ a Hilbert space and T defined on the dense subset $\mathcal{D}$ of $\mathcal{H}$. Then, the operator with domain $\mathcal{D} \otimes \mathcal{H}$ defined on the product vectors as $u \otimes \psi \to Tu \otimes \psi$ is called the **ampliation** of T to $\mathcal{D} \underline{\otimes} \mathcal{H}$.*

Remark 4.3. *For a Hilbert space h we shall denote by $\mathcal{B}(h)$ the *-algebra of bounded operators on the Hilbert space h.*

4.1 Operator-valued processes

As in the classical case, we first define Boson stochastic integrals for simple functions and then we extend it to more general functions. We will denote by $\mathcal{H}$ a Hilbert space.
We denote by $h, h_{t]}$ and $h_{[t}$ the Hilbert spaces $L^2_{\mathcal{H}}[0,\infty)$, $L^2_{\mathcal{H}}[0,t]$ *and* $L^2_{\mathcal{H}}(t,\infty)$, respectively, of square integrable measurable vector-valued functions taking values in $\mathcal{H}$. Furthermore, we will introduce Boson adapted processes.

Remark 4.4. *By applying the Boson exponential property 2.19 to*

$$h = h_{t]} \oplus h_{[t}$$

we make the identification

$$\Gamma_s\left(h_{t]}\right) \otimes \Gamma_s\left(h_{[t}\right) \cong \Gamma_s(h)$$

in which for each exponential vector $e(f)$, $f \in h$, we have

$$e(f) = e\left(f_{t]}\right) \otimes e\left(f_{[t}\right)$$

*where $f_{t]}$ and $f_{[t}$ are the components of f in $h_{t]}$ and $h_{[t}$, respectively. We denote as $\mathcal{E}\left(h_{t]}\right)$ and $\mathcal{E}\left(h_{[t}\right)$ the dense subspaces of $\Gamma_s(h_{t]})$ and $\Gamma_s\left(h_{[t}\right)$, respectively, spanned by the **exponential vectors**.*

[20] See [32].

Remark 4.5. *The operator-valued processes in which we are interested act in the tensor product*

$$\widetilde{\mathcal{H}} = h_0 \otimes \Gamma_s(h) \tag{4.1.1}$$

of $\Gamma_s(h)$ with the Hilbert space h_0 called the "initial space". Furthermore, we denote as

$$\widetilde{\Gamma_s(h_{t]})} := h_0 \otimes \Gamma_s\left(h_{t]}\right)$$

so that

$$\widetilde{\Gamma_s(h)} := \widetilde{\Gamma_s\left(h_{t]}\right)} \otimes \Gamma_s\left(h_{[t}\right).$$

The following lemma is needed to prove Proposition 4.7, which is a generalization of Proposition 2.16.

Lemma 4.6. *Let $\mathcal{H}$ be a finite dimensional Hilbert space, let $y_1, ...y_n \in \mathcal{H}$ be linearly independent and $x_1, ..., x_n \in \mathcal{H}$, and assume that*

$$\sum_{i=1}^{n} x_i \otimes y_i = 0,$$

then it holds that $x_1 = x_2 = ... = x_n = 0$.

Proof. See Propostion 1, page 2 in [31]. □

Proposition 4.7. *Let $u_1, ..., u_n$ be nonzero elements of h_0, a finite dimensional Hilbert space, and let $f_1, , .., f_n$ be distinct elements of h. Thus, the vectors $u_1 \otimes e(f_1), u_2 \otimes e(f_2), ..., u_n \otimes e(f_n)$ are linearly independent in $\widetilde{\Gamma_s(h)}$.*

Proof. From Proposition 2.16 we know that $\{e(f_i) \mid f_i \in \mathcal{H}, i \in \{1, ..., n\}\}$ is linearly independent, let $u_1, ..., u_n \in \mathcal{H}$ and assume that there exists $\alpha_1, ..., \alpha_n \in \mathbb{C}$, such that

$$\sum_{i=0}^{n} \alpha_i (u_i \otimes e(f_i)) = \sum_{i=0}^{n} (\alpha_i u_i \otimes e(f_i)) = 0. \tag{4.1.2}$$

where the equality in (4.1.2) holds by the definition of tensor products. Then, by Lemma 4.6, it follows

$$\alpha_i u_i = 0,$$

$i \in \{1, ..., n\}$ and since the u_i are different from zero we get

$$\alpha_i = 0,$$

$i \in \{1, ..., n\}$, which implies

$$\{u_i \otimes e(f_i) \mid u_i, f_i \in \mathcal{H}, i \in \{1, ..., n\}\}$$

is linearly independent. □

Definition 4.8. *Let $S \subset h$ be a real linear manifold*[21] *closed under all the projections $f \to f_{t]}$, $t \geq 0$ and such that $S + iS$ is dense. We call such S an* ***admissible subspace*** *and denote by $\mathcal{E}_{t]}$ the dense subspaces of $\Gamma_s(h_{t]})$ spanned by the exponential vectors $e(f_{t]})$, respectively, with $f \in S$.*

Definition 4.9. *Let $S \subset h$ be an admissible subspace and $\mathcal{D}$ a linear manifold in h_0. Let $F = (F(t) : t \geq 0)$ be a family of operators in $\widetilde{\Gamma_s(h)}$ such that for arbitrary $t > 0$, $F(t)$ is the ampliation to $\mathcal{D} \underline{\otimes} \mathcal{E} \underline{\otimes} \Gamma_s(h_{[t})$ of an operator in $\widetilde{\Gamma_s(h_{t]})}$ with domain $\mathcal{D} \otimes \mathcal{E}_{t]}$. Then, F is called a* ***Boson adapted process*** *based on $(\mathcal{D}, S)$.*

Example 4.10. *By putting $I = \{I(t) : t \geqq 0\}$, with $I(t) = I_{h_0 \otimes \Gamma_s(\mathcal{H}_{t]})} \otimes I_{\Gamma_s(\mathcal{H}_{[t})}$, we have that the identity operator is a Boson adapted process.*

Remark 4.11. *A family $F = (F(t) : t \geq 0)$ of bounded operators on $\widetilde{\Gamma_s(h)}$ determines a Boson adapted process based on $(\mathcal{D}, S)$ by restricting the domain of each $F(t)$ to $\mathcal{D} \underline{\otimes} \mathcal{E}_{t]} \otimes \Gamma_s(h_{[t})$ if and only if each $F(t)$ belongs to $\mathcal{B}(\widetilde{\Gamma_s(h_{t]})}) \otimes I$. The Boson adapted processes based on $(\mathcal{D}, S)$ form a complex vector space which we denote by $\mathcal{U}(\mathcal{D}, S)$.*

4.2 Stochastic integrals

We now define non-commutative stochastic integrals for simple processes and then we extend the definition to locally square integrable processes.

Definition 4.12. *We call $F \in \mathcal{U}(\mathcal{D}, S)$* ***simple*** *if there exists an increasing sequence $t_n, n = 0, 1, ...$ with t_0 and $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that*

$$F = \sum_{n=0}^{\infty} F_n 1_{[t_n, t_{n+1})},$$

where $F_n = F(t_n)$, to be ***continuous*** *if for each $u \in \mathcal{D}$ and $f \in S$ the map $t \longrightarrow F(t)u \otimes e(f)$ is strongly continuous from $[0, \infty)$ to $\widetilde{\Gamma_s(h)}$, and to be* ***locally square integrable*** *if each such map is strongly measurable*[22] *and satisfies*

$$\int_0^t \|F(s)u \otimes e(f)\| \, \mathrm{d}s < \infty,$$

for all $t > 0$.

Remark 4.13. *We will denote as $\mathcal{U}_0(\mathcal{D}, S)$, $\mathcal{U}_c(\mathcal{D}, S)$, and $L^2(\mathcal{D}, S)$ the subspaces of $\mathcal{U}(\mathcal{D}, S)$ of simple, continuous and locally square integrable processes. It holds*

$$\mathcal{U}_c(\mathcal{D}, S) \subset L^2(\mathcal{D}, S).$$

The following proposition allows us to extend the definition of the stochastic integrals to locally square integrable processes

[21] See Definition E.6.
[22] See Definition 5.24.

Proposition 4.14. *Let $F \in L^2(\mathcal{D}, S)$. Then, there exists a sequence $F^{(n)}$, $n = 1, 2, ...$ of simple processes such that for each $t > 0$, $u \in \mathcal{D}$, and $f \in S$,*

$$\lim_{n\to\infty} \int_0^t \left\| (F(s) - F^{(n)}(s))u \otimes e(f) \right\|^2 \mathrm{d}s = 0.$$

Proof. See Proposition 3.2 in [8]. □

Definition 4.15. *Let $L^{\infty,loc}_{\mathcal{H}}[0,\infty)$ denote the* ***locally bounded measurable vector-valued functions*** *from $[0,\infty)$ into $\mathcal{H}$, let Π be an element of the space $L^{\infty,loc}_{\mathcal{B}(\mathcal{H})}[0,\infty)$ of locally bounded measurable functions from $[0,\infty)$ into the Banach space $\mathcal{B}(\mathcal{H})$. Let f, $g \in L^{\infty,loc}_{\mathcal{H}}[0,\infty)$, then for each $t \geq 0$, $f_t = f1_{[0,t]}$ and $g_t = g1_{[0,t]} \in h$, and $\Pi_t = \Pi\, 1_{[0,t]}$ are elements of $\mathcal{B}(h)$ acting pointwise on h. Then, we call the ampliations in $\widetilde{\Gamma_s(h)} = h_0 \otimes \Gamma_s(h)$ of*

$$A_f(t) = b(f_t), \quad A^\dagger_g(t) = b^\dagger(g_t), \quad \Lambda_\Pi(t) = \lambda(\Pi_t),$$

the ***annihilation, creation, and gauge processes****, of strengths f, g and Π, respectively.*

Proposition 4.16. *$A_f, A^\dagger_g, \Lambda_\Pi$ are continuous, hence locally square integrable.*

Proof. See Proposition 4.1 in [8]. □

Definition 4.17. *Let $E, F, G, H \in \mathcal{U}_0(\mathcal{D}, S)$, so that we may write*

$$E = \sum_{n=0}^{\infty} E_n 1_{[t_n,t_{n+1})}, \quad F = \sum_{n=0}^{\infty} F_n 1_{[t_n,t_{n+1})}, \quad G = \sum_{n=0}^{\infty} G_n 1_{[t_n,t_{n+1})}, \quad H = \sum_{n=0}^{\infty} H_n 1_{[t_n,t_{n+1})},$$

where $0 = t_0 < t_1 < t_2 < ... < t_n \to \infty$ as n tends to infinity. The family of operators $M = (M(t) : t \geq 0)$, with domain $\mathcal{D} \otimes S$ defined by

$$\begin{aligned} M(0) &= 0, \\ M(t) &= M(t_n) + E_n\left(\Lambda_\Pi(t) - \Lambda(t_n)\right) + F_n\left(A_f(t) - A_f(t_n)\right) + \\ & \quad G_n\left(A^\dagger_g(t) - A^\dagger_g(t)\right) + H_n(t - t_n), \end{aligned}$$

for $t_n < t \leqq t_{n+1}$, is called the ***Boson stochastic integral*** *of (E, F, G, H) with respect to Λ_Π, A_f, A_g and Lebesgue measure, and denoted by*

$$M(t) = \int_0^t \left(E\mathrm{d}\Lambda_\Pi + F\mathrm{d}A_f + G\mathrm{d}A^\dagger_g + H\mathrm{d}s\right) + M(0).$$

Theorem 4.18. *Let $E, F, G, h \in \mathcal{U}_{0(\mathcal{D},S)}$, and let M be their* ***Boson stochastic integral****. Then, for arbitrary $u \in h_0$, $l \in h$, $v \in \mathcal{D}$, $d \in S$, and $t \geqq 0$,*

$$\begin{aligned} \langle u \otimes e(l), M(t)v \otimes e(d)\rangle &= \int_0^t \langle u \otimes e(l), \{\langle l(s), \Pi(s)l(s)\rangle_{\mathcal{H}} E(s) + \langle f(s), d(s)\rangle\rangle_{\mathcal{H}} F(s) + \\ & \quad \langle l(s), g(s)\rangle_{\mathcal{H}} G(s) + H(s)\}v \otimes e(d)\rangle \mathrm{d}s. \end{aligned}$$

Proof. See Theorem 4.1 in [8]. □

Theorem 4.19. *Let $E, F, G, H \in \mathcal{U}_0(\mathcal{D}, S)$, and let M be their Boson stochastic integral. Let $0 \leqq s < t$, $\psi \in \Gamma_s(h_{m]})$, $u \in h$, $s \in S$, and $v \in \mathcal{D}$. Then,*

$$\begin{aligned} & \langle \psi \otimes e(u_{[s}), (M(t) - M(s))v \otimes e(s)\rangle \\ = \quad & \int_s^t \langle \psi \otimes e(u_{[s})\rangle \{\langle u(r), \Pi(r)v(r)\rangle_{\mathcal{H}} E(r)\} + \\ & \langle f(r), v(r)\rangle_{\mathcal{H}} F(r) + \langle u(r), g(r)\rangle_{\mathcal{H}} G(r) + H(r)\} d \otimes e(s)\rangle \mathrm{dr}. \end{aligned}$$

Proof. See Theorem 4.2 in [8]. □

Theorem 4.20. *Let $E, F, G, H \in \mathcal{U}_0(\mathcal{D}, S)$, $E', F', G', H' \in \mathcal{U}_0(\mathcal{D}^{'}, S')$, such that*

$$M(t) = \int_0^t \left(E\mathrm{d}\Lambda_{\Pi} + F\mathrm{d}A_f + G\mathrm{d}A_g^{\dagger} + H\mathrm{d}S \right),$$
$$M^{'}(t) = \int_0^t \left(E'\mathrm{d}\lambda_{\Pi'} + F'\mathrm{d}A_{f'} + G^{'}\mathring{d}A_{g'}^{\dagger} + H'\mathrm{d}s \right).$$

Then, for all $u \in \mathcal{D}, u' \in \mathcal{D}^{'}, h \in S, h' \in S'$, and $t \geqq 0$,

$$\begin{aligned} & \langle M(t)u \otimes e(h), M'(t)u' \otimes e(h')\rangle \\ = \quad & \int_0^t \{\langle M(s)u \otimes e(h), [\langle h(s), \Pi'(s)h'(s)\rangle_{\mathcal{H}} E'(s) + \\ & \langle f'(s), h'(s)\rangle_{\mathcal{H}} F'(s) + \langle h(s), g'(s)\rangle_{\mathcal{H}} G'(s) + H'(s)]u' \otimes e(h')\rangle + \\ & \langle [\langle h'(s), \Pi(s)h(s)\rangle_{\mathcal{H}} E(s) + \langle f(s), h(s)\rangle_{\mathcal{H}} F(s) + \\ & \langle h'(s), g(s)\rangle_{\mathcal{H}} G(s) + H(s)]u \otimes e(h), M'(s)u' \otimes e(h')\rangle + \\ & \langle \Pi(s)h(s) \otimes E(s)u \otimes e(h) + g(s) \otimes G(s)u \otimes e(h), \\ & \quad \Pi'(s)h'(s) \otimes E'(s)u' \otimes e'(h) + g'(s) \otimes G'(s)u' \otimes e(h')\rangle_{\mathcal{H}\otimes\Gamma_s(h)}\}\mathrm{ds}. \end{aligned}$$

Proof. See Theorem 4.3 in [8]. □

Corollary 4.21. *Suppose that S consists of locally bounded functions, so that*

$$\begin{aligned} \alpha(T) \quad = \quad & \sup_{0 \leqq s \leqq T} \max\{|\langle h(s), \Pi(s)h(s)\rangle_{\mathcal{H}}|, |\langle f(s), h(s)\rangle_{\mathcal{H}}|, \\ & |\langle h(s), g(s)\rangle_{\mathcal{H}}|, \|\Pi(s)h(s)\|_{\mathcal{H}}^2, \|g(s)\|_{\mathcal{H}}^2\} \end{aligned}$$

is finite for each $T > 0$. Then, for $T > 0$ and $0 \leqq t \leqq T$,

$$\begin{aligned} \|M(t)u \otimes e(h)\|^2 \quad \leqq \quad & 6\alpha(T)^2 \int_0^T \exp(t-s)\{\|E(S)u \otimes e(h)\|^2 + \|F(s)u \otimes e(h)\|^2 + \\ & \|G(s)u \otimes e(h)\|^2 + \|H(s)u \otimes e(h)\|^2\}\mathrm{ds}. \end{aligned}$$

Proof. See Corollary 1 in [8]. □

Corollary 4.22. *Under the hypothesis of Corollary 4.21, for $0 \leqq s \leqq t < T$,*

$$\begin{aligned} \|(M(t) - M(s))u \otimes e(h)\|^2 \quad \leqq \quad & 6\alpha(T)^2 \int_s^t \exp(t-s)\{\|E(r)u \otimes e(h)\|^2 + \|F(r)u \otimes e(h)\|^2\} + \\ & \|G(r)u \otimes e(h)\|^2 + \|H(r)u \otimes e(h)\|^2\}\mathrm{dr}. \end{aligned}$$

In particular M is continuous and thus $M \in L^2(\mathcal{D}, S)$.

Proof. See Corollary 2 in [8]. □

Theorem 4.23. *Provided that S, S' consists of locally bounded functions, the Theorems 4.18, 4.19 and 4.20 remain valid for locally square integrable integrands.*

Proof. See Theorem 4.4 in [8]. □

Remark 4.24. *We use the diferential notation*

$$\mathrm{d}M = E\mathrm{d}\Lambda_\Pi + F\mathrm{d}A_f + G\mathrm{d}A_g^\dagger + H\mathrm{d}t, \tag{4.2.1}$$

to mean that M is a process in $L^2(\mathcal{D}, S)$ (where S consists of locally bounded functions) such that for all $t \geqq 0$,

$$M(t) - M(0) = \int_0^t \left(E\mathrm{d}\Lambda_\Pi + F\mathrm{d}A_f + G\mathrm{d}A_g^\dagger + H\mathrm{d}s \right),$$

where $E, F, G, H \in L^2(\mathcal{D}, S)$.

Remark 4.25. *For a fixed admissible subspace S consisting of locally bounded functions, we denote by $\mathcal{C}(S)$ the set of all processes $M \in \mathcal{U}(h_0, S)$ satisfying*

$$\begin{aligned} \|M(t)u \otimes e(h)\|^2 \quad = \quad & \int_0^t \{2\,\mathrm{Re}\langle M(s)\langle M(s)u \otimes e(h), [\langle h(s), \Pi(s)h(s)\rangle_{\mathcal{H}} E(s) + \\ & \langle f(s), h(s)\rangle_{\mathcal{H}} F(s) + \langle h(s), g(s)\rangle_{\mathcal{H}} G(s) + H(s)]u \otimes e(h)\rangle + \\ & \|\Pi(s)h(s) \otimes E(s)u \otimes e(h) + g(s) \otimes G(s)u \otimes e(h)\|^2\}\mathrm{d}s, \end{aligned}$$

for some $E, F, G, H \in \mathcal{U}_c(h_0, S)$, and some $f, g \in L_t^{\infty,loc}[0,\infty)$ and $\Pi \in L_{\mathcal{B}(\mathcal{H})}^{\infty,loc}[0,\infty)$ such that for all $t > 0$,

$$\sup_{0 \leqq s \leqq t} \max\{\|M(s)\|, \|F(s)\|, \|G(s)\|, \|H(s)\|\} < \infty.$$

4.3 Boson Ito formula

We give a non-commutative generalization of Ito's formula. We denote by $\mathcal{M}(S)$ the linear span of $\mathcal{C}(S)$.

Theorem 4.26. *$\mathcal{M}$ is an algebra, in which multiplication is given by*

$$\mathrm{d}\left(MM'\right) = M\mathrm{d}M' + (\mathrm{d}M)M' + \mathrm{d}M\mathrm{d}M',$$

for $M, M' \in \mathcal{C}(S)$, where M satisfies equation (4.2.1), $(\mathrm{d}M)M'$ is given by

$$(\mathrm{d}M)\, M' = EM'\mathrm{d}\Lambda_\Pi + FM'\mathrm{d}A_f + GM'\mathrm{d}A_g^\dagger + HM'\mathrm{d}t,$$

that is, the differentials $\mathrm{d}\Lambda_\Pi, \mathrm{d}A_f, \mathrm{d}A_g^\dagger$, and $\mathrm{d}t$ commute with the Boson adapted process M', and $\mathrm{d}M\mathrm{d}M'$ is evaluated by combining this with extension by bilinearity of the multiplication rules

	$\mathrm{d}\Lambda_\Pi$	$\mathrm{d}A_{f'}$	$\mathrm{d}A^\dagger_{g'}$	$\mathrm{d}t$
$\mathrm{d}\Lambda_\Pi$	$\mathrm{d}\Lambda_{\Pi\Pi'}$	*0*	$\mathrm{d}A^\dagger_{\Pi g'}$	*0*
$\mathrm{d}A_f$	$\mathrm{d}A^\dagger_{\Pi' f}$	*0*	$\langle f(t), g'(t)\rangle \ \mathrm{d}t$	*0*
$\mathrm{d}A^\dagger_g$	*0*	*0*	*0*	*0*
$\mathrm{d}t$	*0*	*0*	*0*	*0*

Proof. See Theorem 4.5 in [8]. □

5 Fermionic stochastic integration

Here, we provide the concepts and results needed to define Fermion diffusions in the next chapter. That is, we will define Fermion adapted processes, Fermion stochastic integrals and give a Fermion Ito formula. Futhermore, we will use the theory of this chapter to prove the consistency conditions which will allows us to find the coefficients of Fermion diffusions with one, two and three degrees of freedom.

It is worth mentioning, that according to [1] there does not seem to exist analogues of Weyl operators for the asymmetric Fock space. This apparent disadvantage does not seem to make the Fermion stochastic calculus much more difficult to define than the Bosonic calculus. This section summarizes, primarily, results and concepts from [5].

5.1 $\mathbb{Z}_2$-graded Hilbert space

We introduce the concept of Hilbert graded space, since we are interested in grading the antisymmetric Fock space in order to define even and odd operators. As we will see later in Definition 6.3, Fermions are assumed to be odd operators and hence the importance of this section which is one of the main differences between the theory of Boson stochastic integration and Fermion stochastic integration.

Definition 5.1. *A $\mathbb{Z}_2$-**graded Hilbert space** is a Hilbert space direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$, $\mathcal{H}_-$ are called the **even** and **odd** subspaces, respectively. We say that $T \in \mathcal{B}(\mathcal{H})$, is **even** if $T\mathcal{H}_+ \subseteq \mathcal{H}_+$ and $T\mathcal{H}_- \subseteq \mathcal{H}_-$. We say that it is **odd** if $T\mathcal{H}_+ \subseteq \mathcal{H}_-$ and $T\mathcal{H}_- \subseteq \mathcal{H}_+$. Here $\mathcal{B}(\mathcal{H})$ denotes the bounded operators acting on $\mathcal{H}$.*

Remark 5.2. *Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, let T_1, T_2, S_1, $S_2 : \mathcal{H} \to \mathcal{H}$ operators, such that T_1, T_2 are even and S_1, S_2 are odd, with respect to this grading. Then, it holds that T_1T_2 is **even**, T_1S_1 is **odd** and S_1S_2 **even** (assuming that the products are well defined). We shall refer to this properties as the **parity property**.*

We now give some examples of this important concept

Example 5.3. *We grade the bounded operators $\mathcal{B}(\mathcal{H})$ in the following way*

$$\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H})_{odd} \oplus \mathcal{B}(\mathcal{H})_{even}.$$

Here $\mathcal{B}(\mathcal{H})_{odd}$ and $\mathcal{B}(\mathcal{H})_{even}$ denote the odd and even operators with respect to a $\mathbb{Z}_2$-grading of the Hilbert space $\mathcal{H}$.

Example 5.4. *If we let $\mathcal{H} = L^2[0, \infty)$. Denote, exactly as in the Boson case, $\mathcal{H}_{t]}$ and $\mathcal{H}_{[t}$ the Hilbert spaces $L^2[0, t]$ and $L^2(t, \infty)$, and by $\psi_{0t]}$ and $\psi_{0[t}$ their respective vacuum vectors. Then, we have the decomposition*

$$\mathcal{H} = \mathcal{H}_{t]} \oplus \mathcal{H}_{[t}$$

and the vacuum in the Fock space has the form

$$\psi_0 = \psi_{0t]} \otimes \psi_{0[t}.$$

Example 5.5. *From the previous example we have by the Fermion Exponential Property that $\Gamma_a(L^2(\mathbb{R}^+))$ is isomorphic to $\Gamma_a(L^2[0, t)) \otimes \Gamma_a(L^2[t, \infty))$.*

Example 5.6. *We grade the antisymmetric Fock space over a Hilbert space $\mathcal{H}$*

$$\Gamma_a(\mathcal{H}) = \bigoplus_{i=0}^{\infty} \mathcal{H}^{\wedge 2i} \oplus \bigoplus_{i=0}^{\infty} \mathcal{H}^{\wedge 2i+1}.$$

Remark 5.7. *Under the previous $\mathbb{Z}_2$-grading the Fermion creation and annhilitation operators are odd operators.*

Example 5.8. *In the case that the Hilbert space is given by $\mathbb{C}^2$ we can grade it in the following way*

$$\mathbb{C}^2 = \text{span}\begin{pmatrix}1\\0\end{pmatrix} + \text{span}\begin{pmatrix}0\\1\end{pmatrix}.$$

5.2 Operator-Valued Processes

Now we introduce the Fermion annihilation and creation processes, which are odd processes with respect to some $\mathbb{Z}_2$ - grading. Let $a(f)$ and $a^\dagger(f)$ with $f \in L^2(\mathbb{R}^+)$ be the annihilation and creation operators in the Fermion Fock space $\Gamma_a L^2(\mathbb{R}^+)$.

Definition 5.9. *Let $\mathcal{H}$ be a Hilbert space. Then, we call a set the* ***closed linear*** span *of a set of vectors, the smallest closed set containing the linear* span *of the vectors.*

Remark 5.10. *We denote by $\mathcal{E}$, $\mathcal{E}_+$ and $\mathcal{E}_-$, respectively, the closed spans of the* ***Fermion total vectors***

$$\psi_m(f_1, ..., f_m) = a^\dagger(f_m)...a^\dagger(f_1)\psi_0, \tag{5.2.1}$$

with $f_1, ..., f_m \in \ h$, and $m \in \mathbb{N}$, even and odd, respectively. We know from Proposition **??** *that for m unrestricted $\mathcal{E}$ is a dense set in $\mathcal{H}$. In this way we have the $\mathbb{Z}_2$-gradding*

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-.$$

Further, we denote

$$\mathcal{E}_{t]}, \quad \mathcal{E}_{[t}$$

the algebraic span of the vectors (5.2.1) *for arbitrary $m \in \mathbb{N}$ in $\mathcal{H}_{t]}$, $\mathcal{H}_{[t}$, respectively. With $\mathcal{E}_{t]\pm}$ and $\mathcal{E}_{[t\pm}$ we mean the algebraic spans of the vectors* (5.2.1) *in $\mathcal{H}_{t]}, \mathcal{H}_{[t}$, respectively, with m even and odd, respectively. Then, for $t \geqq 0$*

$$\mathcal{E} = \mathcal{E}_{t]} \underline{\otimes} \mathcal{E}_{[t}, \quad \mathcal{E}_+ = \mathcal{E}_{t]+} \underline{\otimes} \mathcal{E}_{[t-} + \mathcal{E}_{t]-} \underline{\otimes} \mathcal{E}_{[t-}, \quad \mathcal{E} = \mathcal{E}_{t]+} \underline{\otimes} \mathcal{E}_{[t-} \otimes \mathcal{E}_{t]-} \underline{\otimes} \mathcal{E}_{[t+},$$

where, as in the previous chapter, $\underline{\otimes}$ denotes the ***algebraic tensor product****. We are interested in operator-valued processes which live in the tensor product*

$$h_0 \otimes \mathcal{H}$$

of $\mathcal{H}$ with the Hilbert space h_0 which we call the ***initial space****. We denote*

$$\widetilde{\mathcal{H}} = h_0 \otimes \mathcal{H}, \quad \widetilde{\mathcal{H}_{t]}} = h_0 \otimes \mathcal{H}_{t]}, \quad \widetilde{\mathcal{E}} = h_0 \underline{\otimes} \mathcal{E}, \quad \widetilde{\mathcal{E}_{t]}} = h_0 \otimes \mathcal{E}_{t]}.$$

Then, for each $t \geqq 0$,

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}_{t]}} \otimes \mathcal{H}_{[t}, \quad \widetilde{\mathcal{E}} = \widetilde{\mathcal{E}_{t]}} \underline{\otimes} \mathcal{E}_{[t}.$$

Remark 5.11. *We shall assume that* h_0 *is* $\mathbb{Z}_2$*-graded, with even and odd subspaces* $h_{0\pm}$*, and we will denote by* θ *the* ***parity operator*** *that is the self-adjoint unitary operator which is* I *on* h_{0+} *and* $-I$ *on* h_{0-}*. Then,* $\widetilde{\mathcal{H}}$ *is* $\mathbb{Z}_2$*-graded by*

$$\widetilde{\mathcal{H}}_+ = h_{0+} \otimes \mathcal{H}_+ + h_{0-} \otimes \mathcal{H}_-, \quad \widetilde{\mathcal{H}_-} = h_{0+} \otimes \mathcal{H}_- + h_{0-} \otimes \mathcal{H}_+,$$

and also,

$$\begin{aligned} \widetilde{\mathcal{E}_+} &= h_{0+}\underline{\otimes}\mathcal{E}_+ + h_{0-}\underline{\otimes}\mathcal{E}_-, & \widetilde{\mathcal{E}}_- &= h_{0+}\underline{\otimes}\mathcal{E}_- + h_{0-}\underline{\otimes}\mathcal{E}_+, \\ \widetilde{\mathcal{E}_{t]+}} &= h_{0+}\underline{\otimes}\mathcal{E}_{t]+} + h_{0-}\underline{\otimes}\mathcal{E}_{t]-}, & \widetilde{\mathcal{E}_{t]-}} &= h_{0+}\underline{\otimes}\mathcal{E}_{t]-} + h_{0-}\underline{\otimes}\mathcal{E}_{t]+}. \end{aligned}$$

Definition 5.12. *We say that an operator* T *in* $\widetilde{\mathcal{H}_{t]}}$ *with domain* $\widetilde{\mathcal{E}_{t]}}$ *is* ***even*** *if* $T\,\widetilde{\mathcal{E}}_{t]\pm} \subset \widetilde{\mathcal{H}}_{t]\pm}$ *and* ***odd*** *if* $T\,\widetilde{\mathcal{E}}_{t]\pm} \subset \widetilde{\mathcal{H}}_{t]\mp}$*.*

Remark 5.13. *Every operator* T *in* $\widetilde{\mathcal{H}}_{t]}$ *with domain* $\widetilde{\mathcal{E}}_{t]}$ *can then be decomposed uniquely into the sum* $T = T_+ + T_-$ *of even and odd parts.*

Remark 5.14. *We denote the parity of vectors and operators by the function* δ *which is 0 on even elements and 1 on odd elements.*

We will use the following definition to prove the consistency conditions and some properties of Fermion diffusions in the following chapter. Particularily, with it, we will define the fundamental processes, which are given by Definition 5.20. Furthermore, it will also allows us to introduce the importat concept of algebraic ampliation, which will help us define adapted processes and later in Section 6 the concept of Fermion diffusions.

Definition 5.15. *We call* $\hat{\otimes}$ *the* ***anticommuting tensor product*** *of operators defined as follows, if* $S \in \mathcal{B}(\mathcal{H}_{t]})$*,* $T \in \mathcal{B}(\mathcal{H}_{[t})$*,* $\psi \in \mathcal{H}_{t]}$ *and* $\phi \in \mathcal{H}_{[t}$*. Then, assuming* T *and* ψ *have definite parity*

$$S\widehat{\otimes}T\psi \otimes \phi := (-1)^{\delta(T)\delta(\psi)} S\psi \otimes T\phi.$$

Definition 5.16. *If* S *is a bounded operator on* $\widetilde{\mathcal{H}}_{[t}$*, its* ***ampliation*** *to* $\widetilde{\mathcal{H}}_{t]} \otimes \mathcal{H}_{[t}$ *is the bounded operator* $I\hat{\otimes}S$ *on* $\widetilde{H}$*. If* T *is not necessarily a bounded operator in* $\widetilde{\mathcal{H}}_{t]}$*, with domain* $\widetilde{\mathcal{E}}_{t]}$*, we define its* ***algebraic ampliation*** *to be the operator in* $\widetilde{\mathcal{H}}$ *with domain* $\widetilde{\mathcal{E}}_{t]-}\underline{\otimes}\mathcal{H}_{[t}$ *which acts on products vectors*

$$T\hat{\underline{\otimes}}I\psi \otimes \phi = T\psi \otimes \phi$$

with $\psi \in \widetilde{\mathcal{E}}_{t]}$*,* $\phi \in \mathcal{H}_{[t}$*. If* T *is of definite parity and* $S \in \mathcal{B}(\mathcal{H}_{[t})$ *is of definite parity, then as operators on* $\widetilde{\mathcal{E}}_{t]-}\underline{\otimes}\mathcal{H}_{[t}$*,*

$$\left(T\widehat{\underline{\otimes}}I\right)\left(I\widehat{\otimes}S\right) = (-1)^{\delta(T)\delta(S)}\left(I\widehat{\otimes}S\right)\left(T\widehat{\underline{\otimes}}I\right). \tag{5.2.2}$$

Remark 5.17. *For each* $f \in \mathcal{H}$ *we have that* $f = f_{t]} \oplus f_{[t}$ *with* $f_{[t} \in \mathcal{H}_{[t}$ *and* $f_{t]} \in \mathcal{H}_{t]}$*,*

$$a(f) = a(f_{t]})\hat{\otimes}I + I\hat{\otimes}a(f_{[t}).$$

Definition 5.18. *Two densely defined operators are* ***mutually adjoint*** *if each is contained in the adjoint of the other.*

Definition 5.19. *The* ***Fermion annihilation and creation processes*** *are the mutually adjoint processes A and $A^\dagger$ defined by*

$$A(t) = I\hat{\otimes}a(1_{[0,t]}), \quad A^\dagger(t) = I\hat{\otimes}a^\dagger(1_{[0,t]}),$$

where, $t \geqq 0$, $a(\cdot)$ and $a^\dagger(\cdot)$ are the Fermion annihilation and creation operators.

Proposition 5.20. *The Fermion annihilation and creation processes are odd and they are continuous.*

Proof. See page 477 in [5]. □

5.3 Fermion Adapted processes

Here, we give, as in the Boson case, the concept of non-commutative adapted processes in the asymmetric Fock space, which we will use to define Fermion diffusions.

Definition 5.21. *A* ***Fermion adapted process*** *is a family $F = (F(t):\ t \geq 0)$ of operators in $h_0 \otimes \Gamma_a(L^2(\mathbb{R}_+))$ such that*

(i) for each $t \geqq 0$, $F(t)$ is the algebraic ampliation to $\widetilde{\mathcal{E}}_{t]}\underline{\otimes}\mathcal{H}_{[t}$ of an operator in $\widetilde{\mathcal{H}}_{t]}$ with domain $\widetilde{\mathcal{E}}_{t]}$,

(ii) there is a family $F^\dagger = \big(F^\dagger(t) : t \geqq 0\big)$ also satisfying the first condition such that each $F^\dagger(t)$ is adjoint to $F(t)$.

Example 5.22. *By putting $I = \{I(t):\ t \geq 0\}$, with $I(t) = I_{h_0\otimes\Gamma_a(\mathcal{H}_{t]})} \otimes I_{\Gamma_a(\mathcal{H}_{[t})}$, we have that the identity operator is a Fermion adapted process.*

Remark 5.23. *$F^\dagger$ is a Fermion adapted process called the* ***adjoint process*** *to F. We denote by $\mathcal{A}$ the complex linear space of all Fermion adapted processes and by $\mathcal{A}_0, \mathcal{A}_c$ and L^{loc} the subspaces of $\mathcal{A}$ of simple, continuous and locally square integrable processes, respectively. Clearly, $\mathcal{A}_0, \mathcal{A}_c \subseteqq L^2_{loc}$.*

Definition 5.24. *Let f be a function on a measure space $(X, \mathcal{A}, \mu)$, taking values in a Banach space E, where $\mathcal{A}$ is a σ-algebra on the set X. Then, f is called* ***strongly measurable*** *if and only if there is a sequence of functions f_n so that $f_n(x) \to f(x)$ in norm for a.e. $x \in X$ and each f_n takes only finitely many values, each value taken on a set in $\mathcal{A}$.*

5.4 Fermion stochastic integrals

Here, we introduce another non-commutative generalization of stochastic integrals by first defining them for simply processes and then extending the definition to a larger set of functions. This extension will allows us to define Fermion diffusions in Definition 6.3. As in the classical case, we start by defining the stochastic integrals for a smaller set of functions and then we extend it to a larger collection of functions.

5.4.1 Simple processes

We will first define stochastic integrals for simple processes.

Definition 5.25. *A Fermion adapted process F is* ***simple*** *if there exists an increasing sequence $t_r, r = 0,1,2,...$ with t_0 and $t_r \longrightarrow \infty$ as $r \longrightarrow \infty$ such that $F = \sum_{r=0}^{\infty} F_r 1_{[t_r,t_{r+1})}$,* ***continuous*** *if for arbitrary $u \in h_0$, $m \geqq 0$, $f_1,...,f_m \in h$ the vector-valued functions $t \longrightarrow F^\sharp(t) \otimes \psi_m(f_1,...,f_m)$, where $F^\sharp$ is either F or $F^\dagger$, are strongly continuous on $[0,\infty)$, and* ***locally square integrable*** *if each such function is strongly measurable and satisfies*

$$\int_0^t \left\| F^\sharp(s) u \otimes \psi_m(f) \right\|^2 \mathrm{d}s < \infty,$$

for all $t > 0$.

We need the following result to extend the definition of Fermion stochastic integrals to a larger collection of functions.

Proposition 5.26. *Let $F \in L^2_{loc}$. Then, there exists a sequence F_n, $n = 1,2,...$ of simple processes such that, for each $t > 0$, $u \in h_0$, $m \geqq 0$, $f_1,...f_m \in h$*

$$\lim_{n\to\infty} \int_0^t \left\| (F^\sharp(s) - F_n^\sharp(s)) u \otimes \psi_m(f) \right\|^2 \, \mathrm{d}s = 0.$$

Proof. See Proposition 3.1 in [5]. □

Definition 5.27. *We say that $F \in \mathcal{A}$ is* ***even*** *(**respectively odd**) if each $F(t)$ is the algebraic ampliation of an* ***even*** *(**respectively odd**) operator in $\tilde{\mathcal{H}}_t$ with domain $\tilde{\mathcal{E}}_t$. For $F \in \mathcal{A}$ we have that $F = F_+ + F_-$ of even and odd parts.*

We start by defining the integrals for Fermion simple processes.

Definition 5.28. *Let $F, G, H \in \mathcal{A}_0$ and suppose that*

$$F = \sum_{r=0}^{\infty} F_r 1_{[t_r,t_{r+1})}, \quad G = \sum_{r=0}^{\infty} G_r 1_{[t_r,t_{r+1})}, \quad H = \sum_{r=0}^{\infty} H_r 1_{[t_r,t_{r+1})}$$

where

$$0 = t_0 < t_1 < ... < t_r \longrightarrow \infty$$

as $r \longrightarrow \infty$. Let $M = (M(t))$ be the family of operators defined inductively by

$$\begin{aligned} M(0) &= 0, \\ M(t) &= M(t_r) + \left(A^\dagger(t) - A^\dagger(t_r)\right) F_r + G_r(A(t) - A(t_r)) + (t - t_r) H_r \end{aligned}$$

for $t_r < t \leq t_{r+1}$. Then, M is called the ***Fermion stochastic integral*** *of (F, G, H) and we write*

$$M(t) = \int_0^t \left(\mathrm{d}A^\dagger F + G \mathrm{d}A + H \mathrm{d}s \right).$$

Remark 5.29. *M is a Fermion adapted process whose **adjoint** is given by*

$$M^\dagger(t) = \int_0^t \left(\mathrm{d}A^\dagger G^\dagger + F^\dagger \mathrm{d}A + H^\dagger \mathrm{d}s \right).$$

We write $\mathrm{d}M = \mathrm{d}A^\dagger F + G\mathrm{d}A + H\mathrm{d}s$ in case that $M = (M(t) : \ t \geqq 0)$ is a Fermion adapted process that satisfies

$$M(t) = M(0) + \int_0^t \left(\mathrm{d}A^\dagger F + G\mathrm{d}A + H\mathrm{d}s \right),$$

where $M(0)$ is the ampliation to $h_0 \otimes \mathcal{H}$ of an element of $B(h_0)$.

Remark 5.30. *If $\mathrm{d}M = \mathrm{d}A^\dagger F + G\mathrm{d}A + H\mathrm{d}t$, then*

$$\mathrm{d}M_+ = \mathrm{d}A^\dagger F_- + G_-\mathrm{d}A + H_+\mathrm{d}t, \quad \mathrm{d}M_- = \mathrm{d}A^\dagger F_+ + G_+\mathrm{d}A + H_-\mathrm{d}t,$$

in particular, if F and G are odd and H is even then their Fermion stochastic integral is even and if F and G are even and H is odd then their Fermion stochastic integral is odd. Furthermore, if M is odd, then F and G need to be even and H odd.

Theorem 5.31. *Let $F, G, H \in \mathcal{A}_0$ and $M(t) = \int_0^t (\mathrm{d}A^\dagger F + G\mathrm{d}A + H\mathrm{d}s)$. Then, for arbitrary $v \in \mathcal{H}_0$, $m, n \geqq 0$, $f_1, ..., f_m, g_1, ..., g_m \in \mathcal{H}$ and $t \geqq 0$,*

$$\begin{aligned} &\langle u \otimes \psi_m(f), M(t) v \otimes \psi_n(g) \rangle \\ = \ &\int_0^t \left\{ \sum_{j=1}^m (-1)^{m-j} \overline{f_j(s)} \right\} \langle \theta u \otimes \psi_{m-1}(f^j), F(s) v \otimes \psi_n(g) \rangle + \\ &\sum_{k=1}^n (-1)^{n-k} \langle u \otimes \psi_m(f), G(s)\theta v \otimes \psi_{n-1}(g^k) \rangle g_k(s) + \\ &\langle u \otimes \psi_m(f), H(s) v \otimes \psi_n(g) \rangle \mathrm{d}s. \end{aligned}$$

Proof. See Theorem 4.1 in [5]. □

The following corollary is used to prove Propositions 6.14, 6.15 and 6.16, which will give us the consistency conditions to find the coefficients of Fermion diffusions.

Corollary 5.32. *If $\mathrm{d}A^\dagger F + G\mathrm{d}A + H\mathrm{d}t = 0$, then F, G and H are all zero.*

Proof. See the last paragraph of page 479 in [5]. □

Definition 5.33. *A real-valued function f on a closed, bounded interval $[a, b]$ is said to be **absolutely continuous**[23] on $[a, b]$ provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $(a_k, b_k)_{k=1}^n$ of open intervals in (a, b), if*

$$\sum_{k=1}^n [b_k - a_k] < \delta,$$

then

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon.$$

[23] See page 119 in [29] for the definition.

Theorem 5.34. *Let $F, G, H, F', G', H' \in \mathcal{A}_0$,*

$$M(t) = \int_0^t \left(\mathrm{d}A^\dagger + G\mathrm{d}A + H\mathrm{d}s \right), \quad M'(t) = \int_0^t \left(\mathrm{d}A^\dagger F' + G'\mathrm{d}A + H'\mathrm{d}s \right).$$

Assume that M and M' have definite parity. Then, for arbitrary $u, v \in h_0, m, n \geqq 0$, $f_1, ..., f_m, g_1, ..., g_n \in h$, the function on $(0, \infty)$, $t \mapsto \langle M(t)u \otimes \psi_m(f), M'(t)v \otimes \psi_n(g)\rangle$, is absolutely continous with derivative

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle M(t)u \otimes \psi_m(f), M'(t)v \otimes \psi_n(g)\rangle \\
= \ & \sum_{j=1}^{m} (-1)^{m-j}\overline{f_j(t)}\{\langle G(t)\theta u \otimes \psi_{m-1}(f^j), M'(t)v \otimes \psi_n(g)\rangle + (-1)^{\delta(M)} \\
& \cdot \langle M(t)\theta u \otimes \psi_{m-1}(f^j), F'(t)v \otimes \psi_n(g)\rangle\} + \\
& \sum_{k=1}^{n} (-1)^{n-k} \left\{ (-1)^{\delta(M')}\langle F(t)u \otimes \psi_m(f), M'(t)\theta v \otimes \psi_{n-1}(g^k)\rangle \right\} g_k(t) + \\
& \langle H(t)u \otimes \psi_m(f), M'(t)v \otimes \psi_n(g)\rangle + \langle M(t)u \otimes \psi_m(f), H'(t)v \otimes \psi_n(g)\rangle + \\
& \langle F(t)u \otimes \psi_m(f), F'(t)v \otimes \psi_n(g)\rangle.
\end{aligned}$$

Proof. See Theorem 4.2 in [5]. □

5.4.2 Locally Square Integrable Processes

Now, we extend as in the classical and Boson case, the definition of a Fermion stochastic integral to a larger set of functions.

Theorem 5.35. *For arbitrary $u \in h_0$, $m \geqq 0$, $f_1, ..., f_m \in h$, and $T \geqq 0$ the sequence $M_n(t)u \otimes \psi_m(f)$, $n = 1, 2, ...$ converges in $\widetilde{\mathcal{H}}$ uniformly for $t \in [0, t]$ to a limit independent of the choice of $F_n, G_n, H_n \in \mathcal{A}_0$, $n = 1, 2, ...$ satisfying*

$$\begin{aligned}
&\int_0^t \left\{ \left\| F^\sharp(s) - F_n^\sharp(s))u \otimes \psi_m(f) \right\|^2 + \left\| (G^\sharp(s) - G_n(s))u \otimes \psi_m(f) \right\|^2 + \right. \\
&\left. \left\| (H^\sharp(s) - H_n^\sharp(s))u \otimes \psi_m(f) \right\|^2 \right\} \mathrm{d}s \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ with

$$M_n(t) = \int_0^t \left(\mathrm{d}A^\dagger F_n + G_n \mathrm{d}A + H_n \mathrm{d}s \right),$$

where $n = 1, 2, ..., t \geqq 0$.

Proof. See Theorem 5.1 in [5]. □

Remark 5.36. *The operator $M(t)$ defined on $\widetilde{\mathcal{E}}$ by*

$$M(t)u \otimes \psi_m(f) = \lim_{n \to \infty} M_n(t)u \otimes \psi_m(f)$$

is clearly linear on u and thus can be extended uniquely as an algebraic ampliation to $\widetilde{\mathcal{E}}_t \underline{\otimes} \mathcal{H}_{[t}$, we denote also the extension by $M(t)$. Then, $M = \{M(t) : t \geqq 0\}$ is a Fermion adapted process

Definition 5.37. *We call the Fermion adapted process of the previous Remark the* ***Fermion stochastic integral*** *of the locally square integrable processes* (F, G, H)*, and we denote it, as before, by*

$$M(t) = \int_0^t \left(\mathrm{d}A^\dagger F + G\mathrm{d}A + H\mathrm{d}s\right),$$

where $t \geqq 0$*. The adjoint process* $M^\dagger(t)$ *by*

$$M^\dagger(t) = \int_0^t \left(\mathrm{d}A^\dagger G^\dagger + F^\dagger \mathrm{d}A + H^\dagger \mathrm{d}s\right),$$

with $t \geqq 0$*.*

Proposition 5.38. *The Fermion stochastic integral* $M(t)$ *is a continuous process.*

Proof. See page 486 in [5]. □

Proposition 5.39. *The Theorems 5.31 and 5.34 hold for integrands* $F, G, H \in L^2_{loc}$*.*

Proof. See page 486 in [5]. □

Remark 5.40. *We denote by* $\mathcal{M}$ *the set of all Fermion adapted processes* M *satisfying*

$$\mathrm{d}M = \mathrm{d}A^\dagger F + G\mathrm{d}A + H\mathrm{d}t$$

for locally square-integrable F*,* G *and* H*, with the further property that for each* $t \geqq 0$*,* $M(t)$*,* $F(t)$*,* $G(t)$*, and* $H(t)$ *are bounded operators, and*

$$\sup_{0 \leqq s \leqq t} \max\{\|M(s)\|, \|F(s)\|, \|G(s)\|, \|H(s)\|\} < \infty.$$

5.5 Fermion Ito formula

Here, we give another generalization of the Ito formula which will help us to prove the consistency conditions given by Propositions 6.14, 6.15 and 6.16. Besides, we provide "differential" rules in Remark 5.44 that will allows us to prove the previously mentioned propositions.

Theorem 5.41. $\mathcal{M}$ *is a *-algebra under pointwise operator multiplication and the involution* $M \longrightarrow M^\dagger$*. Furthermore, for* $M_1, M_2 \in \mathcal{M}$*,*

$$d(M_1 M_2) = dM_1 M_2 + M_1 dM_2 + dM_1 dM_2,$$

where assuming that

$$\mathrm{d}M_j = \mathrm{d}A^\dagger F_j + G_j \mathrm{d}A + H_j \mathrm{d}t$$

and that M_1, M_2*, are of definite parity*

$$\begin{aligned}
\mathrm{d}M_1 M_2 &= \mathrm{d}A^\dagger F_1 M_2 + (-1)^{\delta(M_2)} G_1 M_2 \mathrm{d}A + H_1 M_2 \mathrm{d}t, \\
M_1 \mathrm{d}M_2 &= (-1)^{\delta(M_1)} \mathrm{d}A^\dagger M_1 F_2 + M_1 G_2 \mathrm{d}A + M_1 H_2 \mathrm{d}t, \\
\mathrm{d}M_1 \mathrm{d}M_2 &= G_1 F_2 \mathrm{d}t.
\end{aligned}$$

Proof. See Theorem 5.2 in [5]. □

Theorem 5.42. *Let $L_j, j = 1, 2, 3$ be bounded operators on the initial space $\mathcal{H}_0$ and denote by $\hat{L}_j$, $j = 1, 2, 3$ their ampliation to $\hat{\mathcal{H}}_0$. Assume L_1, L_2 odd and L_3 even. Then, the equation*

$$\mathrm{d}U \;=\; \mathrm{d}A^\dagger U \hat{L}_1 + U\left(\hat{L}_2 \mathrm{d}A + \hat{L}_3 \mathrm{d}t\right), \quad U(0) \;=\; I$$

has a unique solution.

Proof. See Theorem 6.2 in [5]. □

As we will see in Definition 6.3, the initial conditions of Fermion diffusions are different from the one given in Theorem 5.42 and therefore it will not help proving the uniqueness of Fermion diffusions with one degree of freedom. Still, it is an important result that is worth mentioning in the context of Fermion stochastic calculus.

Remark 5.43. *In particular, since $L_1 = L_2 = 0$ are odd operators the equation*

$$\mathrm{d}U = U\hat{L}_3 \mathrm{d}t, \quad U(0) = I,$$

has a unique solution with $\hat{L}_3$ an even operator. By observing that $\widetilde{L}_3 = kI$ with $k \in \mathbb{C}$ is an even operator we get that the linear equation

$$\mathrm{d}U = kU\mathrm{d}t, \quad U(0) = I,$$

has a unique solution.

Remark 5.44. *From Proposition 2.41 we know that the process $A(t)$ acts on the elements of $\left\{a^\dagger(f_n) \cdots a^\dagger(f_1)\Omega : n \in \mathbb{N}, f_1, ..., f_n \in L^2(\mathbb{R}^+)\right\}$ as*

$$\sum_{j=1}^{n} (-1)^{n-j} \int_0^t f_j(\tau)\mathrm{d}\tau \left(\prod_{k=1, k\neq j}^{n} a^\dagger(f_k) \right) \Omega.$$

From which we infer that

$$\mathrm{d}A\psi_m(f_1, ..., f_m) \;=\; \sum_{j=1}^{m} (-1)^{m-j} f_j(t)\mathrm{d}t \psi_{m-1}\left(f_1, ... \overset{j}{\bigwedge} ... f_m\right)$$

and that the differentials of the creation and annihilation processes satisfy

$$0 \;=\; (\mathrm{d}A)^2 \;=\; \left(\mathrm{d}A^\dagger\right)^2 \;=\; \mathrm{d}A^\dagger \mathrm{d}A, \quad \mathrm{d}A\mathrm{d}A^\dagger \;=\; \mathrm{d}t.$$

6 Fermion diffusions

In this section, we generalize the concept of classical diffusions as given in [26] to Fermions and we find the solutions for equations of the form

$$\mathrm{d}a = \mathrm{d}A^{\dagger}F + G\mathrm{d}A + H\mathrm{d}t,$$

where A_t and $A_t^{\dagger}$ are the Fermion creation and annihilation processes. Specifically, we will generalize the consistency conditions given in [3] and the definition of Fermion diffusions to n-degrees of freedom. In this section, we summarize some results and concepts of [3].

6.1 Fermion Von Neumann Uniqueness Theorem.

The Theorems 6.1 and 6.2 tell us, roughly speaking, how many coefficients general odd and even operators that belong to a certain C^*-algebra have.[24] We will use these Theorems to give explicitely the even and odd coefficients of Fermion diffusions with one, two and three degrees of freedom.

Proposition 6.1. *Let $\mathcal{H}$ be a Hilbert space of dimension one, let $(b, b^{\dagger})$ be a* ***Fermion system with one degree of freedom*** *and let $\mathcal{A}$ be the *-algebra generated by the identity operator I acting on $\mathcal{H}$ and the operators $\left(b, b^{\dagger}\right)$. Then, there exists a *-isomorphism to the C^*-algebra $(\mathcal{M}_{2\times 2}(\mathbb{C}), *)$ given by*

$$\pi : \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \to \alpha_1 b^{\dagger}b + \alpha_2 b^{\dagger} + \alpha_3 b + \alpha_4 bb^{\dagger}$$

where each α_i, with $i \in \{1, 2, 3, 4\}$, is a complex number.

Proof. It holds

$$\pi(AB) = \pi(A)\pi(B)$$

with $A, B \in M_2(\mathbb{C})$. Let

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}$$

[24] See Appendix A for the definition of C^*-algebra. The statement and the isomorphism was taken from [33].

then

$$
\begin{aligned}
\pi(A)\pi(B) &= \left(\alpha_1 b^\dagger b + \alpha_2 b^\dagger + \alpha_3 b + \alpha_4 bb^\dagger\right)\left(\beta_1 b^\dagger b + \beta_2 b^\dagger + \beta_3 b + \beta_4 bb^\dagger\right) \\
&= \alpha_1\beta_1 b^\dagger bb^\dagger b + \alpha_1\beta_2 b^\dagger bb^\dagger + \alpha_1\beta_3 b^\dagger bb + \beta_4\alpha_1 b^\dagger bbb^\dagger + \\
&\quad \alpha_2\beta_1 b^\dagger b^\dagger b + \alpha_2\beta_2 b^\dagger b^\dagger + \alpha_2\beta_3 b^\dagger b + \alpha_2\beta_4 b^\dagger bb^\dagger + \\
&\quad \alpha_3\beta_1 bb^\dagger b + \alpha_3\beta_2 bb^\dagger + \alpha_3\beta_3 bb + \alpha_3\beta_4 bbb^\dagger + \\
&\quad \alpha_4\beta_1 bb^\dagger b^\dagger b + \alpha_4\beta_2 bb^\dagger b^\dagger + \alpha_4\beta_3 bb^\dagger b + \alpha_4\beta_4 bb^\dagger bb^\dagger \\
&\overset{\text{CAR}}{=} \alpha_1\beta_1 b^\dagger b\left(I - bb^\dagger\right) + \alpha_1\beta_2 b^\dagger\left(I - b^\dagger b\right) + \\
&\quad \alpha_2\beta_3 b^\dagger b + \alpha_2\beta_4 b^\dagger\left(I - b^\dagger b\right) + \\
&\quad \alpha_3\beta_1 b\left(I - bb^\dagger\right) + \alpha_3\beta_2 bb^\dagger + \\
&\quad \alpha_4\beta_3 b\left(I - bb^\dagger\right) + \alpha_4\beta_4 bb^\dagger\left(I - b^\dagger b\right) \\
&\overset{\text{CAR}}{=} \alpha_1\beta_1 b^\dagger b + \alpha_1\beta_2 b^\dagger + \\
&\quad \alpha_2\beta_3 b^\dagger b + \alpha_2\beta_4 b^\dagger + \\
&\quad \alpha_3\beta_1 b + \alpha_3\beta_2 bb^\dagger + \\
&\quad \alpha_4\beta_3 b + \alpha_4\beta_4 bb^\dagger \\
&= \pi(AB).
\end{aligned}
$$

Finally, we have

$$\pi\left(A^\dagger\right) = \pi(A)^\dagger$$

since

$$\pi(A^\dagger) = \overline{\alpha_1} b^\dagger b + \overline{\alpha_3} b^\dagger + \overline{\alpha_2} b + \overline{\alpha_4} bb^\dagger = \pi(A)^\dagger.$$

Additionally, the morphism is unital

$$\pi(I) = I,$$

since

$$\pi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = b^\dagger b + bb^\dagger = I.$$

Furthermore, we observe that π is clearly a linear, injective and surjective function. Therefore, π is a *-isomorphism. □

We have a more general result that will help us finding the zero terms of the coefficients of Fermion diffusions

Theorem 6.2. *Let h be a pre-Hilbert space with closure $\overline{h}$ and let $\mathcal{U}_i$, $i = 1, 2$, be two C^*-algebras generated by the identity I and elements $a_i(f)$, $f \in h$, satisfying*

(i) $f \to a_i(f)$ is antilinear,

(ii) $\{a_i(f), a_i(g)\} = 0$,

(iii) $\left\{a_i(f), a_i^\dagger(g)\right\} = \langle f, g\rangle I$

for all $f, g \in h$, $i = 1, 2$.
*Then, there exists a unique *-isomorphism $\alpha : \mathcal{U}_1 \to \mathcal{U}_2$ such that*

$$\alpha(a_1(f)) = a_2(f)$$

*for all $f \in h$. Thus there exists a unique, up to *-isomorphism, C^*-algebra $\mathcal{U} = \mathcal{U}(h) = \mathcal{U}(\overline{h})$ generated by elements $a(f)$, satisfying the canonical anticommutation relations over h.*
Furthermore

(i) $\|a(f)\| = \|f\|$ *for all* $f \in \overline{h}$.

(ii) If h is n-dimensional, where $n < \infty$, then $\mathcal{U}(h)$ is isomorphic with the C^*-algebra of $2^n \times 2^n$ complex matrices.

(iii) $\mathcal{U}(h)$ is separable if, and only if, h is separable.

(iv) $\mathcal{U}(h)$ is simple.

(v) If U is a bounded linear operator on h and v a bounded antilinear operator satisfying

$$V^\dagger U + U^\dagger V = 0 = UV^\dagger + VU^\dagger,$$
$$U^\dagger U + V^\dagger V = I = UU^\dagger + VV^\dagger,$$

*then there exists a unique *-automorphism γ of $\mathcal{U}(h)$ such that*

$$\gamma(a(f)) = a(Uf) + a^\dagger(Vf)$$

and in this case

$$\gamma^{-1}(a(f)) = a\left(U^\dagger f\right) + a^\dagger\left(V^\dagger f\right).$$

Proof. See Theorem 5.2.5 in [27]. □

6.2 Fermion Diffusions

We now generalize the concept of Fermion diffusion given in [3].

Definition 6.3 (Fermion diffusions)**.** *Let h_0 be a Hilbert space. For each $t \geqq 0$, let $b^1(t), ..., b^n(t)$ be operators that satisfy* CAR *in $\widetilde{\Gamma_a(\mathcal{H}_{t]})}$. Then, we call a family of adapted processes $B^i = \left(B^i(t) : t \geqq 0\right)$ on $h_0 \otimes \Gamma_a(L^2(\mathbb{R}_+))$, with $i \in \{1, ..., n\}$, a Fermion diffusion with n-degrees of freedom if*

(i) for each $t \geqq 0$, and each $i \in \{1, ..., n\}$, $B^i(t)$ is the algebraic ampliation to $\widetilde{\mathcal{E}}_{t]}\underline{\otimes}\Gamma_a(\mathcal{H}_{[t})$ of $b^i(t)$ in $\widetilde{\Gamma_a(\mathcal{H}_{t]})}$ with domain $\widetilde{\mathcal{E}_{t]}}$;[25]

[25]Notation described in Remark 5.10.

(ii) for each $t \geqq 0$ and each $i \in \{1, ..., n\}$ it satisfies the Fermion stochastic differential equations

$$\mathrm{d}B^i(t) = \mathrm{d}A(t)^\dagger F^i(t) + G^i(t)\mathrm{d}A(t) + H^i(t)\mathrm{d}t,$$
$$\mathrm{d}B^{i\dagger}(t) = \mathrm{d}A(t)F^{i\dagger}(t) + G^{i\dagger}(t)\mathrm{d}A^\dagger(t) + H^{i\dagger}(t)\mathrm{d}t,$$

with initial conditions given by

$$B^i(0) = b^i(0)\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))}, \quad B^{i\dagger}(0) = b^i(0)^\dagger\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))},$$

where the "coefficients" $(F^i(t), G^i(t), H^i(t))$ are algebraic ampliations to $\widetilde{\mathcal{E}_{t]}} \otimes \Gamma_a(\mathcal{H}_{[t})$ of operators $(f^i(t), g^i(t), h^i(t))$ in the C^*-algebra generated by the operators $b^i(t)$ and the identity operator I on $\widetilde{\Gamma_a(\mathcal{H}_{t]})}$. Furthermore, we assume each $b^i(t)$ to be odd as in Definition 5.12.

Remark 6.4. *For each $t \geqq 0$, $i \in \{1, ..., n\}$, a Fermion diffusion with n-degrees of freedom satisfies*

$$B^i(t) = b^i(0)\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))} + \int_0^t F^i(s)\mathrm{d}A^\dagger(s) + \int_0^t G^i(s)\mathrm{d}A(s) + \int_0^t H^i(s)\mathrm{d}s,$$
$$B^{i\dagger}(t) = b^i(0)\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))} + \int_0^t F^{i\dagger}(s)\mathrm{d}A(s) + \int_0^t G^{i\dagger}(s)\mathrm{d}A(s) + \int_0^t H^{i\dagger}(s)\mathrm{d}s.$$

That means $B^i(t) \in \mathcal{M}$ and hence we can apply the Ito formula 5.41 to Fermion diffusions and, in general, the theory developed in Section 5.

We notice that we have the following more general definition, but since we are only interested in Fermion diffusion with a low amount of degrees of freedom, we will use 6.3 to make calculations.

Definition 6.5 (Fermion diffusions with arbitrary degrees of freedom)**.** *Let h_0 be a Hilbert space, $t \geqq 0$, and f an element of $\mathcal{H}$, let $b^f(t)$ be operators that satisfy* CAR *in $\widetilde{\Gamma_a\left(\mathcal{H}_{t]}\right)}$. Then, we call a family of adapted processes $B^f = \left(B^f(t) : t \geqq 0\right)$ on $h_0 \otimes \Gamma_a(L^2(\mathbb{R}_+))$, with $f \in \mathcal{H}$ a Fermion diffusion with arbitrary degrees of freedom if*

(i) for each $t \geqq 0$, and each $f \in \mathcal{H}$, $B^f(t)$ is the algebraic ampliation to $\widetilde{\mathcal{E}}_{t]}\underline{\otimes}\Gamma_a(\mathcal{H}_{[t})$ of $b^f(t)$ in $\widetilde{\Gamma_a(\mathcal{H}_{t]})}$ with domain $\widetilde{\mathcal{E}_{t]}}$;[26]

(ii) for each $t \geqq 0$ and each $f \in \mathcal{H}$ it satisfies the Fermion stochastic differential equations

$$\mathrm{d}B^f(t) = \mathrm{d}A(t)^\dagger D^f(t) + G^f(t)\mathrm{d}A(t) + H^f(t)\mathrm{d}t,$$
$$\mathrm{d}B^{f\dagger}(t) = \mathrm{d}A(t)D^{f\dagger}(t) + G^{f\dagger}(t)\mathrm{d}A^\dagger(t) + H^{f\dagger}(t)\mathrm{d}t,$$

with initial conditions given by

$$B^f(0) = b^f(0)\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))}, \quad B^{f\dagger}(0) = b^f(0)^\dagger\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))},$$

where the "coefficients" $(D^f(t), G^f(t), H^f(t))$ are algebraic ampliations to $\widetilde{\mathcal{E}_{t]}} \otimes \Gamma_a(\mathcal{H}_{[t})$ of operators $(d^f(t), g^f(t), h^f(t))$ in the C^*-algebra generated by the operators $b^f(t)$ and the identity operator I on $\widetilde{\Gamma_a(\mathcal{H}_{t]})}$. Furthermore, we assume that each $b^f(t)$ is odd as in Definition 5.12.

[26]Notation described in Remark 5.10.

Example 6.6 (Fermion diffusion with one degree of freedom)**.** *Let* $\underbrace{h_0 = \mathbb{C}^2}$*, for each* $t \geqq 0$ *let* $\big(b(t), b^\dagger(t)\big)$ *be a Fermion system with one degree of freedom in* $\Gamma_a(L^2(\mathbb{R}))$*, and*

$$B(0) \;=\; b^0 \widehat{\otimes} I_{\Gamma_a(L^2(\mathbb{R}_+))}, \quad B(0)^\dagger \;=\; b^{(0)\dagger} \widehat{\otimes} I_{\Gamma_a(L^2(\mathbb{R}_+))},$$

where

$$b^{(0)} \;=\; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b^{(0)\dagger} \;=\; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then, we have a Fermion diffusion with one degree of freedom with initial conditions given by

$$\begin{aligned} B(t) \;&=\; b^0 \widehat{\otimes} I_{\Gamma_a(L^2(\mathbb{R}_+))} + \int_0^t F(s)\mathrm{d}A^\dagger(s) + \int_0^t G(s)\mathrm{d}A(s) + \int_0^t H(s)\mathrm{d}s, \\ B^\dagger(t) \;&=\; b^{(0)\dagger} \widehat{\otimes} I_{\Gamma_a(L^2(\mathbb{R}_+))} + \int_0^t F^\dagger(s)\mathrm{d}A(s) + \int_0^t G(s)\mathrm{d}A^\dagger(s) + \int_0^t H^\dagger(s)\mathrm{d}s, \end{aligned}$$

where

$$\begin{aligned} F(0) \;&=\; \lambda I + \lambda' B^\dagger(0)B(0) + \lambda'' B(0) + \lambda''' B^\dagger(0), \\ H(0) \;&=\; \mu I + \mu' B^\dagger(0)B(0) + \mu'' B(0) + \mu''' B^\dagger(0), \\ G(0) \;&=\; \tau I + \tau' B^\dagger(0)B(0) + \tau'' B(0) + \tau''' B^\dagger(0) \end{aligned}$$

and

$$\begin{aligned} F(t) \;&=\; \lambda I + \lambda' B(t)^\dagger B(t) + \lambda'' B(t) + \lambda''' B^\dagger(t), \\ H(t) \;&=\; \mu I + \mu' B^\dagger(t)B(t) + \mu'' B(t) + \mu''' B^\dagger(t), \\ G(t) \;&=\; \tau I + \tau' B^\dagger(t)B(t) + \tau'' B(t) + \tau''' B^\dagger(t), \end{aligned}$$

where $\lambda, \lambda', \lambda''', \mu, \mu', \mu'', \mu''', \tau, \tau'', \tau''' \in \mathbb{C}$*. The form of the coefficients is infered from Proposition 6.1.*

Remark 6.7. *Now, that we have defined Fermion diffusions, the importance of Theorems 6.1 and 6.2 becomes clear, which together with the parity property* (*Remark 5.2*) *tell us, specifically, that general odd coefficients of a Fermion diffusion with one degree of freedom have 4 terms and that general even coefficients have, at most, 2 terms different from zero. Furthermore, it implies that the odd coefficients of Fermion diffusions with n-degrees of freedom have, in general,* 2^{2n} *terms and that the even coefficients have, at most,* 2^{2n-1} *terms different from zero.*

Remark 6.8. *For odd operators* C_i *with* $i \in \{1, ..., n\}$ *acting on a* $\mathbb{Z}_2$*-graded Hilbert space it holds, by the Parity Property* (*Remark* 5.2)*, that*

$$C_1 * \cdots * C_n$$

is ***even*** *if n is even and* ***odd*** *if n is odd, respectively. As the sum of odd operators with even operators is odd, we have that general odd operators are the sum of odd and even operators. On the other hand, even operators can only have even operators as summands.*

Example 6.9. *Let* $(B^i(t), B^{i\dagger}(t))$ *be Fermion diffusions with n-degrees of freedom such that n is even and let* $i \in \{1, ..., n\}$*. Then, the operators of the form*

$$H_i = \alpha_i \underbrace{b_1 * \cdots * \hat{b}_i * \cdots b_n}$$

are odd, where $\hat{b_i}$ *means that the* b_i*'s are to be omitted and* $\alpha_i \in \mathbb{C}$.

6.3 General consistency conditions

Now we give consistency conditions that all Fermion diffusions need to satisfy, this will allows us to calculate the zerto terms of the coefficients of Fermion diffusions with one, two degrees and three degrees of freedom. More specifically, we will find the zero terms of the even and odd coefficients of Fermion diffusions.
As intuitively expected, the derivative of a "constant" is zero

Remark 6.10. *For the identity operator in $\widetilde{\Gamma_a(\mathcal{H}_{t]})}$ we conclude $I \in \mathcal{M}$. Since $F = G = H = 0$ are localy bounded square integrable functions it holds that*

$$I(t) = \int_0^t 0\,\mathrm{d}A + \int_0^t 0\,\mathrm{d}A^\dagger + \int_0^t 0\ \mathrm{d}s + I(0),$$

since we have that

$$I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}} \otimes I_{\Gamma_a(\mathcal{H}_{[t})} = I_{h_0 \otimes \Gamma_a(L^2(\mathbb{R}_+))},$$

it follows that

$$\mathrm{d}I(t) = 0.$$

We use Lemmas 6.11 and 6.12[27] *to prove Propositions 6.14, 6.15 and 6.16 which will give us the consistency conditions.*

Lemma 6.11. *For any X, Y and Z operators acting on a Hilbert space $\mathcal{H}$ we have that*

(i) $\{X, Z\} = 0$ *then* $\{XY, Z\} = X[Y, Z]$

(ii) $\{Y, Z\} = 0$ *then* $\{XY, Z\} = -[X, Z]Y$.

Proof. From the first equality we infer that $ZX = -XZ$. Therefore,

$$\{XY, Z\} = XYZ + ZXY = XYZ - XZY = X[Y, Z].$$

The second equality follows similarly from $YZ = -ZY$. □

Lemma 6.12. *Fermion diffusions with n-degrees of freedom satisfy the following anticommutation relations*

$$\left\{B^i(t), I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}} \widehat{\otimes} \mathrm{d}A(t)\right\} = \left\{B^i(t), \mathrm{d}A^\dagger(t)\right\} = \left\{B^{i\dagger}(t), \mathrm{d}A(t)\right\} = \left\{B^i(t),\ \mathrm{d}A^{i\dagger}(t)\right\} = 0.$$

[27] This lemma is not proven in[3].

Proof. We have the following equalities

$$\begin{aligned}
\left\{b^i(t)\widehat{\underline{\otimes}}I_{\Gamma_a(\mathcal{H}_{[t)})}, I_{\widetilde{\Gamma(\mathcal{H}_{t]})}}\widehat{\otimes}a\left(1_{[t,\infty)}\right)\right\} &\overset{2.23}{=} \left(b^i(t)\widehat{\underline{\otimes}}I_{\Gamma_a(\mathcal{H}_{[t)})}\right)\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}a(1_{[t,\infty)})\right) + \\
&\quad \left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}a(1_{[t,\infty)})\right)\left(b^i(t)\widehat{\underline{\otimes}}I_{\Gamma_a(\mathcal{H}_{[t)})}\right) \\
&\overset{5.2.2}{=} (-1)\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}a(1_{[t,\infty)})\right)\left(b^i(t)\widehat{\underline{\otimes}}I_{\Gamma_a(\mathcal{H}_{[t)})}\right) + \\
&\quad \left(b^i(t)\widehat{\underline{\otimes}}I_{\Gamma_a(\mathcal{H}_{[t)})}\right)\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}a(1_{[t,\infty)})\right) - \\
&\overset{5.15}{=} \left(b^i(t)\otimes a(1_{[t,\infty)})\right) + \left(b^i(t)\phi\otimes a\left(1_{[t,\infty)}\right)\right) \\
&= 0.
\end{aligned}$$

The other identities follow similarily. ☐

Remark 6.13. *We observe that the conditions of Proposition 6.14 can be infered from Propositions 6.15 and 6.16 but we put it in an extra proposition in order to facilitate calculations* (*e.g.* (6.3.7)

Proposition 6.14 (Consistency condition with the same index)**.** *Let* $\left(B^i(t), B^{i\dagger}(t)\right), i \in \{1, ..., n\}$ *be Fermion diffusions with* n*-degrees of freedom, then for each* $i \in \{1, ..., n\}$ *it holds that*

$$[F^i(t), B^{i\dagger}(t)] + [G^{i\dagger}(t), B^i(t)] = 0, \tag{6.3.1}$$
$$\{H^i(t), B^{i\dagger}(t)\} + \{B^i(t), H^{i\dagger}(t)\} = -(F^{i\dagger}(t)F(t) + G^i(t)G^{i\dagger}(t)), \tag{6.3.2}$$
$$[F^i(t), B^i(t)] = 0, \tag{6.3.3}$$
$$[G^i(t), B^i(t)] = 0, \tag{6.3.4}$$
$$\{H^i(t), B^i(t)\} = -G^i(t)F^i(t). \tag{6.3.5}$$

Proof. Now, we apply the Fermion Ito formula

$$\begin{aligned}
0 &\overset{6.10}{=} \mathrm{d}I_{h_0\otimes\Gamma_a(\mathbb{R}_+)} \\
&\overset{\text{CAR}}{=} \mathrm{d}\left\{B^i(t), B(t)^{i\dagger}\right\} \\
&\overset{2.23}{=} \mathrm{d}\left(B^i(t)B(t)^{i\dagger} + B(t)^{i\dagger}B(t)\right) \\
&\overset{5.41}{=} \mathrm{d}B^i(t)B^{i\dagger}(t) + B(t)\mathrm{d}B^{i\dagger}(t) + \mathrm{d}B(t)\mathrm{d}B^{i\dagger} + \mathrm{d}B^{i\dagger(t)}B(t) + B(t)^{i\dagger}\mathrm{d}B^i(t) + \\
&\quad \mathrm{d}B^{i\dagger}(t)\mathrm{d}B^i(t) \\
&\overset{2.23}{=} \left\{\mathrm{d}B^i(t), B^{i\dagger}(t)\right\} + \left\{B^i(t), \mathrm{d}B^{i\dagger}(t)\right\} + \left\{\mathrm{d}B^i(t), \mathrm{d}B^{i\dagger}(t)\right\} \\
&\overset{6.3}{=} \left\{\mathrm{d}A^\dagger F^i(t) + G^i(t)\mathrm{d}A + H^i(t)\mathrm{d}t, B^{i\dagger}(t)\right\} + \\
&\quad \left\{B^i(t), \mathrm{d}A(t)^\dagger G^{i\dagger}(t) + F^{i\dagger}(t)\mathrm{d}A(t) + H^{i\dagger}(t)\mathrm{d}t\right\} + \\
&\quad \left\{\mathrm{d}A^\dagger(t)F^i(t) + G^i(t)\mathrm{d}A(t) + H^i(t)\mathrm{d}t, \mathrm{d}A^\dagger F^{i\dagger}(t) + G^{i\dagger}(t)\mathrm{d}A(t) + H^{i\dagger}(t)\mathrm{d}t\right\}.
\end{aligned}$$

We do the same with $\{B^i(t), B^i(t)\} = 0$ and using the fact that $\{X, Y\} = \{Y, X\}$ for all operators X,Y we get

$$\begin{aligned} 0 &= \left\{\mathrm{d}B^i(t), B^i(t)\right\} + \left\{B^i(t), \mathrm{d}B^i(t)\right\} + \left\{\mathrm{d}B^i(t), \mathrm{d}B^i(t)\right\} \\ &\overset{6.3}{=} 2\left\{\mathrm{d}A^\dagger(t)F^i(t) + G^i(t)\mathrm{d}A(t) + H^i(t)\mathrm{d}t, B^i(t)\right\} + \\ &\quad \left\{\mathrm{d}A^\dagger(t)F^i(t) + G^i(t)\mathrm{d}A(t) + H^i(t)\mathrm{d}t, \mathrm{d}A^\dagger(t)F^i(t) + G^i(t)\mathrm{d}A(t) + H^i\mathrm{d}t\right\}. \end{aligned}$$

On the other side, we infer by Lemma 6.12 that

$$\left\{I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A^\dagger, B^{i\dagger}(t)\right\} = 0,$$

then by Lemma 6.11 it holds

$$\left\{\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A^\dagger\right)F^i(t), B^{i\dagger}\right\} = \left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A^\dagger\right)[F^i(t), B^{i\dagger}],$$

hence we have by applying four times Lemma 6.11 that

$$\begin{aligned} \left\{\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A\right)G^i(t), B^i(t)\right\} &= -\left[G^i, B^{i\dagger}\right]\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A\right) \\ &= \left[B^{i\dagger}, G^i\right]\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A\right), \\ \left\{\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A^\dagger\right)G^{i\dagger}, B^i(t)\right\} &= \left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A^\dagger\right)\left[G^{i\dagger}, B^i(t)\right], \\ \left\{F^{i\dagger}\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A\right), B^i\right\} &= \left[B^i, F^{i\dagger}\right]\left(I_{\widetilde{\Gamma_a(\mathcal{H}_{t]})}}\widehat{\otimes}\mathrm{d}A\right), \end{aligned}$$

and since it holds that for any operators that $\{A+B, Y+D\} = \{A, Y\} + \{A, D\} + \{B, Y\} + \{B, D\}$, we get

$$\begin{aligned} &\left\{\mathrm{d}A^\dagger(t)F^i(t) + G^i(t)\mathrm{d}A(t) + H^i(t)\mathrm{d}t, \mathrm{d}A^\dagger F^{i\dagger}(t) + G^{i\dagger}(t)\mathrm{d}A(t) + H^{i\dagger}(t)\mathrm{d}t\right\} \\ =\ &\left\{\mathrm{d}A^\dagger(t)F^i(t), \mathrm{d}A^\dagger(t)F^i(t)\right\} + \left\{\mathrm{d}A^\dagger(t)F^i(t), G^{i\dagger}(t)\mathrm{d}A(t)\right\} + \\ &\left\{\mathrm{d}A^\dagger(t)F^i(t), H^{i\dagger}(t)\mathrm{d}t\right\} + \left\{G^i(t)\mathrm{d}A(t), \mathrm{d}A^\dagger F^i(t)\right\} + \\ &\left\{G^i(t)\mathrm{d}A(t), G^{i\dagger}(t)\mathrm{d}A(t)\right\} + \left\{G^i(t)\mathrm{d}A(t), H^{i\dagger}(t)\mathrm{d}t\right\} \\ =\ &\left\{H^i(t)\mathrm{d}t, \mathrm{d}A^\dagger F^{i\dagger}(t)\right\} + \left\{H^i(t)\mathrm{d}t, G^{i\dagger}\mathrm{d}A(t)\right\} + \left\{H^i(t)\mathrm{d}t, H^i(t)\mathrm{d}t\right\} \\ \overset{5.44}{=}\ &\left(F^{i\dagger}(t)F^i(t) + G^i(t)G^i(t)\right)\mathrm{d}t, \end{aligned}$$

Therefore, we deduce

$$\begin{aligned} 0 &= \mathrm{d}A^\dagger\left(\left[F^i(t), B(t)^{i\dagger}\right] + \left[G^{i\dagger}(t), B^i(t)\right]\right) + \\ &\quad \left(\left[B^i(t), F^{i\dagger}(t)\right] + \left[B^{i\dagger}(t), G^i(t)\right]\right)\mathrm{d}A + \\ &\quad \left(F^{i\dagger}F^i + G^i(t)G^{i\dagger}(t) + \left\{H^i(t), B^{i\dagger}(t)\right\} + \left\{B^i(t), H^{i\dagger}(t)\right\}\right)\mathrm{d}t, \end{aligned}$$

and in the same way we obtain

$$0 = \mathrm{d}A_t^\dagger[F, b_t] - [G_t, b_t]\mathrm{d}A_t + (\{H_t, b_t\} + G_t F)\,\mathrm{d}t,$$

and then we infer from Corollary 5.32 that the desired consistency equations hold. □

The next proposition is proven in a similar way, but we simplify the notation by only considering the operators satisfying CAR, *that is, substituting $B^i(t)$ by b^i.*

Proposition 6.15 (Second consistency conditions)**.** *Let $(B^i(t), B^{i\dagger}(t))$ be a Fermion diffusion with n-degrees of freedom, then for each $i, j \in \{1, ..., n\}$, with $i \neq j$, it holds that*

$$\begin{aligned} [F_i, b_j] + [F_j, b_i] &= 0, \\ [G_i, b_j] + [G_j, b_i] &= 0, \\ \{H_i, b_j\} + \{H_j, b_i\} &= -G_i F_j - G_j F_i. \end{aligned}$$

Proof. From

$$0 \overset{\mathrm{CAR}}{=} \mathrm{d}\{b_i, b_j\},$$

and

$$\begin{aligned} \mathrm{d}(b_i b_j + b_j b_i) &\overset{5.41}{=} \mathrm{d}b_i b_j + b_i \mathrm{d}b_j + \mathrm{d}b_i \mathrm{d}b_j + \mathrm{d}b_j b_i + b_j \mathrm{d}b_i + \mathrm{d}b_j \mathrm{d}b_i \\ &\overset{2.23}{=} \{\mathrm{d}b_i, b_j\} + \{b_i, \mathrm{d}b_j\} + \{\mathrm{d}b_i, \mathrm{d}b_j\} \\ &\overset{6.3}{=} \Big\{\mathrm{d}A^\dagger F_i + G_i \mathrm{d}A + H_i \mathrm{d}t, b_j\Big\} + \Big\{b_i, \mathrm{d}A^\dagger F_j + G_j \mathrm{d}A + H_i \mathrm{d}t\Big\} + \\ &\quad \Big\{\mathrm{d}A^\dagger F_i + G_i \mathrm{d}A + H_i \mathrm{d}t, \mathrm{d}A^\dagger F_j + G_j \mathrm{d}A + H_j \mathrm{d}t\Big\} \\ &\overset{5.44}{=} \mathrm{d}A^\dagger [F_i, b_j] - [G_i, b_j]\,\mathrm{d}A + \{H_i, b_j\}\mathrm{d}t + \\ &\quad \mathrm{d}A^\dagger [F_j, b_i] - [G_j, b_i]\,\mathrm{d}A + \{b_i, H_j\}\,\mathrm{d}t + \\ &\quad G_i F_j + G_j F_i \mathrm{d}t, \end{aligned}$$

we conclude that

$$\begin{aligned} [F_i, b_j] + [F_j, b_i] &= 0, \\ [G_i, b_j] + [G_j, b_i] &= 0, \\ \{H_i, b_j\} + \{H_j, b_i\} &= -G_i F_j - G_j F_i. \end{aligned}$$

which finishes the proof. □

Proposition 6.16 (Third consistency conditions)**.** *Let $(B^i(t), B^{i\dagger}(t))$ be a Fermion diffusion with n-degrees of freedom, then for each $i, j \in \{1, ..., n\}$, with $i \leq j$, it holds that*

$$\begin{aligned} \Big[F_j^\dagger, b_i\Big] + \Big[G_i, b_j^\dagger\Big] &= 0, \\ \Big[F_i, b_j^\dagger\Big] + \Big[G_j^\dagger, b_i\Big] &= 0, \\ \Big\{H_i, b_j^\dagger\Big\} + \Big\{H_j^\dagger, b_i\Big\} &= -F_j^\dagger F_i - G_i G_j^\dagger. \end{aligned}$$

Proof. The proof is almost identical to the proofs of Propositions 6.14 and 6.15. □

Remark 6.17. *Notice that the left-hand side of the second consistency condition of Proposition 6.16, is the adjoint of the left-hand side of the first consistency condition, and therefore the first condition implies the second one and the other way around.*

Remark 6.18. *In particular, Proposition 6.14 holds for the Fermion diffusion with one degree of freedom given in Example 6.6.*

Now, we have the following proposition, that intuitevely, tell us how "complex" are the Fermion diffusions

Proposition 6.19. *For $i \in \{1, ..., n\}$ and t a real number larger or equal than zero, let $B^i(t)$ satisfy the conditions of Definition 6.3. Then, for the odd coefficients of $B^i(t)$, up to adjointness, there are*

$$n + ((n-1) + ... + (n-(n-1)) + n + ((n-1) + ... + (n-(n-1))$$

different consistency conditions corresponding to these operators. For the even coefficients there are

$$2n + 2((n-1) + (n-2) + ... + (n-(n-1))) + n + n(n-1)$$

different consistency conditions corresponding to these operators, where we ignore the adjoint conditions.

Proof. First we prove the formula for the even coeffficients. Let $j \in \{1, ..., n\}$ with j different from i. Then, we observe that there are $2n$ consistency conditions such that

$$[F_i, b_i] = [G_i, b_i] = 0.$$

Further, we have $2((n-1) + (n-2) + ... + (n-(n-1))$ consistency conditions of the form

$$[F_i, b_j] + [F_j, b_i] = [G_i, b_j] + [G_j, b_i] = 0.$$

Besides, we have n equations for which it holds

$$[F_i^\dagger, b_i] + [G_i^\dagger, b_i] = 0.$$

Additionally, we observe that we have $n(n-1)$ equalities satisfying

$$[F_i^\dagger, b_j] + [G_j, b_i^\dagger] = 0.$$

Finally, since these are all the consistency conditions, up to adjointness, we may conclude by adding up the given results.
Similarily, for the odd operators and ignoring the adjoints, we have n different conditions

$$\{H_i, b_i^\dagger\} + \{H_i^\dagger, b_i\} = -F_i^\dagger F_i - G_i G_i^\dagger.$$

Also, there are identities $(n-1) + ... + (n-(n-1))$ that satisfy

$$\{H_i, b_j^\dagger\} + \{H_j^\dagger, b_i\} = -F_j^\dagger F_i - G_i G_j^\dagger.$$

Moreover, there are exactly n consistency conditions

$$\{H^i(t), b_i(t)\} = -G^i(t)F^i(t).$$

Lastly, it remains to consider the $(n-1)+...+(n-(n-1))$ different identities

$$\{H_i, b_j\} + \{H_j, b_i\} = -G_iF_j - G_jF_i.$$

From which conclude by adding the different posisbilities the desired result. □

Remark 6.20. *We observe that the idententy for the number of consistency conditions for even operators, can be simplified by the known formula of the sum of the first n natural numbers to*

$$n(n+1) + n^2. \tag{6.3.6}$$

Similarly, we can simplify the formula for the number of consistency conditions of odd operators to

$$\frac{n(n+1)}{2} + \frac{n(n+1)}{2} = n(n+1). \tag{6.3.7}$$

This means, that as the number, n of degrees of freedom, increases, the number of consistency conditions for both the even and odd operators, does it at a pace $\mathcal{O}(n^2)$. This results tells us, roughly speaking, that it is unfeasible to find the zero terms (with these methods) of general Fermion diffusions with many degrees of freedom.
Our intuition on this is confirmed by the calculations done in Subsection 7.3, where we see that general Fermion diffusions, already, have many non-zero terms, in contrast with the cases with one and two degrees of freedom .
This results also tell us, informally speaking, that contrary to what we would expect by only looking at the results for one, two and three degrees of freedom, that as n increases, both the even and odd coefficients have a similar amount of non-zero terms, meaning that they become similarily "complex".

6.4 More general consistency conditions

In this subsection, we study Fermion diffusions with arbitrary degrees of freedom.

Remark 6.21. *We observe that Fermion diffusions with arbitrary degrees of freedom satisfy Lemma 6.12 as the proof only relies on the oddness of the operators.*

By the previous remark, we can see that it holds

Proposition 6.22 (First general consistency condition)**.** *Let $(B^f(t), B^{f\dagger}(t))$ a Fermion diffusion as in Definition 6.5 and $\mathcal{H}$ a Hilbert space, with $f, g \in \mathcal{H}$. Then, it holds that*

$$\begin{aligned}
[D_f, b_g] + [D_g, b_f] &= 0, \\
[G_f, b_g] + [G_g, b_f] &= 0, \\
\{H_f, b_g\} + \{H_g, b_f\} &= -G_fD_g - G_gF_f.
\end{aligned}$$

Proof. Identically to the proof of Proposition 6.15. □

It also holds

Proposition 6.23 (Second general consistency condition)**.** *Let* $(B_f(t), B_f^\dagger(t))$ *be a Fermion diffusion as in Definition 6.5, and* $\mathcal{H}$ *be a Hilbert space. Then, for each* $f, g \in \mathcal{H}$ *it holds that*

$$\begin{aligned} \left[D_f^\dagger, b_f\right] + \left[G_f, b_g^\dagger\right] &= 0, \\ \left[D_f, b_g^\dagger\right] + \left[G_g^\dagger, b_f\right] &= 0, \\ \left\{H_f, b_g^\dagger\right\} + \left\{H_g^\dagger, b_f\right\} &= -F_g^\dagger F_f - G_f G_g^\dagger. \end{aligned}$$

Proof. As the previous one. □

Remark 6.24. *From the first and second general consistency conditions 6.22 6.23 we can infer both Propositions 6.15 and 6.16.*

7 Coefficients

Now, we will give the explicit coefficients of Fermion diffusions with one and two degrees of freedom, and the even coefficients of Fermion diffusions with three degrees of freedom. We will also give the general form of the odd coefficients of a Fermion diffusion with 3-degrees of freedom without applying the consistency conditions. What we will do, roughly speaking, is to find the scalars that are zero in the linear combination of odd and even operators and to put the scalars of these linear combinations in terms of other scalars, by using the minimum amount of scalars to express the even and odd operators of Fermion diffusions. It means that we will find the zero terms of the Fermion diffusions. This section summarizes some results and concepts of [3].

Remark 7.1. *Throughout this section we will use the fact that linear combinations of products of operators satisfying* CAR *equated to zero, imply that all the coefficients are zero, that is*

$$\sum_{k=0}^{n} \alpha_k C_k = 0$$

implies that $\alpha_k = 0$, *where* C_k *is the product of operators satisfying* CAR. *We can see that this is true as from Theorem 6.2 we know that operators satisfying* CAR *generate a unique* C^**-algebra up to *-isomorphisms.*

7.1 One degree of freedom

Now, we apply the consistency conditions to a Fermion diffusion with one degree of freedom in order to find its explicit form.

7.1.1 Even coefficients

With the following proposition, we get the explicit form of the even coefficients of Fermion diffusions with one degree of freedom.

Proposition 7.2. *For* $i \in \{1, ..., n\}$, $t \geqq 0$. *Let* $(B^i(t))$ *be a Fermion diffusion with* n*-degrees of freedom,* $\lambda, k, \mu, v \in \mathbb{C}$, *such that the even coefficients have the following form:*

$$F^i(t) = \lambda \left(I_{h_0 \otimes \Gamma_a(L^2[0,\infty)})}\right) + \kappa \left(b^{i\dagger}(t) \widehat{\otimes} I_{\Gamma_a(\mathcal{H}_{[t})})(b^i(t) \widehat{\otimes} I_{\Gamma_a(\mathcal{H}_{[t})}\right),$$
$$G^i(t) = \mu \left(I_{h_0 \otimes \Gamma_a(L^2[0,\infty)})}\right) + \nu \left(b^{i\dagger}(t) \widehat{\otimes} I_{\Gamma_a(\mathcal{H}_{[t})}\right) \left(b^i(t) \widehat{\otimes} I_{\Gamma_a(\mathcal{H}_{[t})}\right),$$

for each $i \in \{1, ..., n\}$, *it follows that*

$$\kappa = \nu = 0.$$

Proof. We simplify the notation by dropping the time dependency and the algebraic ampliation, that is

$$F = \lambda I + k b^\dagger b, \quad G = \mu I + v b^\dagger b.$$

Now, we have

$$\begin{aligned}
0 &\overset{6.3.4}{=} \left[\lambda I + kb^\dagger b, b^\dagger\right] + \left[\bar{\mu} + I + \bar{v}b^\dagger b, b\right] \\
&\overset{D.1}{=} \lambda b^\dagger + kb^\dagger bb^\dagger - \lambda b^\dagger - kb^\dagger b^\dagger b + \bar{\mu} b + \bar{v}b^\dagger bb - \bar{\mu}\ b - \bar{v}bb^\dagger b \\
&\overset{\text{CAR}}{=} kb^\dagger bb^\dagger - \bar{v}\ bb^\dagger b \\
&\overset{\text{CAR}}{=} k\left(\left(1 -\ bb^\dagger\right)b^\dagger\right) - \bar{v}\left(\left(1 - b^\dagger b\right)b\right) \\
&\overset{\text{CAR}}{=} kb^\dagger - \bar{v}b.
\end{aligned}$$

Therefore, $k = v = 0$. □

Proposition 7.3. *The coefficients $F(t)$, $G(t)$ of a Fermion diffusion with one degree of freedom have the form*

$$F(t) = \lambda(I_{h_0 \otimes \Gamma_a(L^2([0,\infty))}), \quad G(t) = \mu(I_{h_0 \otimes \Gamma_a(L^2([0,\infty))}).$$

Proof. Since $(\mathrm{d}B(t), \mathrm{d}B^\dagger(t))$ are odd it follows from Remark 5.30 that $F(t)$ and $G(t)$ are even, then they have the form of the hypothesis of Proposition 7.2 by Theorem 6.2 and hence the claim follows. □

7.1.2 Odd coefficients

Here, we use the consistency conditions 6.14, 6.15 and 6.16 to calculate the odd coefficients of Fermion diffusions with one degree of freedom and give its more general form as done in [3].

Remark 7.4. *An admissible Fermion diffusion with one degree of freedom is given by putting $F(t) = G(t) = 0$. In this case, we get the equations*

$$\mathrm{d}B(t) = H(t)\mathrm{d}t, \quad \mathrm{d}B(t)^\dagger = H^\dagger(t)\mathrm{d}t.$$

Remark 7.5. *If $F, G \neq 0$, then, since H is odd, its general form is given by*

$$H\ =\ \mu I + \mu' b + \mu'' b^\dagger + \mu bb^\dagger.$$

Thus, from CAR *we infer*

$$\begin{aligned}
\left\{\mu I + \mu' bb + \mu'' b^\dagger + \mu bb^\dagger, b\right\} &\overset{2.23}{=} \mu b + \mu' b^\dagger b + \mu'' b^\dagger b + \mu''' bb^\dagger b + \\
&\quad\ \mu b + \mu' bb + \mu'' bb^\dagger + \mu''' bbb^\dagger \\
&= 2\mu b + \mu''\left(b^\dagger b + bb^\dagger\right) + \mu'''\left(b\left(I - b^\dagger b\right)\right) \\
&\overset{\text{CAR}}{=} 2\mu b + \mu'' I + \mu''' b.
\end{aligned}$$

which implies that

$$2\mu b + \mu'' I + \mu''' b \overset{(6.3.5)}{=} -\lambda\mu I.$$

Then,

$$\mu'' = -\lambda\mu.$$

Finally, H has the form

$$H \;=\; -\lambda\mu b^\dagger + \alpha b,$$

with $\alpha \in \mathbb{C}$.
Now, we only need to substitute H to get

$$\left\{-\lambda\mu b^\dagger + \alpha b, b^\dagger\right\} + \left\{b, -\overline{\lambda}\overline{\mu} b + \overline{\alpha} b^\dagger\right\} \overset{(6.3.2)}{=} \left(\overline{\lambda}\lambda + \mu\overline{\mu})\right) I \;=\; -\left(|\lambda|^2 + |\mu|^2\right) I. \qquad (7.1.1)$$

On the other side, we have that the equation (7.1.1) *is equal to*

$$-\lambda\mu b^\dagger b^\dagger + \alpha b b^\dagger - \lambda\mu b^\dagger b^\dagger + b^\dagger b - \overline{\lambda}\overline{\mu} + \overline{\alpha} b b^\dagger - \overline{\lambda}\overline{\mu} b b + \overline{\alpha} b^\dagger b \overset{\text{CAR}}{=} \alpha b b^\dagger + \alpha b^\dagger b - \overline{\alpha} b b^\dagger - \overline{\alpha} b^\dagger b,$$

hence, we have

$$\alpha I + \overline{\alpha} I \;=\; \left(|\lambda|^2 + |\mu|^2\right) I,$$

which implies that

$$2\,\mathrm{Re}\,\alpha \;=\; -\left(|\lambda|^2 + |\mu|^2\right).$$

Therefore, we get that the general form of H is given by

$$H \;=\; -\lambda\mu b^\dagger - \frac{1}{2}\left(|\lambda|^2 + |\mu|^2\right) b + i\beta b, \qquad (7.1.2)$$

where $\beta \in \mathbb{R}$ and with $\beta = \mathrm{Im}\,\alpha$.

7.1.3 General form

Now we give the general form of a Fermion diffusion with one degree of freedom.

Remark 7.6. *By lettting $H = H' + H_0$ with $i\,[\mathcal{W}, b]$ and $\mathcal{W}$ self-adjoint, we infer from the general form of H given in equation 7.1.2 that $H_0 = \gamma b + \delta b^\dagger$, with $\gamma, \delta \in \mathbb{C}$. Also, by the von Neumann-uniquness theorem and the hypothesis on H we have that*

$$\mathcal{W} = tI + \omega b + \overline{\omega} b^* + z b^\dagger b,$$

where $t, z \in \mathbb{R}, \omega \in \mathbb{C}$. Therefore, we have that

$$\begin{aligned}
\gamma b + \delta b^\dagger \quad &= \quad i\left[tI + \omega b + \overline{\omega} b^\dagger + z b^\dagger b, b\right] \\
&= \quad i\left(tb + \omega b b + \overline{\omega} b^\dagger b + z b^\dagger b b\right) - \left(tb + \omega b b + \overline{\omega} b b^\dagger + z b b^\dagger b\right) \\
&\overset{\text{CAR}}{=} \quad i\left(\overline{\omega}\left(b^\dagger b - b b^\dagger\right) + z b\right).
\end{aligned}$$

Then $\delta = 0$ and $\gamma = -iz$ and hence

$$H_0 = -iz$$

we call H_0 the "Hamiltonian" aspect of the evolution, we consider equations of the form[28]

[28] See in [3] for the reference on how to "ignore" the "Hamiltonian" term of an equation.

Proposition 7.7. *The general form a Fermion diffusion with one degree of freedom is the following*

$$\mathrm{d}B(t) \;=\; \lambda \mathrm{d}A^\dagger + \mu \mathrm{d}A - \left(\lambda\mu B(t)^\dagger + \frac{1}{2}\left(\mid \lambda \mid^2 + \mid \mu \mid^2\right) B(t)\right) \mathrm{d}t, \tag{7.1.3}$$

$$\mathrm{d}B^\dagger(t) \;=\; \overline{\mu}\mathrm{d}A^\dagger + \overline{\lambda}\mathrm{d}A - \left(\overline{\mu}\overline{\lambda}B(t) + \frac{1}{2}\left(|\lambda|^2 + |\mu\,|^2\right) B^\dagger(t)\right) \mathrm{d}t. \tag{7.1.4}$$

Proof. It follows from the results on odd and even coefficients. □

7.1.4 Uniqueness

We now prove that Fermion diffusions with one degree of freedom, have only one solution.

Definition 7.8. *The **Fermion Brownian Motion** (P_t, Q_t) is given by*

$$P(t) \;=\; -i\left(A(t) - A(t)^\dagger\right), \quad Q(t) \;=\; A(t) + A(t),$$

and the process $(p(t), q(t))$ by

$$p(t) \;=\; -i\left(b(t) - b^\dagger(t)\right), \quad q(t) \;=\; b(t) + b^\dagger(t),$$

for $t \geqq 0$.

Remark 7.9. *$(p(t), q(t))$ satisfies the fixed time relations*

$$q(t)^2 = p(t)^2 = I, \quad \{p(t), q(t)\} = 0.$$

Remark 7.10. *By adding the equation (7.1.3) to equation (7.1.4) we obtain*

$$\mathrm{d}q(t) \;=\; (\lambda + \mu)\mathrm{d}Q - \frac{1}{2}(\lambda + \mu)^2 q \mathrm{d}t, \tag{7.1.5}$$

and substracting the equation (7.1.4) from equation (7.1.3) gives

$$\mathrm{d}p(t) = (\mu - \lambda)\mathrm{d}P - \frac{1}{2}(\mu - \lambda)^2 p \mathrm{d}t. \tag{7.1.6}$$

We give also the explicit form of the solutions

Proposition 7.11. *The following are solutions of the equations (7.1.5) and (7.1.6)*

$$q(t) \;=\; \exp\left(-\frac{1}{2}\rho^2 t\right) q_0 + \int_0^t \exp\left(-\frac{1}{2}\rho^2 (t - \tau)\right) \mathrm{d}Q(\tau),$$

$$p(t) \;=\; \exp\left(-\frac{1}{2}\rho^2 t\right) p_0 + \int_0^t \exp\left(-\frac{1}{2}\rho^2 (t - \tau)\right) \mathrm{d}P(\tau),$$

where $\rho \;=\; \lambda + \mu, \quad \theta \;=\; \mu - \lambda$.

Proof. Follows immediately. □

Proposition 7.12. *There exists a unique Fermion diffusion with one degree of freedom satisfying the conditions of Example 6.6.*

Proof. For notational convenience we drop the time dependency and we avoid writing the algebraic ampliation of $(b, b^\dagger)$ and $(b_1, b_1^\dagger)$ be solutions of the equations (7.1.3) and (7.1.4). Then, we know that

$$p = -i\left(b - b^\dagger\right), \quad p_1 = -i\left(b_1 - b_1^\dagger\right),$$

are solutions of the equation (7.1.6). On the other hand, we have,

$$q = b + b^\dagger, \quad q = b_1 + b_1^\dagger,$$

are solutions of the equation (7.1.5). From [2] we know that the equations

$$d(p - p_1) = -\frac{1}{2}(\mu - \lambda)^2(p - p_1)\mathrm{d}t, \quad d(q - q_1) = -\frac{1}{2}(\lambda + \mu)^2(q - q_1)^2\mathrm{d}t,$$

with initial conditions

$$p(0) - p_1(0) = 0, \quad q(0) - q_1(0) = 0,$$

respectively, have a unique solution, which is the zero operator. Therefore, we have,

$$-i\left(b - b^\dagger\right) = -i\left(b_1 - b_1^\dagger\right) \tag{7.1.7}$$

and

$$b + b^\dagger = b_1 + b_1^\dagger. \tag{7.1.8}$$

Hence, we infer from (7.1.7) the following

$$b - b^\dagger = b_1 - b_1^\dagger \tag{7.1.9}$$

by adding equation (7.1.9) to equation (7.1.8) we obtain

$$2b = 2b_1$$

which implies that $b = b_1$ and $b^\dagger = b_1^\dagger$. Thus, the solutions are unique. □

7.2 Two degrees of freedom

As in the case of one degree of freedom, we use the consistency conditions given by Propositions 6.14, 6.15 and 6.16 in order to find the even and odd coefficients of Fermion diffusions with two degrees of freedom.

Remark 7.13. *In the case of a Fermion diffusion with two degrees of freedom we have equations of the form*

$$\mathrm{d}B^1(t) = F^1(t)\mathrm{d}A^\dagger(t) + G^{1\dagger}(t)\mathrm{d}A(t) + H^{1\dagger}(t)\mathrm{d}t,$$

$$\mathrm{d}B^{1\dagger}(t) = F^{1\dagger}(t)\mathrm{d}A(t) + G^1(t)\mathrm{d}A^\dagger(t) + H^{1\dagger}\mathrm{d}t,$$

$$\mathrm{d}B^2(t) = F^2(t)\mathrm{d}A^\dagger(t) + G^{2\dagger}(t)\mathrm{d}A(t) + H^{2\dagger}(t)\mathrm{d}t,$$

$$\mathrm{d}B^{2\dagger}(t) = F^{2\dagger}(t)\mathrm{d}A(t) + G^{2\dagger}(t)\mathrm{d}A^\dagger(t) + H^{2\dagger}\mathrm{d}t.$$

Remark 7.14. *Let F_1, F_2, G_1 and G_2 be the even coefficients of Remark 7.13. Then, from Theorem 6.2 we infer that general even operators in the* CAR*-algebra has the following form*

$$F_1 = \lambda_1 I + \lambda_2 b_1 b_1^\dagger + \lambda_3 b_1 b_2 + \lambda_4 b_1 b_2^\dagger + \lambda_5 b_1^\dagger b_2 + \lambda_6 b_1^\dagger b_2^\dagger + \lambda_7 b_2 b_2^\dagger + \lambda_8 b_1 b_1^\dagger b_2 b_2^\dagger,$$

$$F_2 = \mu_1 I + \mu_2 b_1 b_1^\dagger + \mu_3 b_1 b_2 + \mu_4 b_1 b_2^\dagger + \mu_5 b_1^\dagger b_2 + \mu_6 b_1^\dagger b_2^\dagger + \mu_7 b_2 b_2^\dagger + \mu_8 b_1 b_1^\dagger b_2 b_2^\dagger,$$

$$G_1 = \kappa_1 I + \kappa_2 b_1 b_1^\dagger + \kappa_3 b_1 b_2 + \kappa_4 b_1 b_2^\dagger + \kappa_5 b_1^\dagger b_2 + \kappa_6 b_1^\dagger b_2^\dagger + \kappa_7 b_2 b_2^\dagger + \kappa_8 b_1 b_1^\dagger b_2 b_2^\dagger,$$

$$G_2 = \nu_1 I + \nu_2 b_1 b_1^\dagger + \nu_3 b_1 b_2 + \nu_4 b_1 b_2^\dagger + \nu_5 b_1^\dagger b_2 + \nu_6 b_1^\dagger b_2^\dagger + \nu_7 b_2 b_2^\dagger + \nu_8 b_1 b_1^\dagger b_2 b_2^\dagger.$$

7.2.1 Even coefficients

Our purpose here, is to find the even coefficients of Fermion diffusions with two degrees of freedom.

Proposition 7.15. *Let F_1, F_2, G_1 and G_2 be given as in Remark 7.14 the coefficients of a Fermion diffusion with two degrees of freedom. Then, they are the multiples of the identity operators.*

Proof. We will apply all the consistency conditions of Propositions 6.15 and 6.16 in order to get equations of the coefficients F_1, F_2, F_3, F_4. Furthermore, we will repeatedly use Remark 7.1 to find the terms of the equations that are zero. We start by computing F_1 and G_2 in the third and fourth equalities of Proposition 6.14, meaning the following

$$[F_1, b_1] = [G_1, b_1] \; = \; 0.$$

By the symmetry of these conditions and Remark 7.1[29] we obtain the zero terms

$$\lambda_2 = \lambda_5 = \lambda_6 = \lambda_8 = \kappa_2 = \kappa_5 = \kappa_6 = \kappa_8 = 0.$$

By substituting F_2 and G_2 in the third and fourth equalities of Proposition 6.14, we obtain

$$[F_2, b_2] = [G_2, b_2] \; = \; 0,$$

and by Remark 7.1 the zeros

$$\mu_4 = \mu_6 = \mu_7 = \mu_8 = \nu_4 = \nu_6 = \nu_7 = \nu_8 = 0.$$

We carry on with this technique and substitute F_1, F_2, G_1, G_2 in the first equality of Proposition 6.15 to get

$$[F_1, b_2] + [F_2, b_1] \; = \; [G_1, b_2] + [G_2, b_1] \; = \; 0,$$

which is equivalent to

$$0 = \left[\lambda_1 I + \lambda_3 b_1 b_2 + \lambda_4 b_1 b_2^+ + \lambda_7 b_2 b_2^+, b_2\right] + \left[\mu_1 I + \mu_2 b_1 b_1^+ + \mu_3 b_1 b_2 + \mu_5 b_1^+ b_2, b_1\right],$$

[29]See the proof of three degrees freedom for a better understanding of the computations or see below for this sort of computations.

applying the definition of the commutator operator and the canonical commutation relations allows us to deduce

$$0 = \lambda_4 b_1 b_2^+ b_2 - \lambda_4 b_2 b_1 b_2^+ + \lambda_7 b_2 b_2^+ b_2 + \mu_2 b_1 b_1^+ b_1 + \mu_5 b_1^+ b_2 b_1 - \mu_5 b_1 b_1^+ b_2.$$

Then, we simplify this equation with the canonical anticommutation relations, and obtain an equivalent identity

$$\lambda_4 b_1 + \lambda_7 b_2 + \mu_2 b_1 + \mu_5 b_2 = 0.$$

With Remark 7.1 we deduce that the equation above implies

$$\lambda_7 + \mu_5 \;=\; \kappa_7 + \nu_5 \;=\; 0, \quad \lambda_4 = -\mu_2, \quad \kappa_4 = -\nu_2.$$

We now put F_1 and $G_1^\dagger$ in the second condition of Proposition 6.16 to get

$$\left[F_1, b_1^\dagger\right] + \left[G_1^\dagger, b_1\right] \;=\; 0,$$

then we only need to compute

$$0 = \left[\lambda_1 I + \lambda_3 b_1 b_2 + \lambda_4 b_1 b_2^+ + \lambda_7 b_2 b_2^+, b_1^\dagger\right] + \left[\overline{\kappa}_1 I + \overline{\kappa}_3 b_2^\dagger b_1^\dagger + \overline{\kappa}_4 b_2 b_1^\dagger + \overline{\kappa}_7 b_2 b_2^+, b_1\right].$$

to find the zero terms, which we can do by using the definition of the commutator operator and the canonical anticommutation relations. From this observation, we see that it holds

$$0 = \lambda_3 b_1 b_2 b_1^\dagger - \lambda_3 b_1^\dagger b_1 b_2 + \lambda_4 b_1 b_2^\dagger b_1^\dagger - \lambda_4 b_1^\dagger b_1 b_2^\dagger + \overline{\kappa}_3 b_2^\dagger b_1^\dagger b_1 - \overline{\kappa}_3 b_1 b_2^\dagger b_1^\dagger + \overline{\kappa}_4 b_2 b_1^\dagger b_1 - \overline{\kappa}_4 b_1 b_2 b_1^\dagger.$$

We may further simplify this equation, by appliyng the canonical anticommutation relations, to get

$$0 \;=\; -\lambda_3 b_2 - \lambda_4 b_2^\dagger + \overline{\kappa}_3 b_2^\dagger + \overline{\kappa}_4 b_2,$$

and find by Remark 7.1 the following conditions

$$\lambda_3 = \overline{\kappa}_3, \quad \lambda_4 = \overline{\kappa}_4.$$

We substitute F_1 and G_2 in the secod condition of Proposition 6.16 in order to obtain the identity

$$\left[F_1^\dagger, b_2\right] + \left[G_2, b_1^\dagger\right] \;=\; 0.$$

More precisely, we have the conditions

$$0 = \left[\overline{\lambda}_1 I + \overline{\lambda}_3 b_2^\dagger b_1^\dagger + \overline{\lambda}_4 b_2 b_1^\dagger + \overline{\lambda}_7 b_2 b_2^+, b_2\right] + \left[\nu_1 I + \nu_2 b_1 b_1^\dagger + \nu_3 b_1 b_2 + \nu_5 b_1^\dagger b_2, b_1^\dagger\right].$$

Again, by applying the definition of the commutator operator and the canonical anticommutation relations, we see that the equation above is the same as

$$0 = \overline{\lambda}_3 b_2^\dagger b_1^\dagger b_2 - \overline{\lambda}_3 b_2 b_2^\dagger b_1^\dagger + \overline{\lambda}_7 b_2 - \nu_2 b_1^\dagger + \nu_3 b_1 b_2 b_1^\dagger - \nu_3 b_1^\dagger b_1 b_2.$$

Continuing with the use of this technique, we simplify the terms more to get an equivalent identity to the one above

$$-\overline{\lambda}_3 b_1^\dagger + \overline{\lambda}_7 b_2 - \nu_2 b_1^\dagger + \nu_3 b_2 = 0,$$

and conclude by Remark 7.1 that

$$-\overline{\lambda}_3 - \nu_2 = \overline{\lambda}_7 + \nu_3 = 0.$$

Since the even coefficients $F_2^\dagger$ and G_1 of the Fermion diffusion with two degrees of freedom satisfy the second equality of Proposition 6.14, we obtain the consistency condition

$$\left[F_2^\dagger, b_1\right] + \left[G_1, b_2^\dagger\right] \;=\; 0,$$

that is the same as having

$$0 \;=\; \left[\overline{\mu}_1 I + \overline{\mu}_2 b_1 b_1^\dagger + \overline{\mu}_3 b_2^\dagger b_1^\dagger + \overline{\mu}_5 b_2^\dagger b_1, b_1\right] + \left[\kappa_1 I + \kappa_3 b_1 b_2 + \kappa_4 b_1 b_2^\dagger + \kappa_7 b_2 b_2^\dagger, b_2^\dagger\right].$$

Using the same techniques that we have used so far to simplify equations with the terms of even operators, we can see that it is true that

$$0 \;=\; \overline{\mu}_2 b_1 + \overline{\mu}_3 b_2^\dagger b_1^\dagger b_1 - \overline{\mu}_3 b_1 b_2^\dagger b_1^\dagger + \kappa_3 b_1 b_2 b_1^\dagger - \kappa_3 b_1^\dagger b_1 b_2 + \kappa_4 b_1 b_2^\dagger b_1^\dagger - \kappa_4 b_1^\dagger b_1 b_2^\dagger.$$

Further reducing the terms of the last equation with help of the canonical anticommutation relations, we conclude that the following equation

$$0 \;=\; \overline{\mu}_2 b_1 + \overline{\mu}_3 b_2^\dagger - \kappa_3 b_2 - \kappa_4 b_2^\dagger.$$

has the same zeros as the last one that we presented. Therefore, by Remark 7.1 we have

$$\overline{\mu}_2 \;=\; \overline{\mu}_3 - \kappa_4 = -\kappa_3 \;=\; 0.$$

Together with the previous found conditions, we can see that the terms below are also zero

$$\lambda_4 \;=\; \kappa_4 \;=\; \lambda_3 \;=\; \nu_2 \;=\; \mu_3 \;=\; 0.$$

The even operators F_2, $G_2^\dagger$, with their current values, satisfy the second consistency condition of Proposition 6.16 and this condition is given by

$$\left[F_2, b_2^\dagger\right] + \left[G_2^\dagger, b_2\right] \;=\; 0,$$

specifically, it means

$$0 = \left[\mu_1 I + \mu_2 b_1 b_1^\dagger + \mu_5 b_1^\dagger b_2, b_1\right] + \left[\overline{\nu}_1 + \overline{\nu}_3 b_2^\dagger b_1^\dagger + \overline{\nu}_5 b_2^\dagger b_1, b_2\right],$$

hence, by applying the canonical anticommutation relations we cann see that the identity below

$$0 \;=\; \mu_2 b_1 + \mu_5 b_1^\dagger b_2 b_1 - \mu_5 b_1 b_1^\dagger b_2 + \overline{\nu}_3 b_2^\dagger b_1^\dagger b_2 - \overline{\nu}_3 b_2 b_2^\dagger b_1^\dagger + \overline{\nu}_5 b_2^\dagger b_1 b_2 - \overline{\nu}_5 b_2 b_2^\dagger b_1.$$

is equivalent to the previous one. Carrying on, as previously done, we have the equivalent identity

$$-\mu_2 b_1 - \mu_5 b_2 - \overline{\nu}_3 b_1^\dagger - \overline{\nu}_5 b_1 = 0.$$

Hence, by Remark 7.1 we have that the following terms of F_2 and G_2 are also zero

$$\mu_2 - \overline{\nu}_5 \; = \; \mu_5 \; = \; \overline{\nu}_3 \; = \; 0.$$

Furthermore, from $\lambda_4 = -\overline{\nu}_5 = 0$, we find the last zeros

$$\lambda_7 \; = \; \mu_2 \; = \; \kappa_7 \; = \; 0.$$

Finally, we see that the even operators are the identity, consequently, we may put

$$G_1 = \kappa I, \quad G_2 = \nu_1 I, \quad F_1 = \lambda_1 I, \quad F_2 = \mu_1 I,$$

which finishes the proof. □

7.2.2 Odd coefficients

Now, we only need to apply the consistency conditions to H_1 and H_2 to give explicitely a Fermion diffusion with two degrees of freedom. In order to achieve that, we calculate the terms of the odd coefficients that are zero.

Remark 7.16. *Since H_1 and H_2 are odd operators, we deduce from Theorem 6.2 that its more general form is given by*

$$\begin{aligned} H_1 \;&=\; \theta_1 I + \theta_2 b_1 + \theta_3 b_2 + \theta_4 b_1^\dagger + \theta_5 b_2^\dagger + \theta_6 b_1 b_1^\dagger + \theta_7 b_1 b_2 + \theta_8 b_1 b_2^\dagger + \theta_9 b_1^\dagger b_2 + \theta_{10} b_1^\dagger b_2^\dagger + \\ &\quad \theta_{11} b_2 b_2^\dagger + \theta_{12} b_1 b_1^\dagger b_2 + \theta_{13} b_1 b_1^\dagger b_2^\dagger + \theta_{14} b_2 b_2^\dagger b_1 + \theta_{15} b_2 b_2^\dagger b_1^\dagger + \theta_{16} b_1 b_1^\dagger b_2 b_2^\dagger, \end{aligned}$$

$$\begin{aligned} H_2 \;&=\; l_1 I + l_2 b_1 + l_3 b_2 + l_4 b_1^\dagger + l_5 b_2^\dagger + l_6 b_1 b_1^\dagger + l_7 b_1 b_2 + l_8 b_1 b_2^\dagger + l_9 b_1^\dagger b_2 + l_{10} b_1^\dagger b_2^\dagger + \\ &\quad l_{11} b_2 b_2^\dagger + l_{12} b_1 b_1^\dagger b_2 + l_{13} b_1 b_1^\dagger b_2^\dagger + l_{14} b_2 b_2^\dagger b_1 + l_{15} b_2 b_2^\dagger b_1^\dagger + l_{16} b_1 b_1^\dagger b_2 b_2^\dagger. \end{aligned}$$

We observe that from Propositions 6.14, 6.15 and 6.16 we infer the following consistency conditions for H_1 and H_2

$$\left\{H_1, b_1^\dagger\right\} + \left\{b_1, H_1^\dagger\right\} = \left(-|\lambda_1|^2 - |k_1|^2\right), \tag{7.2.1}$$

$$\{H_1, b_1\} = -\lambda_1 \kappa_1 I, \tag{7.2.2}$$

$$\{H_1, b_2\} + \{H_2, b_1\} = -\kappa_1 \mu_1 I - \nu_1 \lambda_1 I, \tag{7.2.3}$$

$$\{H_2, b_2\} = -\mu_1 \nu_1 I, \tag{7.2.4}$$

$$\left\{H_2, b_2^\dagger\right\} + \left\{b_2, H_2^\dagger\right\} = \left(-|\mu_1|^2 - |\nu|^2\right), \tag{7.2.5}$$

$$\left\{H_1, b_2^\dagger\right\} + \left\{H_2^\dagger, b_1\right\} = \overline{\mu}_1 \lambda_1 I - \kappa_1 \overline{\nu} I. \tag{7.2.6}$$

Proposition 7.17. *The odd coefficients, H_1 and H_2, of a Fermion diffusion with two degrees of freedom, given as in Remark 7.16, can be reduced to*

$$\begin{aligned} H_1 &= \left(-|\lambda_1|^2 - |\kappa_1|^2\right) b_1 + i\beta b_1 + (-\overline{\mu_1}\lambda_1 - \kappa_1\overline{\nu_1} - \bar{l}_2)I - \lambda_1\kappa_1 b_1^\dagger + \\ &\quad (-\kappa_1\mu_1 - \nu_1\lambda_1 - l_4)\, b_2^\dagger - \bar{l}_7 b_2 b_2^\dagger + i\alpha b_2 b_2^\dagger b_1 + 2\overline{l_7} b_1 b_1^\dagger b_2 b_2^\dagger, \end{aligned}$$

$$\begin{aligned} H_2 &= l_1 I + l_2 b_1 + \left(-|\mu_1|^2 - |\nu_1|^2\right) b_2 + i\rho b_2 + l_4 b_1^\dagger + (-\mu\nu) b_2^\dagger + l_6 b_1 b_1^\dagger + \\ &\quad l_7 b_1 b_2 - \bar{l}_7 b_1^\dagger b_2 + l_6 b_2 b_2^\dagger + l_{12} b_1 b_1^\dagger b_2 + -2l_6 b_1 b_1^\dagger b_2 b_2^\dagger, \end{aligned}$$

where β, α, ρ denotes the imaginary part of θ_2, θ_7 and l_3, respectively.

Proof. We will use repeatedly Remark 7.1 and the consistency conditions to find the zero terms of the coefficients of Fermion diffusions. The consistency condition (7.2.2) implies

$$\begin{aligned} \{H_1, b_1\} &= 2b_1\theta_1 + \theta_4 + \theta_6 b_1 + 2\theta_9 b_1 b_1^\dagger b_2 - \theta_9 b_2 - 2\theta_{10} b_1^\dagger b_1 b_2^\dagger + \theta_{10} b_2^\dagger + \\ &\quad 2\theta_{11} b_1 b_2 b_2^\dagger - \theta_{12} b_1 b_2 - \theta_{13} b_1 b_2^\dagger + \theta_{15} b_2 b_2^\dagger + \theta_{16} b_1 b_2 b_2^\dagger \\ &= -\lambda_1\kappa_1 I. \end{aligned}$$

Hence, it is true by Remark 7.1 that

$$\theta_4 = -\lambda_1\kappa_1, \tag{7.2.7}$$

and

$$2\theta_1 + \theta_6 = \theta_9 = \theta_{10} = 2\theta_{11} + \theta_{16} = \theta_{12} = \theta_{13} = \theta_{15} = 0. \tag{7.2.8}$$

From the consistency conditions (7.2.1) and (7.2.7) we get

$$\begin{aligned} \left(-|\lambda_1|^2 - |\kappa_1|^2\right) I &= \Big\{\theta_1 I + \theta_2 b_1 + \theta_3 b_2 + \theta_4 b_1^\dagger + \theta_5 b_2^\dagger + \theta_6 b_1 b_1^\dagger + \theta_7 b_1 b_2 + \\ &\quad \theta_8 b_1 b_2^\dagger + \theta_{11} b_2 b_2^\dagger + \theta_{14} b_2 b_2^\dagger b_1 + \theta_{16} b_1 b_1^\dagger b_2 b_2^\dagger, b_1^\dagger\Big\} + \\ &\quad \Big\{b_1, \overline{\theta_1} I + \overline{\theta_2} b_1^\dagger + \overline{\theta_3} b_2^\dagger + \overline{\theta_4} b_1 + \\ &\quad \overline{\theta_5} b_2 + +\overline{\theta_7} b_2^\dagger b_1^\dagger + \overline{\theta_6} b_1 b_1^\dagger + \overline{\theta_8} b_2 b_1^\dagger + \overline{\theta_{11}} b_2 b_2^\dagger + \\ &\quad \overline{\theta_{14}} b_1^\dagger b_2 b_2^\dagger + \overline{\theta_{16}} b_2 b_2^\dagger b_1 b_1^\dagger\Big\} \\ &= 2\theta_1 b_1^\dagger + \theta_2 I + \theta_6 b_1^\dagger - 2\theta_7 b_1 b_1^\dagger b_2 + \theta_7 b_2 + \\ &\quad 2\theta_8 b_1^\dagger b_1 b_2^\dagger - \theta_8 b_2^\dagger + 2\theta_{11} b_1^\dagger b_2 b_2^\dagger + \theta_{14} b_2 b_2^\dagger + \theta_{16} b_1^\dagger b_2 b_2^\dagger + \\ &\quad 2\overline{\theta_1} b_1 + \overline{\theta_2} + \overline{\theta_6} b_1 + \overline{\theta_7} b_2^\dagger - 2\overline{\theta_7} b_2^\dagger b_1 b_1^\dagger - 2\overline{\theta_8} b_1 b_1^\dagger b_2 + \\ &\quad \overline{\theta_8} b_2 + 2\overline{\theta_{11}} b_1 b_2 b_2^\dagger + \overline{\theta_{14}} b_2 b_2^\dagger + \overline{\theta_{16}} b_1 b_2 b_2^\dagger \\ &= 2\,\mathrm{Re}(\theta_2) + 2\theta_8 b_1^\dagger b_1 b_2^\dagger - 2\overline{\theta_7} b_2^\dagger b_1 b_1^\dagger + \overline{\theta_7} b_2^\dagger - \theta_8 b_2^\dagger - 2\theta_7 b_1 b_1^\dagger b_2 - \\ &\quad 2\overline{\theta_8} b_1 b_1^\dagger b_2 + \overline{\theta_8} b_2 + \theta_7 b_2 + 2\,\mathrm{Re}(\theta_{14}) b_2 b_2^\dagger. \end{aligned}$$

As a result of this, Remark 7.1 allows us to put

$$2\,\mathrm{Re}(\theta_2) = -|\lambda_1|^2 - |\kappa_1|^2, \quad \overline{\theta}_8 + \theta_7 = 2Re(\theta_{14}) = 0.$$

The consistency condition (7.2.4) leads to the equation

$$\begin{aligned}
-\mu_1\nu_1 I &= \{H_2, b_2\} \\
&= 2b_2l_1 + l_5I + 2l_6b_2b_1b_1^\dagger + l_8b_1 - l_8b_2b_2^\dagger b_1 + l_{10}b_1^\dagger - \\
&\quad 2l_{10}b_1^\dagger b_2b_2^\dagger + l_{11}b_2 + l_{13}b_1b_1^\dagger - l_{14}b_2b_1 - l_{15}b_2b_1^\dagger + l_{16}b_2b_1b_1^\dagger.
\end{aligned}$$

Thus, by Remark 7.1 we obtain

$$2l_1 + l_{11} = 0, \quad l_5 = -\mu_1\nu_1, \quad 2l_6 + l_{16} = l_8 = l_{10} = l_{13} = l_{14} = l_{15} = 0. \tag{7.2.9}$$

From the equation (7.2.5) and the consistency condition (7.2.9) we infer

$$\begin{aligned}
&\Big\{l_1I + l_2b_1 + l_3b_2 + l_4b_1^\dagger + (-\mu\nu)b_2^\dagger + l_6b_1b_1^\dagger + l_7b_1b_2 + l_9b_1^\dagger b_2 + \\
&l_{11}b_2b_2^\dagger + l_{12}b_1b_1^\dagger b_2 + l_{16}b_1b_1^\dagger b_2b_2^\dagger, b_2^\dagger\Big\} + \\
&\Big\{b_2, \overline{l_1}I + \overline{l_2}b_1^\dagger + \overline{l_3}b_2^\dagger + \overline{l_4}b_1 + (-\overline{\mu\nu})b_2 + \overline{l_6}b_1b_1^\dagger + \overline{l_7}b_2^\dagger b_1^\dagger + \overline{l_9}b_2^\dagger b_1 + \\
&\overline{l_{11}}b_2b_2^\dagger + \overline{l_{12}}b_2^\dagger b_1b_1^\dagger + \overline{l_{16}}b_2b_2^\dagger b_1b_1^\dagger\Big\} \\
= \ &2b_2^\dagger l_1 + l_3I + 2l_6b_2^\dagger b_1b_1^\dagger + 2l_7b_2b_2^\dagger b_1 - l_7b_1 - l_9b_1^\dagger + 2l_9b_1^\dagger b_2b_2^\dagger + l_{11}b_2^\dagger + \\
&l_{12}b_1b_1^\dagger + l_{16}b_1b_1^\dagger b_2^\dagger + 2b_2\bar{l}_1 + \bar{l}_3 + 2\bar{l}_6b_2b_1b_1^\dagger + 2\bar{l}_7b_2b_2^\dagger b_1^\dagger - \bar{l}_7b_1^\dagger - \\
&2\bar{l}_9b_2^\dagger b_2b_1 + 2\bar{l}_9b_1 + \bar{l}_{11}b_2 + \bar{l}_{12}b_1b_1^\dagger + \bar{l}_{16}b_2b_1b_1^\dagger \\
= \ &2l_7b_2b_2^\dagger b_1 - l_7b_1 - l_9b_1^\dagger + 2l_9b_1^\dagger b_2b_2^\dagger + l_{12}b_1b_1^\dagger + l_3I + \overline{l_3}I + \\
&2\bar{l}_7b_2b_2^\dagger b_1^\dagger - \bar{l}_7b_1^\dagger - 2\bar{l}_9b_2^\dagger b_2b_1 + \bar{l}_9b_1 + \bar{l}_{12}b_1b_1^\dagger.
\end{aligned}$$

Therefore, by Remark 7.1 we get

$$2\,\mathrm{Re}(l_3) \ = \ -|\mu_1|^2 - |\nu_1|^2, \quad 2\,\mathrm{Re}(l_{12}) \ = \ 0, \quad l_7 + \bar{l}_9 \ = \ 0.$$

Substituting H_1 and H_2 in (7.2.3) gives us the relations

$$\begin{aligned}
&\Big\{\theta_1I + \theta_2b_1 + \theta_3b_2 + \theta_4b_1^\dagger + \theta_5b_2^\dagger + \theta_6b_1b_1^\dagger + \theta_8b_1b_2^\dagger + \\
&\theta_{11}b_2b_2^\dagger + \theta_{14}b_2b_2^\dagger b_1 + \theta_{16}b_1b_1^\dagger b_2b_2^\dagger, b_2\Big\} + \\
&\Big\{l_1I + l_2b_1 + l_3b_2 + l_4b_1^\dagger + (-\mu_1\nu_1)b_2^\dagger + l_6b_1b_1^\dagger - l_9b_1b_2 + l_9b_1^\dagger b_2 + l_{11}b_2b_2^\dagger + \\
&l_{12}b_1b_1^\dagger b_2 + l_{16}b_1b_1^\dagger b_2b_2^\dagger, b_1\Big\} \\
= \ &2\theta_1b_2 + \theta_5I + 2\theta_6b_2b_1b_1^\dagger + \theta_8b_1 - 2\theta_8b_1b_2b_2^\dagger + \theta_{11}b_2 - \theta_{14}b_2b_1 + \theta_{16}b_1b_1^\dagger b_2 + \\
&2l_1b_1 + l_4I + l_6b_1 - l_9b_2 + 2l_9b_2b_1b_1^\dagger \ + 2l_{11}b_1b_2b_2^\dagger - l_{12}b_1b_2 + l_{16}b_1b_2b_2^\dagger.
\end{aligned}$$

Then, by Remark 7.1 it follows

$$\begin{aligned}
&2\theta_1 + \theta_{11} - l_9 = 0, \quad \theta_5 + l_4 = -\kappa_1\mu_1 - \nu_1\lambda_1, \quad \theta_8 + 2l_1 + l_6 = 0, \\
&0 = -2\theta_8 + 2l_{11} + l_{16} = 2\theta_6 + \theta_{16} + 2l_9 = \theta_{11} - l_9 = -\theta_{14} + l_{12}.
\end{aligned}$$

By applying (7.2.6) to H_2 we obtain

$$\begin{aligned}
&\Big\{\theta_1 I + \theta_2 b_1 + \theta_3 b_2 + \theta_4 b_1^\dagger + \theta_5 b_2^\dagger + \theta_6 b_1 b_1^\dagger + \theta_8 b_1 b_2^\dagger + \\
&\theta_{11} b_2 b_2^\dagger + \theta_{14} b_2 b_2^\dagger b_1 + \theta_{16} b_1 b_1^\dagger b_2 b_2^\dagger, b_2^\dagger\Big\} + \\
&\Big\{\overline{l_1} I + \overline{l_2} b_1^\dagger + \overline{l_3} b_2^\dagger + \overline{l_4} b_1 + (-\overline{\mu\nu}) b_2 + \overline{l_6} b_1 b_1^\dagger + \overline{l_7} b_2^\dagger b_1^\dagger + \overline{l_9} b_2^\dagger b_1 + \\
&\overline{l_{11}} b_2 b_2^\dagger + \overline{l_{12}} b_2^\dagger b_1 b_1^\dagger + \overline{l_{16}} b_2 b_2^\dagger b_1 b_1^\dagger, b_1\Big\} \\
= \; & 2\theta_1 b_2^\dagger + \theta_3 I + 2\theta_6 b_2^\dagger b_1 b_1^\dagger + \theta_{11} b_2^\dagger + \theta_{14} b_2^\dagger b_1 + \theta_{16} b_1 b_1^\dagger b_2^\dagger + \\
&2 b_1 \bar{l}_1 + \bar{l}_2 I + \bar{l}_6 b_1 + \bar{l}_7 b_2^\dagger - 2\bar{l}_7 b_1 b_1^\dagger b_2^\dagger + 2\bar{l}_{11} b_1 b_2 b_2^\dagger + \\
&\bar{l}_{12} b_2^\dagger b_1 + \bar{l}_{16} b_2 b_2^\dagger b_1,
\end{aligned}$$

and consequently we find with Remark 7.1 the zero term relations

$$\begin{aligned}
0 &= 2\theta_1 + \theta_{11} + \bar{l}_7 = \theta_{11} + \bar{l}_7 = \theta_{14} + \bar{l}_{12} = 2\theta_6 + \theta_{16} - 2\bar{l}_7 \\
&= 2\bar{l}_1 + \bar{l}_6, \quad \bar{l}_2 + \theta_3 = -\bar{\mu}\lambda_1 - \kappa_1\bar{\nu}_1, \quad 2\bar{l}_{11} + \bar{l}_{16} = 0.
\end{aligned} \tag{7.2.10}$$

By substracting the second equation from the first equation in the consistency condition (7.2.10) we find another zero of H_1

$$\theta_1 = 0,$$

combining it with (7.2.7) means that the following is true

$$\theta_6 = 0.$$

By seeing that $2\theta_1 + \theta_{11} + \bar{l}_7 = 0$, we may put

$$\theta_{11} = -\bar{l}_7.$$

From the third equality of the consistency condition (7.2.7) and $2\theta_{11} + \theta_{16} = 0$, together with the fourth equation (7.2.10) give us

$$\theta_{16} = 2\bar{l}_7 = -2l_9 = -2\theta_{11}.$$

Because we have $\mathrm{Re}(l_{12}) = 0$, it follows $\bar{l}_{12} = -l_{12}$, which implies

$$\mathrm{Re}(\theta_{14}) = 0.$$

From the fifth equality of the equation (7.2.10) we infer

$$\theta_8 + 2\bar{l}_1 + \bar{l}_6 = 0,$$

and as a consequence, we have

$$\theta_8 = 0.$$

Furthermore, since it is true that $\theta_8 = -\theta_7$, we find the zero term

$$\theta_7 = 0.$$

Finally, the equalities

$$2l_1 + l_{11} = 2l_6 + l_{16} = 2l_{11} + l_{16} = 2l_1 + l_6 = 0,$$

lead to

$$l_{11} = l_6, \quad 2l_1 = -l_6, \quad l_{16} = -2l_{11},$$

which finishes the proof. □

7.2.3 General form

Finally, we give the general form of a Fermion diffusion with two degrees of freedom.

Proposition 7.18. *The general form of a Fermion differential equation with two degrees of freedom, is the following*

$$\begin{aligned}
\mathrm{d}B_1(t) &= \lambda \mathrm{dA} + \kappa \mathrm{dA}^\dagger + \Big(\left(-|\lambda|^2 - |\kappa|^2\right) B_1 + i\beta B_1 + (-\overline{\mu}\lambda - \kappa\overline{\nu} - \overline{\rho})I - \\
&\quad \lambda\kappa B_1^\dagger + (-\kappa\mu - \nu\lambda - \gamma)B_2^\dagger - \overline{\theta}B_2B_2^\dagger + i\alpha B_2B_2^\dagger B_1 + 2\overline{\theta}B_1B_1^\dagger B_2B_2^\dagger \Big)\mathrm{dt},
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}B_2(t) &= \mu \mathrm{d}A^\dagger + \nu \mathrm{d}A + \Big(lI + \rho B_1 + (-|\mu|^2 - |\nu|^2)B_2 + i\sigma B_2 + \gamma B_1^\dagger + (-\mu\nu)B_2^\dagger + \\
&\quad \phi B_1B_1^\dagger + \theta B_1B_2 - \overline{\theta}B_1^\dagger B_2 + \phi B_2B_2^\dagger + \Phi B_1B_1^\dagger B_2 + -2\phi B_1B_1^\dagger B_2B_2^\dagger \Big)\mathrm{dt},
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}B_1^\dagger(t) &= \overline{\lambda}\mathrm{dA}^\dagger + \overline{\kappa}\mathrm{dA} + \Big((-|\lambda|^2 - |\kappa|^2)B_1^\dagger - i\beta B_1^\dagger + (-\mu\overline{\lambda} - \overline{\kappa}\nu - \rho)I - \\
&\quad \overline{\lambda}\overline{\kappa}B_1 + (-\overline{\kappa\mu} - \overline{\nu}\overline{\lambda} - \overline{\gamma})B_2 - \theta B_2B_2^\dagger - i\alpha B_1^\dagger B_2B_2^\dagger + 2\theta B_2B_2^\dagger B_1B_1^\dagger \Big)\mathrm{dt},
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}B_2^\dagger(t) &= \overline{\mu}\mathrm{d}A + \nu \mathrm{d}A^\dagger + \Big(\bar{l}I + \overline{\rho}B_1^\dagger + (-|\mu|^2 - |\nu|^2)B_2^\dagger - i\sigma B_2^\dagger + \overline{\gamma}B_1 + (-\overline{\mu\nu})B_2 + \\
&\quad \overline{\phi}B_1B_1^\dagger + \overline{\theta}B_2^\dagger B_1^\dagger - \theta B_2^\dagger B_1 + \overline{\phi}B_2B_2^\dagger + \overline{\Phi}B_2^\dagger B_1B_1^\dagger + -2\overline{\phi}B_2B_2^\dagger B_1B_1^\dagger \Big)\mathrm{dt},
\end{aligned}$$

where $\lambda, \kappa, \beta, \mu, \rho, \nu, \sigma, \phi, \Phi, \sigma, l \in \mathbb{C}$. *with initial conditions*

$$\begin{aligned}
B_1(0) &= b_1(0)\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))}, \quad & B_1^\dagger(0) &= b_1(0)^\dagger\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))}, \\
B_2(0) &= b_2(0)\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))}, \quad & B_2^\dagger(0) &= b_2(0)^\dagger\widehat{\otimes}I_{\Gamma_a(L^2(\mathbb{R}_+))}.
\end{aligned}$$

Proof. It follows from Propositions 7.15 and 7.17. □

7.3 Three degrees of freedom

In this section, we calculate the zeros terms of the even coefficients of Fermion diffusions with three degrees of freedom and we give the general form of the odd coefficients(but we do not find the zeros of this operators) of a Fermion diffusion with three degrees of freedom. By doing this, we see that, cotrary to the cases of one and two degree of freedom, the terms of the even coefficients (and hence those of the odd operators) are considerably more complex, meaning that they have more non-zero terms. More precisely, the main motivation of the computations done in this subsection is to illustrate how the Fermion diffusions grow in “complexity” as the number of degrees of freedom increases.

7.3.1 Even coefficients

Remark 7.19. *From Theorem 6.2 we infer that the even coefficients of a Fermion diffusion with three degrees of freedom have the following form*

$$\begin{aligned} F_1 \;=\;& \lambda_1 I + \lambda_2 b_1 b_1^\dagger + \lambda_3 b_1 b_2 + \lambda_4 b_1 b_2^\dagger + \lambda_5 b_1 b_3 + \lambda_6 b_1 b_3^\dagger + \lambda_7 b_2 b_2^\dagger + \lambda_8 b_2 b_3 + \\ & \lambda_9 b_2 b_3^\dagger + \lambda_{10} b_3 b_3^\dagger + \lambda_{11} b_1^\dagger b_2 + \lambda_{12} b_1^\dagger b_2^\dagger + \lambda_{13} b_1^\dagger b_3 + \lambda_{14} b_1^\dagger b_3^\dagger + \lambda_{15} b_2^\dagger b_3 + \\ & \lambda_{16} b_2^\dagger b_3^\dagger + \lambda_{17} b_1 b_1^\dagger b_2 b_2^\dagger + \lambda_{18} b_1 b_1^\dagger b_2 b_3 + \lambda_{19} b_1 b_1^\dagger b_2 b_3^\dagger + \lambda_{20} b_1 b_1^\dagger b_3 b_3^\dagger + \\ & \lambda_{21} b_1 b_1^\dagger b_2^\dagger b_3 + \lambda_{22} b_1 b_1^\dagger b_2^\dagger b_3^\dagger + \lambda_{23} b_2 b_2^\dagger b_1 b_3 + \lambda_{24} b_2 b_2^\dagger b_1 b_3^\dagger + \lambda_{25} b_2 b_2^\dagger b_1^\dagger b_3 + \\ & \lambda_{26} b_2 b_2^\dagger b_1^\dagger b_3^\dagger + \lambda_{27} b_2 b_2^\dagger b_3 b_3^\dagger + \lambda_{28} b_3 b_3^\dagger b_1 b_2 + \lambda_{29} b_3 b_3^\dagger b_1 b_2^\dagger + \lambda_{30} b_3 b_3^\dagger b_1^\dagger b_2 + \\ & \lambda_{31} b_3 b_3^\dagger b_1^\dagger b_2^\dagger + \lambda_{32} b_1 b_1^\dagger b_2 b_2^\dagger b_3 b_3^\dagger, \end{aligned}$$

$$\begin{aligned} F_2 \;=\;& \mu_1 I + \mu_2 b_1 b_1^\dagger + \mu_3 b_1 b_2 + \mu_4 b_1 b_2^\dagger + \mu_5 b_1 b_3 + \mu_6 b_1 b_3^\dagger + \mu_7 b_2 b_2^\dagger + \mu_8 b_2 b_3 + \\ & \mu_9 b_2 b_3^\dagger + \mu_{10} b_3 b_3^\dagger + \mu_{11} b_1^\dagger b_2 + \mu_{12} b_1^\dagger b_2^\dagger + \mu_{13} b_1^\dagger b_3 + \mu_{14} b_1^\dagger b_3^\dagger + \mu_{15} b_2^\dagger b_3 + \\ & \mu_{16} b_2^\dagger b_3^\dagger + \mu_{17} b_1 b_1^\dagger b_2 b_2^\dagger + \mu_{18} b_1 b_1^\dagger b_2 b_3 + \mu_{19} b_1 b_1^\dagger b_2 b_3^\dagger + \mu_{20} b_1 b_1^\dagger b_3 b_3^\dagger + \\ & \mu_{21} b_1 b_1^\dagger b_2^\dagger b_3 + \mu_{22} b_1 b_1^\dagger b_2^\dagger b_3^\dagger + \mu_{23} b_2 b_2^\dagger b_1 b_3 + \mu_{24} b_2 b_2^\dagger b_1 b_3^\dagger + \mu_{25} b_2 b_2^\dagger b_1^\dagger b_3 + \\ & \mu_{26} b_2 b_2^\dagger b_1^\dagger b_3^\dagger + \mu_{27} b_2 b_2^\dagger b_3 b_3^\dagger + \mu_{28} b_3 b_3^\dagger b_1 b_2 + \mu_{29} b_3 b_3^\dagger b_1 b_2^\dagger + \mu_{30} b_3 b_3^\dagger b_1^\dagger b_2 + \\ & \mu_{31} b_3 b_3^\dagger b_1^\dagger b_2^\dagger + \mu_{32} b_1 b_1^\dagger b_2 b_2^\dagger b_3 b_3^\dagger, \end{aligned}$$

$$\begin{aligned} F_3 \;=\;& \kappa_1 I + \kappa_2 b_1 b_1^\dagger + \kappa_3 b_1 b_2 + \kappa_4 b_1 b_2^\dagger + \kappa_5 b_1 b_3 + \kappa_6 b_1 b_3^\dagger + \kappa_7 b_2 b_2^\dagger + \kappa_8 b_2 b_3 + \\ & \kappa_9 b_2 b_3^\dagger + \kappa_{10} b_3 b_3^\dagger + \kappa_{11} b_1^\dagger b_2 + \kappa_{12} b_1^\dagger b_2^\dagger + \kappa_{13} b_1^\dagger b_3 + \kappa_{14} b_1^\dagger b_3^\dagger + \kappa_{15} b_2^\dagger b_3 + \\ & \kappa_{16} b_2^\dagger b_3^\dagger + \kappa_{17} b_1 b_1^\dagger b_2 b_2^\dagger + \kappa_{18} b_1 b_1^\dagger b_2 b_3 + \kappa_{19} b_1 b_1^\dagger b_2 b_3^\dagger + \kappa_{20} b_1 b_1^\dagger b_3 b_3^\dagger + \\ & \kappa_{21} b_1 b_1^\dagger b_2^\dagger b_3 + \kappa_{22} b_1 b_1^\dagger b_2^\dagger b_3^\dagger + \kappa_{23} b_2 b_2^\dagger b_1 b_3 + \kappa_{24} b_2 b_2^\dagger b_1 b_3^\dagger + \kappa_{25} b_2 b_2^\dagger b_1^\dagger b_3 + \\ & \kappa_{26} b_2 b_2^\dagger b_1^\dagger b_3^\dagger + \kappa_{27} b_2 b_2^\dagger b_3 b_3^\dagger + \kappa_{28} b_3 b_3^\dagger b_1 b_2 + \kappa_{29} b_3 b_3^\dagger b_1 b_2^\dagger + \kappa_{30} b_3 b_3^\dagger b_1^\dagger b_2 + \\ & \kappa_{31} b_3 b_3^\dagger b_1^\dagger b_2^\dagger + \kappa_{32} b_1 b_1^\dagger b_2 b_2^\dagger b_3 b_3^\dagger, \end{aligned}$$

$$\begin{aligned} G_1 \;=\;& \lambda'_1 I + \lambda'_2 b_1 b_1^\dagger + \lambda'_3 b_1 b_2 + \lambda'_4 b_1 b_2^\dagger + \lambda'_5 b_1 b_3 + \lambda'_6 b_1 b_3^\dagger + \lambda'_7 b_2 b_2^\dagger + \lambda'_8 b_2 b_3 + \\ & \lambda'_9 b_2 b_3^\dagger + \lambda'_{10} b_3 b_3^\dagger + \lambda'_{11} b_1^\dagger b_2 + \lambda'_{12} b_1^\dagger b_2^\dagger + \lambda'_{13} b_1^\dagger b_3 + \lambda'_{14} b_1^\dagger b_3^\dagger + \lambda'_{15} b_2^\dagger b_3 + \\ & \lambda'_{16} b_2^\dagger b_3^\dagger + \lambda'_{17} b_1 b_1^\dagger b_2 b_2^\dagger + \lambda'_{18} b_1 b_1^\dagger b_2 b_3 + \lambda'_{19} b_1 b_1^\dagger b_2 b_3^\dagger + \lambda'_{20} b_1 b_1^\dagger b_3 b_3^\dagger + \\ & \lambda'_{21} b_1 b_1^\dagger b_2^\dagger b_3 + \lambda'_{22} b_1 b_1^\dagger b_2^\dagger b_3^\dagger + \lambda'_{23} b_2 b_2^\dagger b_1 b_3 + \lambda'_{24} b_2 b_2^\dagger b_1 b_3^\dagger + \lambda'_{25} b_2 b_2^\dagger b_1^\dagger b_3 + \\ & \lambda'_{26} b_2 b_2^\dagger b_1^\dagger b_3^\dagger + \lambda'_{27} b_2 b_2^\dagger b_3 b_3^\dagger + \lambda'_{28} b_3 b_3^\dagger b_1 b_2 + \lambda'_{29} b_3 b_3^\dagger b_1 b_2^\dagger + \lambda'_{30} b_3 b_3^\dagger b_1^\dagger b_2 + \\ & \lambda'_{31} b_3 b_3^\dagger b_1^\dagger b_2^\dagger + \lambda'_{32} b_1 b_1^\dagger b_2 b_2^\dagger b_3 b_3^\dagger, \end{aligned}$$

$$
\begin{aligned}
G_2 \quad = \quad & \mu'_1 I + \mu'_2 b_1 b_1^\dagger + \mu'_3 b_1 b_2 + \mu'_4 b_1 b_2^\dagger + \mu'_5 b_1 b_3 + \mu'_6 b_1 b_3^\dagger + +\mu'_7 b_2 b_2^\dagger + \mu'_8 b_2 b_3 + \\
& \mu'_9 b_2 b_3^\dagger + \mu'_{10} b_3 b_3^\dagger + \mu'_{11} b_1^\dagger b_2 + \mu'_{12} b_1^\dagger b_2^\dagger + \mu'_{13} b_1^\dagger b_3 + \mu'_{14} b_1^\dagger b_3^\dagger + \mu'_{15} b_2^\dagger b_3 + \\
& \mu'_{16} b_2^\dagger b_3^\dagger + \mu'_{17} b_1 b_1^\dagger b_2 b_2^\dagger + \mu'_{18} b_1 b_1^\dagger b_2 b_3 + \mu'_{19} b_1 b_1^\dagger b_2 b_3^\dagger + \mu'_{20} b_1 b_1^\dagger b_3 b_3^\dagger + \\
& \mu'_{21} b_1 b_1^\dagger b_2^\dagger b_3 + \mu'_{22} b_1 b_1^\dagger b_2^\dagger b_3^\dagger + \mu'_{23} b_2 b_2^\dagger b_1 b_3 + \mu'_{24} b_2 b_2^\dagger b_1 b_3^\dagger + \mu'_{25} b_2 b_2^\dagger b_1^\dagger b_3 + \\
& \mu'_{26} b_2 b_2^\dagger b_1^\dagger b_3^\dagger + \mu'_{27} b_2 b_2^\dagger b_3 b_3^\dagger + \mu'_{28} b_3 b_3^\dagger b_1 b_2 + \mu'_{29} b_3 b_3^\dagger b_1 b_2^\dagger + \mu'_{30} b_3 b_3^\dagger b_1^\dagger b_2 + \\
& \mu'_{31} b_3 b_3^\dagger b_1^\dagger b_2^\dagger + \mu'_{32} b_1 b_1^\dagger b_2 b_2^\dagger b_3 b_3^\dagger ,
\end{aligned}
$$

$$
\begin{aligned}
G_3 \quad = \quad & \kappa'_1 I + \kappa'_2 b_1 b_1^\dagger + \kappa'_3 b_1 b_2 + \kappa'_4 b_1 b_2^\dagger + \kappa'_5 b_1 b_3 + \kappa'_6 b_1 b_3^\dagger + \kappa'_7 b_2 b_2^\dagger + \kappa'_8 b_2 b_3 + \\
& \kappa'_9 b_2 b_3^\dagger + \kappa'_{10} b_3 b_3^\dagger + \kappa'_{11} b_1^\dagger b_2 + \kappa'_{12} b_1^\dagger b_2^\dagger + \kappa'_{13} b_1^\dagger b_3 + \kappa'_{14} b_1^\dagger b_3^\dagger + \kappa'_{15} b_2^\dagger b_3 + \\
& \kappa'_{16} b_2^\dagger b_3^\dagger + \kappa'_{17} b_1 b_1^\dagger b_2 b_2^\dagger + \kappa'_{18} b_1 b_1^\dagger b_2 b_3 + \kappa'_{19} b_1 b_1^\dagger b_2 b_3^\dagger + \kappa'_{20} b_1 b_1^\dagger b_3 b_3^\dagger + \\
& \kappa'_{21} b_1 b_1^\dagger b_2^\dagger b_3 + \kappa'_{22} b_1 b_1^\dagger b_2^\dagger b_3^\dagger + \kappa'_{23} b_2 b_2^\dagger b_1 b_3 + \kappa'_{24} b_2 b_2^\dagger b_1 b_3^\dagger + \kappa'_{25} b_2 b_2^\dagger b_1^\dagger b_3 + \\
& \kappa'_{26} b_2 b_2^\dagger b_1^\dagger b_3^\dagger + \kappa'_{27} b_2 b_2^\dagger b_3 b_3^\dagger + \kappa'_{28} b_3 b_3^\dagger b_1 b_2 + \kappa'_{29} b_3 b_3^\dagger b_1 b_2^\dagger + \kappa'_{30} b_3 b_3^\dagger b_1^\dagger b_2 + \\
& \kappa'_{31} b_3 b_3^\dagger b_1^\dagger b_2^\dagger + \kappa'_{32} b_1 b_1^\dagger b_2 b_2^\dagger b_3 b_3^\dagger .
\end{aligned}
$$

7.3.2 Consistency conditions

Now, we reduce the even coefficients of a Fermion diffusion with three degrees of freedom with the consistency conditions given by the Propositions 6.14, 6.15 and 6.16.

Remark 7.20. *We will use in the next propostion that*

$$\left[b_i^\dagger b_j, b_i\right] = -b_j, \quad \left[b_j b_i^\dagger, b_i\right] = b_j, \quad \left[b_j^\dagger b_i, b_i^\dagger\right] = b_j^\dagger, \quad \left[b_i b_j^\dagger, b_i^\dagger\right] = -b_j^\dagger.$$

Furthermore, it holds

$$\left[b_i b_i^\dagger b_j^\dagger b_l, b_i\right] = b_i b_j^\dagger b_l, \quad \left[b_i^\dagger b_j b_j^\dagger b_l, b_j\right] = -b_i^\dagger b_j b_l, \quad \left[b_i b_i^\dagger b_j^\dagger b_l, b_j\right] = -b_i b_i^\dagger b_l$$

and

$$\left[b_i b_i^\dagger b_j^\dagger b_l, b_i^\dagger\right] = -b_i^\dagger b_j^\dagger b_l, \quad \left[b_i b_j b_j^\dagger b_l, b_i^\dagger\right] = -b_j b_j^\dagger b_l.$$

We have similar identities by interchanging the sign on the right side.

Proposition 7.21. *Let F_1, F_2, F_3, G_1, G_2 and G_3, be the even coefficients given as in Remark 7.19, of a Fermion diffusion with three degrees of freedom. Then, by expressing the even operators in terms of the non-zero terms, we get*

$$\begin{aligned} F_1 &= \lambda_1 I + \lambda_3 b_1 b_2 + \lambda_4 b_1 b_2^\dagger + \lambda_5 b_1 b_3 + \lambda_6 b_1 b_3^\dagger + \\ & \lambda_7 b_2 b_2^\dagger + \lambda_8 b_2 b_3 + \lambda_9 b_2 b_3^\dagger + \lambda_{10} b_3 b_3^\dagger, \end{aligned}$$

$$\begin{aligned} F_2 &= \mu_1 I - \lambda_4 b_1 b_1^\dagger + \mathrm{Re}(\mu_3) b_1 b_2 + \mu_5 b_1 b_3 + \mu_6 b_1 b_3^\dagger + \\ & \mu_8 b_2 b_3 + \mu_{10} b_3 b_3^\dagger + \lambda_7 b_1^\dagger b_2, \end{aligned}$$

$$\begin{aligned} F_3 &= \kappa_1 I - \lambda_6 b_1 b_1^\dagger + \kappa_3 b_1 b_2 + \mu_6 b_1 b_2^\dagger + \kappa_5 b_1 b_3 + \\ & \kappa_8 b_2 b_3 + \kappa_{11} b_1^\dagger b_2 + \mu_{10} b_3 b_3^\dagger \end{aligned}$$

$$\begin{aligned} G_1 &= \lambda_1' I + \overline{\lambda}_4 b_1 b_2 + \overline{\lambda}_3 b_1 b_2^\dagger + \overline{\lambda}_6 b_1 b_3 + \overline{\lambda}_5 b_1 b_3^\dagger + \\ & \mathrm{Re}(\mu_3) b_2 b_2^\dagger - \overline{\mu}_6 b_2 b_3 + \mu_5 b_2 b_3^\dagger + \kappa_5 b_3 b_3^\dagger \end{aligned}$$

$$\begin{aligned} G_2 &= \mu_1' I + \overline{\lambda}_3 b_1 b_1^\dagger + \overline{\lambda}_7 b_1 b_2 + \overline{\lambda}_9 b_1 b_3 + \overline{\lambda}_8 b_1 b_3^\dagger + \overline{\mu}_8 b_2 b_3^\dagger + \\ & \overline{\kappa}_8 b_3 b_3^\dagger + \mathrm{Re}(\mu_3) b_1^\dagger b_2 - \kappa_3 b_1^\dagger b_3 + \lambda_{16}' b_1^\dagger b_3^\dagger \end{aligned}$$

$$\begin{aligned} G_3 &= \kappa_1' I - \overline{\lambda}_5 b_1 b_1^\dagger - \overline{\lambda}_8 b_1 b_2^\dagger + \overline{\lambda}_{10} b_1 b_3 - \mu_8 b_2 b_2^\dagger + \overline{\mu}_{10} b_2 b_3 + \\ & \overline{\mu}_5 b_1^\dagger b_2 + \lambda_{16}' b_1^\dagger b_2^\dagger + \kappa_5 b_1^\dagger b_3 + \overline{\kappa}_8 b_2^\dagger b_3. \end{aligned}$$

Proof. We observe that the commutator of b_i or $b_i^\dagger$ with an operator that is the product of an even amount of operators that does not include b_i or $b_i^\dagger$ is zero. Furthermore, we will use, as before, Remark 7.1. We start as in the previous cases by applying the consistency conditions given by Proposition 6.14.
The third and fourth conditions of Proposition 6.14 imply

$$[F_1, b_1] = [G_1, b_1] = 0,$$

and

$$\begin{aligned} 0 &= \lambda_2 b_1 - \lambda_{11} b_1^\dagger b_1 b_2 - \lambda_{11} b_1 b_1^\dagger b_2 - \lambda_{12} b_1^\dagger b_1 b_2^\dagger - \lambda_{12} b_1 b_1^\dagger b_2^\dagger - \lambda_{13} b_1^\dagger b_1 b_3 - \lambda_{13} b_1 b_1^\dagger b_3 - \\ & \lambda_{14} b_1^\dagger b_1 b_3^\dagger - \lambda_{14} b_1 b_1^\dagger b_3^\dagger + \lambda_{17} b_1 b_2 b_2^\dagger + \lambda_{18} b_1 b_2 b_3 + \lambda_{19} b_1 b_2 b_3^\dagger + \\ & \lambda_{20} b_1 b_3 b_3^\dagger + \lambda_{21} b_1 b_2^\dagger b_3 + \lambda_{22} b_1 b_2^\dagger b_3^\dagger - \lambda_{25} b_2 b_2^\dagger b_1^\dagger b_1 b_3 - \lambda_{25} b_2 b_2^\dagger b_1 b_1^\dagger b_3 - \\ & \lambda_{26} b_2 b_2^\dagger b_3^\dagger - \lambda_{30} b_3 b_3^\dagger b_2 - \lambda_{31} b_3 b_3^\dagger b_2^\dagger + \lambda_{32} b_1 b_2 b_2^\dagger b_3 b_3^\dagger. \end{aligned}$$

Thus, by Remark 7.20 we get that

$$\begin{aligned} 0 &= \lambda_2 b_1 - \lambda_{11} b_2 - \lambda_{12} b_2^\dagger - \lambda_{13} b_3 - \lambda_{14} b_3^\dagger + \lambda_{17} b_1 b_2 b_2^\dagger + \lambda_{18} b_1 b_2 b_3 + \lambda_{19} b_1 b_2 b_3^\dagger + \\ & \lambda_{20} b_1 b_3 b_3^\dagger + \lambda_{21} b_1 b_2^\dagger b_3 + \lambda_{22} b_1 b_2^\dagger b_3^\dagger - \lambda_{25} b_2 b_2^\dagger b_3 - \lambda_{26} b_2 b_2^\dagger b_3^\dagger - \lambda_{30} b_3 b_3^\dagger b_2 - \\ & \lambda_{31} b_3 b_3^\dagger b_2^\dagger + \lambda_{32} b_1 b_2 b_2^\dagger b_3 b_3^\dagger. \end{aligned}$$

Hence, by the symmetry of the conditions and Remark 7.1 it holds[30]

$$\lambda_i = \lambda'_i = 0, \tag{7.3.1}$$

for $i \in \{2, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 25, 26, 30, 31, 32\}$. Now, the symmetric conditions obtained by susbstituting F_2, G_2 in the third and fourth identities of Proposition 6.14

$$[F_2, b_2] = [G_2, b_2] = 0,$$

together with Remark 7.20, give us

$$\begin{aligned} 0 \quad = \quad & \mu_4 b_1 + \mu_7 b_2 + \mu_{12} b_1^\dagger - \mu_{15} b_3 - \mu_{16} b_3^\dagger + \mu_{17} b_1 b_1^\dagger b_2 - \mu_{21} b_1 b_1^\dagger b_3 - \mu_{22} b_1 b_1^\dagger b_3^\dagger + \\ & \mu_{23} b_2 b_1 b_3 + \mu_{24} b_2 b_1 b_3^\dagger + \mu_{25} b_2 b_1^\dagger b_3 + \mu_{26} b_2 b_1^\dagger b_3^\dagger + \\ & \mu_{27} b_2 b_3 b_3^\dagger + \mu_{29} b_3 b_3^\dagger b_1 + \mu_{31} b_3 b_3^\dagger b_1^\dagger + \mu_{32} b_1 b_1^\dagger b_2 b_3 b_3^\dagger. \end{aligned}$$

Then, we have for $i \in \{4, 7, 12, 15, 16, 17, 21, 22, 23, 24, 25, 26, 27, 29, 31, 32\}$ and Remark 7.1 that it holds

$$\mu_i = \mu'_i = 0. \tag{7.3.2}$$

Now, by considering the last identical consistency conditions with only one even operator for the operators F_3 and G_3 we obtain

$$[F_3, b_3] = [G_3, b_3] = 0,$$

and by Remark 7.20

$$\begin{aligned} 0 \quad = \quad & \kappa_6 b_1 + \kappa_9 b_2 + \kappa_{10} b_3 + \kappa_{14} b_1^\dagger + \kappa_{16} b_2^\dagger + \kappa_{19} b_1 b_1^\dagger b_2 + \kappa_{20} b_1 b_1^\dagger b_3 + \kappa_{22} b_1 b_1^\dagger b_2^\dagger + \\ & \kappa_{24} b_2 b_2^\dagger b_1 + \kappa_{26} b_2 b_2^\dagger b_1^\dagger + \kappa_{27} b_2 b_2^\dagger b_3 + \kappa_{28} b_3 b_1 b_2 + \kappa_{29} b_3 b_1 b_2^\dagger + \\ & \kappa_{30} b_3 b_1^\dagger b_2 + \kappa_{31} b_3 b_1^\dagger b_2^\dagger + \kappa_{32} b_1 b_1^\dagger b_2 b_2 b_3. \end{aligned}$$

Therefore, we have for $i \in \{6, 9, 10, 14, 16, 19, 20, 22, 24, 26, 27, 28, 29, 30, 31, 32\}$, by Remark 7.1, the following equality

$$\kappa_i = \kappa'_i = 0. \tag{7.3.3}$$

By the symmetry of the conditions obtained by substituting F_1, F_2 in the first consistency condition of Proposition 6.15 and G_1, G_2 in the second one, we get

$$[F_1, b_2] + [F_2, b_1] = [G_1, b_2] + [G_2, b_2] = 0,$$

and by Remark 7.20 the following

$$\begin{aligned} 0 \quad = \quad & \lambda_4 b_1 + \lambda_7 b_2 - \lambda_{15} b_3 - \lambda_{16} b_3^\dagger + \lambda_{23} b_2 b_1 b_3 + \lambda_{24} b_2 b_1 b_3^\dagger + \\ & \lambda_{27} b_2 b_3 b_3^\dagger + \lambda_{29} b_3 b_3^\dagger b_1 + \mu_2 b_1 - \mu_{11} b_2 - \mu_{13} b_3 - \mu_{14} b_3^\dagger + \\ & \mu_{18} b_1 b_2 b_3 + \mu_{19} b_1 b_2 b_3^\dagger + \mu_{20} b_1 b_3 b_3^\dagger - \mu_{30} b_3 b_3^\dagger b_2. \end{aligned}$$

[30]Notice that half of the coefficients are zero, exactly as in the case with two degrees of freedom.

Which means by Remark 7.1 that the following terms must be zero

$$\begin{aligned} \lambda_4+\mu_2&=0, & \lambda'_4+\mu'_2&=0,\\ \lambda_7-\mu_{11}&=0, & \lambda'_7-\mu'_{11}&=0,\\ \lambda_{15}-\mu_{13}&=0, & \lambda'_{15}-\mu'_{13}&=0,\\ -\lambda_{16}-\mu_{14}&=0, & -\lambda'_{16}-\mu'_{14}&=0,\\ -\lambda_{23}-\mu_{18}&=0, & -\lambda'_{23}-\mu'_{18}&=0,\\ -\lambda_{24}+\mu_{19}&=0, & -\lambda'_{24}+\mu'_{19}&=0,\\ \lambda_{27}-\mu_{30}&=0, & \lambda'_{27}-\mu'_{30}&=0,\\ \lambda_{29}-\mu_{20}&=0, & \lambda'_{29}-\mu'_{20}&=0. \end{aligned} \tag{7.3.4}$$

The identical symmetric consistency conditions obtained from substituting F_1, F_3 in the first consistency condition of Proposition 6.15 and G_1, G_3 in the second consistency condition imply

$$[F_1,b_3]+[F_3,b_1]=[G_1,b_3]+[G_3,b_1]=0,$$

which together with Remark 7.20 allows to put

$$\begin{aligned} 0 \quad = \quad & \lambda_6 b_1+\lambda_9 b_2+\lambda_{10}b_3+\lambda_{16}b_2^\dagger+\lambda_{24}b_2b_2^\dagger b_1+\lambda_{27}b_2b_2^\dagger b_3+\\ & \lambda_{28}b_3b_1b_2+\lambda_{29}b_3b_1b_2^\dagger+\kappa_2b_1-\kappa_{11}b_2-\kappa_{12}b_2^\dagger+\kappa_{13}b_3+\kappa_{17}b_1b_2b_2^\dagger+\\ & \kappa_{18}b_1b_2b_3+\kappa_{21}b_1b_2^\dagger b_3-\kappa_{25}b_2b_2^\dagger b_3. \end{aligned}$$

Thus, by Remark 7.1 we deduce

$$\begin{aligned} \lambda_6+\kappa_2&=0, & \lambda'_6+\kappa'_2&=0,\\ \lambda_9-\kappa_{11}&=0, & \lambda'_9+\kappa'_{11}&=0,\\ \lambda_{10}-\kappa_{13}&=0, & \lambda'_{10}-\kappa'_{13}&=0,\\ \lambda_{16}-\kappa_{12}&=0, & \lambda'_{16}-\kappa'_{12}&=0,\\ \lambda_{24}-\kappa_{17}&=0, & \lambda'_{24}-\kappa'_{17}&=0,\\ \lambda_{27}-\kappa_{25}&=0, & \lambda'_{27}-\kappa'_{25}&=0,\\ \lambda_{28}+\kappa_{18}&=0, & \lambda'_{28}+\kappa'_{18}&=0,\\ \lambda_{29}-\kappa_{21}&=0, & \lambda'_{29}-\kappa'_{21}&=0. \end{aligned} \tag{7.3.5}$$

The equalities obtained by putting F_2, F_3 in the first consistency condition of Proposition 6.15 and G_2, G_3 in the second one lead us to the following identities

$$[F_2,b_3]+[F_3,b_2]=[G_2,b_3]+[G_3,b_2]=0.$$

Applying Remark 7.20 to the equation above gives us the conditions

$$\begin{aligned} 0 \quad = \quad & -\mu_6b_1-\mu_9b_2+\mu_{10}b_3-\mu_{14}b_1^\dagger+\mu_{19}b_1b_1^\dagger b_2+\mu_{20}b_1b_1^\dagger b_3+\\ & \mu_{28}b_3b_1b_2+\mu_{30}b_3b_1^\dagger b_2+\kappa_4b_1+\kappa_7b_2+\kappa_{12}b_1^\dagger-\kappa_{15}b_3+\\ & \kappa_{17}b_1b_1^\dagger b_2-\kappa_{21}b_1b_1^\dagger b_3+\kappa_{23}b_2b_1b_3+\kappa_{25}b_2b_1^\dagger b_3. \end{aligned}$$

The previoius equation together with Remark 7.1 gives us the zero terms

$$\begin{aligned}
-\mu_6 + \kappa_4 &= 0, & \mu'_6 + \kappa'_4 &= 0,\\
-\mu_9 + \kappa_7 &= 0, & \mu'_9 + \kappa'_7 &= 0,\\
\mu_{10} - \kappa_{15} &= 0, & \mu'_{10} - \kappa'_{15} &= 0,\\
-\mu_{14} + \kappa_{12} &= 0, & -\mu'_{14} + \kappa'_{12} &= 0,\\
\mu_{19} - \kappa_{17} &= 0, & \mu'_{19} - \kappa'_{17} &= 0,\\
\mu_{20} - \kappa_{21} &= 0 & \mu'_{20} - \kappa'_{21} &= 0,\\
\mu_{28} - \kappa_{23} &= 0, & \mu'_{28} - \kappa'_{23} &= 0,\\
\mu_{30} - \kappa_{25} &= 0, & \mu'_{30} - \kappa'_{25} &= 0.
\end{aligned} \tag{7.3.6}$$

Substituting the even coefficients, $F_1, G_1^\dagger$ in second identity of Proposition 6.16 allows to equate

$$\left[F_1, b_1^\dagger\right] + \left[G_1^\dagger, b_1\right] = 0,$$

and Remark 7.20 lead us to conclude

$$\begin{aligned}
0 \quad = \quad & -\lambda_3 b_2 - \lambda_4 b_2^\dagger - \lambda_5 b_3 - \lambda_6 b_3^\dagger - \lambda_{23} b_2 b_2^\dagger b_3 - \lambda_{24} b_2 b_2^\dagger b_3^\dagger - \\
& \lambda_{28} b_3 b_3^\dagger b_2 - \lambda_{29} b_3 b_3^\dagger b_2^\dagger + \overline{\lambda}'_3 b_2^\dagger + \overline{\lambda}'_4 b_2 + \overline{\lambda}'_5 b_3^\dagger + \overline{\lambda}'_6 b_3 - \\
& \overline{\lambda}'_{23} b_3^\dagger b_2 b_2^\dagger + \overline{\lambda}'_{24} b_2 b_2^\dagger b_3 + \overline{\lambda}'_{28} b_3 b_3^\dagger b_2^\dagger + \overline{\lambda}'_{29} b_3 b_3^\dagger b_2.
\end{aligned}$$

Hence, by Remark 7.1 we may put

$$\begin{aligned}
0 \quad &= \quad -\lambda_3 + \overline{\lambda}'_4 \;=\; -\lambda_4 + \overline{\lambda}'_3 \;=\; -\lambda_5 + \overline{\lambda}'_6 \;=\; -\lambda_6 + \overline{\lambda}'_5 \;=\; -\lambda_{23} + \overline{\lambda}'_{24} \\
&= \quad -\lambda_{24} - \overline{\lambda}'_{23} \;=\; -\lambda_{28} + \overline{\lambda}'_{29} \;=\; -\lambda_{29} + \overline{\lambda}'_{28}.
\end{aligned} \tag{7.3.7}$$

Finding the zeros of the operators F_1, $G_2^\dagger$ in the second equality of Proposition 6.16 implies the validity of

$$\left[F_1, b_2^\dagger\right] + \left[G_2^\dagger, b_1\right] = 0,$$

and together with Remark 7.20 the correctness of

$$\begin{aligned}
0 \quad = \quad & \lambda_3 b_1 - \lambda_7 b_2^\dagger - \lambda_8 b_3 - \lambda_9 b_3^\dagger + -\lambda_{23} b_2^\dagger b_1 b_3 - \lambda_{24} b_2^\dagger b_1 b_3^\dagger - \\
& \lambda_{27} b_2^\dagger b_3 b_3^\dagger + \lambda_{28} b_3 b_3^\dagger b_2 + \overline{\mu}'_2 b_1 + \overline{\mu}'_3 b_2^\dagger + \overline{\mu}'_5 b_3^\dagger + \overline{\mu}'_6 b_3 + \\
& \overline{\mu}'_{18} b_3^\dagger b_2^\dagger b_1 + \overline{\mu}'_{19} b_3 b_2^\dagger b_1 + \overline{\mu}'_{20} b_3 b_3^\dagger b_1 + \overline{\mu}'_{28} b_2^\dagger b_3 b_3^\dagger.
\end{aligned}$$

Therefore, Remark 7.1 means that the following

$$\begin{aligned}
0 \quad &= \quad \lambda_3 + \overline{\mu}'_2 \;=\; -\lambda_7 + \overline{\mu}'_3 \;=\; -\lambda_8 + \overline{\mu}'_6 \;=\; -\lambda_9 + \overline{\mu}'_5 \;=\; -\lambda_{23} + \overline{\mu}'_{19} \\
&= \quad -\lambda_{24} + \overline{\mu}'_{18} = -\lambda_{27} + \overline{\mu}'_{28} \;=\; \lambda_{28} \;=\; \overline{\mu}'_{20},
\end{aligned} \tag{7.3.8}$$

is true. Thus, from the equations (7.3.7), (7.3.6) and (7.3.5), respectively, it follows

$$\lambda'_{29} \;=\; \kappa'_{21} \;=\; \kappa_{18} \;=\; 0.$$

Considering the second identity of Proposition 6.16 for F_1, $G_3^\dagger$ give us

$$\left[F_1, b_3^\dagger\right] + \left[G_3^\dagger, b_1\right] = 0,$$

consequently, by Remark 7.20 we have

$$\begin{aligned} 0 \quad = \quad & \lambda_5 b_1 + \lambda_8 b_2 - \lambda_{10} b_3^\dagger + \lambda_{15} b_2^\dagger + \lambda_{23} b_2 b_2^\dagger b_1 - \lambda_{27} b_2 b_2^\dagger b_3^\dagger - \\ & \lambda_{29} b_3^\dagger b_1 b_2^\dagger + \overline{\kappa}_2' b_1 + \overline{\kappa}_3' b_2^\dagger + \overline{\kappa}_4' b_2 + \overline{\kappa}_5' b_3^\dagger + \overline{\kappa}_{17}' b_2 b_2^\dagger b_1 + \\ & \overline{\kappa}_{18}' b_3^\dagger b_2^\dagger b_1 + \overline{\kappa}_{23}' b_3^\dagger b_2 b_2^\dagger. \end{aligned}$$

Therefore, by Remark 7.1 we deduce

$$\begin{aligned} 0 \quad &= \quad \lambda_5 + \overline{\kappa}_2' = \lambda_8 + \overline{\kappa}_4' = -\lambda_{10} + \overline{\kappa}_5' = \lambda_{15} + \overline{\kappa}_3' \\ &= \quad \lambda_{23} + \overline{\kappa}_{17}' = -\lambda_{27} + \overline{\kappa}_{23}' = -\lambda_{29} = \overline{\kappa}_{17}'. \end{aligned} \tag{7.3.9}$$

Hence, we have by the equations (7.3.4), (7.3.5), (7.3.6), (7.3.7) and (7.3.8) the zero terms

$$\mu_{20} \;=\; \kappa_{21} \;=\; \lambda_{24}' \;=\; \mu_{19}' \;=\; \lambda_{23} = 0, \tag{7.3.10}$$

By considering equation (7.3.4) we find the zeros

$$\mu_{18} \;=\; -\lambda_{24}' \;=\; 0. \tag{7.3.11}$$

A consequence of the second result of Proposition 6.16 for the operators F_2, G_2 is

$$\left[F_2, b_2^\dagger\right] + \left[G_2^\dagger, b_2\right] = 0.$$

This being equivalent, by Remark 7.20 to

$$\begin{aligned} 0 \quad = \quad & \mu_3 b_1 - \mu_8 b_3 - \mu_9 b_3^\dagger + \mu_{11} b_1^\dagger - \mu_{13} b_1 b_1^\dagger b_3 - \mu_{19} b_1 b_1^\dagger b_3^\dagger + \mu_{28} b_3 b_3^\dagger b_2 + \mu_{30} b_3 b_3^\dagger b_1^\dagger - \\ & \overline{\mu}_3' b_1^\dagger + \overline{\mu}_8' b_3^\dagger + \overline{\mu}_9' b_3 - \overline{\mu}_{11}' b_1 + \overline{\mu}_{18}' b_3^\dagger b_1 b_1^\dagger - \overline{\mu}_{28}' b_2^\dagger b_3 b_3^\dagger - \overline{\mu}_{30}' b_3 b_3^\dagger b_1. \end{aligned}$$

Thus, from Remark 7.1 we infer

$$\begin{aligned} 0 \quad &= \quad \mu_3 - \overline{\mu}_{11}' = -\mu_8 + \overline{\mu}_9' = \mu_9 - \overline{\mu}_8' = \mu_{11} - \overline{\mu}_3' \\ &= \quad -\mu_{13} = -\mu_{19} + \overline{\mu}_{18}' = \mu_{28} = \overline{\mu}_{30}' = \overline{\mu}_{28}' = \mu_{30}. \end{aligned} \tag{7.3.12}$$

From the equalities (7.3.9), (7.3.10), (7.3.8), (7.3.6), we deduce

$$-\lambda_{27} \;=\; \kappa_{23}' \;=\; \kappa_{25}' \;=\; \kappa_{23} \;=\; \lambda_{27}' \;=\; \lambda_{15} \;=\; -\lambda_{24} = 0, \tag{7.3.13}$$

Furthermore, by (7.3.11), (7.3.4), we obtain

$$\kappa_3' \;=\; \lambda_{23}' \;=\; \mu_{19} \;=\; 0. \tag{7.3.14}$$

That together with (7.3.11) and (7.3.6), allows us to find the zeros

$$\mu_{18}' \;=\; \kappa_{25} \;=\; 0,$$

of F_3, G_2. The equality with adjoints infered from Proposition 6.16

$$\left[F_2, b_3^\dagger\right] + \left[G_3^\dagger, b_2\right] = 0,$$

give us

$$0 = \mu_5 b_1 + b_2\mu_8 - \mu_{10} b_3^\dagger + \mu_{14} b_2^\dagger - \overline{\kappa}_3' b_1^\dagger + \overline{\kappa}_7' b_2 + \overline{\kappa_8}' b_3^\dagger - \overline{\kappa_{11}}' b_1 + \overline{\kappa}_{18}' b_1 b_1^\dagger b_3^\dagger,$$

which implies by Remark 7.1 the following

$$\mu_5 - \overline{\kappa}_{11}' = \mu_8 + \overline{\kappa}_7' = -\mu_{10} + \overline{\kappa}_8' = \mu_{14} = \kappa_3' = \kappa_{18}' = 0. \tag{7.3.15}$$

The identities (7.3.15), (7.3.4), (7.3.6) and (7.3.9) lead to

$$\lambda_{16} = \kappa_{12} = \lambda_{15} = 0, \tag{7.3.16}$$

Doing the same as before with F_3, $G_3^\dagger$ in Proposition 6.16, we consider

$$\left[F_3, b_3^\dagger\right] + \left[G_3^\dagger, b_3\right] = 0,$$

and by Remark 7.20 we conclude

$$0 = \kappa_5 b_1 + \kappa_8 b_2 + \kappa_{13} b_1^\dagger + \kappa_{15} b_2^\dagger + \kappa_{25} b_2 b_2^\dagger b_1^\dagger - \overline{\kappa}_5' b_1^\dagger - \overline{\kappa}_8' b_2^\dagger - \overline{\kappa}_{13}' b_1 - \overline{\kappa}_{15}' b_2.$$

As a result of this, we find, by Remark, 7.1 the following zero terms for the even coefficients F_3 and G_3

$$\kappa_5 - \overline{\kappa}_{13}' = \kappa_8 - \overline{\kappa}_{15}' = \kappa_{13} - \overline{\kappa}_5' = \kappa_{15} - \overline{\kappa}_8' = \kappa_{25} = 0. \tag{7.3.17}$$

By solving one of the conditions of Proposition 7.1 for F_2, $G_1^\dagger$

$$\left[F_2, b_1^\dagger\right] + \left[G_1^\dagger, b_2\right] = 0$$

and Remark 7.20 we have

$$0 = -\mu_2 b_1^\dagger - \mu_3 b_2 - \mu_5 b_3 - \mu_6 b_3^\dagger + \overline{\lambda}_7' b_2 + \overline{\lambda}_8' b_3^\dagger + \overline{\lambda}_9' b_3 - \overline{\lambda}_{28}' b_3 b_3^\dagger b_1^\dagger.$$

This gives us the following terms

$$-\mu_2 - \overline{\lambda}_3' = -\mu_3 + \overline{\lambda}_7' = -\mu_5 + \overline{\lambda}_9' = -\mu_6 + \overline{\lambda}_8' = -\overline{\lambda}_{28}' = 0 \tag{7.3.18}$$

of the even coefficients, that are zero by Remark 7.1. Remark 7.20 implies the identities

$$\begin{aligned}
\left[\kappa_2 b_1 b_1^\dagger, b_1^\dagger\right] &= -\kappa_2 b_1^\dagger, & \left[\overline{\lambda}_5' b_3^\dagger b_1^\dagger, b_3\right] &= -\overline{\lambda}_5' b_1^\dagger,\\
\left[\kappa_3 b_1 b_2, b_1^\dagger\right] &= -\kappa_3 b_2, & \left[\overline{\lambda}_8' b_3^\dagger b_2^\dagger, b_3\right] &= -\overline{\lambda}_8' b_2^\dagger,\\
\left[\kappa_4 b_1 b_2^\dagger, b_1^\dagger\right] &= -\kappa_4 b_2^\dagger, & \left[\overline{\lambda}_{10}' b_3 b_3^\dagger, b_3\right] &= \overline{\lambda}_{10}' b_3,\\
\left[\kappa_5 b_1 b_3, b_1^\dagger\right] &= -\kappa_5 b_3, & \left[\overline{\lambda}_{15} b_3^\dagger b_2, b_3\right] &= -\overline{\lambda}_{15} b_2,
\end{aligned}$$

and

$$\left[\kappa_{17}b_1b_1^\dagger b_2b_2^\dagger, b_1^\dagger\right] \;=\; -\kappa_{17}b_1^\dagger b_2b_2^\dagger,$$

which by the condition

$$\left[F_3, b_1^\dagger\right] + \left[G_1^\dagger, b_3\right] \;=\; 0,$$

obtained from Proposition 6.16 allows us to deduce

$$-\kappa_2 - \overline{\lambda}_5' \;=\; -\kappa_3 - \overline{\lambda}_{15}' \;=\; -\kappa_4 - \overline{\lambda}_8' \;=\; -\kappa_5 + \overline{\lambda}_{10}' \;=\; -\kappa_{17} \;=\; 0. \tag{7.3.19}$$

Then Remark 7.20 has as a consequence

$$\begin{aligned}
\left[\kappa_3b_1b_2, b_2^\dagger\right] &= \kappa_3b_1, & \left[\overline{\mu}_5'b_3^\dagger b_1^\dagger, b_3\right] &= -\overline{\mu}_5'b_1^\dagger,\\
\left[\kappa_7b_2b_2^\dagger, b_2^\dagger\right] &= -\kappa_7b_2, & \left[\overline{\mu}_8'b_3^\dagger b_2^\dagger, b_3\right] &= -\overline{\mu}_8'b_2^\dagger,\\
\left[\kappa_8b_2b_3, b_2^\dagger\right] &= -\kappa_8b_3, & \left[\overline{\mu}_{10}'b_3b_3^\dagger, b_3\right] &= \overline{\mu}_{10}'b_3,\\
\left[\kappa_{11}b_1^\dagger b_2, b_2^\dagger\right] &= -\kappa_{11}b_1^\dagger, & \left[\overline{\mu}_{13}'b_3^\dagger b_1, b_3\right] &= -\overline{\mu}_{13}'b_1.
\end{aligned}$$

Furthermore,

$$\left[F_3, b_2^\dagger\right] + \left[G_2^\dagger, b_3\right] \;=\; 0,$$

give us by Remark 7.1 that

$$\kappa_3 - \overline{\mu}_{13}' = -\kappa_7 = -\kappa_8 + \overline{\mu}_{10}' = -\kappa_{11} + \overline{\mu}_5' = -\mu_8' = 0. \tag{7.3.20}$$

Lastly, by Remark 7.3.6 we have found the last term that is zero

$$\mu_9 = 0.$$

Finally, by expressing the coefficients in the hirarchical order, $\lambda, \mu, \kappa, \lambda', \mu', \kappa'$ we get the even coefficients. That is, we express F_1 only in terms of λ's, F_2 with λ's and μ's, F_3 just with λ's, μ's and κ's and so on. □

Remark 7.22. *We observe that in contrast with the cases of one and two degrees of freedom, the even coefficients are not trivial. Intuitively, this makes sense as in the case of Fermion diffusions with three degrees of freedom we have 192 terms. On the other hand, we have only 21 consistency conditions. Furthermore, we notice that the amount of terms of the even coefficients increases a lot "faster" than the amount of consistency conditions, which is a good reason to believe that the number of non-zero terms of the even coefficients of a Fermion diffusion with many degrees of freedom is also large. For general Fermion diffusions with more than three degrees of freedom the amount of non-zero terms of the even and odd coefficients should be considerably larger.*

7.3.3 Odd coefficients

Here, we give the general form of odd coefficient but we do not apply the consistency conditions due to the large amount of calculations required.

Remark 7.23. *We observe, that exactly as for the even coefficients, the odd coefficients of a Fermion diffusion with 3 degrees of freedom have 192 terms. On the other hand, there are only 12 consistency conditions for the odd coefficients of Fermion diffusions with three degrees of freedom. Finally, we point out that the computations of this subsection are considerable more detailed than the ones that we did for Fermion diffusions with lower degrees of freedom.*

Remark 7.24. *From Theorem 6.2 we infer that the general form of an odd operator with three degrees of freedom has the following form*

$$\begin{aligned}
H_1 \quad = \quad & \theta'_1 I + \theta'_2 b_1 b_1^\dagger + \theta'_3 b_1 b_2 + \theta'_4 b_1 b_2^\dagger + \theta'_5 b_1 b_3 + \theta'_6 b_1 b_3^\dagger + \theta'_7 b_2 b_2^\dagger + \theta'_8 b_2 b_3 + \\
& \theta'_9 b_2 b_3^\dagger + \theta'_{10} b_3 b_3^\dagger + \theta'_{11} b_1^\dagger b_2 + \theta'_{12} b_1^\dagger b_2^\dagger + \theta'_{13} b_1^\dagger b_3 + \theta'_{14} b_1^\dagger b_3^\dagger + \theta'_{15} b_2^\dagger b_3 + \\
& \theta'_{16} b_2^\dagger b_3^\dagger + \theta'_{17} b_1 b_1^\dagger b_2 b_2^\dagger + \theta'_{18} b_1 b_1^\dagger b_2 b_3 + \theta'_{19} b_1 b_1^\dagger b_2 b_3^\dagger + \theta'_{20} b_1 b_1^\dagger b_3 b_3^\dagger + \\
& \theta'_{21} b_1 b_1^\dagger b_2^\dagger b_3 + \theta'_{22} b_1 b_1^\dagger b_2^\dagger b_3^\dagger + \theta'_{23} b_2 b_2^\dagger b_1 b_3 + \theta'_{24} b_2 b_2^\dagger b_1 b_3^\dagger + \theta'_{25} b_2 b_2^\dagger b_1^\dagger b_3 + \\
& \theta'_{26} b_2 b_2^\dagger b_1^\dagger b_3^\dagger + \theta'_{27} b_2 b_2^\dagger b_3 b_3^\dagger + \theta'_{28} b_3 b_3^\dagger b_1 b_2 + \theta'_{29} b_3 b_3^\dagger b_1 b_2^\dagger + \theta'_{30} b_3 b_3^\dagger b_1^\dagger b_2 + \\
& \theta'_{31} b_3 b_3^\dagger b_1^\dagger b_2^\dagger + \theta'_{32} b_1 b_1^\dagger b_2 b_2^\dagger b_3 b_3^\dagger + \theta'_{33} b_1 + \theta'_{34} b_1^\dagger + \theta'_{35} b_2 + \theta'_{36} b_2^\dagger \theta'_{37} b_3 + \theta'_{38} b_3^\dagger + \\
& \theta'_{39} b_1 b_1^\dagger b_2 + \theta'_{40} b_1 b_1^\dagger b_2^\dagger + \theta'_{41} b_1 b_1^\dagger b_3 + \theta'_{42} b_1 b_1^\dagger b_3^\dagger + \theta'_{43} b_1 b_2 b_2^\dagger + \theta'_{44} b_1 b_2 b_3 + \\
& \theta'_{45} b_1 b_2 b_3^\dagger + \theta'_{46} b_1 b_2^\dagger b_3 + \theta'_{47} b_1 b_2^\dagger b_3^\dagger + \theta'_{48} b_1 b_3 b_3^\dagger + \theta'_{49} b_1^\dagger b_2 b_3 + \theta'_{50} b_1^\dagger b_2 b_3^\dagger + \\
& \theta'_{51} b_1^\dagger b_2^\dagger b_3 + \theta'_{52} b_1^\dagger b_2^\dagger b_3^\dagger + \theta'_{53} b_1^\dagger b_3 b_3^\dagger + \theta'_{54} b_1^\dagger b_3 b_3^\dagger + \theta'_{55} b_2 b_2^\dagger b_3 + \theta'_{56} b_2 b_2^\dagger b_3^\dagger + \\
& \theta'_{57} b_2 b_3 b_3^\dagger + \theta'_{58} b_2^\dagger b_3 b_3^\dagger + \theta'_{59} b_1 b_1^\dagger b_2 b_2^\dagger b_3 + \theta'_{60} b_1 b_1^\dagger b_2 b_2^\dagger b_3^\dagger + \theta'_{61} b_1 b_1^\dagger b_3 b_3^\dagger b_2 + \\
& \theta'_{62} b_1 b_1^\dagger b_3 b_3^\dagger b_2^\dagger + \theta'_{63} b_2 b_2^\dagger b_3 b_3^\dagger b_1 + \theta'_{64} b_2 b_2^\dagger b_3 b_3^\dagger b_1^\dagger,
\end{aligned}$$

$$\begin{aligned}
H_2 \quad = \quad & \theta''_1 I + \theta''_2 b_1 b_1^\dagger + \theta''_3 b_1 b_2 + \theta''_4 b_1 b_2^\dagger + \theta''_5 b_1 b_3 + \theta''_6 b_1 b_3^\dagger + \theta''_7 b_2 b_2^\dagger + \theta''_8 b_2 b_3 + \\
& \theta''_9 b_2 b_3^\dagger + \theta''_{10} b_3 b_3^\dagger + \theta''_{11} b_1^\dagger b_2 + \theta''_{12} b_1^\dagger b_2^\dagger + \theta''_{13} b_1^\dagger b_3 + \theta''_{14} b_1^\dagger b_3^\dagger + \theta''_{15} b_2^\dagger b_3 + \\
& \theta''_{16} b_2^\dagger b_3^\dagger + \theta''_{17} b_1 b_1^\dagger b_2 b_2^\dagger + \theta''_{18} b_1 b_1^\dagger b_2 b_3 + \theta''_{19} b_1 b_1^\dagger b_2 b_3^\dagger + \theta''_{20} b_1 b_1^\dagger b_3 b_3^\dagger + \\
& \theta''_{21} b_1 b_1^\dagger b_2^\dagger b_3 + \theta''_{22} b_1 b_1^\dagger b_2^\dagger b_3^\dagger + \theta''_{23} b_2 b_2^\dagger b_1 b_3 + \theta''_{24} b_2 b_2^\dagger b_1 b_3^\dagger + \theta''_{25} b_2 b_2^\dagger b_1^\dagger b_3 + \\
& \theta''_{26} b_2 b_2^\dagger b_1^\dagger b_3^\dagger + \theta''_{27} b_2 b_2^\dagger b_3 b_3^\dagger + \theta''_{28} b_3 b_3^\dagger b_1 b_2 + \theta''_{29} b_3 b_3^\dagger b_1 b_2^\dagger + \theta''_{30} b_3 b_3^\dagger b_1^\dagger b_2 + \\
& \theta''_{31} b_3 b_3^\dagger b_1^\dagger b_2^\dagger + \theta''_{32} b_1 b_1^\dagger b_2 b_2^\dagger b_3 b_3^\dagger + \theta''_{33} b_1 + \theta''_{34} b_1^\dagger + \theta''_{35} b_2 + \theta''_{36} b_2^\dagger \theta''_{37} b_3 + \theta''_{38} b_3^\dagger + \\
& \theta''_{39} b_1 b_1^\dagger b_2 + \theta''_{40} b_1 b_1^\dagger b_2^\dagger + \theta''_{41} b_1 b_1^\dagger b_3 + \theta''_{42} b_1 b_1^\dagger b_3^\dagger + \theta''_{43} b_1 b_2 b_2^\dagger + \theta''_{44} b_1 b_2 b_3 + \\
& \theta''_{45} b_1 b_2 b_3^\dagger + \theta''_{46} b_1 b_2^\dagger b_3 + \theta''_{47} b_1 b_2^\dagger b_3^\dagger + \theta''_{48} b_1 b_3 b_3^\dagger + \theta''_{49} b_1^\dagger b_2 b_3 + \theta''_{50} b_1^\dagger b_2 b_3^\dagger + \\
& \theta''_{51} b_1^\dagger b_2^\dagger b_3 + \theta''_{52} b_1^\dagger b_2^\dagger b_3^\dagger + \theta''_{53} b_1^\dagger b_3 b_3^\dagger + \theta''_{54} b_1^\dagger b_3 b_3^\dagger + \theta''_{55} b_2 b_2^\dagger b_3 + \theta''_{56} b_2 b_2^\dagger b_3^\dagger + \\
& \theta''_{57} b_2 b_3 b_3^\dagger + \theta''_{58} b_2^\dagger b_3 b_3^\dagger + \theta''_{59} b_1 b_1^\dagger b_2 b_2^\dagger b_3 + \theta''_{60} b_1 b_1^\dagger b_2 b_2^\dagger b_3^\dagger + \theta''_{61} b_1 b_1^\dagger b_3 b_3^\dagger b_2 + \\
& \theta''_{62} b_1 b_1^\dagger b_3 b_3^\dagger b_2^\dagger + \theta''_{63} b_2 b_2^\dagger b_3 b_3^\dagger b_1 + \theta''_{64} b_2 b_2^\dagger b_3 b_3^\dagger b_1^\dagger,
\end{aligned}$$

$$
\begin{aligned}
H_3 \quad = \quad & \theta_1'''I + \theta_2'''b_1b_1^\dagger + \theta_3'''b_1b_2 + \theta_4'''b_1b_2^\dagger + \theta_5'''b_1b_3 + \theta_6'''b_1b_3^\dagger + \theta_7'''b_2b_2^\dagger + \theta_8'''b_2b_3 + \\
& \theta_9'''b_2b_3^\dagger + \theta_{10}'''b_3b_3^\dagger + \theta_{11}'''b_1^\dagger b_2 + \theta_{12}'''b_1^\dagger b_2^\dagger + \theta_{13}'''b_1^\dagger b_3 + \theta_{14}'''b_1^\dagger b_3^\dagger + \theta_{15}'''b_2^\dagger b_3 + \\
& \theta_{16}'''b_2^\dagger b_3^\dagger + \theta_{17}'''b_1b_1^\dagger b_2b_2^\dagger + \theta_{18}'''b_1b_1^\dagger b_2b_3 + \theta_{19}'''b_1b_1^\dagger b_2b_3^\dagger + \theta_{20}'''b_1b_1^\dagger b_3b_3^\dagger + \\
& \theta_{21}'''b_1b_1^\dagger b_2^\dagger b_3 + \theta_{22}'''b_1b_1^\dagger b_2^\dagger b_3^\dagger + \theta_{23}'''b_2b_2^\dagger b_1b_3 + \theta_{24}'''b_2b_2^\dagger b_1b_3^\dagger + \theta_{25}'''b_2b_2^\dagger b_1^\dagger b_3 + \\
& \theta_{26}'''b_2b_2^\dagger b_1^\dagger b_3^\dagger + \theta_{27}'''b_2b_2^\dagger b_3b_3^\dagger + \theta_{28}'''b_3b_3^\dagger b_1b_2 + \theta_{29}'''b_3b_3^\dagger b_1b_2^\dagger + \theta_{30}'''b_3b_3^\dagger b_1^\dagger b_2 + \\
& \theta_{31}'''b_3b_3^\dagger b_1^\dagger b_2^\dagger + \theta_{32}'''b_1b_1^\dagger b_2b_2^\dagger b_3b_3^\dagger + \theta_{33}'''b_1 + \theta_{34}'''b_1^\dagger + \theta_{35}'''b_2 + \theta_{36}'''b_2^\dagger \theta_{37}'''b_3 + \theta_{38}'''b_3^\dagger + \\
& \theta_{39}'''b_1b_1^\dagger b_2 + \theta_{40}'''b_1b_1^\dagger b_2^\dagger + \theta_{41}'''b_1b_1^\dagger b_3 + \theta_{42}'''b_1b_1^\dagger b_3^\dagger + \theta_{43}'''b_1b_2b_2^\dagger + \theta_{44}'''b_1b_2b_3 + \\
& \theta_{45}'''b_1b_2b_3^\dagger + \theta_{46}'''b_1b_2^\dagger b_3 + \theta_{47}'''b_1b_2^\dagger b_3^\dagger + \theta_{48}'''b_1b_3b_3^\dagger + \theta_{49}'''b_1^\dagger b_2b_3 + \theta_{50}'''b_1^\dagger b_2b_3^\dagger + \\
& \theta_{51}'''b_1^\dagger b_2^\dagger b_3 + \theta_{52}'''b_1^\dagger b_2^\dagger b_3^\dagger + \theta_{53}'''b_1^\dagger b_3b_3^\dagger + \theta_{54}'''b_1^\dagger b_3b_3^\dagger + \theta_{55}'''b_2b_2^\dagger b_3 + \theta_{56}'''b_2b_2^\dagger b_3^\dagger + \\
& \theta_{57}'''b_2b_3b_3^\dagger + \theta_{58}'''b_2^\dagger b_3b_3^\dagger + \theta_{59}'''b_1b_1^\dagger b_2b_2^\dagger b_3 + \theta_{60}'''b_1b_1^\dagger b_2b_2^\dagger b_3^\dagger + \theta_{61}'''b_1b_1^\dagger b_3b_3^\dagger b_2 + \\
& \theta_{62}'''b_1b_1^\dagger b_3b_3^\dagger b_2^\dagger + \theta_{63}'''b_2b_2^\dagger b_3b_3^\dagger b_1 + \theta_{64}'''b_2b_2^\dagger b_3b_3^\dagger b_1^\dagger .
\end{aligned}
$$

where $\theta_i', \theta_i'', \theta_i''' \in \mathbb{C}$ *and* $i \in \{1, 2, 3, 4, 5, ..., 64\}$.

8 Some general results

The purpose of this section is mainly to show, what we believe, should be the line of work of this theory. By playing with the different possibilities for the terms of the odd and even coefficients or by imposing conditions derived from the consistency conditions (Propositions 6.15 and 6.15) we get some interesting results.
Here, we find the even coefficients of Fermion diffusions with certain properties. We will like to point out, that it might be that some of the results presented here, could probably be part of a more general theorem. Exactly, as the first and second results are a consequence of the third proposition.

Proposition 8.1. *Let $(B^i(t), B^{i\dagger}(t))$ as in Definition 6.11, let n be even, assume that all the even coefficients are the identity operator multiplied by some constant, that is, for*

$$\lambda_1, ..., \lambda_n, \mu_1, ..., \mu_n \in \mathbb{C}$$

it holds

$$F_1 = \lambda_1 I, ..., F_n = \lambda_n I,\ G_1 = \mu_1 I, ..., G_n = \mu_n I.$$

Furthermore, that the odd coefficients, $H_1, ..., H_n$, are given exactly as in Example 6.9. Then,

$$\lambda_1 = \cdots = \lambda_n = \mu_1 \cdots = \mu_n = 0.$$

Proof. From the second condition of Proposition 6.14 we have

$$\{H_i(t), B^{i\dagger}(t)\} + \{H_i^\dagger(t), B^i(t)\} = -|\lambda_i|^2 - |\mu_i|^2$$

Then, the left handside is zero, since we have

$$(\alpha_1 b_1 * \cdots \alpha_i \hat{b}_i * \cdots * \alpha_n b_n) * b_i^\dagger + b_i^\dagger * (\alpha_1 b_1 * \cdots \alpha_i \hat{b}_i * \cdots * \alpha_n b_n)$$

is equal, by the canonical anticommutation relations, to

$$(-1)^{n-1} b_i^\dagger * (\alpha_1 b_1 * \cdots \alpha_i \hat{b}_i * \cdots * \alpha_n b_n) + b_i^\dagger * (\alpha_1 b_1 * \cdots \alpha_i \hat{b}_i * \cdots * \alpha_n b_n)$$

but since it holds that n is even it follow that $n-1$ is odd and this implies

$$\{H_i, b_i^\dagger\} = 0.$$

Similarily, we can see that $\{H_i^\dagger, b_i\} = 0$, hence we have $-|\lambda_i|^2 - |\mu_i|^2 = 0$. Then, it follows, since both numbers need to be larger or equal to zero, that $\lambda_i = \mu_i = 0$. □

We observe, that by symmetry, we have the following result

Proposition 8.2. *Let $(B^i(t), B^{i\dagger}(t))$ as in Definition 6.11, with n even, assume that all the even coefficients are the identity operator, that is, for $\lambda_1, ..., \lambda_n, \mu_1, ..., \mu_n \in \mathbb{C}$, it holds*

$$F_1 = \lambda_1 I, ..., F_n = \lambda_n I,\ G_1 = \mu_1 I, ..., G_n = \mu_n I.$$

Furthermore, that the odd coefficients, $H_1, ..., H_n$, are given in the following way

$$H_i = \alpha_1 b_1^\dagger * \cdots * \alpha_i \hat{b_i}^\dagger \cdots \alpha_n b_n^\dagger,$$

where $\widehat{b}_i$ means that b_i is to be omitted. Then,

$$\lambda_1 = \cdots = \lambda_n = \mu_1 \cdots = \mu_n = 0.$$

Proof. By symmetry, similar to the proof of Proposition 8.1. □

Furthermore, these results inspire a generalization

Proposition 8.3. *Let* $(B^i(t), B^{i\dagger}(t))$ *as in Definition 6.11, with* $i \in \{1, ..., n\}$*, assume that all the even coefficients are the identity operator, that is, for* $\lambda_1, ..., \lambda_n, \mu_1, ..., \mu_n \in \mathbb{C}$*, it holds*

$$F_1 = \lambda_1 I, ..., F_n = \lambda_n I,\ G_1 = \mu_1 I, ..., G_n = \mu_n I.$$

Furthermore, that the odd coefficients, $H_1, ..., H_n$*, satisfy*

$$\{H_i(t), B_i^\dagger(t)\} + \{H_i^\dagger(t), B_i(t)\} = 0$$

Then,

$$\lambda_1 = \cdots = \lambda_n = \mu_1 \cdots = \mu_n = 0.$$

Proof. It follows by equating the condition of the hypothesis to $-|\lambda_i|^2 - |\mu_i|^2$ □

By considering even coefficients with the same term, we can find a similar result

Proposition 8.4. *Let* $(B^i(t), B^{i\dagger}(t))$ *as in Definition 6.11, with* $i \in \{1, ..., n\}$ *and with* n *larger or equal than two, assume that all the even coefficients are the linear combination of* I *and* $b_i b_i^\dagger$*. Let* $\lambda_1, ..., \lambda_n, \lambda'_1, ..., \lambda'_n \in \mathbb{C}$*. Furthermore, we assume that they have the same terms, that is*

$$F_1 = \lambda_1 I + \lambda'_1 b_1 b_1^\dagger, ..., F_n = \lambda_n I + \lambda'_n b_n b_n^\dagger,\ G_1 = \lambda_1 I + \lambda'_n b_n b_n^\dagger, ..., G_n = \lambda_n I + \lambda'_n b_n b_n^\dagger.$$

Additionally, that the odd coefficients, $H_1, ..., H_n$*, satisfy*

$$\{H_i(t), B_i^\dagger(t)\} + \{H_i^\dagger(t), B_i(t)\} = 0$$

Then,

$$\lambda_1 = \cdots = \lambda_n = \lambda'_1 \cdots = \lambda'_n = 0.$$

Proof. From the third equality of Proposition 6.16 and the hypothesis, we deduce the following equation

$$0 = -(\lambda_i I + \lambda'_i b_i b_i^\dagger)(\overline{\lambda}_i I + \overline{\lambda}'_i b_i b_i^\dagger) - (\lambda_i + \lambda'^\dagger b_i b_i^\dagger)(\overline{\lambda}_i I - \overline{\lambda}'_i b_i b_i^\dagger).$$

Hence,

$$0 = (\lambda_i I + \lambda'_i b_i b_i^\dagger)(\overline{\lambda}_i I + \overline{\lambda}'_i b_i b_i^\dagger),$$

which implies, by the canonical anticommutation relations, that

$$0 = |\lambda_i|^2 I + \overline{\lambda}_i \lambda'_i b_i b_i^\dagger + \lambda_i \overline{\lambda}'_i b_i b_i^\dagger + |\lambda'_i|^2 b_i b_i^\dagger.$$

Therefore, by Remark 7.1 we deduce

$$\lambda_i = \overline{\lambda}_i \lambda'_i + \lambda_i \overline{\lambda}_i + |\lambda'_i|^2 = 0,$$

this last equation being equivalent to

$$|\lambda'_i|^2 = 0,$$

and we conclude the proof. □

Remark 8.5. *We notice, by considering the odd operators of Example 6.9, that for $i \neq j$, we have by the third consistency condition given by Proposition 6.16 and that by Proposition 8.1 (assuming these hypothesis) we have*

$$\{H_i, b_j^\dagger\} + \{H_j^\dagger, b_i\} = 0 \tag{8.0.1}$$

and by the third property of Proposition 6.15 we have

$$\{H_i, b_j\} + \{H_j, b_i\} = 0.$$

The left handside of the last equality can be seen to be zero, without considering Proposition 8.1. On the other hand, (8.0.1) *implies for the case of a Fermion diffusion with two degrees of freedom, the following*

$$l^2 b_2 b_2^\dagger + l^2 b_2^\dagger b_2 + l^2 b_2 b_2^\dagger + l^2 b_2^\dagger b_2 = 0,$$

which implies

$$l = 0,$$

meaning that, the odd coefficients can only be zero. Which lead us to conjecture that, by assuming

$$\{H_i(t), B_i^\dagger(t)\} + \{H_i^\dagger(t), B_i(t)\} = 0$$

and that the even operators are a multiple of the identity, that the odd operators need to be zero. Contradicting the hypothesis of a Fermion diffusion and showing that no such Fermion diffusion can exist as the zero operator can not satisfy the canonical anticommutation relations.

Proposition 8.6. *Considering the operators of Example 6.9 and the hypothesis of Proposition 8.1 we get that there can not be a Fermion diffusion satisfying the hypothesis of Proposition 8.1*

sketch. By Proposition 8.1 we have

$$\{H_i, b_j^\dagger\} + \{H_j^\dagger, b_i\} = 0$$

but

$$\alpha_i(b_1 * \cdots * b_j^\dagger b_j * \cdots \widehat{b_i} * \cdots * b_n)$$

has the same sign as

$$\alpha_i(b_1 * \cdots * \alpha_j b_j b_j^\dagger * \cdots * \widehat{b_i} * \cdots * b_n)$$

which implies by the canonical anticommutation relations the following

$$\alpha_i(b_1 * \cdots * b_j b_j^\dagger * \cdots \hat{b}_i * \cdots * b_n = 0$$

that only can happen if $\alpha_i = 0$. $\square$

9 Conclusion

From equations (6.3.6) and 6.3.7 we know that as the number of degrees of freedom of a Fermion diffusion, with n-degrees of freedom, increases, the number of the consistency conditions of the odd and even coefficients grow at a pace of $\mathcal{O}(n^2)$. On the other hand, we infer from Theorem 6.2 that, in general, the coefficients of a Fermion diffusion with n-degrees of freedom have 2^{2n} terms and the odd coefficients have $2^{2n-1} + 2^{2n-1}$ in total. As we could corroborate in the case of three degrees of freedom (Subsection 7.3), the amount of non-zero terms of the even coefficients increases considerably in comparison with the cases of one and two degrees of freedom. In this sense, the results presented here seem to only be useful to find the zeros of non-general coefficients with only a small amount of non-zero terms or with certain properties imposed on the coefficients. For instance, we could calculate Fermion diffusions with 4 degrees of freedom such that the even operators are the identity and the odd operators have only 2 or three non-zero terms. Moreover, we can apply the consistency conditions to Fermion diffusions that satisfy certain imposed conditions, such as having only trivial even coefficients, as we did in Section 8. In this sense, by using combinatorial arguments and studying "families" of Fermion diffusions rater than general ones, it might be possible to get a lot of information on the nature of these quantum differential equations.

Appendices

A C*-algebras

Here, we make a short review of C*-algebras. We shall use the concepts presented here in order to define Fermion diffusions (Definition 6.11). We use the definition of a C*-algebra several times throughout this work (*e.g.* Theorem 6.2). We take the definitions and concepts from [28].

Definition A.1. *Let A be an algebra and let $x, y, z \in A$. Then A is said to be* ***associative*** *if*

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

where $\cdot$ denotes the multiplication.

Definition A.2. *A Banach algebra is an associative algebra A over the real or complex numbers that is also a normed space and complete in the metric induced by the norm and such that the norm satisfies*

$$\|xy\| \leq \|x\| \|y\|,$$

for alll $x, y \in A$.

Definition A.3. *A C^*-algebra, A, is a Banach algebra over the field of complex numbers, together with a map $* : A \longrightarrow A$, such that, for all $x, y \in A$,*

(i) $(x)^* := x^*$

(ii) $(xy)^* := y^* x^*$

(iii) $(x + y)^* = x^* + y^*$

(iv) $(x^*)^* := x$

(v) for every $\lambda \in \mathbb{C}$ it holds

$$(\lambda x)^* := \overline{\lambda} x^*.$$

(vi) the C^* property holds

$$\|x^* x\| = \|x\| \, \|x^*\| \, .$$

Example A.4. *The bounded operators $\mathcal{B}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ with the *-map given by the adjoint $\dagger$.*

Definition A.5. *A bounded linear map $\pi : A \longrightarrow B$, between C^*-algebras A and B is called a *-homomorphism if*

(i) for x and y in A

$$\pi(xy) = \pi(x)\pi(y),$$

(ii) for $x \in A$

$$\pi(x^*) = \pi(x)^*. \tag{A.0.1}$$

The following definition is a formalization of the concept of quantum sothcastic processes and it is taken from [6]

Definition A.6. *Let $\mathcal{B}$ be a C^*-algebra with identity and let T be a set; a stochastic process over $\mathcal{B}$ indexed by T is a triple $(\mathcal{A}, \{j_t : t \in T\}, \omega)$ where $\mathcal{A}$ is a C^*-algebra with identity and, for each $t \in T$, j_t is a *-homomorphism of $\mathcal{B}$ into $\mathcal{A}$ with $j_t(1_{\mathcal{B}}) = 1_{\mathcal{A}}$, $\mathcal{A}$ is generated by the image algebras $\{\mathcal{A}_t = j_t(\mathcal{B}) : t \in T\}$ and ω is a state on $\mathcal{A}$.*

B Quantum probability

Quantum probability is a non-commutative generalization of probability theory, where the role of random variables is played by self-adjoint operators acting on a Hilbert space. Probability measures are replaced by normal states, i.e., positive weakly continous linear functionals on a von Neuman algebra and such that $\omega(I) = 1$.[31] This section summarizes some results and concepts from [4] and [19].

Definition B.1. *Let $\mathcal{H}$ be a finite dimensional Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle$, and $A : \mathcal{H} \longrightarrow \mathcal{H}$ an operator. Then, if $\langle u, Av \rangle = \langle Au,\ v \rangle$ for all $u, v \in \mathcal{H}$, then we say that A is a self-adjoint operator and we write $A = A^*$.*

Definition B.2. *We call a pair $(\mathcal{H}, \rho)$ a **quantum probability space** if $\mathcal{H}$ is a Hilbert space and $\rho : \mathcal{H} \longrightarrow \mathcal{H}$, satisfies $\mathrm{Tr}(\rho) = 1$ and $\langle u, \rho u \rangle > 0$ for all $u \neq 0$ in $\mathcal{H}$. We call any operator satisfying the previous conditions a **state**.*

Definition B.3. *For a quantum probability space $(\mathcal{H}, \rho)$ we call the **expected value** of a self-adjoint operator A with respect to the state ρ the trace of the product of the state ρ with A and we denote it by*

$$\mathbb{E}(A) = \mathrm{Tr}(\rho A).$$

Definition B.4. *The law of a self-adjoint operator A is the measure μ on the spectrum $\sigma(A)$ given by*

$$\mu = \mathrm{Tr}(\rho f(A))$$

for all bounded measurable functions on $\sigma(A)$.

The following results are standard, the proofs are therefore omitted.

Theorem B.5. *If $\mathcal{H}$ is a Hilbert space of finite dimension and A is Hermitian, then there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. Each eigenvalue is real.*

Theorem B.6. *If we let $(\varphi_i)_{i=1,\ldots,n}$ be an orthonormal base of eigenvectors of a finite dimensional Hilbert Space $\mathcal{H}$ corresponding to eigenvalues λ_i of a self-adjoint operator A. Then we have*

$$A = \sum_{i=0}^{n} \lambda_i |\varphi_i\rangle\langle\varphi_i|,$$

where the operator $|u\rangle\langle v|$ is defined as $|u\rangle\langle v|\varphi = \langle v, \varphi\rangle u$.

Remark B.7. *If u is any vector in a Hilbert space $\mathcal{H}$ then we put $\Omega = \{1, ...n\}$, $X(i) = \lambda_i$ and $\mu(i) = \langle \varphi_i, u \rangle$. Then we get*

$$\mathbb{E}(X) = \sum_i \lambda_i \mu(i) = \sum_i \lambda_i \langle u|\varphi_i\rangle\langle\varphi_i|u\rangle = \langle u, Au\rangle,$$

[31] See [4].

hence we see that the expected value of a random variable with probability measure μ coincides with the expected value of a self-adjoint operator. More generally, if f is any bounded function, then we have that

$$\mathbb{E}(f(X)) = \sum_i f(\lambda_i)\mu(i) = \sum_i f(\lambda_i)\langle u|\varphi_i\rangle\langle\varphi_i|u\rangle = \langle u, f(A)u\rangle.$$

Here the operator $f(A)$ is defined as $f(A)\ \varphi_i = f(\lambda_i)\varphi_i$ for all $i \in \{1, ..., n\}$.

Remark B.8. *If we have a collection of elements u_i of a Hilbert space $\mathcal{H}$ such that $\|u_i\| = 1$ and a collection of numbers such that $\sum_i p_i = 1$ then a general state is a convex combination of the previous construction:*

$$\sum_i p_i \langle u_i, f(A)u_i\rangle = \mathrm{Tr}(\rho f(A)),$$

with

$$\rho = \sum_i p_i |u_i\rangle\langle u_i|.$$

States are also known as density matrices.

Definition B.9. *We say that a state is a **pure state** if it has the form $\rho_u =\mid u\rangle\langle u \mid$.*

Theorem B.10. *Pure states are extremals in the convex set of all states $\mathcal{S}(\mathcal{H}) = \{\rho \in \mathcal{L}(\mathcal{H}) | \rho \geq 0, \mathrm{Tr}(\rho) = 1\}$ where $\mathcal{L}(\mathcal{H})$ denotes the linear operators acting on a Hilbert space $\mathcal{H}$.*

Remark B.11. *It is an easy task to define the law of a family of commuting self-adjoint operators, for example if A and B are both commuting observables then their law is given by the measure μ on $\sigma(A) \times \sigma(B)$. But if A and B are non-commuting, it is not clear how to define $f(A, B)$. Take for example $f(x, y) = x^2y^2$, then it clearly holds that $f(x, y) = xyxy$ but it does not necessarily hold that*

$$A^2B^2 = ABAB.$$

We have the more abstract definition

Definition B.12. *A **quantum probability space** is a pair (A, P), where A is a *-algebra and P is a normal state. The orthogonal projections of A are the events in A and $P(p)$ is the probability that the event p occurs.*

B.1 Example of Quantum Probability Spaces

Now, we give the explicit form of certain quantum probability spaces. We prove the following lemma to prove Proposition $(B.14)$,

Lemma B.13. *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be an operator acting on a complex Hilbert space $\mathcal{H}$ such that*

$$\langle Tu, u\rangle = 0$$

for all $u \in \mathcal{H}$. It holds that

$$T = 0.$$

Proof. Let $u, v \in \mathcal{H}$, then

$$0 = \langle T(u+v), (u+v) \rangle = \langle Tu, u \rangle + \langle Tu, v \rangle + \langle Tv, u \rangle + \langle Tv, v \rangle = \langle Tu, v \rangle + \langle Tv, u \rangle.$$

On the other hand, we have that

$$0 = \langle T(u+iv), u+iv \rangle = \langle Tu, u \rangle + \langle Tu, iv \rangle + \langle Tiu, v \rangle + \langle Tiu, Tiu \rangle = \langle Tu, iv \rangle + \langle Tiv, u \rangle.$$

Therefore,

$$\langle Tu, v \rangle = -\langle Tv, u \rangle, \; \langle Tu, v \rangle = \langle Tv, u \rangle,$$

which implies that

$$\langle Tu, v \rangle = 0$$

for all $u, v \in \mathcal{H}$. Therefore, T is the zero operator. □

Proposition B.14. *Consider the Hilbert space of dimension 2:*

$$\mathcal{H} = \operatorname{span}_{\mathbb{C}}(|\, 0 \,\rangle\langle\, 1 \,|).$$

Then, all the quantum probability spaces have the form $(\mathcal{H}, \rho)$ *with*

$$\rho = \begin{bmatrix} \alpha & r \\ \bar{r} & 1-\alpha \end{bmatrix},$$

$\alpha \in [0,1]$, $r \in \mathbb{C}, \alpha(1-\alpha) - |r|^2 \geq 0$.

Proof. Let

$$\rho = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$$

be a state, we observe that the condition $\rho \geq 0$ implies that ρ is self-adjoint. Since $\langle u, \rho u \rangle \in \mathbb{R}$, we have that

$$\langle u, \rho u \rangle = \overline{\langle \rho u, u \rangle} = \langle \rho u, u \rangle = \langle u, \rho^* u \rangle.$$

Then

$$\langle (\rho^* - \rho) u, u \rangle = 0,$$

which by Lemma $(B.13)$ implies that $\rho = \rho^*$, from which we have that $\alpha_2 = \overline{\alpha_3}$. From $\operatorname{Tr}(\rho) = 1$ it follows that $\alpha_4 = 1 - \alpha_1$. On the other side, we have that

$$\left\langle \begin{bmatrix} \alpha_1 & \alpha_2 \\ \overline{\alpha_2} & 1-\alpha_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \geq 0$$

which implies that $\alpha_1 \geq 0$ and from

$$\left\langle \begin{bmatrix} \alpha_1 & \alpha_2 \\ \overline{\alpha_2} & 1-\alpha_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \geq 0$$

we infer that $1 \geq \alpha_1$. Now let λ be an eigenvalue of ρ and let u be an eigenvector corresponding to this eigenvalue, then we have that

$$\langle u, \rho u \rangle = \overline{\lambda} \langle u, u \rangle \geq 0,$$

which implies that $\lambda \geq 0$. On the other hand, we have that

$$\lambda = \frac{1 \pm \sqrt{1 - 4((1-\alpha_1)\alpha_1 - |\alpha_2|^2)}}{2} \geq 0,$$

which in particular implies

$$\alpha_1(1-\alpha_1) - |\alpha_2|^2 \geq 0.$$

Which finishes the proof. □

B.2 Borel Calculus

In this section we make a summary of some results and concepts of operator theory taken from [14], furthermore we make a short review of Borel functional calculus, as given in [19], we mantain the informality of that paper for communicative reasons. Borel functional calculus, roughly speaking, allows to apply a general Borel function to self-adjoint operators.

Definition B.15. *Suppose A is an operator defined on a dense subspace $\mathrm{Dom}(A) = \mathcal{H}$. Let $\mathrm{Dom}(A^*)$ to be the space of all $\phi \in \mathcal{H}$ for which the linear functional*

$$\phi \to \langle \phi, A\psi \rangle,$$

$\phi \in \mathrm{Dom}(A)$ is bounded. For $\phi \in \mathrm{Dom}(A^*)$, define $A^*\phi$ to be the unique vector such that $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$ for all $\psi \in \mathrm{Dom}(A)$.

Definition B.16. *An **unbounded operator** A on a Hilbert space $\mathcal{H}$ is a linear map from a dense subspace $\mathrm{Dom}(A) \subset \mathcal{H}$ into $\mathcal{H}$.*

Definition B.17. *Let A be an unbounded operator on $\mathcal{H}$ and let $\mathrm{Dom}(A) \subset \mathcal{H}$ be the domain of A, then **the graph** of A is the linear manifold $G(A) \subset \mathcal{H} \oplus \mathcal{H}$ defined by*

$$G(A) = \{(u, Au) | u \in \mathrm{Dom}(A)\}.$$

Remark B.18. *A linear functional $\langle \phi, A\cdot \rangle$ is bounded if there exists a constant C such that $|\phi, A\psi| \leq C\|\psi\|$ for all $\psi \in \mathrm{Dom}(A)$. A^* is linear on its domain, and is called the adjoint of A.*

Definition B.19. *An unbounded operator A on H is **self-adjoint** if*

$$\mathrm{Dom}(A^*) = \mathrm{Dom}(A)$$

and $A^*\phi = A\phi$ for all $\phi \in \mathrm{Dom}(A)$.

Definition B.20. *A family $\{P_\Omega\}$ of projections on a Hilbert space $\mathcal{H}$ such that*[32]

[32]see page 235 in [25] for the definition.

(i) each P_Ω *is an orthogonal projection.*

(ii) $P_\emptyset = 0$, $P_{(-a,a)} = I$, *for some a.*

(iii) if $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, *with* $\Omega_n \cap \Omega_m = \emptyset$, *for all¸* $n \neq m$, *then*

$$P_\Omega = s - \lim_{N\to\infty} \left(\sum_{n=1}^{N} P_{\Omega_n} \right)$$

is called a ***bounded projection valued measure*** *(p.v.m.).*[33]

Definition B.21. *An unbounded operator A on $\mathcal{H}$ is said to be* ***closed*** *if the graph of A is a closed subset of $\mathcal{H} \times \mathcal{H}$. An unbounded operator A on $\mathcal{H}$ is said to be* ***closable*** *if the closure in $\mathcal{H} \times \mathcal{H}$ of the graph of A is the graph of a function. If A is closable, then the* ***closure*** A^{cl} *of A is the operator with graph equal to the closure of the graph A.*

Definition B.22. *An unbounded operator A on a Hilbert space $\mathcal{H}$ is* ***symmetric*** *if*

$$\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$$

for all $\phi, \psi \in \mathrm{Dom}(A)$.

Definition B.23. *An unbounded operator A on $\mathcal{H}$ is said to be* ***essentially self-adjoint*** *if A is symmetric and closable and A^{cl} is self-adjoint.*

Proposition B.24. *Suppose μ is a projection valued measure on (X, Ω) with values in $\mathcal{B}(\mathcal{H})$ and $f : X \to \mathbb{C}$ is a measurable function (not necessarily bounded). Define a subspace W_f of $\mathcal{H}$ by*

$$W_f = \left\{ \psi \in \mathcal{H} \,\middle|\, \int_X |f(\lambda)|^2 \mathrm{d}\mu_\psi(\lambda) < \infty \right\}.$$

Then, there exists a unique unbounded operator on $\mathcal{H}$ with domain W_f which is denoted by $\int_X f \mathrm{d}\mu$ with the property that

$$\left\langle \psi, \left(\int_X f \mathrm{d}\mu \right) \psi \right\rangle = \int_X f(\lambda) \mathrm{d}\mu_\psi(\lambda)$$

for all $\psi \in W_f$. This operator satisfies

$$\left\langle \left(\int_X f \mathrm{d}\mu \right) \psi, \left(\int_X f \mathrm{d}\mu \right) \psi \right\rangle$$
$$= \int_x |f|^2 \, \mathrm{d}\mu_\psi.$$

Proof. See Proposition 10.1 in [14]. □

[33] limit taken in the strong sense.

Theorem B.25 (Von Neumann spectral theorem)**.** *Suppose A is a self-adjoint operator on $\mathcal{H}$. Then, there is a unique projection-valued measure μ^A on $\sigma(A)$ with values in $\mathcal{B}(\mathcal{H})$ such that*

$$\int_{\sigma(A)} \lambda \mathrm{d}\mu^A(\lambda) = A$$

where $\sigma(A)$ denotes the spectrum of A.

Proof. See in [19]. □

Definition B.26 (functional calculus)**.** *For any measurable function f on $\sigma(A)$, define a (possible unbounded) operator, denoted $f(A)$, by*

$$f(A) := \int_{\sigma(A)} f(\lambda)\mathrm{d}\mu^A(\lambda).$$

Definition B.27. *If A is a self-adjoint operator on $\mathcal{H}$, then for any unit vector $\psi \in \mathcal{H}$, define a probability measure μ^A_ψ on $\mathbb{R}$ by the formula*

$$\mu^A_\psi(E) := \left\langle \psi, \mu^A(E)\psi \right\rangle.$$

Remark B.28. *For a normal state ω we have that $\omega^A = \omega\xi^A$ is a Borel probability measure on $\mathbb{R}$ and that*

$$\omega(f(A)) = \int_{\sigma(A)} f(a)d\omega^A(a)$$

We call ω^A the ***law of the observable*** *A, where $\sigma(A)$ is the spectrum of the operator A.*

In general, we have

Theorem B.29. *Let $A_1, ..., A_n$ be a commuting family of self-adjoint operators, then there exists a spectral measure $\xi^{A_1,...,A_n}$ on $\mathbb{R}^n$, such that*

$$\langle u, f_1(A_1) \cdots f_n(A_n)u\rangle = \int f_1(a_1) \cdots f_n(a_n)\mathrm{d}\left\langle u, \xi^{A_1 \cdots A_n}\right\rangle$$

for all $u \in \mathcal{H}$ and for any bounded Borel functions $f_1, ..., f_n$ and we call the Borel probability measure $\omega \circ \xi^{A_1,...,A_n}$ on $\mathbb{R}^n$ ***the joint law of the n****-tuple $(A_1, ..., A_n)$.*

Proof. See in [19]. □

Remark B.30. *Therefore, when we consider commuting families of self-adjoint operators, quantum probability coincides with classical probability. But, in case that we have a family of noncommuting self-adjoint operators A and B then, in general, there exists no measure μ on $\mathbb{R}^2$ (in the case of having a family of two observables) such that*

$$\omega(f(A)f(B)) = \int f(a)g(b)d\mu(a,b)$$

for all bounded Borel functions f and g.

C Stone's Theorem

Stone's theorem allows to consider **strongly continuous** (one parameter) **unitary groups** instead of self-adjoint operators. Stone's theorem is extensively used in proofs of results concerning the Weyl operators and hence its importance. This section summarizes some results and concepts of [8].

Definition C.1. *Let $\mathcal{H}$ be a Hilbert space. Let $\{U(t)\}_{t\in\mathbb{R}}$ be a family of unitary operators such that $U(t)U(s) = U(t+s)$ and such that they are **strongly continuous**, i.e. for all $t_0 \in \mathbb{R}, \psi \in \mathcal{H}$:*

$$\lim_{t\to t_0} \|U_t(\psi) - U_{t_0}(\psi)\| = 0.$$

*Then, we call $\{U_t\}_{t\in\mathbb{R}}$ a **strongly continuous** (one parameter) **unitary group**.*

Theorem C.2 (Stone's theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and let $(U_t)_{t\in\mathbb{R}}$ be a strongly continous one-parameter unitary group. Hence, there exists a either a bounded or unbounded self-adjoint operator $A : D_A \to \mathcal{H}$, that is self-adjoint on D_A and such that*

$$U_t = \exp(itA), t \in \mathbb{R}.$$

The domain D_A of A is explicitely given

$$D_A = \left\{ \psi \in \mathcal{H} \,\middle|\, \lim_{t\to 0} \frac{-i}{t}(U_t(\psi) - \psi) \;\; exists \right\}. \tag{C.0.1}$$

Conversely, let $A : D_A \longrightarrow \mathcal{H}$ be either a bounded or unbounded self-adjoint operator on $D_A \subset \mathcal{H}$. Then, the one-parameter family $(U_t)_{t\in\mathbb{R}}$ of unitary operators given by

$$U_t := \exp(itA), t \in \mathbb{R}.$$

is a strongly continuous one-parameter group.

Proof. See 10.14 and 10.15 in [14]. □

Definition C.3. *If $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group on a complex Hilbert space $\mathcal{H}$, then the unique self-adjoint operator A such that $U(t) = \exp(iAt)$ is called the **infinitesimal generator** of $U(t)$ or stone generator.*

Remark C.4. *The **infinitesimal generator** of $(U_t)_{t\in\mathbb{R}}$. may be calculated as*

$$A\psi = -i \lim_{t\to 0} \frac{U_t\psi - \psi}{t},$$

with the domain given in $(C.0.1)$ and with the limit calculated with respect to the norm in the Hilbert space $\mathcal{H}$.

D Realization of the commutation relations

This section summarizes some results and concepts of [14].

Definition D.1. *Let A, B be two operators acting on a Hilbert space $\mathcal{H}$, then we define the* ***commutator operator***

$$[A, B] := AB - BA.$$

Definition D.2. *We say that two operators A, B satisfy the* ***canonical commutation relations*** *if $[A, B] = ihI$, where I denotes the identity operator and h is the Max-Planck constant.*

Example D.3. *Let $\mathcal{H} = L^2(\mathbb{R})$, then the position and momentum operators satisfy the canonical commutation relations.*[34]

Remark D.4. *It can be proven that operators satisfying the canonical commutation relations can't be simultaneously bounded.*[35]

Definition D.5. *We define a* ***representation*** *to be a a group homomorphism U from G to the space $\mathcal{B}(\mathcal{H})$ of bounded operators on some Hilbert space $\mathcal{H}$.*

Definition D.6. *We say that two one-parameter unitary groups $\{U(t)\}_{t\in\mathbb{R}_+}$ and $\{V(t)\}_{t\in\mathbb{R}_+}$ satisfy the* ***Weyl-commutation*** *relations if*

$$U(t)V(s) = \exp(-ist)V(s)U(t)$$

$\forall\ s, t \in \mathbb{R}_+$.

Theorem D.7 (Neuman)**.** *Let $U(t)$ and $V(s)$ be one-parameter, continuous, unitary groups on a separable*[36] *Hilbert space $\mathcal{H}$ satisfying the Weyl relations. Then, there are closed subspaces $\mathcal{H}_t$ so that*

(i) $\mathcal{H} = \oplus_{t=1}^{N} \mathcal{H}_t$, *$N$ a positive integer or infinity,*

(ii) $U(t) : \mathcal{H}_t \to \mathcal{H}_t, V(s) : \mathcal{H}_t \to \mathcal{H}_t\ s, t \in \mathbb{R}$

and for each t, there is a unitary operator $T_t : \mathcal{H}_t \to L^2(\mathbb{R})$ such that $T_t U(t) T_t^{-1}$ is translation to the left by t and $T_t V(s) T_t^{-1}$ is multiplication by $\exp(isx)$.

Corollary D.8. *Let $U(t)$ and $V(s)$ be continuous one-parameter unitary groups satisfying the Weyl relations on a separable Hilbert space $\mathcal{H}$. Let P be the generator of $U(t)$, Q the generator of $V(s)$. Then, there is a dense domain*[37] *$D \subset \mathcal{H}$ so that*

(i) $P : D \to D,\ \ Q : D \to D,$

(ii) $PQ\phi - QP\phi = -i\phi, \quad \text{for all} \quad \phi \in D$

(iii) P and Q are essentially self-adjoint on D.

[34] see Proposition 3.8 in [14]
[35] see Example 2 in page 274 of [25].
[36] see page 275 in [25].
[37] see page 275 in [25].

E General Boson integration

Here, we give a short presentation of a more general approach to the Boson stochastic integration defined in Section 4, in which instead of considering Lebesgue measures we consider measures with bounded variation. This section summarizes some results and concepts of Chapter 24 and 25 of [1].

E.1 Adapted processes

We are interested in operators acting in the Hilbert space (4.1.1) and we shall use here the same notation as in Section 4.

Definition E.1. *Let $\mathcal{H}$ be an arbitrary complex separable Hilbert space and define for any fixed measurable space $(\Omega, \mathcal{F})$ an Ω-**valued observable** ξ as a mapping $\xi : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ satisfying*

(i) $\xi(\Omega) = I$,

(ii) $\xi(\cup_j F_j) = \sum_j \xi(F_j)$ *if $F_i \cap F_j = \emptyset$, where $j = 1, ...$ and the infinite series of projection operators on the right hand side is interpreted as a strongly convergent sum,*

where $\mathcal{P}(\mathcal{H})$ denotes the projection operators acting on a Hilbert space $\mathcal{H}$, and $\mathcal{F}$ is a σ-algebra of subsets of Ω.

Example E.2. *Let $(\Omega, \mathcal{F}, \mu)$ be any σ-finite measure space where $\mathcal{F}$ is countable generated. In the complex separable Hilbert space $L^2(\mu)$ define $\xi^\mu : \mathcal{F} \to \mathcal{P}(\mathcal{H})$ by*

$$(\xi^\mu(E) f)(\omega) = I_E(\omega) f(\omega),\ f \in L^2(\mu)$$

where I_E denotes the indicator function of E.[38]

Remark E.3. *We call a function $\xi : \mathcal{F}_{\mathbb{R}_+} \to \mathcal{P}(\mathcal{H})$ a $\mathbb{R}$-valued observable with no jump points if $\xi(\{t\}) = 0\ \forall t \geqq 0$, where $\mathcal{F}_{\mathbb{R}_+}$ denotes a σ-algebra of subsets of $\mathbb{R}_+$.*

Remark E.4. *We denote as $\mathcal{H}_{t]}$, $\mathcal{H}_{[t,s]}$ and $\mathcal{H}_{[t}$ the range of the projections $\xi([0,t])$, $\xi([s,t])$ and $\xi([\ t, \infty])$, respectively. In general we can decompose a Hilbert space into orthogonal subspaces given by the range of the projections of spectral measures:*

$$\mathcal{H} = \mathcal{H}_{t_1]} \oplus \mathcal{H}_{[t_1,t_2]} \oplus \cdots \oplus \mathcal{H}_{[t_{j-1},t_j]} \oplus \cdots \oplus \mathcal{H}_{[t_{n-1},t_n]} \oplus \cdots \oplus \mathcal{H}_{[t_n},$$

where $0 <\ t_1 < t_2 < ... < t_n < \infty$.

Example E.5. *Given $\mathcal{H} = L^2(\mathbb{R}_+)$ and taking the orthogonal projections*

$$\mathbb{1}_{[0,t_1]}, \mathbb{1}_{[t_1,t_2]}, ..., \mathbb{1}_{[t_n,\infty]}$$

we get the decomposition

$$L^2(\mathbb{R}_+) = L^2([0,t_1]) \oplus L^2([t_1,t_2]) \oplus \cdots \oplus L^2([t_n,\infty))$$

for $0 < t_1 < ... <\ t_n < \infty$.

[38] Example 7.1 in [1].

Definition E.6. *A subset M of a Hilbert space $\mathcal{H}$ is a **linear manifold** if it is closed under adition of vectors and scalar multiplication.*

Remark E.7. *We will write for any $u \in \mathcal{H}$*

$$u_{t]} = \xi([0,t])u, \quad u_{[t} = \xi([t,\infty])u, \quad u_{[s,t]} = \xi([s,t])u.$$

where ξ is a spectral measure and $\mathcal{H}$ is a Hilbert space.

Definition E.8. *Let $D_0 \subseteq \mathcal{H}_0$, $\mathcal{M} \subset \mathcal{H}$ be linear manifolds such that $\xi([s,t])u \in \mathcal{M}$ whenever $u \in \mathcal{M}$ for all $0 \leq\ s < t < \infty$. We denote by $D_0 \underline{\otimes} \xi(\mathcal{M})$ the linear manifold generated by all vectors of the form $f \underline{\otimes} e(u), f \in D_0, u \in \mathcal{M}$. A family $X = \{X_t \mid t \geqq 0\}$ of operators in $\widetilde{\mathcal{H}}$ is called an **adapted process** with respect to the triple $(\xi, D_0, \mathcal{M})$ if*

- *(i)* $D_0 \underline{\otimes} \xi(\mathcal{M}) \subset D(X_t)$.
- *(ii)* $X_t f \otimes e(u_{t]}) \in \widetilde{\mathcal{H}_{t]}}$ *and* $X_t f \otimes e(u) = \{X_t f \otimes e(u_{t]})\} \otimes e(u_{[t})$ *for all* $t \geqq 0$, $u \in \mathcal{M}, f \in D_0$. *It is said to be regular if, in addition, the map $t \longrightarrow X_t f \otimes e(u)$ from $\mathbb{R}_+$ into $\overline{\mathcal{H}}$ is continuous for every $f \in D_0, u \in \mathcal{M}$.*

Definition E.9. *A map $m : t \longrightarrow m_t$ from $\mathbb{R}_+$ into $\mathcal{H}$ is called a ξ-**martingale** if $m_t \in \mathcal{H}_{t]}$ for every t and $\xi([0,t])m_t = m_s$ for all $s < t$.*

Definition E.10. *We define the **Boson creation** $\{A_m^\dagger \mid t \geqq 0\}$ and **annihilation** $\{A_m \mid t \geqq 0\}$ **processes** associated with the ξ-martingale m by*

- *(i)* $D(A_m^\dagger(t)) = D(A_m(t)) = \xi(\mathcal{H})$;
- *(ii)* $A_m^\dagger(t)e(u) = a^\dagger(m_t)e(u) = \left\{\frac{\mathrm{d}}{\mathrm{d}\epsilon} e(u_{t]} + \epsilon m_t) \mid_{t=0}\right\} \otimes e(u_{[t})$;
- *(iii)* $A_m(t)e(u) = a(m_t)e(u) = \left\{\ll m, u \gg ([0,t])e(u_{[t})\right\} \otimes e(u_{[t})$ *for all* $u \in \mathcal{H}$.

where $\ll \cdot, \cdot \gg$ is a complex valued measure in $\mathbb{R}_+$ which has finite variation in every bounded interval, for details see in [1].

Definition E.11. *Let m, m' be two ξ-martingales and define the conservation process associated with the pair (m,m') of ξ-martingales $\Lambda_{|m\rangle\langle m'|}(t)$ in $\Gamma_s(\mathcal{H})$ by putting*

- *(i)* $D(\Lambda_{|m\rangle\langle m'|}(t)) = \mathcal{E}(\mathcal{H})$;
- *(ii)* $\Lambda_{|m\rangle\langle m'|}(t)e(u) = \lambda(| \, m_t\rangle\langle m_t' \, |) \otimes e(u)$.

*where $\lambda(| \, m\rangle\langle m' \, |)$ is the **second differential quantization** given in Definition 3.12.*
Let $H \in \mathcal{B}(\mathcal{H})$ and $H\xi([0,t])H = H_t$ for all t. Define the operators $\Lambda_H(t)$ in $\Gamma_s(\mathcal{H})$ by putting

- *(i)* $D(\Lambda_H(t)) = \mathcal{E}(\mathcal{H})$;
- *(ii)* $\Lambda_H(t)e(u) = \left\{\lambda(H_t)e(u_{t]})\right\} \otimes e(u_{[t})$ *for all $u \in \mathcal{H}$ where $\lambda(H_t)$ is again the **second differential quantization**.*
  We call
  $$\{\Lambda_H(t) \mid t \geqq 0\}$$
  *the **conservation process** in $\Gamma_s(\mathcal{H})$ associated with H.*

Proposition E.12. *The creation $\left\{A_m^\dagger(t) \mid t \geqq 0\right\}$, annihilation $\{A_m(t) \mid t \geqq 0\}$ and the conservation $\{\Lambda_H(t) \mid t \geqq 0\}$ processes are regular adapted processes with respect to $(\xi, \mathcal{H})$.*

Proof. See Examples 24.1 and 24.2 in [1]. □

E.2 Stochastic integration

Now, we define the stochastic integration with respect to annihilation, creation and conservation processes.

Definition E.13. *We call the creation* $\{A^\dagger_m(t) \mid t \geqq 0\}$*, annihilation* $\{A_m(t) \mid t \geqq 0\}$ *and the conservation processes* $\{\Lambda_H(t) \mid t \geqq 0\}$ *the **fundamental processes**.*

Proposition E.14. *Let* M *denote one of the fundamental processes then it holds that*

$$(M_t - M_s)e(u) = e(u_{s]}) \left\{(M_t - M_s)e\left(u_{[s,t]}\right)\right\} e(u_{[t})$$

for all $0 \leq \ s < \infty, u \in \mathcal{H}$.

Proof. Claimed in page 183 in [1]. □

Definition E.15. *Let* L *be an adapted process in* $\mathcal{H}$ *with respect to* $(\xi, D_0, \mathcal{M})$ *and simple as in Definition* (4.12) *and let* M *be one of the processes* $A^\dagger_m, \Lambda_H, A_m$*. Define the linear operators*

$$X_t = \int_0^t LdM = \int_0^t L_s dM_s$$

by putting

(i) $D(X_t) = D_0 \otimes \xi(\mathcal{M})$

(ii) $X_t f \otimes e(u) = \{L_0 f \otimes e(0)\} \otimes M(t)e(u)\ \ if\ 0 \leq t \leq t_1,$
$= X_{t_n} f \otimes e(u) + \{L_{t_n} f \otimes e(u_{t_n})\} \otimes (M(t) - M(t_n))e(u_{t_{[t_n}})$
if $t_n < t \leq t_{n+1}$, $n = 1, 2, ...$

where the right hand side is determined inductively in n *for each* $f \in D_0$, $u \in \mathcal{M}$.

Proposition E.16. $\{X_t \mid \ t \geq 0\}$ *is a regular adapted process with respect to* (ξ, D_0, M).

Proof. See page 183 in [1]. □

Remark E.17. *The correctness of Definition* $(E.15)$ *can be infered from Proposition* (2.16).

Remark E.18. *The stochastic integral can be defined for stochastically integrable functions, see pages 188-190 in [1].*

For this more general setting we have a Boson Ito formula

Theorem E.19. *Let* M_1, M_2 *be fundamental processes in* $\Gamma_s(\mathcal{H})$*. Then,* M_1, M_2 *is a* $(\xi, \mathcal{H})$*-adapted process satisfying the relation*

$$\mathrm{d}M_1 M_2 = M_1 \mathrm{d}M_2 + M_2 \mathrm{d}M_1 \mathrm{d}M_1 \mathrm{d}M_2$$

where $dM_1 dM_2$ *is given in the following table*

	$\mathrm{d}A^\dagger_{m_2}$	$\mathrm{d}\Lambda_{H_2}$	$\mathrm{d}A_{m_2}$
$\mathrm{d}A^\dagger_{m_1}$	*0*	*0*	*0*
$\mathrm{d}\Lambda_{H_1}$	$\mathrm{d}A^\dagger_{H_1 m_2}$	$d\Lambda_{H_1 H_2}$	*0*
$\mathrm{d}A_{m_1}$	$\ll m_1, m_2 \gg$	$\mathrm{d}A_{H_2^* m_1}$	*0*

Proof. See Proposition 25.25 in [1]. □

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List of Symbols

$\underline{\otimes}$	algebraic tensor product, 19
$\{\cdot,\cdot\}$	anticommutator operator, 8
$\widehat{\otimes}$	anticommuting tensor product, 28
$A(t)$	Fermion annihilation process in Chapter 5 and 6, 29, Boson annihilation process in Chapter 4, 22
$A^\dagger(t)$	Fermion creation process, 29 in Chapters 5 and 6, Boson creation process in Chapter 4, 22
$\mathcal{B}(\mathcal{H})$	the bounded operators acting on a Hilbert space $\mathcal{H}$
CAR	canonical anticommutation relations, 9
$[\cdot,\cdot]$	commutator operator, 84
$\dagger$	The adjoint of an operator
$\mathcal{E}$	Boson total vectors in Chapters 2, 3 and 4, 7, Fermion total vectors in Chapter 5 and 6, 27
$\mathcal{E}_+$, $\mathcal{E}_-$	Fermion total vectors with m even, odd, respectively, 27
$e(f)$	coherent or exponential vector associated to f, 6
$\Gamma_a(\mathcal{H})$	asymmetric Fockspace, 5
$\Gamma_s(\mathcal{H})$	symmetric Fock space, 5
$\widetilde{\mathcal{H}}$	The Hilbert space $\mathcal{H}$ tensored with the Hilbert space h_0, 27
$\mathcal{H}_{t]}$, $\mathcal{H}_{[t}$,	square integrable functions in the intervals $[0,t]$, $[t,\infty]$, 26
$\lambda(H)$	the conservation operator associated with H, 18
Ω	Vacuum vector, 5

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