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Flow approach for the stochastic Burgers equation

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Chapter 1

Introduction

This thesis aims to present in a simple way the recent flow approach technique to study singular stochastic partial differential equations originally developed by Paweł Duch in [Duc21] and [Duc22].

As an example to showcase this technique, we will prove the local existence of the one-dimensional stochastic Burgers equations

$$\partial_t f = \Delta f + \partial_x \left(f^2\right) + \partial_x \xi, \tag{1.1}$$

on the space $\mathbb{H} := \mathbb{R} \times \mathbb{T}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the torus and ξ is a space-time white noise, i.e. it is a centred Gaussian distribution such that for every test functions φ and ψ it holds

$$\mathbb{E}\left[\xi\left(\varphi\right)\xi\left(\psi\right)\right] = \int_{\mathbb{H}} \varphi\psi dxdt.$$

Actually, we will prove the existence in a subset of $\mathbb{H}$ of the form $\mathbb{H}_T := [0,T] \times \mathbb{T}$ for a random $T > 0$.

One of the main difficulties in studying this equation, and all the singular equations in general, is the irregularity of the noise. Because of this, we expect the solution to be a distribution rather than a regular function, so it is unclear what the non-linear operation on the right-hand side means.

To solve this, we introduce a regularisation. We take a mollifier $\theta(t,x)$ in $\mathbb{R}^2$, we set $\theta_\kappa(t,x) = \kappa^{-3/2}\theta\left(\frac{t}{\kappa}, \frac{x}{\kappa^{1/2}}\right)$ for any κ in $(0,1]$ and we use this to define the regular function $\xi_\kappa := \theta_\kappa * \xi$. The choice of the scaling in the definition of θ_κ is to take advantage of the parabolic one of the differential operator $\partial_t - \Delta$. With this, we can consider the regularised stochastic Burgers equation with an initial condition constantly equal to 0

$$\partial_t f_\kappa = \Delta f_\kappa + \partial_x \left(f_\kappa^2\right) + \partial_x \xi_\kappa. \tag{1.2}$$

It is straightforward to prove that this equation is well-posed for every positive value of κ. However, we will not rely on this result, as we will construct a solution independently. Now the hope is that, after proving some a priori estimates, one can show that f_κ converges to some distribution f_0 in a suitable Holder norm, and so f_0 can be seen as the solution of equation (1.1).

The Burgers equation and the related KPZ equation (first introduced in [KPZ86]) are classic test cases for all methods aiming to develop a pathwise theory of singular SPDEs. We quickly note that the two main approaches are: regularity structures

[Hai14b], which allows us to obtain a local series expansion using singulars objects as a basis, and paracontrolled distributions [GIP15], which imposes an Ansatz using Bony's paraproduct. Other than these two, we point out that a flow approach method has already been developed by Kupiainen some years ago [Kup16]. The main difference is that Kupianien's flow is discrete, while Duch's is continuous in the parameter μ we will soon introduce.

A notion of solution to the KPZ equation has already been given using the Cole-Hopf transform by Bertini and Giacomin in the seminal paper [BG97]. After that, a solid pathwise solution has been studied in [Hai14a] using rough paths, an embryonic form of regularity structure; in [GP17] using paracontrolled distributions and very recently (while writing this thesis) in [CF24] using the flow approach, also based on Duch's approach.

Now, let us see how the flow approach works. First, let us consider the following mild form of equation (1.2)

$$f_\kappa = G * F_\kappa(f_\kappa), \tag{1.3}$$

where $F_\kappa(\phi) := \mathbb{1}_{t>0}\left(\phi^2 + \xi_\kappa\right)$ and G is the spatial derivative of the fundamental solution of the parabolic differential operator $\partial_t - \Delta$ (see appendix C).

Now we introduce a further regularising parameter μ and define $f_{\kappa,\mu}$ with $\mu \in [0,1]$ such that $f_{\kappa,0} = f_\kappa$. Duch's idea is to get a closed equation for $f_{\kappa,\mu}$ by changing the functional F_κ in equation (1.3).

To be more explicit, as in Duch's articles, we introduce a scale decomposition of G to define $f_{\kappa,\mu}$. To do this, we fix a smooth non-negative cut-off function $\chi \in C^\infty\left(\mathbb{R}_{\geq 0}\right)$ which vanish in $[0,1]$ and is identically equal to 1 in $[2,\infty)$.
With this, we define $G_\mu(t,x) = \chi\left(t/\mu\right) G(t,x)$ for $\mu > 0$, $G_0 = G$ and

$$f_{\kappa,\mu} := G_\mu * F_\kappa(f_\kappa). \tag{1.4}$$

In order to get a closed equation for $f_{\kappa,\mu}$, one might look for some new functionals $F_{\kappa,\mu}$ such that $F_{\kappa,\mu}(f_{\kappa,\mu}) = F_\kappa(f_\kappa)$.

Let us find out which condition guarantees the latter. First, we will derive in μ, so that

$$\begin{aligned}
&\partial_\mu\left(F_{\kappa,\mu}(f_{\kappa,\mu})\right) = 0 \\
&\partial_\mu F_{\kappa,\mu}(f_{\kappa,\mu}) + DF_{\kappa,\mu}(f_{\kappa,\mu})\left[\partial_\mu f_{\kappa,\mu}\right] = 0,
\end{aligned}$$

where $DF_{\kappa,\mu}(\phi)[\psi]$ is the Gâteaux derivative of $F_{\kappa,\mu}$ in ϕ in direction ψ.

To find $\partial_\mu f_{\kappa,\mu}$, we derive in μ equation (1.4) obtaining

$$\partial_\mu f_{\kappa,\mu} = \dot{G}_\mu * F_\kappa(f_\kappa) = \dot{G}_\mu * F_{\kappa,\mu}(f_{\kappa,\mu}),$$

where $\dot{G}_\mu = \partial_\mu G_\mu$.

Now we substitute this expression in the previous one obtaining

$$\partial_\mu F_{\kappa,\mu}(f_{\kappa,\mu}) + DF_{\kappa,\mu}(f_{\kappa,\mu})\left[\dot{G}_\mu * F_{\kappa,\mu}(f_{\kappa,\mu})\right] = 0.$$

If we set

$$H_{\kappa,\mu}(\phi) := \partial_\mu F_{\kappa,\mu}(\phi) + DF_{\kappa,\mu}(\phi)\left[\dot{G}_\mu * F_{\kappa,\mu}(\phi)\right],$$

we might impose $H_{\kappa,\mu}(\phi) = 0$ for each ϕ. This is essentially the flow equation.

This equation might be solvable in several cases (and it is in [Duc21] and [Duc22]). But, as it is suggested in [DGR24], it might be possible to obtain the desired a priori estimate even with an approximate version of the flow equation (see also [Duc23]). This allows more flexibility in the approach and we think using this variation makes the method even easier to understand.

With this generalization, we set

$$F_{\kappa,\mu}(f_{\kappa,\mu}) + R_{\kappa,\mu} = F_\kappa(f_\kappa)$$

for some reminder $R_{\kappa,\mu}$, which will be small in a suitable norm.

Now we can do similar computations as before.

$$\begin{aligned}
&\partial_\mu \left(F_{\kappa,\mu}(f_{\kappa,\mu}) + R_{\kappa,\mu}\right) = 0 \\
&\partial_\mu F_{\kappa,\mu}(f_{\kappa,\mu}) + DF_{\kappa,\mu}(f_{\kappa,\mu}) \left[\partial_\mu f_{\kappa,\mu}\right] + \partial_\mu R_{\kappa,\mu} = 0.
\end{aligned}$$

To calculate $\partial_\mu f_{\kappa,\mu}$ we derive (1.4)

$$\partial_\mu f_{\kappa,\mu} = \dot{G}_\mu * F_\kappa(f_\kappa) = \dot{G}_\mu * \left(F_{\kappa,\mu}(f_{\kappa,\mu}) + R_{\kappa,\mu}\right).$$

So that

$$\begin{aligned}
&\partial_\mu F_{\kappa,\mu}(f_{\kappa,\mu}) + DF_{\kappa,\mu}(f_{\kappa,\mu}) \left[\dot{G}_\mu * \left(F_{\kappa,\mu}(f_{\kappa,\mu}) + R_{\kappa,\mu}\right)\right] + \partial_\mu R_{\kappa,\mu} = 0 \\
&H_{\kappa,\mu}(f_{\kappa,\mu}) + F_{\kappa,\mu}(f_{\kappa,\mu}) \left[\dot{G}_\mu * R_{\kappa,\mu}\right] + \partial_\mu R_{\kappa,\mu} = 0,
\end{aligned}$$

where $H_{\kappa,\mu}$ is defined as before.

Using that $R_{\kappa,0} = 0$, we obtain the system of equations

$$\begin{cases}
f_{\kappa,\mu} = -\int_\mu^1 \dot{G}_\eta * \left(F_{\kappa,\eta}(f_{\kappa,\eta}) + R_{\kappa,\eta}\right) d\eta + f_{\kappa,1}, \\
R_{\kappa,\mu} = -\int_0^\mu \left(H_{\kappa,\eta}(f_{\kappa,\eta}) + DF_{\kappa,\eta}(f_{\kappa,\eta}) \left[\dot{G}_\eta * R_{\kappa,\eta}\right]\right) d\eta.
\end{cases}$$

Since $f_{\kappa,1} \equiv 0$ in $[0,1] \times \mathbb{T}$, we will neglect it as we will restrict to a short time interval. To conclude the study of the equation, we will do a fixed-point argument in chapter 4, which will give us some a priori estimates and allow us to take the limit as μ and κ which goes to 0.

It is time to choose the shape of $F_{\kappa,\mu}$.

We exploit the fact that $F_{\kappa,0}(\phi)$ is polynomial in ϕ imposing the following form for the functional

$$F_{\kappa,\mu}(\phi)(z) = \sum_{i=0}^{\bar{\imath}} \sum_{m=0}^{2i} \int_{\mathbb{M}^m} F_{\kappa,\mu}^{i,m}(z; dz_1, \ldots, dz_m) \prod_{j=1}^m \phi(z_j), \tag{1.5}$$

where $\mathbb{M} = \mathbb{R}^2$, $\bar{\imath}$ is a natural number that we will choose later and $F_{\kappa,\mu}^{i,m}(z; dz_1, \ldots, dz_m)$ are functions on $\mathbb{H}$ with values in the space of measures on $\mathbb{M}^m$. We also ask for the symmetry of these measures under the permutation of their m components.

To understand where this form of the functional comes from, we rewrite it as

$$F_{\kappa,\mu}(\phi)(z) = \sum_{i=0}^{\bar{\imath}} \sum_{m=0}^{2i} \lambda^i \int_{\mathbb{M}^m} F_{\kappa,\mu}^{i,m}(z; dz_1, \ldots, dz_m) \prod_{j=1}^m \phi(z_j)$$

for $\lambda = 1$. In this way $F_{\kappa,\mu}$ is the truncated series expansion in λ of the solution of $H_{\kappa,\mu} \equiv 0$, when in the stochastic Burgers equation (1.2) the nonlinear term is tuned by a factor λ, that is:

$$\partial_t f_\kappa = \Delta f_\kappa + \lambda \partial_x \left(f_\kappa^2 \right) + \partial_x \xi_\kappa.$$

With the above choice, we have $F_{\kappa,0}^{1,2}(z; dz_1, dz_2) = \mathbb{1}_{\mathring{z}>0} \delta_z(dz_1) \delta_z(dz_2)$, $F_{\kappa,0}^{0,0}(z) = \mathbb{1}_{\mathring{z}>0} \xi_k(z)$ where $\mathring{z}$ is the time component of z and $F_{\kappa,0}^{i,m} = 0$ for all the other choices of the couple (i, m). However, to deal with a renormalisation type problem in section 2, we add $F_{\kappa,0}^{i,0}(z) = \mathbb{1}_{\mathring{z}>0} c_\kappa^i(\mathring{z})$ for $1 \le i \le \bar{\imath}$ to the equation, where c_κ^i are some functions on the time variable. This will not change the equation because only the space derivatives of $F_\kappa(\phi)$ play a role, so we can add arbitrary functions which are constant in the space variable.

Now let us calculate the shape of $H_{\kappa,\mu}$. The first term is easy to find

$$\partial_\mu F_{\kappa,\mu}(\phi)(z) = \sum_{i=0}^{\bar{\imath}} \sum_{m=0}^{2i} \lambda^i \int_{\mathbb{M}^m} \partial_\mu F_{\kappa,\mu}^{i,m}(z; dz_1, \dots, dz_m) \prod_{j=1}^{m} \phi(z_j).$$

The other requires some more computations. First, we observe that

$$\begin{aligned} DF_{\kappa,\mu}(\phi)[\psi] &= \partial_\tau F_{\kappa,\mu}(\phi + \tau\psi)_{|\tau=0} \\ &= \sum_{i=0}^{\bar{\imath}} \sum_{m=0}^{2i} \lambda^i \partial_{\tau|\tau=0} \int_{\mathbb{M}^m} F_{\kappa,\mu}^{i,m}(z; dz_1, \dots, dz_m) \prod_{j=1}^{m} (\phi(z_j) + \tau\psi(z_j)) \\ &= \sum_{i=0}^{\bar{\imath}} \sum_{m=1}^{2i} \lambda^i \int_{\mathbb{M}^m} F_{\kappa,\mu}^{i,m}(z; dz_1, \dots, dz_m) \sum_{l=1}^{m} \psi(z_l) \prod_{\substack{1 \le j \le m \\ j \ne l}} \phi(z_j). \end{aligned}$$

Moreover

$$\dot{G}_\mu * F_{\kappa,\mu}(\phi)(z) = \sum_{l=0}^{\bar{\imath}} \sum_{k=0}^{2l} \lambda^l \int_{\mathbb{M}^{k+1}} \dot{G}_\mu(z - y) F_{\kappa,\mu}^{l,k}(y; dy_1, \dots, dy_k) \prod_{j=1}^{k} \phi(y_j) dy.$$

This, combined with the previous equation, leads to

$$\begin{aligned} DF_{\kappa,\mu}(\phi) \left[\dot{G}_\mu * F_{\kappa,\mu}(\phi) \right](z) &= \sum_{i=0}^{\bar{\imath}} \sum_{m=1}^{2i} \lambda^i \int_{\mathbb{M}^m} F_{\kappa,\mu}^{i,m}(z; dz_1, \dots, dz_m) \\ \times \sum_{r=1}^{m} \sum_{l=0}^{\bar{\imath}} \sum_{k=0}^{2l} \lambda^l \int_{\mathbb{M}^{k+1}} &\dot{G}_\mu(z_r - y) F_{\kappa,\mu}^{l,k}(y; dy_1, \dots, dy_k) \prod_{j=1}^{k} \phi(y_j) \prod_{\substack{1 \le j \le m \\ j \ne l}} \phi(z_j) dy. \end{aligned}$$

This proves that $H_{\kappa,\mu}$ has a form similar to that of the functional $F_{\kappa,\mu}$. Indeed

$$H_{\kappa,\mu}(\phi)(z) = \sum_{i=0}^{2\bar{\imath}} \sum_{m=0}^{2i} \lambda^i \int_{\mathbb{M}^m} H_{\kappa,\mu}^{i,m}(z; dz_1, \dots, dz_m) \prod_{j=1}^{m} \phi(z_j), \tag{1.6}$$

with

$$H_{\kappa,\mu}^{i,m}(z; dz_1, \dots, dz_m) = \partial_\mu F_{\kappa,\mu}^{i,m} + \sum_{l=0}^{i} \sum_{j=0}^{m} (j+1) B \left(\dot{G}_\mu, F_{\kappa,\mu}^{l,j+1}, F_{\kappa,\mu}^{i-l,m-j} \right)$$

if $i \leq \bar{\iota}$ and

$$H^{i,m}_{\kappa,\mu}(z; dz_1, \dots, dz_m) = \sum_{l=0}^{i} \sum_{j=0}^{m} (j+1) B\left(\dot{G}_\mu, F^{l,j+1}_{\kappa,\mu}, F^{i-l,m-j}_{\kappa,\mu}\right) \tag{1.7}$$

if $i > \bar{\iota}$. Where the function B is defined as

$$\begin{aligned} &B(G, W, U)(x; dy_1, \dots, dy_m) \\ &\quad = \frac{1}{m!} \sum_{\pi \in \mathcal{P}_m} \int_{\mathbb{M}^2} G(y-z) W(x; dy, dy_{\pi_1}, \dots, dy_{\pi_k}) U(z; dy_{\pi_{k+1}}, \dots, dy_{\pi_m}) dz, \end{aligned} \tag{1.8}$$

where $\mathcal{P}_m$ is the set of the permutation of m elements. We remark that we have averaged over $\mathcal{P}_m$ to obtain a symmetric $H^{i,m}_{\kappa,\mu}$.

As we want $H_{\kappa,\mu}$ to be as small as possible, we impose $H^{i,m}_{\kappa,\mu} = 0$ for each $i \leq \bar{\iota}$. In this way, we obtain the following flow equation for the coefficients of the functionals

$$\partial_\mu F^{i,m}_{\kappa,\mu} = -\sum_{l=0}^{i} \sum_{j=0}^{m} (j+1) B\left(\dot{G}_\mu, F^{l,j+1}_{\kappa,\mu}, F^{i-l,m-j}_{\kappa,\mu}\right). \tag{1.9}$$

Note that these equations together with the condition $F^{i,m}_{\kappa,\mu} = 0$ if $m > 2i$, define inductively all the coefficients of the functional. In fact, we have assigned the values of $F^{i,m}_{\kappa,0}$ and, if we know $F^{l,k}_{\kappa,\mu}$ for all $l < i$ and for all (l,k) such that $l = i$ and $k > m$, then we have

$$\begin{aligned} F^{i,m}_{\kappa,\mu} &= F^{i,m}_{\kappa,0} + \int_0^\mu \partial_\eta F^{i,m}_{\kappa,\eta} d\eta \\ &= F^{i,m}_{\kappa,0} - \int_0^\mu \sum_{l=0}^{i} \sum_{j=0}^{m} (j+1) B\left(\dot{G}_\eta, F^{l,j+1}_{\kappa,\eta}, F^{i-l,m-j}_{\kappa,\eta}\right) d\eta, \end{aligned} \tag{1.10}$$

and now all the terms on the right-hand side are given.

An important property to observe that follows from induction on the above relation is that $F^{i,m}_{\kappa,\mu}(z; dz_1, \dots, dz_m)$ is equal to zero on $\mathring{z} \leq 0$ and on $\mathring{z}_j > \mathring{z}$ for any j where $\mathring{z}_j$ is the time component of z_j.

We will analyse a regularised version of the system to take advantage of the estimate we will prove in chapter 3. To do so, we first introduce the following notation. Given a function $f(x_0, x_1)$ defined on $\mathbb{H}$, we set $\mathbb{1}_{0,T} f(x_0, x_1) := \mathbb{1}_{[0,T]}(x_0) f(x_0, x_1)$. Now we set

$$\begin{aligned} \tilde{F}^T_{\kappa,\mu}(\phi) &= K_\mu * \mathbb{1}_{0,T} F_{\kappa,\mu}(K_\mu * \phi), \\ \tilde{H}^T_{\kappa,\mu}(\phi) &= K_\mu * \mathbb{1}_{0,T} H_{\kappa,\mu}(K_\mu * \phi) \end{aligned}$$

for a suitable convolution kernel K_μ which is defined in (A.1).

Using these new functionals, we can write two closed equations for

$$\tilde{R}_{\kappa,\mu} = K_\mu * R_{\kappa,\mu} \quad \text{and} \quad \tilde{f}_{\kappa,\mu} = P_\mu f_{\kappa,\mu},$$

where P_μ is defined in appendix A.

Indeed, we have

$$\begin{aligned}\tilde{f}_{\kappa,\mu} &= -P_\mu \int_\mu^1 \dot{G}_\eta * \left(P_\eta \tilde{F}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right) + P_\eta \tilde{R}_{\kappa,\eta}\right) d\eta \\ &= -\int_\mu^1 P_\mu P_\eta \dot{G}_\eta * \left(\tilde{F}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right) + \tilde{R}_{\kappa,\eta}\right) d\eta \\ &= -\int_\mu^1 P_\mu K_\eta * P_\eta^2 \dot{G}_\eta * \left(\tilde{F}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right) + \tilde{R}_{\kappa,\eta}\right) d\eta,\end{aligned}$$

where in the last equality we used that $P_\mu K_\mu$ is the Dirac distribution.

Performing computations similar to those we have just done, we obtain the following regularised system

$$\begin{cases} \tilde{f}_{\kappa,\mu} = -\int_\mu^1 P_\mu K_\eta * \tilde{G}_\eta * \left(\tilde{F}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right) + \tilde{R}_{\kappa,\eta}\right) d\eta, \\ \tilde{R}_{\kappa,\mu} = -\int_0^\mu P_\eta K_\mu * \left(\tilde{H}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right) + D\tilde{F}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right)\left[\tilde{G}_\eta * \tilde{R}_{\kappa,\eta}\right]\right) d\eta, \end{cases} \tag{1.11}$$

where $\tilde{G}_\mu = P_\mu^2 \dot{G}_\mu$.

To solve the regularised system (1.11) and to obtain some estimates, we firstly prove some inequalities on $\tilde{F}^T_{\kappa,\mu}(\phi)$. In particular, we will focus on the supremum norm. Note that

$$\begin{aligned}\left\|\tilde{F}^T_{\kappa,\mu}(\phi)\right\|_{L^\infty(\mathbb{H})} &= \left\|K_\mu * \mathbb{1}_{0,T} F_{\kappa,\mu}(K_\mu * \phi)\right\|_{L^\infty(\mathbb{H})} \\ &= \left\|\sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \int_{\mathbb{M}^m} K_\mu * \mathbb{1}_{0,T} F^{i,m}_{\kappa,\mu}(z; dz_1, \ldots, dz_m) \prod_{j=1}^m K_\mu * \phi(z_j)\right\|_{L^\infty(\mathbb{H})} \\ &= \left\|\sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \int_{\mathbb{M}^m} K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F^{i,m}_{\kappa,\mu}(z; z_1, \ldots, z_m) \prod_{j=1}^m \phi(z_j) dz_1 \ldots dz_m\right\|_{L^\infty(\mathbb{H})} \\ &\le \sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F^{i,m}_{\kappa,\mu}\right\|_{L^\infty(\mathbb{H}; L^1(\mathbb{M}^m))} \|\phi\|^m_{L^\infty(\mathbb{H})},\end{aligned} \tag{1.12}$$

where the indicator function always acts on the first component of the first variable. We set $\|\cdot\|_{\mathcal{V}} = \|\cdot\|_{L^\infty(\mathbb{H}; L^1(\mathbb{M}^m))}$ without specifying the m.

It will be the purpose of section 3 to prove some estimates on

$$\left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}}$$

(see theorem 3.3). We will use these in section 4 to conclude the presentation by proving the existence of the solution of the original equation. To achieve such inequality, we will first introduce the cumulants of the coefficients $F^{i,m}_{\kappa,\mu}$ and we will prove a version of the above inequality for these. After that, we will improve it to the desired inequality.

We conclude this chapter emphasizing that thanks to the above argument we have defined $F^{i,m}_{\kappa,\mu}$ for every $\mu \in [0,1]$ and $\kappa \in (0,1]$. However, as we will see at the end of chapter 3, we will define them for $\kappa = 0$ and $\mu \in (0,1]$. Thanks to this, we will solve the system (1.11) for all $\kappa \in [0,1]$ and get a good candidate for f_0.

Chapter 2

Cumulants of the force coefficients

This chapter aims to study the cumulants of the coefficients of the force. As we have seen in the introduction, we need to bound $\left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F_{\kappa,\mu}^{i,m}\right\|_{\mathcal{V}}$. Here, we will prove a version of the estimate that involves the cumulants instead of the coefficient itself, in short, we will have an averaged version of the desired inequality (theorem 2.7). In the next chapter, we will use it to prove the pointwise one.

2.1 Bounds on the cumulants

We start by defining what cumulants are.

Definition 2.1. *Given a vector $(\zeta_1, \dots, \zeta_p)$ of p random variables, we define its cumulant as*

$$\mathbb{E}\left(\zeta_1; \cdots; \zeta_p\right) := (-i)^p \partial_{t_1} \dots \partial_{t_p} \log \mathbb{E}\left[\exp\left(i(t_1\zeta_1 + \cdots + t_p\zeta_p)\right)\right]_{|t_1=\cdots=t_p=0} .$$

We will also indicate it with $\mathbb{E}\left(\zeta_i\right)_I$, where $I = \{1, \dots, p\}$.

The above definition can be extended by duality to the case where $\{\zeta_j\}_j$ are distributions, as in [Duc21]. It is sufficient to impose

$$\mathbb{E}\left(\zeta_1; \cdots; \zeta_p\right)\left(\phi_1 \otimes \cdots \otimes \phi_p\right) = \mathbb{E}\left(\zeta_1(\phi_1); \cdots; \zeta_p(\phi_p)\right).$$

Thanks to the following proposition, the knowledge of some bounds on the cumulants allows us to estimate the moments.

Proposition 2.2. *Let X be a random variable with finite $n-$th moment. Then,*

$$\mathbb{E}\left(X^n\right) = \sum_{l=1}^{n} \sum_{\substack{I_1 \sqcup \dots \sqcup I_l = [n] \\ I_j \neq \varnothing \text{ for each } j}} \prod_{j=1}^{l} \mathbb{E}\left(X\right)_{I_j},$$

where the sum is taken over all partitions of $[n] = \{1, \dots, n\}$.

The proof of the above proposition can be found in [PT11] (Proposition 3.2.1).

Estimating cumulants instead of standard moments has several advantages. First, they are easier to bound when studying a Gaussian field like white noise. Secondly, they satisfy a simple inductive property stated in the following lemma. Thanks to the latter and the flow equation for force coefficients, we can obtain a flow equation for force cumulants.

Lemma 2.3. *Given a family of random variables $(X, Y, \zeta_1, \dots, \zeta_p)$, the following holds*

$$\mathbb{E}\left(XY; \zeta_1; \dots; \zeta_p\right) = \mathbb{E}\left(X; Y; \zeta_1; \dots; \zeta_p\right) + \sum_{I_1 \cup I_2 = \{1, \dots, p\}} \mathbb{E}\left(X; (\zeta_j)_{j \in I_1}\right) \mathbb{E}\left(Y; (\zeta_j)_{j \in I_1}\right).$$

Like the previous proposition, this lemma is contained in Proposition 3.2.1 of [PT11] as it is the third point of the cited result, the only passage we have done is to explicitly state, using the book's notation, all the partitions τ such that $\tau \vee \sigma = \hat{1}$ where $\sigma = \{\{1, 2\}, \{3\}, \dots, \{p + 2\}\}$.

We now generalise the force coefficients for a technical reason that will be clear in the localisation section 2.2.

Definition 2.4. *Given m a natural numbers, we consider $a = \{\mathring{a}_1, \overline{a}_1, \dots, \mathring{a}_m, \overline{a}_m\}$ a vector of $2m$ non-negative integers such that $|a| = \mathring{a}_1 + \overline{a}_1 + \dots + \mathring{a}_m + \overline{a}_m \leq 1$ (i.e. a has only zeroes or one element is equal to 1 and all the others are equal to 0). Now let i be a natural number, $\mu \in [0, 1]$ and $\kappa \in (0, 1]$. Then we set*

$$F^{i,m,a}_{\kappa,\mu}(z, dz_1, \dots, dz_m) = \prod_{l=1}^{m} (\mathring{z} - \mathring{z}_l)^{\mathring{a}_l} (\overline{z} - \overline{z}_l)^{\overline{a}_l} \cdot F^{i,m}_{\kappa,\mu}(z, dz_1, \dots, dz_m),$$

where $\mathring{z}_j$ and $\overline{z}_j$ (resp. $\mathring{z}$ and $\overline{z}$) are the time and the space components of z_j (resp. z).

Moreover, given any such a, we set $[a] = \mathring{a}_1 + \frac{\overline{a}_1}{2} + \dots + \mathring{a}_m + \frac{\overline{a}_m}{2}$.

The asymmetric definition of $[a]$ is justified by the parabolic scaling of the equation, where the time variable counts twice as much as the space variable.

Before going on, we note that even this generalisation of the force coefficients satisfies a flow equation similar to (1.9). In fact, it holds

$$\partial_\mu F^{i,m,a}_{\kappa,\mu} = -\sum_{l=0}^{i} \sum_{j=0}^{m} \sum_{b,c,d \in \mathcal{F}(a)} (j + 1) B\left(\dot{G}^c_\mu, F^{l,j+1,b}_{\kappa,\mu}, F^{i-l,m-j,d}_{\kappa,\mu}\right), \tag{2.1}$$

where the third sum is over some family of triples of vectors (b, c, d) such that b and d are two vectors of the form of the previous definition and $c \in \{(0, 0), (0, 1), (1, 0)\}$ as in appendix C and such that $[b] + [c] + [d] = [a]$. We remark that the definition of $\dot{G}^c_\mu$ is given in appendix C.

We can now introduce the cumulants that we want to bound.

Definition 2.5. *Let I be an index of the form*

$$I = ((i_1, m_1, a^1, s_1, r_1), \dots, (i_n, m_n, a^n, s_n, r_n)),$$

where i_j, m_j are natural numbers, $s_j \in \{0, 1\}$, $r_j \in \{0, 1, 2\}$ and

$$a^j = \left\{\mathring{a}^j_1, \overline{a}^j_1, \dots, \mathring{a}^j_{m_j}, \overline{a}^j_{m_j}\right\} \text{ are such that } |a^j| \leq 1 \text{ for each } j.$$

We define

$$E^I_{\kappa,\mu}(x_1; dy^1_1, \dots, dy^1_{m_1}; \dots; x_n; dy^n_1, \dots, dy^n_{m_n})$$

$$:= \mathbb{E}\left(\partial_\kappa^{r_1}\partial_\mu^{s_1}F_{\kappa,\mu}^{i_1,m_1,a^1}(x_1;dy_1^1,\dots,dy_{m_1}^1);\dots;\partial_\kappa^{r_n}\partial_\mu^{s_n}F_{\kappa,\mu}^{i_n,m_n,a^n}(x_n;dy_1^n,\dots,dy_{m_n}^n)\right).$$

Moreover, we set

$$\begin{aligned} n(I) &= n;\\ i(I) &= i_1+\cdots+i_n;\\ m(I) &= m_1+\cdots+m_n;\\ |a| &= |a^1|+\cdots+|a^n|;\\ [a] &= [a^1]+\cdots+[a^n];\\ s(I) &= s_1+\cdots+s_n;\\ r(I) &= r_1+\cdots+r_n. \end{aligned}$$

In the sequel, we will have to use a generalisation of the norm $\mathcal{V}$ for these kinds of objects. In general, consider $V(x_1;d\mathbf{y}_1;\dots;x_n;d\mathbf{y}_n)$ where $\mathbf{y}_j\in\mathbb{R}^{m_j}$ for some naturals m_j and $x_j\in\mathbb{H}$. Then, without changing the notation, we set

$$\|V\|_{\mathcal{V}} := \sup_{x_1\in\mathbb{H}}\int_{\mathbb{H}^{n-1}}\int_{\mathbb{M}^m}|V(x_1;d\mathbf{y}_1;\dots;x_n;d\mathbf{y}_n)|\,dx_2\dots dx_n,$$

where $m=\sum_{j=1}^n m_j$ and the integrals in $\mathbb{H}^{n-1}$ is taken with respect to the variables $x_2,\dots,x_n$ and the integral in $\mathbb{M}^m$ is taken with respect to the variables $(\mathbf{y}_j)_j$

We can now write the flow equation for the cumulants. In the following theorem, we state the existence of two operators whose expressions are somewhat convoluted. We think it is better not to focus on their specific form, but only on their existence and estimates, which we will see in (2.2) and (2.3).

Theorem 2.6. *There exist two operators A and B such that for every I index of the form of the previous definition such that $s_1=1$, the term $E^I_{\kappa,\mu}$ can be expressed as a sum of terms $A(\dot G^c_\mu,E^K_{\kappa,\mu})$ and $B(\dot G^c_\mu,E^L_{\kappa,\mu},E^M_{\kappa,\mu})$ where K is such that*

$$\begin{aligned} n(K) &= n(I)+1;\\ i(K) &= i(L);\\ m(K) &= m(I)+1;\\ a(K)+[c] &= a(I);\\ s(K) &= s(I)-1;\\ r(K) &= r(I) \end{aligned}$$

and L and M are such that

$$\begin{aligned} n(L)+n(M) &= n(I)+1;\\ i(L)+i(M) &= i(I);\\ m(L)+m(M) &= m(I)+1;\\ a(L)+a(M)+[c] &= a(I);\\ s(L)+s(M) &= s(I)-1; \end{aligned}$$

$$r(L) + r(M) = r(I).$$

Moreover, given $K = ((i_1, m_1, a^1, s_1, r_1), \dots, (i_{n(K)}, m_{n(K)}, a^{n(K)}, s_{n(K)}, r_{n(K)}))$*, the operator*

$$A(\dot{G}^c_\mu, E^K_{\kappa,\mu}) \Big(x_1; dy^1_2, \dots, dy^1_{m_1}, dy^{n(K)}_1, \dots, dy^{n(K)}_{m_{n(K)}}; x_2; dy^2_1, \dots, dy^2_{m_2}; \dots; \\ x_{n(K)-1}; dy^{n(K)-1}_1, \dots, dy^{n(K)-1}_{m_{n(K)-1}} \Big)$$

is given by

$$\int_{\mathbb{M}^2} \dot{G}^c_\mu(y^1_1 - x_{n(K)}) E^K_{\kappa,\mu}(x_1; dy^1_1, \dots, dy^1_{m_1}; \dots; x_{n(K)}; dy^{n(K)}_1, \dots, dy^{n(K)}_{m_{n(K)}}) dy^1_1 dx_{n(K)}.$$

And, given $L = ((i^L_1, m^L_1, a^{L,1}, s^L_1, r^L_1), \dots, (i^L_{n(L)}, m^L_{n(L)}, a^{L,n(L)}, s^L_{n(L)}, r^L_{n(L)}))$ *and* $M = ((i^M_1, m^M_1, a^{M,1}, s^M_1, r^M_1), \dots, (i^M_{n(M)}, m^M_{n(M)}, a^{M,n(M)}, s^M_{n(M)}, r^M_{n(M)}))$*, if we set* $\bar{m} := m^L_1 + m^M_{n(M)} - 1$*, then the operator*

$$B(\dot{G}^c_\mu, E^L_{\kappa,\mu}, E^M_{\kappa,\mu}) \Bigg(x_1; dy^1_1, \dots dy^1_{\bar{m}}; x_2; dy^2_1, \dots dy^2_{m^L_2}; \dots; x_{n(L)}; dy^{n(L)}_1; \dots; dy^{n(L)}_{m^L_{n(L)}}; \\ x_{n(L)+1}; dy^{n(L)+1}_1 \dots dy^{n(L)+1}_{m^M_1}; \dots; x_{n(L)+n(M)-1}; dy^{n(L)+n(M)-1}_1 \dots dy^{n(L)+n(M)-1}_{m^M_{n(M)}} \Bigg)$$

is given by

$$\frac{1}{(\bar{m})!} \sum_{\pi \in \mathcal{P}_{\bar{m}}} \int_{\mathbb{M}^2} \dot{G}^c_\mu(y - x) E^L_{\kappa,\mu} \Bigg(x_1; dy, dy^1_{\pi(1)}, \dots, dy^1_{\pi\left(m^L_1 - 1\right)}; x_2; dy^2_1, \dots, dy^2_{m^L_2}; \dots; \\ x_{n(L)}; dy^{n(L)}_1 \dots; dy^{n(L)}_{m^L_{n(M)}} \Bigg) \\ E^M_{\kappa,\mu} \Bigg(x; dy^1_{\pi\left(m^L_1\right)} \dots dy^1_{\pi(\bar{m})}; x_{n(L)+1}; dy^{n(L)+1}_1, \dots, dy^{n(L)+1}_{m^M_2}; \\ \dots; x_{n(L)+n(M)-1}; dy^{n(L)+n(M)-1}_1 \dots dy^{n(L)+n(M)-1}_{m^M_{n(M)}} \Bigg) dy dx.$$

We remark that the operator B defined in the previous theorem is a generalisation of the one defined in 1.8 and present in the flow equations (1.9) and (2.1).

The proof of the above important result consists only in a cumbersome computation. In fact, it is sufficient to consider the term $\partial^{r_1}_\kappa \partial_\mu F^{i_1, m_1, a^1}_{\kappa,\mu}$ in the first component of $E^I_{\kappa,\mu}$, expand it with relation (2.1) and conclude with lemma 2.3.

The previous theorem is crucial in this strategy. Thanks to it, we will be able to prove the estimate by induction. To do this, we first need to see how the convolution of the kernels and the norm $\mathcal{V}$ behave with the operators defined in the previous theorem. The former is easy to study. Indeed, directly from the definitions of A and B, for any $g \geq 0$ we obtain

$$\tilde{K}^{*g, \otimes n(K)+m(K)-2}_\mu * A\left(\dot{G}^c_\mu, E^M_{\kappa,\mu}\right) = A\left(\tilde{P}^{2g}_\mu \dot{G}^c_\mu, \tilde{K}^{*g, \otimes n(K)+m(K)}_\mu * E^K_{\kappa,\mu}\right)$$

and

$$\tilde{K}_\mu^{*g,\otimes n(L)+n(M)+m(L)+m(M)-2} * B\left(\dot{G}_\mu^c, E_{\kappa,\mu}^L, E_{\kappa,\mu}^M\right) = B\left(\tilde{P}_\mu^{2g}\dot{G}_\mu^c, \tilde{K}_\mu^{*g,\otimes n(L)+m(L)} * E_{\kappa,\mu}^L, \tilde{K}_\mu^{*g,\otimes n(M)+m(M)} * E_{\kappa,\mu}^M\right).$$

For the latter, consider in general a function $\hat{G}$ defined on $\mathbb{M}$ and two functions $V_1\left(x_1^1; d\mathbf{y}_1^1; \ldots; x_{n_1}^1; d\mathbf{y}_{n_1}^1\right)$ and $V_2\left(x_1^2; d\mathbf{y}_1^2; \ldots; x_{n_2}^2; d\mathbf{y}_{n_2}^2\right)$ where $\mathbf{y}_j^l \in \mathbb{R}^{m_j^l}$ for some naturals m_j^l and $x_j^l \in \mathbb{H}$. Then, we immediately get from the definition of B that

$$\left\|B\left(\hat{G}, V_1, V_2\right)\right\|_{\mathcal{V}} \leq \left\|\hat{G}\right\|_{L^1(\mathbb{M})} \|V_1\|_{\mathcal{V}} \|V_2\|_{\mathcal{V}} \tag{2.2}$$

and from the definition of A, using the change of variable formula with translations, we obtain

$$\left\|A\left(\hat{G}, V_1\right)\right\|_{\mathcal{V}} \leq \left\|T\left|\hat{G}\right|\right\|_{L^\infty(\mathbb{H})} \|V_1\|_{\mathcal{V}}, \tag{2.3}$$

where T is the periodisation operator in the space variable (see proposition A.2). The above special treatment of the operator A is needed as the variable which has been called $x_{n(K)}$ on his definition is integrated in $\mathbb{M}$, while for our estimate we would like to integrate it only over $\mathbb{H}$. To solve this, we had to periodise one of the terms in the integral. More details about the above estimates can be found in [Duc21].

Now notice that for any smooth function h defined on $\mathbb{M}$, we have

$$\begin{aligned}
\|Th\|_{L^\infty(\mathbb{H})} &= \left\|T\left(\tilde{K}_\mu * \tilde{P}_\mu h\right)\right\|_{L^\infty(\mathbb{H})} = \sup_{x\in\mathbb{H}}\left|\sum_{y\in\mathbb{Z}} \tilde{K}_\mu * \tilde{P}_\mu h(x+y)\right| \\
&= \sup_{x\in\mathbb{H}}\left|\sum_{y\in\mathbb{Z}} \int_{\mathbb{M}} \tilde{K}_\mu(x+y-z)\tilde{P}_\mu h(z)dz\right| \\
&= \sup_{x\in\mathbb{H}}\left|\int_{\mathbb{M}} T\tilde{K}_\mu(x-z)\tilde{P}_\mu h(z)dz\right| = \left\|T\left(\tilde{K}_\mu\right) * \tilde{P}_\mu h\right\|_{L^\infty(\mathbb{M})} \\
&\leq \left\|T\tilde{K}_\mu\right\|_{L^\infty(\mathbb{H})} \left\|\tilde{P}_\mu h\right\|_{L^1(\mathbb{M})} \lesssim \mu^{-3/2}\left\|\tilde{P}_\mu h\right\|_{L^1(\mathbb{M})}
\end{aligned}$$

thanks to proposition A.2 (F).

Combining all of the above and using proposition C.1 (A), we obtain

$$\begin{aligned}
\left\|\tilde{K}_\mu^{*g,\otimes n(J)+m(J)-2} * A\left(\dot{G}_\mu, E_{\kappa,\mu}^K\right)\right\|_{\mathcal{V}} &\leq \left\|T\left|\tilde{P}^{2g}\dot{G}_\mu^c\right|\right\|_{L^\infty(\mathbb{H})} \left\|\tilde{K}_\mu^{*g,\otimes n(J)+m(J)} * E_{\kappa,\mu}^K\right\|_{\mathcal{V}} \\
&\lesssim \mu^{-2+[c]}\left\|\tilde{K}_\mu^{*g,\otimes n(J)+m(J)} * E_{\kappa,\mu}^K\right\|_{\mathcal{V}}
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&\left\|\tilde{K}_\mu^{*g,\otimes n(L)+n(M)+m(L)+m(M)-2} * B\left(\dot{G}_\mu^c, E_{\kappa,\mu}^L, E_{\kappa,\mu}^M\right)\right\|_{\mathcal{V}} \\
&\quad\lesssim \mu^{-1/2+[c]}\left\|\tilde{K}_\mu^{*g,\otimes n(L)+m(L)} * E_{\kappa,\mu}^L\right\|_{\mathcal{V}} \left\|\tilde{K}_\mu^{*g,\otimes n(M)+m(M)} * E_{\kappa,\mu}^M\right\|_{\mathcal{V}}.
\end{aligned} \tag{2.5}$$

We can finally state and prove the main estimate.

Theorem 2.7. *Let $\epsilon > 0$ be a sufficiently small real number. Then, there exists an integer $g \geq 1$ such that for every index I as before, we have*

$$\left\| \tilde{K}_\mu^{*g,\otimes n(I)+m(I)} * E^I_{\kappa,\mu} \right\|_{\mathcal{V}} \lesssim \kappa^{(\epsilon-1)r(I)} \mu^{\sigma(I)},$$

where $\sigma(I) = \left(\frac{3}{4} - \epsilon\right) n(I) + \left(\frac{1}{4} + \epsilon\right) m(I) + \left(\frac{1}{4} - 2\epsilon\right) i(I) - \frac{3}{2} - s(I) + a(I)$.

The above σ is found by taking $\sigma(I) = b_1 n(I) + b_2 m(I) + b_3 i(I) + b_4 - s(I) + a(I)$, asking for the above inequality for some special values of I (in particular, for some values that exploit our knowledge of some coefficients for $\mu = 0$) and lastly, by imposing that the inequality can be carried on by recursion thanks to theorem 2.6.

An important observation is that some cumulants converge to zero in the above norm as μ goes to zero, those such that $\sigma(I) > 0$. If we restrict ourselves to the case $n(I) = 1$ and $|a| = 0$, we can characterise the force coefficients such that $\sigma((i, m, 0, 0, 0)) > 0$ and those such that $\sigma((i, m, 0, 0, 0)) \leq 0$. The former are called irrelevant, the latter relevant. We will not do this in this remark, as it will be part of the proof of the above theorem. Here, we only anticipate that we will see that the only relevant force coefficients satisfy $i \leq 3$. In particular, since in our heuristic of chapter 1 we can think of $R_{\kappa,\mu}$ as a small remainder for small values of μ, we must include all the relevant terms in the series expansion of $F_{\kappa,\mu}$. Given this, it makes sense to take $\bar{\imath} = 3$ so that the remainder contains only irrelevant terms.

Remark 2.8. *Note that if the above bound holds for a couple (I, g), it also holds with the same I and any $h > g$. In fact, a simple explicit computation that just uses $\left\| \tilde{K}_\mu \right\|_{L^1(\mathbb{M})} = 1$, yields $\left\| \tilde{K}_\mu^{*(g+1),\otimes n(I)+m(I)} * E^I_{\kappa,\mu} \right\|_{\mathcal{V}} \lesssim \left\| \tilde{K}_\mu^{*g,\otimes n(I)+m(I)} * E^I_{\kappa,\mu} \right\|_{\mathcal{V}}$. Thanks to this, we allow ourselves to change the value of g if necessary, taking a higher value.*

Proof. Let us prove the theorem by induction.

First, consider $i(I) = 0$. In this case, to have a non-trivial term, we must have $m(I) = 0$ and $s(I) = 0$ as the only coefficient of the force appearing in this term is $F^{0,0}_{\kappa,\mu}$ which does not depend on μ. Finally, note that as the characteristic function of a Gaussian vector is an exponential of a quadratic function, their cumulants of order 3 or greater are equal to zero. This proves that the only interesting cases are $n = 1$ and $n = 2$. The first one of these is zero as the white noise is centred and so we are left with only $I = ((0, 0, 0, 0, r_1), (0, 0, 0, 0, r_2))$.

Thanks to all these arguments, we can focus on the last term, which is

$$\sup_{z \in \mathbb{H}} \int_{\mathbb{M}} \left| \mathbb{E}\left(\tilde{K}_\mu^{*2} * \mathbb{1}_{0,\infty} [\partial_\kappa^{r_1} \theta_\kappa * \xi(z)] \cdot \tilde{K}_\mu^{*2} * \mathbb{1}_{0,\infty} [\partial_\kappa^{r_2} \theta_\kappa * \xi(y)] \right) \right| dy$$
$$\leq \left\| \partial_\kappa^{r_1} \theta_\kappa * \tilde{K}_\mu^{*2} \right\|_{L^1(\mathbb{M})} \left\| \partial_\kappa^{r_2} \theta_\kappa * \tilde{K}_\mu^{*2} \right\|_{L^1(\mathbb{M})}.$$

It is enough to prove that

$$\left\| \partial_\kappa^{r} \theta_\kappa * \tilde{K}_\mu^{*2} \right\|_{L^1(\mathbb{M})} \lesssim \kappa^{(\hat{\epsilon}-1)r} \mu^{-\epsilon}$$

for any $r \in \{0, 1, 2\}$. If $r = 0$, it holds easily, as the left-hand side is bounded uniformly in κ and μ. Now let us study the case where $r \geq 1$.

If $\mu \leq \kappa$, the above follows from

$$\left\|\partial_\kappa^r \theta_\kappa * \tilde{K}_\mu^{*2}\right\|_{L^1(\mathbb{M})} \leq \|\partial_\kappa^r \theta_\kappa\|_{L^1(\mathbb{M})} \left\|\tilde{K}_\mu^{*2}\right\|_{L^1(\mathbb{M})} \leq \kappa^{-r} \leq \kappa^{(\epsilon-1)r}\mu^{-\epsilon}.$$

Now consider the case $\kappa < \mu$. In this part, let us use (t, x) to indicate the components of $\mathbb{M}$. Let S_κ be the parabolic scaling operator defined by $S_\kappa v(t,x) = \kappa^{-3/2} v\left(t/\kappa, x/\kappa^{1/2}\right)$. Then observe that

$$\partial_\kappa \theta_k = -\partial_t S_\kappa [t\theta] - \frac{1}{2\kappa^{1/2}} \partial_z S_\kappa [x\theta]$$

which, thanks to proposition A.2 (B) and (C), gives

$$\begin{aligned}\left\|\partial_\kappa^r \theta_\kappa * \tilde{K}_\mu^{*2}\right\|_{L^1(\mathbb{M})} &\leq \left\|\partial_t \tilde{K}_\mu\right\|_{TV} \|t\theta\|_{L^1(\mathbb{M})} + \frac{1}{2\kappa^{1/2}} \left\|\partial_x \tilde{K}_\mu\right\|_{L^1(\mathbb{M})} \|x\theta\|_{L^1(\mathbb{M})} \\ &\lesssim \mu^{-1} + \kappa^{-1/2}\mu^{-1/2} \leq \kappa^{\epsilon-1}\mu^{-\epsilon}.\end{aligned}$$

Which solves the case $r = 1$. If $r = 2$, we have to do a similar expansion with the scaling operator. In this case, the two derivatives that appear must be distributed between the two kernels $\tilde{K}_\mu$ so that each has only one derivative.

For the inductive step, let I be an index of the form given in definition 2.5 and assume that the thesis of the theorem holds for all indices J such that $i(J) < i(I)$ or $i(J) = i(I)$ and $m(J) > m(I)$. We divide this into multiple cases.

First, assume $s(I) \neq 0$. We can use theorem 2.6 to bound the term

$$\left\|\tilde{K}_\mu^{*g,\otimes n(I)+m(I)} * E_{\kappa,\mu}^I\right\|_{\mathcal{V}}$$

with terms of the form

$$\left\|\tilde{K}_\mu^{*g,\otimes n(I)+m(I)} * A\left(\dot{G}_\mu, E_{\kappa,\mu}^J\right)\right\|_{\mathcal{V}} \text{ and } \left\|K_\mu^{*g,\otimes n(I)+m(I)} * B\left(\dot{G}_\mu^c, E_{\kappa,\mu}^L, E_{\kappa,\mu}^M\right)\right\|_{\mathcal{V}}.$$

The first can be estimated using the inductive hypothesis and inequality (2.4). Indeed

$$\begin{aligned}\left\|\tilde{K}_\mu^{*g,\otimes n(I)+m(I)} * A\left(\dot{G}_\mu, E_{\kappa,\mu}^J\right)\right\|_{\mathcal{V}} &\lesssim \mu^{-2+[c]} \left\|\tilde{K}_\mu^{*g,\otimes n(J)+m(J)} * E_{\kappa,\mu}^J\right\|_{\mathcal{V}} \\ &\lesssim \mu^{-2+[c]}\kappa^{(\hat{\epsilon}-1)r(J)}\mu^{\sigma(J)} = \kappa^{(\epsilon-1)r(I)}\mu^{\sigma(I)},\end{aligned}$$

where we used the relations given in theorem 2.6 to infer $\sigma(I) = -2 + [c] + \sigma(J)$.

Whereas for the second term, we have

$$\begin{aligned}&\left\|K_\mu^{*g,\otimes n(I)+m(I)} * B\left(\dot{G}_\mu^c, E_{\kappa,\mu}^L, E_{\kappa,\mu}^M\right)\right\|_{\mathcal{V}} \\ &\qquad\lesssim \mu^{-1/2+[c]} \left\|\tilde{K}_\mu^{*g,\otimes n(L)+m(L)} * E_{\kappa,\mu}^L\right\|_{\mathcal{V}} \left\|K_\mu^{*g,\otimes n(M)+m(M)} * E_{\kappa,\mu}^M\right\|_{\mathcal{V}} \\ &\qquad\lesssim \mu^{-1/2+[c]} \cdot \kappa^{(\epsilon-1)r(L)}\mu^{\sigma(L)} \cdot \kappa^{(\epsilon-1)r(M)}\mu^{\sigma(M)} = \kappa^{(\epsilon-1)r(I)}\mu^{\sigma(I)},\end{aligned}$$

where we have used inequality (2.5) and the fact that from theorem 2.6, it holds $r(I) = r(L) + r(M)$ and $\sigma(I) = \sigma(L) + \sigma(M) - 1/2 + [c]$.

We now consider the case where $s(I) = 0$ and divide it into two more subcases. Suppose $\sigma(I) > 0$, i.e. we assume that $E^I_{\kappa,\mu}$ is irrelevant.

Let us first notice that, in this case, $E_{\kappa,0} = 0$. In fact, if $n(I) \geq 2$, then at least one term in the definition of the cumulant $E_{\kappa,0}$ is deterministic, and so it is equal to zero. If $n(I) = 1$, the only terms with $i(I) > 0$ for which this can be non-zero, are $I = ((i, 0, 0, 0, r))$ and $I = ((1, 2, a, 0, r))$.

For the first one, we have $\sigma((i, 0, 0, s, r)) = \frac{3}{4} - \epsilon + i\left(\frac{1}{4} - 2\epsilon\right) - \frac{3}{2}$, which is negative for every $i \leq 3 = \bar{\iota}$.

For the second one, if $|a| = 0$, we have $\sigma((1, 2, 0, s, r)) \leq \frac{3}{4} - \epsilon + 2\left(\frac{1}{4} + \epsilon\right) + \frac{1}{4} - 2\epsilon - \frac{3}{2} = -\epsilon < 0$, while $F^{1,2,a}_{\kappa,\mu} = 0$ by direct inspection if $|a|$ is non-zero as by (1.9) we have $F^{1,2}_{\kappa,\mu}(z; dz_1, dz_2) = \mathbb{1}_{\mathring{z}>0}\delta_z(dz_1)\delta_z(dz_2)$.

To resume all of this, we have proved that if $\sigma(I) > 0$, then $E^I_{\kappa,0} = 0$. Now, given $q \in \{1, \dots, n(I)\}$ let I_q be defined as equal to I except that $s_q = 1$ in I_q while $s_q = 0$ in I. In particular, we already have

$$\left\|\tilde{K}^{*g,\otimes n(I_q)+m(I_q)}_{\mu} * E^{I_q}_{\kappa,\eta}\right\|_{\mathcal{V}} \lesssim \kappa^{(\epsilon-1)\sigma(I_q)}\mu^{\sigma(I_p)} = \kappa^{(\epsilon-1)\sigma(I)}\mu^{\sigma(I)-1}.$$

So it is enough to consider

$$E^I_{\kappa,\mu} = E^I_{\kappa,0} + \sum_{q=1}^{n(I)} \int_0^{\mu} E^{I_q}_{\kappa,\eta} d\eta = \sum_{q=1}^{n(I)} \int_0^{\mu} E^{I_q}_{\kappa,\eta} d\eta$$

and to take the norm to obtain

$$\begin{aligned}
&\left\|\tilde{K}^{*g,\otimes n(I)+m(I)}_{\mu} * E^I_{\kappa,\eta}\right\|_{\mathcal{V}} \\
&\leq \sum_{q=1}^{n(I)} \left\|\tilde{K}^{*g,\otimes n(I)+m(I)}_{\mu} * \int_0^{\mu} E^{I_q}_{\kappa,\eta} d\eta\right\|_{\mathcal{V}} \leq \sum_{q=1}^{n(I)} \int_0^{\mu} \left\|\tilde{K}^{*g,\otimes n(I)+m(I)}_{\mu} * E^{I_q}_{\kappa,\eta}\right\|_{\mathcal{V}} d\eta \\
&= \sum_{q=1}^{n(I)} \int_0^{\mu} \left\|\tilde{P}^g_{\eta}\tilde{K}^{*g,\otimes n(I)+m(I)}_{\mu} * \tilde{K}^{*g,\otimes n(I)+m(I)}_{\eta} * E^{I_q}_{\kappa,\eta}\right\|_{\mathcal{V}} d\eta \\
&\leq \sum_{q=1}^{n(I)} \int_0^{\mu} \left\|\tilde{P}^g_{\eta}\tilde{K}^{*g,\otimes n(I)+m(I)}_{\mu}\right\|_{TV} \left\|\tilde{K}^{*g,\otimes n(I)+m(I)}_{\eta} * E^{I_q}_{\kappa,\eta}\right\|_{\mathcal{V}} d\eta \\
&\leq \sum_{q=1}^{n(I)} \int_0^{\mu} \left\|\tilde{K}^{*g,\otimes n(I)+m(I)}_{\eta} * E^{I_q}_{\kappa,\eta}\right\|_{\mathcal{V}} d\eta \lesssim \sum_{q=1}^{n(I)} \int_0^{\mu} \kappa^{(\epsilon-1)r(I)}\eta^{\sigma(I_q)} d\eta \\
&= \sum_{q=1}^{n(I)} \int_0^{\mu} \kappa^{(\epsilon-1)r(I)}\eta^{\sigma(I)-1} d\eta \lesssim \kappa^{(\epsilon-1)r(I)}\mu^{\sigma(I)},
\end{aligned}$$

where we used proposition A.2 (C).

Finally, we can consider the last case: $\sigma(I) \leq 0$, i.e. $E^I_{\kappa,\mu}$ is relevant.

Note that, if $n(I) \geq 2$, it holds

$$\sigma(I) \geq 2\left(\frac{3}{4} - \epsilon\right) + \left(\frac{1}{4} - 2\epsilon\right) - \frac{3}{2} = \frac{1}{4} - 4\epsilon > 0,$$

because $i(I) \geq 1$. So assume $n(I) = 1$.

If $m(I) \geq 3$, we have

$$\sigma(I) \geq \left(\frac{3}{4} - \epsilon\right) + 3\left(\frac{1}{4} + \epsilon\right) + \left(\frac{1}{4} - 2\epsilon\right) - \frac{3}{2} = \frac{1}{4} > 0.$$

We are left with 3 cases depending on $m(I)$. Let us find out which are the cumulants left to be analysed.

If $m(I) = 2$:

$$\begin{aligned} 0 \geq \sigma(I) &= \left(\frac{3}{4} - \epsilon\right) + 2\left(\frac{1}{4} + \epsilon\right) + \left(\frac{1}{4} - 2\epsilon\right) i(I) - \frac{3}{2} + a(I) \\ &= -\frac{1}{4} + \epsilon + \left(\frac{1}{4} - 2\epsilon\right) i(I) + a(I), \end{aligned}$$

which is true only if $i(I) = 1$ and $|a(I)| = 0$.

If $m(I) = 1$:

$$\begin{aligned} 0 \geq \sigma(I) &= \left(\frac{3}{4} - \epsilon\right) + \left(\frac{1}{4} + \epsilon\right) + \left(\frac{1}{4} - 2\epsilon\right) i(I) - \frac{3}{2} + a(I) \\ &= -\frac{1}{2} + \left(\frac{1}{4} - 2\epsilon\right) i(I) + a(I), \end{aligned}$$

which is true only if $|a(I)| = 0$ and $i(I) \in \{1, 2\}$.

If $m(I) = 0$:

$$0 \geq \sigma(I) = \left(\frac{3}{4} - \epsilon\right) + \left(\frac{1}{4} - 2\epsilon\right) i(I) - \frac{3}{2} = -\frac{3}{4} - \epsilon + \left(\frac{1}{4} - 2\epsilon\right) i(I),$$

which is true only if $1 \leq i(I) \leq \bar{\imath}$. So we are left with the terms with $(i(I), m(I))$ such that $(i(I), m(I)) \in \{(1,0), (2,0), (3,0), (1,1), (2,1), (1,2)\}$. Let us bound them. The case $m(I) = 2$ is easy, as we have already seen that $F^{1,2}_{\kappa,\mu} = \mathbb{1}_{\mathring{z}>0}\delta_z(dz_1)\delta_z(dz_2)$ and so

$$\left\| \tilde{K}^{*g,\otimes 3}_{\mu} * F^{1,2}_{\kappa,\mu} \right\|_{\mathcal{V}} \lesssim 1$$

that is less than $\mu^{-\epsilon} = \mu^{\sigma(1,2,0,0,r)}$.

If $m(I) = 0$, we have to exploit the arbitrariness of the functions c^i_κ. Note that

$$\mathbb{E}\left(F^{i,0}_{\kappa,\mu}\right) = c^i_\kappa + \int_0^\mu \mathbb{E}\left(\partial_\eta F^{i,0}_{\kappa,\eta}\right) d\eta.$$

To solve the problem of the non-integrability in 0, we can take

$$c^i_\kappa = -\int_0^{\frac{1}{2}} \mathbb{E}\left(\partial_\eta F^{i,0}_{\kappa,\eta}\right) d\eta,$$

which is constant in space for the stationarity of the white noise. With this choice, it holds

$$\mathbb{E}\left(F^{i,0}_{\kappa,\mu}\right) = \int_{\frac{1}{2}}^\mu \mathbb{E}\left(\partial_\eta F^{i,0}_{\kappa,\eta}\right) d\eta.$$

With this, we obtain

$$\begin{aligned}
&\left\| \tilde{K}^{*g}_{\mu} * \mathbb{E}\left(\partial^r_\kappa F^{i,0}_{\kappa,\mu}\right) \right\|_{L^\infty(\mathbb{H})} \\
&\quad \le \left| \int_{\frac{1}{2}}^{\mu} \left\| \tilde{K}^{*g}_{\mu} * \mathbb{E}\left(\partial^r_\kappa \partial_\eta F^{i,0}_{\kappa,\eta}\right) \right\|_{L^\infty(\mathbb{H})} d\eta \right| \\
&\quad \le \left| \int_{\frac{1}{2}}^{\mu} \left\| \tilde{P}^g_\eta \tilde{K}^{*g}_{\mu} * \tilde{K}^{*g}_{\eta} * \mathbb{E}\left(\partial^r_\kappa \partial_\eta F^{i,0}_{\kappa,\eta}\right) \right\|_{L^\infty(\mathbb{H})} d\eta \right| \\
&\quad \le \left| \int_{\frac{1}{2}}^{\mu} \left\| \tilde{P}^g_\eta \tilde{K}^{*g}_{\mu} \right\|_{TV} \left\| \tilde{K}^{*g}_{\eta} * \mathbb{E}\left(\partial^r_\kappa \partial_\eta F^{i,0}_{\kappa,\eta}\right) \right\|_{L^\infty(\mathbb{H})} d\eta \right| \\
&\quad \le \left| \int_{\frac{1}{2}}^{\mu} \left\| \tilde{K}^{*g}_{\eta} * \mathbb{E}\left(\partial^r_\kappa \partial_\eta F^{i,0}_{\kappa,\eta}\right) \right\|_{L^\infty(\mathbb{H})} d\eta \right| \lesssim \left| \int_{\frac{1}{2}}^{\mu} \kappa^{(\epsilon-1)r} \eta^{\sigma((i,0,0,1,r))} d\eta \right| \\
&\quad = \left| \int_{\frac{1}{2}}^{\mu} \kappa^{(\epsilon-1)r} \eta^{\sigma((i,0,0,0,r))-1} d\eta \right| \lesssim \kappa^{(\epsilon-1)r} \left| \mu^{\sigma((i,0,0,0,r))} - 2^{-\sigma((i,0,0,0,r))} \right| \\
&\quad \lesssim \kappa^{(\epsilon-1)r} \mu^{\sigma((i,0,0,0,r))},
\end{aligned}$$

where we used proposition A.2 (C).

We are left with $m(I) = 1$. The term with $i(I) = 1$ is easy to bound as, from (1.9), we have

$$\partial_\mu F^{1,1}_{\kappa,\mu} = -2B\left(\dot{G}_\mu, F^{1,2}_{\kappa,\mu}, F^{0,0}_{\kappa,\mu}\right)$$

and therefore

$$F^{1,1}_{\kappa,\mu}(z, dz_1) = -2\int_0^\mu \int \dot{G}_\eta(x-y)\xi_\kappa(y)\delta_z(dx)\delta_z(dz_1)dyd\eta,$$

which has zero average because the white noise is centred.

The last term to be estimated is $E^{((2,1,0,0,r))}_{\kappa,\mu}$, and as it requires some more technical tools, we will study it in the following section. □

2.2 Localisation

In this section, we will develop a part of the theory that justifies the need for the generalisation introduced in definition 2.4.

We already know that we can bound $E^{((2,1,0,0,r))}_{\kappa,\mu}$ multiplied by a linear polynomial of the above type (as $\sigma((2,1,a,0,r)) > 0$ for both non-empty choices of a). So, it is reasonable that some version of the Taylor expansion might do the job.

We start with a symmetry argument. First, let S be the reflection operator in the space variable, i.e. given $(x_1, x_2) \in \mathbb{M}$ we set $S(x_1, x_2) = (x_1, -x_2)$. Similarly, if $x = (z, z_1, \dots, z_m) \in \mathbb{H}\times\mathbb{M}^m$, with an abuse of notation, we set $Sx := (Sz, Sz_1, \dots, Sz_m)$. Then observe that using the flow equation, we can infer that

$$\mathbb{E}\left(F^{i,m}_{\kappa,\mu}(\cdot)\right) = (-1)^m \mathbb{E}\left(F^{i,m}_{\kappa,\mu}(S\cdot)\right). \tag{2.6}$$

This is obvious for $\mu = 0$; therefore, we conclude by induction using equation (1.10). This reflects the fact that equation (1.2) is invariant under the transformation $(f_\kappa, \xi_\kappa) \mapsto (-f_\kappa \circ S, \xi_\kappa \circ S)$ and $\xi_\kappa \circ S$ has the same law of ξ_κ.

Now we introduce the operator I defined as follows, given $V(z; dz_1)$, we set $I(V)(x) = \int_{\mathbb{M}} V(x; dy)$. Then, if we consider $x = (x_1, x_2) \in \mathbb{H}$, by the stationarity in the space variable of the white noise, we obtain

$$I\left(\mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}\right)(x_1, x_2) = I\left(\mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}\right)(x_1, 0).$$

Given all this, we have the following easy computation

$$\begin{aligned} I\left(\mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}\right)(x_1, 0) &= \int \mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}(x_1, 0; dz_1) = -\int \mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}(x_1, 0, Sdz_1) \\ &= -\int \mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}(x_1, 0, dz_1) = -I\left(\mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}\right)(x_1, 0), \end{aligned}$$

where we used the antisymmetry given by equation (2.6) and the change of variable formula. This argument implies $I\left(\mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}\right) \equiv 0$. Let us now see that this, together with the fact that we are already able to bound $E_{\kappa,\mu}^{((2,1,a,0,r))}$ with $|a| = 1$, gives the desired estimate for $E_{\kappa,\mu}^{((2,1,0,0,r))}$.

To do this, let L_b with $b \in \{1, 2\}$ be the operator given by $L_b V(x; dy) := (y_b - x_b)V(x; dy)$. We moreover set $[1] := 1$ and $[2]{:=}1/2$ similarly to what we have done in definition 2.5. Let us assume that $I(V) \equiv 0$.

By the Taylor formula, for any ϕ function in $\mathbb{M}$, we get

$$\phi(y) = \phi(x) + \int_0^1 (y - x) \cdot \nabla(\phi)(x + \tau(y - x))d\tau.$$

So that

$$\begin{aligned} \int_{\mathbb{M}} \phi(y)V(x; dy) &= \int_{\mathbb{M}} \phi(x)V(x; dy) + \int_{\mathbb{M}} \int_0^1 V(x; dy)(y - x) \cdot \nabla(\phi)(x + \tau(y - x))d\tau \\ &= \phi(x)I(V)(x) + \int_{\mathbb{M}} \int_0^1 V(x; dy)(y - x) \cdot \nabla(\phi)(x + \tau(y - x))d\tau \\ &= \int_{\mathbb{M}} \int_0^1 V(x; dy)(y - x) \cdot \nabla(\phi)(x + \tau(y - x))d\tau. \end{aligned} \tag{2.7}$$

Using the above relation with the regularising kernels, we obtain

$$\begin{aligned} &\tilde{K}_\mu^{*g+1,\otimes 2} * V(z, z_1) \\ &= \int_{\mathbb{M}^2} \tilde{K}_\mu^{*g+1}(z - x)\tilde{K}_\mu^{*g+1}(z_1 - y)V(x; dy)dx \\ &= -\int_{\mathbb{M}^2} \int_0^1 \tilde{K}_\mu^{*g+1}(z - x)V(x; dy)(y - x) \cdot \nabla\left(\tilde{K}_\mu^{*g+1}\right)(z_1 - [x + \tau(y - x)])d\tau dx \\ &= -\int_{\mathbb{M}^3} \int_0^1 \tilde{K}_\mu^{*g+1}(z - x)\tilde{K}_\mu^{*g}(z_1 - [x + \tau(y - x)] - v)V(x; dy)(y - x) \cdot \nabla\tilde{K}_\mu(v)d\tau dx dv. \end{aligned}$$

We use the substitution in the integral in dv given by $w = z_1 - v + (1-\tau)(y-x)$

$$
\begin{aligned}
&= -\sum_{b\in\{1,2\}} \int_{\mathbb{M}^3}\int_0^1 \tilde{K}_\mu^{*g+1}(z-x)\tilde{K}_\mu^{*g}(w-y) \\
&\qquad\qquad \times V(x;dy)(y_b-x_b)\partial^b\tilde{K}_\mu(z_1-w+(1-\tau)(y-x))d\tau dx dw \\
&= -\sum_{b\in\{1,2\}} \int_{\mathbb{M}^4}\int_0^1 \tilde{K}_\mu^{*g}(z-x-u)\tilde{K}_\mu^{*g}(w-y)V(x;dy)(y_b-x_b) \\
&\qquad\qquad \times\tilde{K}_\mu(u)\partial^b\tilde{K}_\mu(z_1-w+(1-\tau)(y-x))d\tau dx dw du \\
&= -\sum_{b\in\{1,2\}} \int_{\mathbb{M}^2}\int_0^1 \left[\tilde{K}_\mu^{*g,\otimes 2} * L_b V\right](z-u,w)\tilde{K}_\mu(u)\partial^b\tilde{K}_\mu(z_1-w+(1-\tau)(y-x))d\tau dw du.
\end{aligned}
$$

Now if we take the norm and use proposition A.2 (A), (B), and (C), we obtain

$$
\begin{aligned}
&\left\|\tilde{K}_\mu^{*g+1,\otimes 2} * V\right\|_{\mathcal{V}} \\
&\quad\le \sup_{z\in\mathbb{H}}\sum_{b\in\{1,2\}}\int_{\mathbb{M}^3}\int_0^1 \left|\left[\tilde{K}_\mu^{*g,\otimes 2} * L_b V\right](z-u,w)\right. \\
&\qquad\qquad \left.\times\tilde{K}_\mu(u)\partial^b\tilde{K}_\mu(z_1-w+(1-\tau)(y-x))\right| d\tau dw du dz_1 \\
&\quad= \sup_{z\in\mathbb{H}}\sum_{b\in\{1,2\}}\left\|\partial^b\tilde{K}_\mu\right\|_{TV}\int_{\mathbb{M}^2}\int_0^1\left|\left[\tilde{K}_\mu^{*g,\otimes 2} * L_b V\right](z-u,w)\tilde{K}_\mu(u)\right| d\tau dw du \\
&\quad\lesssim \sup_{z\in\mathbb{H}}\sum_{b\in\{1,2\}}\mu^{-[b]}\int_{\mathbb{M}^2}\left|\left[\tilde{K}_\mu^{*g,\otimes 2} * L_b V\right](z-u,w)\tilde{K}_\mu(u)\right| dw du \\
&\quad= \sup_{z\in\mathbb{H}}\sum_{b\in\{1,2\}}\mu^{-[b]}\int_{\mathbb{M}}\left\|\tilde{K}_\mu^{*g,\otimes 2} * L_b V(z-u,\cdot)\right\|_{L^1(\mathbb{R}^2)}\tilde{K}_\mu(u)du \\
&\quad\le \sum_{b\in\{1,2\}}\mu^{-[b]}\left\|\tilde{K}_\mu^{*g,\otimes 2} * L_b V\right\|_{\mathcal{V}}\left\|\tilde{K}_\mu\right\|_{L^1(\mathbb{R}^2)} = \sum_{b\in\{1,2\}}\mu^{-[b]}\left\|\tilde{K}_\mu^{*g,\otimes 2} * L_b V\right\|_{\mathcal{V}}.
\end{aligned}
$$

It is now sufficient to take $V = \mathbb{E}\partial_\kappa^r F_{\kappa,\mu}^{2,1}$ to conclude that

$$
\begin{aligned}
\left\|\tilde{K}_\mu^{*g+1,\otimes 2} * E_{\kappa,\mu}^{((2,1,0,0,r))}\right\|_{\mathcal{V}} &\lesssim \sum_{b\in\{1,2\}}\mu^{-[b]}\left\|\tilde{K}_\mu^{*g,\otimes 2} * E_{\kappa,\mu}^{((2,1,(b),0,r))}\right\|_{\mathcal{V}} \\
&\lesssim \sum_{b\in\{1,2\}}\mu^{-[b]}\kappa^{(\epsilon-1)r}\mu^{\sigma((2,1,(b),0,0))} \\
&= \sum_{b\in\{1,2\}}\mu^{-[b]}\kappa^{(\epsilon-1)r}\mu^{\sigma((2,1,0,0,0))+[b]} \lesssim \kappa^{(\epsilon-1)r}\mu^{\sigma((2,1,0,0,0))},
\end{aligned}
$$

where $(b) = (1,0)$ if $b = 1$ and $(b) = (0,1)$ if $b = 2$.

Remark 2.9. *Finally, we point out that we can take $g = 3$ in the above theorem because we started with two kernels for the base case and added one more in the localisation part.*

Chapter 3

Bounds on the coefficients

This chapter is divided into two parts. In the first one, we will use theorem 2.7 to obtain a bound on the moments of the coefficients of the functional. We will use the latter in the second part to prove the desired pointwise estimate (theorem 3.3).

3.1 Moments of the force coefficients

We let $\bar{F}^{i,m,a}_{\kappa,\mu}(z) = I\left(F^{i,m,a}_{\kappa,\mu}\right)(z)$ for each (i,m,a) such that $\sigma(i,m,a,0,0) \leq 0$ where I is, as in definition 2.5, given by

$$\bar{F}^{i,m,a}_{\kappa,\mu}(z) = I\left(F^{i,m,a}_{\kappa,\mu}\right)(z) = \int_{\mathbb{M}^m} F^{i,m,a}_{\kappa,\mu}(z, dx_1, \ldots, dx_m).$$

Then we have the following proposition.

Proposition 3.1. *In the above setting, for each $n \in \mathbb{Z}^+$ even, the following holds*

$$\mathbb{E}\left[\left(\tilde{K}^{*4}_\mu * \partial^r_\kappa \partial^s_\mu \bar{F}^{i,m,a}_{\kappa,\mu}(x)\right)^n\right] \lesssim \kappa^{n(\epsilon-1)r} \mu^{n\sigma'(i,m,a)-ns},$$

where $\sigma'(i,m,a) := -\frac{3}{4} - \epsilon + m\left(\frac{1}{4} + \epsilon\right) + i\left(\frac{1}{4} - 2\epsilon\right) + [a]$, $s \in \{0,1\}$ *and* $r \in \{0,1,2\}$.

Proof. We already know by 2.7 that a similar estimate holds for the cumulants of the coefficients. At this point, it is sufficient to use the relation between cumulants and moments given by proposition 2.2. Thanks to the latter, we have

$$\mathbb{E}\left[\left(\tilde{K}^{*4}_\mu * \partial^r_\kappa \partial^s_\mu \bar{F}^{i,m,a}_{\kappa,\mu}(x)\right)^n\right] = \sum_{l=1}^{n} \sum_{I_1 \sqcup \cdots \sqcup I_l = [n]} \prod_{j=1}^{l} \mathbb{E}\left(\tilde{K}_\mu * \tilde{K}^{*3}_\mu * \partial^r_\kappa \partial^s_\mu \bar{F}^{i,m,a}_{\kappa,\mu}\right)_{I_j}.$$

Let us estimate the right-hand side.

$$\begin{aligned}
&\mathbb{E}\left(\tilde{K}_\mu * \tilde{K}^{*3}_\mu * \partial^r_\kappa \partial^s_\mu \bar{F}^{i,m,a}_{\kappa,\mu}\right)_{I_j} \\
&\quad = \mathbb{E}\left(\int_{\mathbb{M}} \tilde{K}_\mu\left(x - z^h\right) \tilde{K}^{*3}_\mu * \partial^r_\kappa \partial^s_\mu \bar{F}^{i,m,a}_{\kappa,\mu}\left(z^h\right) dz^h\right)^{|I_j|}_{h=1} \\
&\quad = \int_{\mathbb{M}^{|I_j|}} \prod_{h=1}^{|I_j|} \tilde{K}_\mu\left(x - z^h\right) \mathbb{E}\left(\tilde{K}^{*3}_\mu * \partial^r_\kappa \partial^s_\mu \bar{F}^{i,m,a}_{\kappa,\mu}\left(z^h\right)\right)^{|I_j|}_{h=1} dz^1 \ldots dz^{|I_j|}.
\end{aligned}$$

Which, together with proposition A.2 (A) and the already cited theorem 2.7, gives

$$\begin{aligned}
&\left\|\mathbb{E}\Big(\tilde{K}_\mu*\tilde{K}_\mu^{*3}*\partial_\kappa^r\partial_\mu^s\bar{F}_{\kappa,\mu}^{i,m,a}\Big)_{I_j}\right\|_{L^\infty(\mathbb{H})}\\
&\quad\le\left\|\tilde{K}_\mu\right\|_{L^1(\mathbb{R}^2)}\left\|\tilde{K}_\mu\right\|_{L^\infty(\mathbb{R}^2)}^{|I_j|-1}\left\|\mathbb{E}\left(\tilde{K}_\mu^{*3}*\partial_\kappa^r\partial_\mu^s\bar{F}_{\kappa,\mu}^{i,m,a}\right)_{I_j}\right\|_{\mathcal{V}}\\
&\quad=\left\|\tilde{K}_\mu\right\|_{L^1(\mathbb{R}^2)}\left\|\tilde{K}_\mu\right\|_{L^\infty(\mathbb{R}^2)}^{|I_j|-1}\left\|I\left(\mathbb{E}\left(\tilde{K}_\mu^{*3,\otimes 1+m}*\partial_\kappa^r\partial_\mu^s F_{\kappa,\mu}^{i,m,a}\right)_{I_j}\right)\right\|_{\mathcal{V}}\\
&\quad\le\left\|\tilde{K}_\mu\right\|_{L^1(\mathbb{R}^2)}\left\|\tilde{K}_\mu\right\|_{L^\infty(\mathbb{R}^2)}^{|I_j|-1}\left\|\mathbb{E}\left(\tilde{K}_\mu^{*3,\otimes 1+m}*\partial_\kappa^r\partial_\mu^s F_{\kappa,\mu}^{i,m,a}\right)_{I_j}\right\|_{\mathcal{V}}\\
&\quad\lesssim\mu^{-\frac{3}{2}(|I_j|-1)}\cdot\kappa^{(\epsilon-1)r|I_j|}\mu^{|I_j|\left(\frac{3}{4}-\epsilon\right)+m|I_j|\left(\frac{1}{4}+\epsilon\right)+i|I_j|\left(\frac{1}{4}-2\epsilon\right)-\frac{3}{2}-s|I_j|+[a]|I_j|}\\
&\quad=\kappa^{(\epsilon-1)r|I_j|}\mu^{|I_j|\left(-\frac{3}{4}-\epsilon\right)+m|I_j|\left(\frac{1}{4}+\epsilon\right)+i|I_j|\left(\frac{1}{4}-2\epsilon\right)-s|I_j|+[a]|I_j|}\\
&\quad=\kappa^{(\epsilon-1)r|I_j|}\mu^{|I_j|\left(-\frac{3}{4}-\epsilon+m\left(\frac{1}{4}+\epsilon\right)+i\left(\frac{1}{4}-2\epsilon\right)-s+[a]\right)}.
\end{aligned}$$

This allows us to conclude as

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\tilde{K}_\mu^{*4}*\partial_\kappa^r\partial_\mu^s\bar{F}_{\kappa,\mu}^{i,m,a}(x)\Big)^n\Big]\\
&\quad\le\sum_{l=1}^n\sum_{I_1\sqcup\cdots\sqcup I_l=[n]}\prod_{j=1}^l\left\|\mathbb{E}\left(\tilde{K}_\mu^{*4}*\partial_\kappa^r\partial_\mu^s\bar{F}_{\kappa,\mu}^{i,m,a}\right)_{I_j}\right\|_{L^\infty(\mathbb{H})}\\
&\quad\lesssim\sum_{l=1}^n\sum_{I_1\sqcup\cdots\sqcup I_l=[n]}\prod_{j=1}^l\kappa^{|I_j|(\epsilon-1)r}\mu^{|I_j|\left(-\frac{3}{4}-\epsilon+m\left(\frac{1}{4}+\epsilon\right)+i\left(\frac{1}{4}-2\epsilon\right)-s+[a]\right)}\\
&\quad\lesssim\kappa^{n(\epsilon-1)r}\mu^{n\sigma'(i,m,a)-ns}.
\end{aligned}$$

□

3.2 Pointwise estimates

Let us fix $\Xi_{\kappa,\mu}:=\tilde{K}_\mu^{*4}*\mathbb{1}_{0,1}\partial_\kappa^r\bar{F}_{\kappa,\mu}^{i,m,a}$ for an $r\in\{0,1\}$ and

$$\hat{\Xi}_{\kappa,\mu}:=\mu^\Theta\tilde{K}_\mu*\Xi_{\kappa,\mu}=\mu^\Theta\tilde{K}_\mu^{*5}*\mathbb{1}_{0,1}\partial_\kappa^r\bar{F}_{\kappa,\mu}^{i,m,a}$$

for a fixed Θ.

We will bound $\mathbb{E}\left(\left\|\partial_\kappa^l\partial_\mu^s\hat{\Xi}_{\kappa,\mu}\right\|_{L^\infty(\mathbb{H})}^n\right)$ for any $s,l\le 1$. To shorten this part, we set $\sigma':=\sigma'(i,m,a)$.

First, observe that given a function v defined on $\mathbb{H}$, if we let $h=\tilde{K}_\mu*v$, then we have

$$\begin{aligned}
\mathbb{1}_{0,1}h=\tilde{K}_\mu*\tilde{P}_\mu(\mathbb{1}_{0,1}h)&=\tilde{K}_\mu*\left[\left(\mu\partial_t\mathbb{1}_{0,1}\right)\left(1-\mu\Delta\right)h\right]+\tilde{K}_\mu*\mathbb{1}_{0,1}\tilde{P}_\mu h\\
&=\mu(1-\mu\Delta)\tilde{K}_\mu*[\delta_1(t)h]+\tilde{K}_\mu*\mathbb{1}_{0,1}\tilde{P}_\mu h
\end{aligned}$$

and therefore

$$\left\|\tilde{K}_\mu*\mathbb{1}_{0,1}v\right\|_{L^\infty(\mathbb{H})}\lesssim\left\|\mathbb{1}_{0,1}\tilde{K}_\mu*v\right\|_{L^\infty(\mathbb{H})}.$$

Similarly, the previous holds with $\tilde{K}_\mu^{*g}$ for any $g \geq 1$. Now we can bound the desired term.

Let us start with $s = 0$. In this case, we have

$$\begin{aligned}
\mathbb{E}\left(\left\|\partial_\kappa^l \hat{\Xi}_{\kappa,\mu}\right\|_{L^\infty(\mathbb{H})}^n\right) & \\
&= \mu^{\Theta n}\mathbb{E}\left(\left\|\tilde{K}_\mu * \partial_\kappa^l \Xi_{\kappa,\mu}\right\|_{L^\infty(\mathbb{H})}^n\right) = \mu^{\Theta n}\mathbb{E}\left(\left\|T\tilde{K}_\mu \star \partial_\kappa^l \Xi_{\kappa,\mu}\right\|_{L^\infty(\mathbb{H})}^n\right) \\
&\leq \mu^{\Theta n}\mathbb{E}\left(\left\|T\tilde{K}_\mu\right\|_{L^{\frac{n}{n-1}}(\mathbb{H})}^n \left\|\partial_\kappa^l \Xi_{\kappa,\mu}\right\|_{L^n(\mathbb{H})}^n\right) \\
&= \mu^{\Theta n}\left\|T\tilde{K}_\mu\right\|_{L^{\frac{n}{n-1}}(\mathbb{H})}^n \int_{\mathbb{H}} \mathbb{E}\left(\left(\partial_\kappa^l \Xi_{\kappa,\mu}\right)^n(x)\right) dx \\
&\lesssim \mu^{\Theta n} \cdot \mu^{-3/2} \cdot \kappa^{n(\epsilon-1)(l+r)} \mu^{n\sigma'} = \kappa^{n(\epsilon-1)(l+r)} \mu^{n\left(\sigma'+\Theta-\frac{3}{2n}\right)},
\end{aligned}$$

where we used proposition A.2 (F), that $\star$ is the convolution on $\mathbb{H}$ and T is the periodisation operator (see proposition A.2).

Now let us consider $s = 1$. In this case, it holds

$$\partial_\mu \hat{\Xi}_{\kappa,\mu} = \Theta\mu^{\Theta-1}\tilde{K}_\mu^{*5} * \mathbb{1}_{0,1}\partial_\kappa^r \bar{F}_{\kappa,\mu}^{i,m,a} + 5\mu^\Theta \partial_\mu \tilde{K}_\mu * \tilde{K}_\mu^{*4} * \mathbb{1}_{0,1}\partial_\kappa^r \bar{F}_{\kappa,\mu}^{i,m,a} + \mu^\Theta \tilde{K}_\mu^{*5} * \mathbb{1}_{0,1}\partial_\kappa^r \partial_\mu \bar{F}_{\kappa,\mu}^{i,m,a}.$$

As before, we have

$$\mathbb{E}\left(\left\|\Theta\mu^{\Theta-1}\tilde{K}_\mu^{*5} * \partial_\kappa^{r+l}\bar{F}_{\kappa,\mu}^{i,m,a}\right\|_{L^\infty(\mathbb{H})}^n\right) \lesssim \kappa^{n(\epsilon-1)(l+r)}\mu^{n\left(\sigma'+\Theta-1-\frac{3}{2n}\right)}$$

and

$$\begin{aligned}
\mathbb{E}\left(\left\|\mu^\Theta \tilde{K}_\mu^{*5} * \partial_\kappa^{r+l}\partial_\mu \bar{F}_{\kappa,\mu}^{i,m,a}\right\|_{L^\infty(\mathbb{H})}^n\right) &\lesssim \mu^{\Theta n} \cdot \mu^{-3/2} \cdot \int_{\mathbb{H}} \mathbb{E}\left(\left(\partial_\mu \partial_\kappa^l \Xi_{\kappa,\mu}(x)\right)^n\right) dx \\
&\lesssim \mu^{\Theta n - 3/2} \cdot \kappa^{n(\epsilon-1)(r+l)} \mu^{n\sigma'-n} \\
&= \kappa^{n(\epsilon-1)(l+r)} \mu^{n\left(\sigma'+\Theta-1-\frac{3}{2n}\right)}.
\end{aligned}$$

Furthermore, we notice that

$$\mu^\Theta \partial_\mu \tilde{K}_\mu * \tilde{K}_\mu^{*4} * \mathbb{1}_{0,1}\partial_\kappa^r \bar{F}_{\kappa,\mu}^{i,m,a} = \mu^\Theta \tilde{P}_\mu \partial_\mu \tilde{K}_\mu * \tilde{K}_\mu^{*5} * \mathbb{1}_{0,1}\partial_\kappa^r \bar{F}_{\kappa,\mu}^{i,m,a} = \tilde{P}_\mu \partial_\mu \tilde{K}_\mu * \hat{\Xi}_{\kappa,\mu}.$$

So, using A.2 (E), we obtain

$$\begin{aligned}
\mathbb{E}\left(\left\|\mu^\Theta \tilde{P}_\mu \partial_\mu \tilde{K}_\mu * \tilde{K}_\mu^{*5} * \mathbb{1}_{0,1}\partial_\kappa^{l+r}\bar{F}_{\kappa,\mu}^{i,m,a}\right\|_{L^\infty(\mathbb{H})}^n\right) &\leq \mathbb{E}\left(\left\|\tilde{P}_\mu \partial_\mu \tilde{K}_\mu\right\|_{TV}^n \cdot \left\|\partial_\kappa^l \hat{\Xi}_{\kappa,\mu}\right\|_{L^\infty(\mathbb{H})}^n\right) \\
&\lesssim \mu^{-n} \cdot \kappa^{n(\epsilon-1)(l+r)} \mu^{n\left(\sigma'+\Theta-\frac{3}{2n}\right)} \\
&= \kappa^{n(\epsilon-1)(l+r)} \mu^{n\left(\sigma'+\Theta-1-\frac{3}{2n}\right)}.
\end{aligned}$$

To summarise, we have proved

$$\mathbb{E}\left(\left\|\partial_\kappa^l \partial_\mu^s \hat{\Xi}_{\kappa,\mu}\right\|_{L^\infty(\mathbb{H})}^n\right) \lesssim \kappa^{n(\epsilon-1)(l+r)} \mu^{n\left(\sigma'+\Theta-s-\frac{3}{2n}\right)}. \tag{3.1}$$

This easily implies a pointwise estimate. Indeed, if $r = 0$, we have

$$\begin{aligned}\hat{\Xi}_{\kappa,\mu} &= \hat{\Xi}_{1,\mu} - \int_\kappa^1 \partial_\nu \hat{\Xi}_{\nu,\mu} d\nu = \\ &= \left(\hat{\Xi}_{1,1} - \int_\mu^1 \partial_\eta \hat{\Xi}_{1,\eta} d\eta\right) - \int_\kappa^1 \partial_\nu \left(\hat{\Xi}_{\nu,1} - \int_\mu^1 \partial_\eta \hat{\Xi}_{\nu,\eta} d\eta\right) d\nu \\ &= \hat{\Xi}_{1,1} - \int_\mu^1 \partial_\eta \hat{\Xi}_{1,\eta} d\eta - \int_\kappa^1 \partial_\nu \hat{\Xi}_{\nu,1} d\nu + \int_\kappa^1 \int_\mu^1 \partial_\nu \partial_\eta \hat{\Xi}_{\nu,\eta} d\eta d\nu.\end{aligned}$$

That gives

$$\begin{aligned}\left\|\hat{\Xi}_{\kappa,\mu}\right\|^n_{L^\infty(\mathbb{H})} \lesssim \left\|\hat{\Xi}_{1,1}\right\|^n_{L^\infty(\mathbb{H})} + \int_\mu^1 \left\|\partial_\eta \hat{\Xi}_{1,\eta}\right\|^n_{L^\infty(\mathbb{H})} d\eta + \int_\kappa^1 \left\|\partial_\nu \hat{\Xi}_{\nu,1}\right\|^n_{L^\infty(\mathbb{H})} d\nu \\ + \int_\kappa^1 \int_\mu^1 \left\|\partial_\nu \partial_\eta \hat{\Xi}_{\nu,\eta}\right\|^n_{L^\infty(\mathbb{H})} d\eta d\nu.\end{aligned}$$

Taking the expected value, this will result in

$$\begin{aligned}\mathbb{E}\left(\sup_{\kappa,\mu} \left\|\hat{\Xi}_{\kappa,\mu}\right\|^n_{L^\infty(\mathbb{H})}\right)^{1/n} \lesssim \mathbb{E}\left(\left\|\hat{\Xi}_{1,1}\right\|^n_{L^\infty(\mathbb{H})}\right)^{1/n} + \int_0^1 \mathbb{E}\left(\left\|\partial_\eta \hat{\Xi}_{1,\eta}\right\|^n_{L^\infty(\mathbb{H})}\right)^{1/n} d\eta \\ + \int_0^1 \mathbb{E}\left(\left\|\partial_\nu \hat{\Xi}_{\nu,1}\right\|^n_{L^\infty(\mathbb{H})}\right)^{1/n} d\nu + \int_0^1 \int_0^1 \mathbb{E}\left(\left\|\partial_\nu \partial_\eta \hat{\Xi}_{\nu,\eta}\right\|^n_{L^\infty(\mathbb{H})}\right)^{1/n} d\eta d\nu.\end{aligned}$$

Using (3.1), we know that the right-hand side is finite if the two functions $\eta^{\sigma' + \Theta - 1 - \frac{3}{2n}}$, $\nu^{\epsilon - 1}$ are integrable near 0. The latter is because $\epsilon - 1 > -1$. For the first one, we take n big enough so that $\frac{3}{2n} < \epsilon/2$ and $\Theta = \frac{3}{4} + 2\epsilon - m\left(\frac{1}{4} + 2\epsilon\right) - i\left(\frac{1}{4} - 4\epsilon\right) - [a]$. With this choice, we obtain $\sigma' + \Theta - 1 - \frac{3}{2n} = \epsilon + (2i - m)\epsilon - 1 - \frac{3}{2n} \geq \epsilon/2 - 1 > -1$ if $2i \geq m$ (but this is the only interesting regime because, in the other one, the left-hand side is identically 0).

If $r = 1$, we have to repeat the last estimates with $\kappa^{1-\epsilon/2}\hat{\Xi}_{\kappa,\mu}$ instead of $\hat{\Xi}_{\kappa,\mu}$ to prove

$$\mathbb{E}\left(\sup_{\kappa,\mu} \left\|\kappa^{1-\epsilon/2}\hat{\Xi}_{\kappa,\mu}\right\|^n_{L^\infty(\mathbb{H})}\right)^{1/n} < \infty.$$

To summarise, let us define $\alpha = \frac{1}{4} + 2\epsilon$, $\beta = \frac{1}{4} - 4\epsilon$, $\gamma = -\frac{3}{4} - 2\epsilon$ and $\rho(i, m, a) = \alpha m + \beta i + \gamma + [a]$. Then we have proved the following proposition.

Proposition 3.2. *Let (i, m, a) such that $\sigma(i, m, a, 0, 0) \leq 0$ and $r \in \{0, 1\}$. Then*

$$\left\|\tilde{K}^{*5}_\mu * \mathbb{1}_{0,1} \partial^r_\kappa \bar{F}^{i,m,a}_{\kappa,\mu}\right\|_{L^\infty(\mathbb{H})} \lesssim \kappa^{(\epsilon/2-1)r} \mu^{\rho(i,m,a)}$$

holds uniformly in $(\mu, \kappa) \in (0, 1]^2$.

We can finally state and prove the general theorem that gives a pointwise estimate for any coefficients of the force.

Theorem 3.3. *There exists an integer $g \geq 1$ such that for any (i, m, a) with i, m natural numbers and a as in definition 2.5, and $r \in \{0, 1\}$, the following holds uniformly in $(\mu, \kappa) \in (0, 1]^2$*

$$\left\| \tilde{K}_\mu^{*g,\otimes 1+m} * \mathbb{1}_{0,1} \partial_\kappa^r F_{\kappa,\mu}^{i,m,a} \right\|_{\mathcal{V}} \lesssim \kappa^{(\epsilon/2-1)r} \mu^{\rho(i,m,a)}.$$

We remark that the indicator function in the above theorem acts only on the first component of the coefficients of the force.

Proof. We will prove this by induction, similarly to the proof of theorem 2.7.

First, note that if $m = 0$, it holds $F_{\kappa,\mu}^{i,0,0} = \bar{F}_{\kappa,\mu}^{i,0,0}$ and so we already have the thesis by theorem 3.2. Now assume $i > 0$ and $m > 0$. As in the abovementioned proof, we first treat the case $\rho(i, m, a) > 0$.

We notice that in this case $F_{\kappa,0}^{i,m,a} = 0$ as otherwise, we would have $(i, m) = (1, 2)$ but $\rho(1, 2, 0) = -2\epsilon < 0$ and if $a \neq 0$, it holds $F_{\kappa,\mu}^{1,2,a} = 0$ as in the proof of theorem 2.7. Now, by the flow equation (2.1), we have

$$\mathbb{1}_{0,1} \partial_\mu F_{\kappa,\mu}^{i,m,a} = -\sum_{l=0}^{i} \sum_{j=0}^{m} \sum_{b,c,d \in \mathcal{F}(a)} (j+1) B\left(\dot{G}_\mu^c, \mathbb{1}_{0,1} F_{\kappa,\mu}^{l,j+1,b}, \mathbb{1}_{0,1} F_{\kappa,\mu}^{i-l,m-j,d}\right),$$

where we were able to bring the indicator functions inside the operator thanks to the support property of $F_{\kappa,\mu}^{i,m}$ remarked in chapter 1 after their inductive definition. Now, using relation (2.2), proposition C.1 (A) and the induction hypothesis, we obtain

$$\begin{aligned}
&\left\| \tilde{K}_\mu^{*g,\otimes 1+m} * \mathbb{1}_{0,1} \partial_\mu F_{\kappa,\mu}^{i,m,a} \right\|_{\mathcal{V}} \\
&\quad \lesssim \sum_{l=0}^{i} \sum_{j=0}^{m} \sum_{b,c,d \in \mathcal{F}(a)} \left\| \tilde{P}_\mu^{2g} \dot{G}_\mu^c \right\|_{L^1(\mathbb{M})} \left\| \tilde{K}_\mu^{*g,\otimes j+2} * \mathbb{1}_{0,1} F_{\kappa,\mu}^{l,j+1,b} \right\|_{\mathcal{V}} \\
&\qquad\qquad \left\| \tilde{K}_\mu^{*g,\otimes m-j+1} * \mathbb{1}_{0,1} F_{\kappa,\mu}^{i-l,m-j,d} \right\|_{\mathcal{V}} \\
&\quad \lesssim \sum_{l=0}^{i} \sum_{j=0}^{m} \sum_{b,c,d \in \mathcal{F}(a)} \mu^{-1/2+[c]} \cdot \mu^{\rho(l,j+1,b)} \cdot \mu^{\rho(i-l,m-j,d)} \\
&\quad = \sum_{l=0}^{i} \sum_{j=0}^{m} \sum_{b,c,d \in \mathcal{F}(a)} \mu^{-1/2+\alpha(m+1)+\beta i+2\gamma+[a]} \lesssim \mu^{\rho(i,m,a)-1},
\end{aligned}$$

where we used $\alpha + \gamma = -1/2$.

Deriving equation (2.1) in κ, we obtain

$$\begin{aligned}
\mathbb{1}_{0,1} \partial_\kappa \partial_\mu F_{\kappa,\mu}^{i,m,a} = -\sum_{l=0}^{i} \sum_{j=0}^{m} \sum_{b,c,d \in \mathcal{F}(a)} (j+1) \Big[&B\left(\dot{G}_\mu^c, \mathbb{1}_{0,1} \partial_\kappa F_{\kappa,\mu}^{l,j+1,b}, \mathbb{1}_{0,1} F_{\kappa,\mu}^{i-l,m-j,d}\right) + \\
&+ B\left(\dot{G}_\mu^c, \mathbb{1}_{0,1} F_{\kappa,\mu}^{l,j+1,b}, \mathbb{1}_{0,1} \partial_\kappa F_{\kappa,\mu}^{i-l,m-j,d}\right) \Big].
\end{aligned}$$

Taking the norm and using inequality (2.2), we have

$$\left\| \tilde{K}_\mu^{*g,\otimes 1+m} * \mathbb{1}_{0,1} \partial_\kappa \partial_\mu F_{\kappa,\mu}^{i,m,a} \right\|_{\mathcal{V}}$$

$$\lesssim \sum_{l=0}^{i}\sum_{j=0}^{m}\sum_{b,c,d\in\mathcal{F}(a)} \left\|\tilde{P}_\mu^{2g}\dot{G}_\mu^c\right\|_{L^1(\mathbb{M})} \big(\kappa^{(\epsilon/2-1)r}\mu^{\rho(l,j+1,b)}\cdot\mu^{\rho(i-l,m-j,d)} + \mu^{\rho(l,j+1,b)}\cdot\kappa^{(\epsilon/2-1)r}\mu^{\rho(i-l,m-j,d)}\big)$$

$$\lesssim \sum_{l=0}^{i}\sum_{j=0}^{m}\sum_{b,c,d\in\mathcal{F}(a)} \kappa^{\epsilon/2-1}\mu^{-1/2+\alpha(m+1)+\beta i+2\gamma+[a]} \lesssim \kappa^{\epsilon/2-1}\mu^{\rho(i,m,a)-1}.$$

To summarise

$$\left\|\tilde{K}_\mu^{*g,\otimes 1+m} * \mathbb{1}_{0,1}\partial_\kappa^r\partial_\mu F_{\kappa,\mu}^{i,m,a}\right\|_{\mathcal{V}} \lesssim \kappa^{(\epsilon/2-1)r}\mu^{\rho(i,m,a)-1}$$

for any $r\in\{0,1\}$. Using

$$\tilde{K}_\mu^{*g,\otimes 1+m} * \mathbb{1}_{0,1}\partial_\kappa^r F_{\kappa,\mu}^{i,m,a} = \int_0^\mu \tilde{K}_\mu^{*g,\otimes 1+m} * \mathbb{1}_{0,1}\partial_\kappa^r\partial_\eta F_{\kappa,\eta}^{i,m,a}d\eta$$

and $F_{\kappa,0}^{i,m,a}=0$, we obtain

$$\begin{aligned}\left\|\tilde{K}_\mu^{*g,\otimes 1+m} * \mathbb{1}_{0,1}\partial_\kappa^r F_{\kappa,\mu}^{i,m,a}\right\|_{\mathcal{V}} &\le \int_0^\mu \left\|\tilde{K}_\mu^{*g,\otimes 1+m} * \mathbb{1}_{0,1}\partial_\kappa^r\partial_\eta F_{\kappa,\eta}^{i,m,a}\right\|_{\mathcal{V}} d\eta\\ &= \int_0^\mu \left\|\tilde{P}_\eta^g\tilde{K}_\mu^{*g,\otimes 1+m} * \tilde{K}_\eta^{*4,\otimes 1+m} * \mathbb{1}_{0,1}\partial_\kappa^r\partial_\eta F_{\kappa,\eta}^{i,m,a}\right\|_{\mathcal{V}} d\eta\\ &\le \int_0^\mu \left\|\tilde{P}_\eta\tilde{K}_\mu^{\otimes 1+m}\right\|_{TV}^g \left\|\tilde{K}_\eta^{*g,\otimes 1+m} * \mathbb{1}_{0,1}\partial_\kappa^r\partial_\eta F_{\kappa,\eta}^{i,m,a}\right\|_{\mathcal{V}} d\eta\\ &\le \int_0^\mu \left\|\tilde{K}_\eta^{*g,\otimes 1+m} * \mathbb{1}_{0,1}\partial_\kappa^r\partial_\eta F_{\kappa,\eta}^{i,m,a}\right\|_{\mathcal{V}} d\eta\\ &\lesssim \int_0^\mu \kappa^{(\epsilon/2-1)r}\eta^{\rho(i,m,a)-1}d\eta \lesssim \kappa^{(\epsilon/2-1)r}\mu^{\rho(i,m,a)},\end{aligned}$$

where we used proposition A.2 (D).

We are left with the case $\rho(i,m,a)\le 0$. Again, as in the proof of theorem 2.7, we characterise all these (i,m,a).

If $m\ge 3$, we have

$$\begin{aligned}\rho(i,m,a) &= \alpha m+\beta i+\gamma+[a] \ge 3\alpha+\beta+\gamma\\ &= 3\left(\frac{1}{4}+2\epsilon\right)+\left(\frac{1}{4}-4\epsilon\right)-\frac{3}{4}-2\epsilon = \frac{1}{4} > 0.\end{aligned}$$

As we already know the theorem for $m=0$, we have to study only the cases $m\in\{1,2\}$.

If $m=2$:

$$\begin{aligned}0 \ge \rho(i,m,a) &= 2\alpha+\beta i+\gamma+[a]\\ &= 2\left(\frac{1}{4}+2\epsilon\right)+i\left(\frac{1}{4}-4\epsilon\right)-\frac{3}{4}-2\epsilon+[a]\\ &= 2\epsilon+i\left(\frac{1}{4}-4\epsilon\right)-\frac{1}{4}+[a],\end{aligned}$$

which is true only if $i = 1$ and $|a| = 0$.

If $m = 1$:

$$0 \geq \rho(i, m, a) = \alpha + \beta i + \gamma + [a] = -\frac{1}{2} + i\left(\frac{1}{4} - 4\epsilon\right) + [a],$$

which is true only if $|a| = 0$ and $i \in \{1, 2\}$.

So we just have to study all (i, m, a) in $\{(1, 2, 0), (1, 1, 0), (2, 1, 0)\}$. As $F^{1,2}_{\kappa,\mu}$ is deterministic, in this case the theorem follows from 2.7. We are now left with $F^{1,1}_{\kappa,\mu}$ and $F^{2,1}_{\kappa,\mu}$. To estimate those two, we have to use the localisation strategy as in section 2.2.

Remember that it holds

$$\int_{\mathbb{M}} \phi(y)V(x; dy) = \phi(x)I(V)(x) + \int_{\mathbb{M}}\int_0^1 V(x; dy)(y - x) \cdot \nabla(\phi)(x + \tau(y - x))d\tau.$$

If we convolve with the regularising kernel, we obtain that $\left\|\tilde{K}^{*g+1,\otimes 2}_{\mu} * V\right\|_{\mathcal{V}}$ can be bounded by the sum of

$$\left\|\int_{\mathbb{M}} \tilde{K}^{*g+1}_{\mu}(z - x)\tilde{K}^{*g+1}_{\mu}(z_1 - x)I(V)(x)dx\right\|_{\mathcal{V}}$$

and

$$\left\|\int_{\mathbb{M}}\int_0^1 \tilde{K}^{*g+1}_{\mu}(z - x)V(x; dy)(y - x)\nabla\left(\tilde{K}^{*g+1}_{\mu}\right)(z_1 - [x + \tau(y - x)])d\tau dx\right\|_{\mathcal{V}}.$$

As we have seen in section 2.2, the latter can be bounded up to a multiplicative constant by $\sum_{b\in\{1,2\}} \mu^{-[b]}\left\|\tilde{K}^{*g,\otimes 2}_{\mu} * L_b V\right\|_{\mathcal{V}}$. Whereas for the first one, we have

$$\begin{aligned}
&\left\|\int_{\mathbb{M}} \tilde{K}^{*g+1}_{\mu}(z - x)\tilde{K}^{*g+1}_{\mu}(z_1 - x)I(V)(x)dx\right\|_{\mathcal{V}} \\
&\qquad = \sup_{z\in\mathbb{T}}\int_{\mathbb{M}}\left|\int_{\mathbb{M}} \tilde{K}^{*g+1}_{\mu}(z - x)\tilde{K}^{*g+1}_{\mu}(z_1 - x)I(V)(x)dx\right|dz_1 \\
&\qquad = \sup_{z\in\mathbb{T}}\int_{\mathbb{M}}\left|\int_{\mathbb{M}} \tilde{K}^{*g+1}_{\mu}(z - x)\tilde{K}^{*g+1}_{\mu}(w)I(V)(x)dx\right|dw \\
&\qquad = \sup_{z\in\mathbb{T}}\left|\int_{\mathbb{M}} \tilde{K}^{*g+1}_{\mu}(z - x)I(V)(x)dx\right| = \left\|\tilde{K}^{*g+1}_{\mu} * I(V)\right\|_{\mathcal{V}}.
\end{aligned}$$

Now, it is enough to take $V = \mathbb{1}_{0,1}\partial^r_\kappa F^{i,1}_{\kappa,\mu}$ with $r \in \{0, 1\}$ and $i \in \{1, 2\}$ and to use the inductive hypothesis together with theorem 2.7 to conclude that

$$\begin{aligned}
\left\|\tilde{K}^{*g+1,\otimes 2}_{\mu} * \mathbb{1}_{0,1}\partial^r_\kappa F^{i,1}_{\kappa,\mu}\right\|_{\mathcal{V}} &\lesssim \left\|\tilde{K}^{*g+1}_{\mu} * I\left(\mathbb{1}_{0,1}\partial^r_\kappa F^{i,1}_{\kappa,\mu}\right)\right\|_{\mathcal{V}} + \sum_{b\in\{1,2\}} \mu^{-[b]}\left\|\tilde{K}^{*g,\otimes 2}_{\mu} * \mathbb{1}_{0,1}L_b\partial^r_\kappa F^{i,1}_{\kappa,\mu}\right\|_{\mathcal{V}} \\
&= \left\|\tilde{K}^{*g+1}_{\mu} * \mathbb{1}_{0,1}\partial^r_\kappa \bar{F}^{i,1}_{\kappa,\mu}\right\|_{\mathcal{V}} + \sum_{b\in\{1,2\}} \mu^{-[b]}\left\|\tilde{K}^{*g,\otimes 2}_{\mu} * \mathbb{1}_{0,1}\partial^r_\kappa F^{i,1,b}_{\kappa,\mu}\right\|_{\mathcal{V}}
\end{aligned}$$

$$\lesssim \kappa^{(\epsilon/2-1)r}\mu^{\rho(i,1,0)} + \sum_{b\in\{1,2\}} \mu^{-[b]}\kappa^{(\epsilon/2-1)r}\mu^{\rho(i,1,b)}$$
$$\lesssim \kappa^{(\epsilon/2-1)r}\mu^{\rho(i,1,0)}.$$

That concludes the last two cases. □

Remark 3.4. *We note that in the above theorem, we can choose $g = 7$ because we started with 5 kernels to apply proposition 3.2, and we used the localisation argument to bound 2 terms, so we added 2 kernels.*

Moreover, we point out that all the above arguments lead to a similar inequality with $\mathbb{1}_{0,T}$ for any $T \in (0,1]$. In particular, it holds

$$\left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T}\partial_\kappa^r F_{\kappa,\mu}^{i,m,a}\right\|_{\mathcal{V}} \lesssim \kappa^{(\epsilon/2-1)r}\mu^{\rho(i,m,a)}, \tag{3.2}$$

where $K_\mu = \tilde{K}_\mu^{*7}$ has been defined in definition A.1.

We conclude by noting that thanks to the last theorem, it is possible to define $K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F_{0,\mu}^{i,m,a}(z; dz_1, \ldots, dz_m)$ for any $\mu \in (0,1]$ integrating the above inequality in κ as it is integrable near 0. Moreover, the following holds uniformly in $\mu \in (0,1]$ and $\kappa \in [0,1]$

$$\left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F_{\kappa,\mu}^{i,m,a}\right\|_{\mathcal{V}} \lesssim \mu^{\rho(i,m,a)} \tag{3.3}$$

and a similar consideration can be done with $H_{\kappa,\mu}$.

Chapter 4

Fixed-point argument

This chapter is devoted to the proof of the existence of the solution of the stochastic Burgers equation.

We will prove the following theorem, which is the main result of the thesis.

Theorem 4.1. *Given any $\bar{\alpha} \in \left(\frac{1}{2}, 1\right)$, there exists a (random) time $T > 0$ such that the regularised Burgers equation (1.2) has a solution $f_\kappa \in C^{-\bar{\alpha}}(\mathbb{H}_T)$ with the constant 0 as initial condition for every $\kappa > 0$, and there exists a distribution $f_0 \in C^{-\bar{\alpha}}(\mathbb{H}_T)$ such that f_κ converges to f_0 in $C^{-\bar{\alpha}}(\mathbb{H}_T)$ for κ that goes to 0.*

We recall that we want to study the system (1.11) defined as

$$\begin{cases} \tilde{f}_{\kappa,\mu} = -\int_\mu^1 P_\mu K_\eta * \tilde{G}_\eta * \left(\tilde{F}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right) + \tilde{R}_{\kappa,\eta}\right) d\eta, \\ \tilde{R}_{\kappa,\mu} = -\int_0^\mu P_\eta K_\mu * \left(\tilde{H}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right) + D\tilde{F}^T_{\kappa,\eta}\left(\tilde{f}_{\kappa,\eta}\right)\left[\tilde{G}_\eta * \tilde{R}_{\kappa,\eta}\right]\right) d\eta. \end{cases}$$

We will solve it using the contraction principle in a suitable Banach space. Let us define the latter. First, let us fix $\delta \in (0, \beta/2)$ small enough (recall that $\beta = \frac{1}{4} - 4\epsilon$).

Definition 4.2. *Given $\{f_\mu \in C(\mathbb{H}_T)\}_{\mu\in(0,1]}$ and $\{g_\mu \in C(\mathbb{H}_T)\}_{\mu\in(0,1]}$ both continuous in μ, we define*

$$|||f, g||| := \left(\sup_{\mu\in(0,1]} \mu^{\alpha+\delta} \|f_\mu\|_{L^\infty(\mathbb{H}_T)}\right) \vee \left(\sup_{\mu\in(0,1]} \mu^c \|g_\mu\|_{L^\infty(\mathbb{H}_T)}\right)$$

with $c < 0$ that we will determine later.

We moreover define

$$\mathscr{B}_{T,M} := \left\{(f_\bullet, g_\bullet) \in (C((0,1] \times \mathbb{H}_T))^2 \mid |||f, g||| \leq M\right\}.$$

To solve the system, we would like to define a contraction

$$S_k : \mathscr{B}_{T,M} \to \mathscr{B}_{T,M}$$

such that

$$(S_k(f, g))_1 = -\int_\mu^1 P_\mu K_\eta * \tilde{G}_\eta * \left(\tilde{F}^T_{\kappa,\eta}(f_\eta) + g_\eta\right) d\eta,$$

$$(S_k(f,g))_2 = -\int_0^\mu P_\eta K_\mu * \left(\tilde{H}^T_{\kappa,\eta}(f_\eta) + D\tilde{F}^T_{\kappa,\eta}(f_\eta)\left[\tilde{G}_\eta * g_\eta\right]\right) d\eta$$

to obtain a fixed point. Now we show this is the case for a suitable choice of T and M. The proof relies heavily on inequality (3.3) and requires a lot of calculations, although it is quite easy.

We start by proving a bound on the regularised force.

Lemma 4.3. *Let $(f,g) \in B_{T,M}$ with $T \le 1$ and $M \ge 1$. Then the following hold uniformly in $\mu \in (0,1]$ and $\kappa \in [0,1]$*

$$(A)\ \left\|\tilde{F}^T_{\kappa,\mu}(f_\mu)\right\|_{L^\infty(\mathbb{H}_T)} \lesssim \mu^\gamma M^{2\bar{\iota}},$$

$$(B)\ \left\|\tilde{H}^T_{\kappa,\mu}(f_\mu)\right\|_{L^\infty(\mathbb{H}_T)} \lesssim T^\delta \mu^{-1-\delta+\gamma+(\beta-2\delta)(\bar{\iota}+1)} M^{4\bar{\iota}},$$

$$(C)\ \left\|D\tilde{F}^T_{\kappa,\eta}(f_\eta)\left[\tilde{G}_\eta * g_\eta\right]\right\|_{L^\infty(\mathbb{H}_T)} \lesssim T^\delta \mu^{-1-c} M^{2\bar{\iota}}.$$

Proof. This lemma is a straightforward application of inequality (3.3).

For the first, we use (1.12). Thanks to it, we have

$$\begin{aligned}
&\left\|\tilde{F}^T_{\kappa,\mu}(f_\mu)\right\|_{L^\infty(\mathbb{H}_T)} \\
&\qquad \le \sum_{i=0}^{\bar{\iota}}\sum_{m=0}^{2i} \left\|K_\mu * \mathbb{1}_{0,T} F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}} \|f_\mu\|^m_{L^\infty(\mathbb{H}_T)} \lesssim \sum_{i=0}^{\bar{\iota}}\sum_{m=0}^{2i} \mu^{\alpha m+\beta i+\gamma} \|f_\mu\|^m_{L^\infty(\mathbb{H}_T)} \\
&\qquad \le \mu^\gamma M^{2\bar{\iota}} \sum_{i=0}^{\bar{\iota}} \mu^{\beta i} \sum_{m=0}^{2i} \mu^{-\delta m} \le \mu^\gamma M^{2\bar{\iota}} \sum_{i=0}^{\bar{\iota}} \mu^{(\beta-2\delta)i} \lesssim \mu^\gamma M^{2\iota},
\end{aligned}$$

where we used that $\beta - 2\delta > 0$.

Similarly, we just need to expand the coefficients of $H_{\kappa,\mu}$ and use that $H^{i,m}_{\kappa,\mu} = 0$ if $i \le \bar{\iota}$ or $i > 2\bar{\iota}$ to obtain

$$\begin{aligned}
&\left\|\tilde{H}^T_{\kappa,\mu}(f_\mu)\right\|_{L^\infty(\mathbb{H}_T)} \\
&\quad \le \sum_{i=\bar{\iota}+1}^{2\bar{\iota}}\sum_{m=0}^{2i}\sum_{l=i-\bar{\iota}}^{\bar{\iota}}\sum_{k=0}^{m} \left\|K_\mu^{\otimes m+1} * B\left(\dot{G}_\mu, \mathbb{1}_{0,T} F^{l,k+1}_{\kappa,\mu}, \mathbb{1}_{0,T} F^{i-l,m-k}_{\kappa,\mu}\right)\right\|_{\mathcal{V}} \|f\|^m_{L^\infty(\mathbb{H}_T)} \\
&\quad \le \sum_{i,m,l,k} \left\|P^2_\mu \dot{G}_\mu\right\|_{L^1(\mathbb{H}_T)} \left\|K_\mu^{\otimes k+2} * \mathbb{1}_{0,T} F^{l,k+1}_{\kappa,\mu}\right\|_{\mathcal{V}} \\
&\qquad\qquad \times \left\|K_\mu^{\otimes m-k+1} * \mathbb{1}_{0,T} F^{i-l,m-k}_{\kappa,\mu}\right\|_{\mathcal{V}} \|f\|^m_{L^\infty(\mathbb{H}_T)} \\
&\quad \lesssim \sum_{i,m,l,k} T^\delta \mu^{-1/2-\delta} \cdot \mu^{\alpha(k+1)+\beta l+\gamma} \cdot \mu^{\alpha(m-k)+\beta(i-l)+\gamma} \mu^{-\alpha m-\delta m} M^m \\
&\quad \le T^\delta \mu^{\alpha-1/2-\delta+2\gamma} M^{4\bar{\iota}} \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \mu^{\beta i} \sum_{m=0}^{2i} \mu^{-\delta m} \lesssim T^\delta \mu^{-1-\delta+\gamma} M^{4\bar{\iota}} \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \mu^{(\beta-2\delta)i} \\
&\quad \lesssim T^\delta \mu^{-1-\delta+\gamma+(\beta-2\delta)(\bar{\iota}+1)} M^{4\bar{\iota}},
\end{aligned}$$

where we used inequality (2.2) and proposition C.1 (B).

Finally

$$
\begin{aligned}
&\left\| K_\mu * D\mathbb{1}_{0,T} F_{\kappa,\mu}(K_\mu * f)[K_\mu * \tilde{G}_\mu * g] \right\|_{L^\infty(\mathbb{H}_T)} \\
&\qquad \lesssim \sum_{i=1}^{\bar{\iota}} \sum_{m=1}^{2i} \left\| K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F^{i,m}_{\kappa,\mu} \right\|_{\mathcal{V}} \|f\|^{m-1}_{L^\infty(\mathbb{H}_T)} \left\| \tilde{G}_\mu \right\|_{L^1(\mathbb{H}_T)} \|g\|_{L^\infty(\mathbb{H}_T)} \\
&\qquad \lesssim \sum_{i,m} \mu^{\alpha m + \beta i + \gamma} \cdot \mu^{-(\alpha+\delta)(m-1)} M^{m-1} \cdot T^\delta \mu^{-1/2-\delta} \cdot \mu^{-c} M \\
&\qquad = T^\delta \mu^{-1-c} \sum_{i,m} \mu^{\beta i} \cdot \mu^{-\delta m} M^m \le T^\delta \mu^{-1-c} M^{2\bar{\iota}} \sum_{i=1}^{\bar{\iota}} \mu^{\beta i} \sum_{m=1}^{2i} \mu^{-\delta m} \\
&\qquad \lesssim T^\delta \mu^{-1-c} M^{2\bar{\iota}} \sum_{i=1}^{\bar{\iota}} \mu^{(\beta-2\delta)i} \lesssim T^\delta \mu^{-1-c} M^{2\bar{\iota}}.
\end{aligned}
$$

□

We can now prove that the image of S_κ is contained in $\mathscr{B}_{T,M}$.

Theorem 4.4. *For every $T \le 1$, $M \ge 1$ such that $T^\delta M^{4\bar{\iota}-1}$ is sufficiently small, $S_k : \mathscr{B}_{T,M} \to \mathscr{B}_{T,M}$ is well-defined for every $\kappa \in [0,1]$.*

Before proving it, we stress that the choices of T and M are uniform in κ. This allows us to obtain an a priori estimate independent of the regularisation parameter.

Proof. We will estimate the two components independently. Let us start with the first one.

$$
\begin{aligned}
&\|(S_k(f,g))_1\|_{L^\infty(\mathbb{H}_T)} \\
&\qquad \le \int_\mu^1 \|P_\mu K_\eta\|_{TV} \left\| \tilde{G}_\eta \right\|_{L^1(\mathbb{H}_T)} \left(\left\| \tilde{F}^T_{\kappa,\eta}(f_\eta) \right\|_{L^\infty(\mathbb{H}_T)} + \|g_\eta\|_{L^\infty(\mathbb{H}_T)} \right) d\eta \\
&\qquad \lesssim \int_\mu^1 1 \cdot T^\delta \eta^{-1/2-\delta} \cdot \left(\eta^\gamma M^{2\bar{\iota}} + \eta^{-c} M \right) d\eta \\
&\qquad \lesssim T^\delta M^{2\bar{\iota}} \int_\mu^1 \eta^{-1/2-\delta+\gamma} d\eta \\
&\qquad \lesssim T^\delta M^{2\bar{\iota}} \mu^{-\delta+1/2+\gamma} = T^\delta M^{2\bar{\iota}} \mu^{-\alpha-\delta},
\end{aligned}
$$

where we used propositions A.2 (D) and C.1 (B) and lemma 4.3 (A). So

$$
\sup_{\mu \in (0,1]} \mu^{\alpha+\delta} \|(S_k(f,g))_1\|_{L^\infty(\mathbb{H}_T)} \lesssim T^\delta M^{2\bar{\iota}}.
$$

Whereas for the second component

$$
\begin{aligned}
&\|(S_k(f,g))_2\|_{L^\infty(\mathbb{H}_T)} \\
&\qquad \le \int_0^\mu \|P_\eta K_\mu\|_{TV} \cdot \left(\left\| \tilde{H}^T_{\kappa,\eta}(f_\eta) \right\|_{L^\infty(\mathbb{H}_T)} + \left\| D\tilde{F}^T_{\kappa,\eta}(f_\eta) \left[\tilde{G}_\eta * g_\eta \right] \right\|_{L^\infty(\mathbb{H}_T)} \right) d\eta
\end{aligned}
$$

$$\lesssim \int_0^\mu 1 \cdot \left[T^\delta \eta^{-1-\delta+\gamma+(\beta-2\delta)(\bar\iota+1)} M^{4\bar\iota} + T^\delta \eta^{-1-c} M^{2\bar\iota}\right] d\eta$$
$$\lesssim T^\delta \left(\mu^{-c} + \mu^{-\delta+\gamma+(\beta-2\delta)(\bar\iota+1)}\right) M^{4\bar\iota}.$$

Now it makes sense to impose $c = \delta - \gamma - (\beta - 2\delta)(\bar\iota + 1)$, which is negative if ϵ and δ are sufficiently small (because $\bar\iota = 3$).

This gives us

$$\sup_{\mu\in(0,1]} \mu^c \left\|(S_k(f,g))_2\right\|_{L^\infty(\mathbb{H}_T)} \lesssim T^\delta M^{4\bar\iota},$$

and it allows us to conclude as

$$\left\|\left\|S_\kappa(f,g)\right\|\right\| \lesssim T^\delta M^{4\bar\iota}$$

can be made smaller than M if $T^\delta M^{4\bar\iota-1}$ is sufficiently small. □

Now that we have proved the well-posedness of the map S_κ, we want to show that it is a contraction for a good choice of the two parameters. We stress once more that even for this property the choices of T and M are independent of κ.

Theorem 4.5. *For every $T \leq 1$ and $M \geq 1$ such that $T^\delta M^{4\bar\iota-1}$ is sufficiently small, the family $\{S_k : \mathcal{B}_{T,M} \to \mathcal{B}_{T,M}\}_{\kappa\in[0,1]}$ is 1/2-uniformly Lipschitz.*

Proof. Let (f,g), $\left(\bar f, \bar g\right) \in \mathcal{B}_{T,M}$. We start by proving a result similar to lemma 4.3 suitable for the Lipschitzianity.

$$(A)\ \left\|\tilde F^T_{\kappa,\mu}(f_\mu) - \tilde F^T_{\kappa,\eta}\left(\bar f_\mu\right)\right\|_{L^\infty(\mathbb{H}_T)}$$
$$\leq \sum_{i=1}^{\bar\iota}\sum_{m=1}^{2i} \left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}} \left\|f_\mu - \bar f_\mu\right\|_{L^\infty(\mathbb{H}_T)} \left(\left\|f_\mu\right\|^{m-1}_{L^\infty(\mathbb{H}_T)} \vee \left\|\bar f_\mu\right\|^{m-1}_{L^\infty(\mathbb{H}_T)}\right)$$
$$\lesssim \sum_{i,m} \mu^{\alpha m+\beta i+\gamma} \cdot \mu^{-\alpha-\delta} \left\|\left\|f-\bar f, g-\bar g\right\|\right\| \cdot \mu^{-(\alpha+\delta)(m-1)} M^{m-1}$$
$$\leq \mu^\gamma M^{2\bar\iota-1} \left\|\left\|f-\bar f, g-\bar g\right\|\right\| \sum_{i=1}^{\bar\iota} \mu^{\beta i} \sum_{m=1}^{2i} \mu^{-\delta m}$$
$$\lesssim \mu^\gamma M^{2\bar\iota-1} \left\|\left\|f-\bar f, g-\bar g\right\|\right\| \sum_{i=1}^{\bar\iota} \mu^{(\beta-2\delta)i} \lesssim \mu^\gamma M^{2\bar\iota-1} \left\|\left\|f-\bar f, g-\bar g\right\|\right\|. \quad (4.1)$$

While for the difference of two H, we have

$$(B)\ \left\|\tilde H^T_{\kappa,\mu}\left(f_\mu\right) - \tilde H^T_{\kappa,\mu}\left(\bar f_\mu\right)\right\|_{L^\infty(\mathbb{H}_T)}$$
$$\leq \sum_{i=\bar\iota+1}^{2\bar\iota}\sum_{m=1}^{2i}\sum_{l=i-\bar\iota}^{\bar\iota}\sum_{k=0}^{m} \left\|K_\mu^{\otimes m+1} * B\left(\dot G_\mu, \mathbb{1}_{0,T} F^{l,k+1}_{\kappa,\mu}, \mathbb{1}_{0,T} F^{i-l,m-k}_{\kappa,\mu}\right)\right\|_{\mathcal{V}}$$
$$\times \left\|f_\mu - \bar f_\mu\right\|_{L^\infty(\mathbb{H}_T)} \left(\left\|f_\mu\right\|^{m-1}_{L^\infty(\mathbb{H}_T)} \vee \left\|\bar f_\mu\right\|^{m-1}_{L^\infty(\mathbb{H}_T)}\right)$$
$$\leq \sum_{i,m,l,k} \left\|P_\mu^{2g} \dot G_\mu\right\|_{L^1(\mathbb{H}_T)} \left\|K_\mu^{\otimes k+2} * \mathbb{1}_{0,T} F^{l,k+1}_{\kappa,\mu}\right\|_{\mathcal{V}} \left\|K_\mu^{\otimes m-k+1} * \mathbb{1}_{0,T} F^{i-l,m-k}_{\kappa,\mu}\right\|_{\mathcal{V}}$$
$$\times \left\|f_\mu - \bar f_\mu\right\|_{L^\infty(\mathbb{H}_T)} \left(\left\|f_\mu\right\|^{m-1}_{L^\infty(\mathbb{H}_T)} \vee \left\|\bar f_\mu\right\|^{m-1}_{L^\infty(\mathbb{H}_T)}\right)$$

$$\lesssim \sum_{i,m,l,k} T^{\delta}\mu^{-1/2-\delta}\cdot\mu^{\alpha(k+1)+\beta l+\gamma}\cdot\mu^{\alpha(m-k)+\beta(i-l)+\gamma}\mu^{-\alpha m-\delta m}M^{m-1}|||f-\bar f, g-\bar g|||$$

$$\lesssim T^{\delta}\mu^{\alpha-1/2-\delta+2\gamma}M^{4\bar\iota-1}|||f-\bar f, g-\bar g|||\sum_{i=\bar\iota+1}^{2\bar\iota}\mu^{\beta i}\sum_{m=0}^{2i}\mu^{-\delta m}$$

$$\lesssim T^{\delta}\mu^{-1-\delta+\gamma}M^{4\bar\iota-1}|||f-\bar f, g-\bar g|||\sum_{i=\bar\iota+1}^{2\bar\iota}\mu^{(\beta-2\delta)i}$$

$$\lesssim T^{\delta}\mu^{-1-\delta+\gamma+(\beta-2\delta)(\bar\iota+1)}M^{4\bar\iota-1}|||f-\bar f, g-\bar g||| = T^{\delta}\mu^{-1-c}M^{4\bar\iota-1}|||f-\bar f, g-\bar g|||.$$

Lastly

$$(C)\ \left\|D\tilde F^T_{\kappa,\eta}(f_\mu)\left[\tilde G_\mu * g_\mu\right] - D\tilde F^T_{\kappa,\mu}(\bar f_\mu)\left[\tilde G_\mu * \bar g_\mu\right]\right\|_{L^\infty(\mathbb{H}_T)}$$

$$\leq \sum_{i=1}^{\bar\iota}\sum_{m=1}^{2i}\left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T}F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}}\left\|\tilde G_\mu\right\|_{L^1(\mathbb{H}_T)}\left[\left(\|f_\mu\|^{m-1}_{L^\infty(\mathbb{H}_T)}\|g_\mu-\bar g_\mu\|_{L^\infty(\mathbb{H}_T)}\right)\right.$$
$$\left.\vee\left(\|f_\mu\|^{m-2}_{L^\infty(\mathbb{H}_T)}\|g_\mu\|_{L^\infty(\mathbb{H}_T)}\left\|f_\mu-\bar f_\mu\right\|_{L^\infty(\mathbb{H}_T)}\right)\right]$$

$$\lesssim \sum_{i,m}\mu^{\alpha m+\beta i+\gamma}\cdot T^{\delta}\mu^{-1/2-\delta}\cdot\mu^{-(\alpha+\delta)(m-1)}\mu^{-c}M^{m-1}|||f-\bar f, g-\bar g|||$$

$$\leq \mu^{\alpha+\gamma-1/2-c}T^{\delta}M^{2\bar\iota-1}|||f-\bar f, g-\bar g|||\sum_{i=1}^{\bar\iota}\mu^{\beta i}\sum_{m=1}^{2i}\mu^{-\delta m}$$

$$\lesssim \mu^{-1-c}T^{\delta}M^{2\bar\iota-1}|||f-\bar f, g-\bar g|||\sum_{i=1}^{\bar\iota}\mu^{(\beta-2\delta)i}$$

$$\lesssim \mu^{-1-c}T^{\delta}M^{2\bar\iota-1}|||f-\bar f, g-\bar g|||.$$

We now have all the tools to prove the theorem. Indeed, for the first component, we have

$$\left\|\left(S_k(f,g)-S_k\left(\bar f,\bar g\right)\right)_1\right\|_{L^\infty(\mathbb{H}_T)}$$
$$\leq \int_\mu^1\|P_\mu K_\eta\|_{TV}\left\|\tilde G_\eta\right\|_{L^1(\mathbb{H}_T)}\left(\left\|\tilde F^T_{\kappa,\eta}(f_\eta)-\tilde F^T_{\kappa,\eta}\left(\bar f_\eta\right)\right\|_{L^\infty(\mathbb{H}_T)}+\|g_\eta-\bar g_\eta\|_{L^\infty(\mathbb{H}_T)}\right)d\eta$$
$$\lesssim \int_\mu^1 1\cdot T^{\delta}\eta^{-1/2-\delta}\cdot\left(\eta^{\gamma}M^{2\bar\iota-1}|||f-\bar f, g-\bar g|||+\eta^{-c}|||f-\bar f, g-\bar g|||\right)d\eta$$
$$\lesssim T^{\delta}M^{2\bar\iota-1}|||f-\bar f, g-\bar g|||\int_\mu^1\eta^{-1/2-\delta+\gamma}d\eta \lesssim T^{\delta}M^{2\bar\iota-1}|||f-\bar f, g-\bar g|||\mu^{-\delta+1/2+\gamma}$$
$$= T^{\delta}M^{2\bar\iota-1}|||f-\bar f, g-\bar g|||\mu^{-\alpha-\delta}.$$

While for the second one

$$\left\|\left(S_k(f,g)-S_k\left(\bar f,\bar g\right)\right)_2\right\|_{L^\infty(\mathbb{H}_T)}$$
$$\leq \int_0^\mu\|P_\eta K_\mu\|_{TV}\cdot\left[\left\|\tilde H^T_{\kappa,\eta}(f_\eta)-\tilde H^T_{\kappa,\eta}\left(\bar f_\eta\right)\right\|_{L^\infty(\mathbb{H}_T)}+\right.$$
$$\left.+\left\|D\tilde F^T_{\kappa,\eta}(f_\eta)\left[\tilde G_\eta * g_\eta\right]-D\tilde F^T_{\kappa,\eta}\left(\bar f_\eta\right)\left[\tilde G_\eta * \bar g_\eta\right]\right\|_{L^\infty(\mathbb{H}_T)}\right]d\eta$$

$$
\begin{aligned}
&\lesssim \int_0^{\mu} 1 \cdot \left[T^\delta \eta^{-1-c} M^{4\bar{\iota}-1} \left|\left|\left| f-\bar{f}, g-\bar{g} \right|\right|\right| + \eta^{-1-c} T^\delta M^{2\bar{\iota}-1} \left|\left|\left| f-\bar{f}, g-\bar{g} \right|\right|\right|\right] d\eta \\
&= T^\delta \left|\left|\left| f-\bar{f}, g-\bar{g} \right|\right|\right| \left(M^{4\bar{\iota}-1} + M^{2\bar{\iota}-1}\right) \int_0^{\mu} \eta^{-1-c} d\eta \\
&\lesssim T^\delta \mu^{-c} \left|\left|\left| f-\bar{f}, g-\bar{g} \right|\right|\right| \left(M^{4\bar{\iota}-1} + M^{2\bar{\iota}-1}\right) \lesssim T^\delta \mu^{-c} M^{4\bar{\iota}-1} \left|\left|\left| f-\bar{f}, g-\bar{g} \right|\right|\right|.
\end{aligned}
$$

The last two together give us

$$
\left|\left|\left| S_\kappa(f,g) - S_\kappa\left(\bar{f},\bar{g}\right) \right|\right|\right| \lesssim T^\delta M^{4\bar{\iota}-1} \left|\left|\left| f-\bar{f}, g-\bar{g} \right|\right|\right|
$$

that concludes the proof. □

Now that we have the last two theorems, for each κ we can define the fixed point of S_κ. But to get a solution to the original equation, we need a stability result: the convergence of these fixed points as κ goes to 0.

To do so, we will use the following easy lemma.

Lemma 4.6. *Let X be a complete metric space, and for every κ in $[0,1]$ let $T_\kappa : X \to X$ be a $L-$Lipschitz function with $L < 1$. Assume that $T_\kappa(\phi) \to T_0(\phi)$ for κ that goes to 0 for each ϕ in X. Moreover, let $\{\phi_\kappa\}_{\kappa\in[0,1]}$ be their unique fixed point given by the contraction principle. Then $\phi_\kappa \to \phi_0$ for κ that goes to 0.*

Proof. Using the triangle inequality and the Lipschitz condition, we have

$$
\begin{aligned}
d(\phi_\kappa, \phi_0) &\leq d(T_\kappa(\phi_\kappa), T_\kappa(\phi_0)) + d(T_\kappa(\phi_0), T_0(\phi_0)) \\
&\leq L d(\phi_k, \phi_0) + d(T_\kappa(\phi_0), T_0(\phi_0)).
\end{aligned}
$$

This concludes the proof as

$$
d(\phi_\kappa, \phi_0) \leq \frac{d(T_\kappa(\phi_0), T_0(\phi_0))}{1-L} \to 0
$$

□

To use the latter, we still need to prove the stability of S_κ as κ goes to 0. This is exactly the content of the following theorem.

Theorem 4.7. *For every T and M given by theorem 4.4, it holds*

$$
S_\kappa(f,g) \xrightarrow{\mathscr{B}_{T,M}} S_0(f,g)
$$

for κ that goes to 0 and for every (f,g) in $\mathscr{B}_{T,M}$.

Like the previous results of this chapter, the proof of this theorem is quite easy, but it requires some lengthy calculations.

Proof. First, we will find a relation between $\tilde{F}^T_{\kappa,\mu}(f_\mu)$ and $\tilde{F}^T_{0,\mu}(f_\mu)$. This is easy to obtain as

$$
\tilde{F}^T_{\kappa,\eta}(f_\mu) = \sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \int_{\mathbb{M}^{m+1}} K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} F^{i,m}_{\kappa,\mu}(z; dz_1, \ldots, dz_m) \prod_{j=1}^{m} f_\mu(z_j) dz
$$

$$= \sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \int_{\mathbb{M}^{m+1}} K_{\mu}^{\otimes 1+m} * \mathbb{1}_{0,T} F_{0,\mu}^{i,m}(z; dz_1, \ldots, dz_m) \prod_{j=1}^{m} f_{\mu}(z_j) dz$$
$$+ \sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \int_0^k \int_{\mathbb{M}^{m+1}} K_{\mu}^{\otimes 1+m} * \mathbb{1}_{0,T} \partial_l F_{l,\mu}^{i,m}(z; dz_1, \ldots, dz_m) \prod_{j=1}^{m} f_{\mu}(z_j) dz dl.$$

Thanks to this, we have the following inequality

$$\begin{aligned}
\left\| \tilde{F}_{\kappa,\eta}^T (f_\mu) - \tilde{F}_{0,\mu}^T (f_\mu) \right\|_{L^\infty(\mathbb{H}_T)} &\le \sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \int_0^k \left\| K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} \partial_l F_{l,\mu}^{i,m} \right\|_{\mathcal{V}} \| f_\mu \|_{L^\infty(\mathbb{H}_T)}^m dl \\
&\lesssim \sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \int_0^k l^{\epsilon/2-1} \mu^{\alpha m + \beta i + \gamma} \| f_\mu \|_{L^\infty(\mathbb{H}_T)}^m dl \\
&\lesssim \sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \kappa^{\epsilon/2} \mu^{\alpha m + \beta i + \gamma} \| f_\mu \|_{L^\infty(\mathbb{H}_T)}^m \\
&\le \kappa^{\epsilon/2} \mu^{\gamma} \sum_{i=0}^{\bar{\iota}} \sum_{m=0}^{2i} \mu^{\alpha m + \beta i} \cdot \mu^{-(\alpha+\delta)m} M^m \\
&\le \kappa^{\epsilon/2} \mu^{\gamma} M^{2\bar{\iota}} \sum_{i=0}^{\bar{\iota}} \mu^{\beta i} \sum_{m=0}^{2i} \mu^{-\delta m} \\
&\lesssim \kappa^{\epsilon/2} \mu^{\gamma} M^{2\bar{\iota}} \sum_{i=0}^{\bar{\iota}} \mu^{(\beta - 2\delta) i} \le \kappa^{\epsilon/2} \mu^{\gamma} M^{2\bar{\iota}}, \qquad (4.2)
\end{aligned}$$

where we used theorem 3.3.

This immediately gives us the convergence of the first component, in fact

$$\begin{aligned}
&\| (S_k(f,g) - S_0(f,g))_1 \|_{L^\infty(\mathbb{H}_T)} \\
&\qquad \le \int_\mu^1 \| P_\mu K_\eta \|_{TV} \left\| \tilde{G}_\eta \right\|_{L^1(\mathbb{H}_T)} \left\| \tilde{F}_{\kappa,\eta}^T (f_\eta) - \tilde{F}_{0,\eta}^T (f_\eta) \right\|_{L^\infty(\mathbb{H}_T)} d\eta \\
&\qquad \lesssim \int_\mu^1 1 \cdot \eta^{-1/2} \cdot \kappa^{\epsilon/2} \eta^{\gamma} M^{2\bar{\iota}} d\eta \lesssim \kappa^{\epsilon/2} \mu^{1/2+\gamma} M^{2\bar{\iota}},
\end{aligned}$$

which concludes because

$$\begin{aligned}
\sup_{\mu \in (0,1]} \mu^{\alpha+\delta} \| (S_k(f,g) - S_0(f,g))_1 \|_{L^\infty(\mathbb{H}_T)} &\lesssim \sup_{\mu \in (0,1]} \mu^{\alpha+\delta} \kappa^{\epsilon/2} \mu^{1/2+\gamma} M^{2\bar{\iota}} \\
&= \sup_{\mu \in (0,1]} \mu^{\delta} \kappa^{\epsilon/2} M^{2\bar{\iota}} = \kappa^{\epsilon/2} M^{2\bar{\iota}} \xrightarrow{\kappa \to 0} 0.
\end{aligned}$$

We can now do similar computations for the second component.

First

$$\tilde{H}_{\kappa,\mu}^T (f_\mu) = \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \sum_{m=0}^{2i} \int_{\mathbb{M}^{m+1}} K_{\mu}^{\otimes 1+m} * \mathbb{1}_{0,T} H_{\kappa,\mu}^{i,m}(z; dz_1, \ldots, dz_m) \prod_{j=1}^{m} f_{\mu}(z_j) dz$$

$$= \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \sum_{m=0}^{2i} \int_{\mathbb{M}^{m+1}} K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} H_{0,\mu}^{i,m}(z; dz_1, \dots, dz_m) \prod_{j=1}^{m} f_\mu(z_j) dz$$
$$+ \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \sum_{m=0}^{2i} \int_0^k \int_{\mathbb{M}^{m+1}} K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} \partial_l H_{\kappa,\mu}^{i,m}(z; dz_1, \dots, dz_m) \prod_{j=1}^{m} f_\mu(z_j) dz dl.$$

Let us compute the term inside the integral. Deriving expression (1.7) in κ, we obtain

$$\mathbb{1}_{0,T} \partial_\kappa H_{\kappa,\mu}^{i,m} = \sum_{l=1-\bar{\iota}}^{\bar{\iota}} \sum_{j=0}^{m} (j+1) \Big[B\left(\dot{G}_\mu, \mathbb{1}_{0,T} \partial_\kappa F_{\kappa,\mu}^{l,j+1}, \mathbb{1}_{0,T} F_{\kappa,\mu}^{i-l,m-j}\right) +$$
$$+ B\left(\dot{G}_\mu, \mathbb{1}_{0,T} F_{\kappa,\mu}^{l,j+1}, \mathbb{1}_{0,T} \partial_\kappa F_{\kappa,\mu}^{i-l,m-j}\right)\Big].$$

After convolving with $K_\mu^{\otimes 1+m}$, the last series can be bounded term by term using inequality (3.3), proposition C.1 (A) and theorem 3.3) to get

$$\left\| K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} \partial_\kappa H_{\kappa,\mu}^{i,m} \right\|_{\mathcal{V}}$$
$$\lesssim \sum_{l=1-\bar{\iota}}^{\bar{\iota}} \sum_{j=0}^{m} \left\| \dot{G}_\mu \right\|_{L^1(\mathbb{H}_T)} \Big[\left\| K_\mu^{\otimes j+2} * \mathbb{1}_{0,T} \partial_\kappa F_{\kappa,\mu}^{l,j+1} \right\|_{\mathcal{V}} \left\| K_\mu^{\otimes m-j+1} * \mathbb{1}_{0,T} F_{\kappa,\mu}^{i-l,m-j} \right\|_{\mathcal{V}} +$$
$$+ \left\| K_\mu^{\otimes j+1} * \mathbb{1}_{0,T} F_{\kappa,\mu}^{l,j+1} \right\|_{\mathcal{V}} \left\| K_\mu^{\otimes m-j+1} * \mathbb{1}_{0,T} \partial_\kappa F_{\kappa,\mu}^{i-l,m-j} \right\|_{\mathcal{V}} \Big]$$
$$\lesssim \sum_{l=1-\bar{\iota}}^{\bar{\iota}} \sum_{j=0}^{m} \mu^{-1/2} \big[\kappa^{\epsilon/2-1} \mu^{\alpha(j+1)+\beta l+\gamma} \cdot \mu^{\alpha(m-j)+\beta(i-l)+\gamma}$$
$$+ \mu^{\alpha(j+1)+\beta l+\gamma} \cdot \kappa^{\epsilon-1} \mu^{\alpha(m-j)+\beta(i-l)+\gamma} \big]$$
$$\lesssim \sum_{l=1-\bar{\iota}}^{\bar{\iota}} \sum_{j=0}^{m} \mu^{-1/2} \kappa^{\epsilon/2-1} \mu^{\alpha(m+1)+\beta i+2\gamma} \lesssim \kappa^{\epsilon/2-1} \mu^{\alpha m+\beta i+\gamma-1}.$$

Which finally brings us to an estimate on $\tilde{H}_{\kappa,\mu}^T(\bullet) - \tilde{H}_{0,\mu}^T(\bullet)$. Indeed the previous results give

$$\left\| \tilde{H}_{\kappa,\mu}^T(f_\mu) - \tilde{H}_{0,\mu}^T(f_\mu) \right\|_{L^\infty(\mathbb{H}_T)}$$
$$\leq \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \sum_{m=0}^{2i} \int_0^k \left\| K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T} \partial_l H_{l,\mu}^{i,m} \right\|_{\mathcal{V}} \left\| f_\mu \right\|_{L^\infty(\mathbb{H}_T)}^m dl$$
$$\lesssim \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \sum_{m=0}^{2i} \int_0^k l^{\epsilon/2-1} \mu^{\alpha m+\beta i+\gamma-1} \left\| f_\mu \right\|_{L^\infty(\mathbb{H}_T)}^m dl$$
$$\lesssim \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \sum_{m=0}^{2i} \kappa^{\epsilon/2} \mu^{\alpha m+\beta i+\gamma-1} \left\| f_\mu \right\|_{L^\infty(\mathbb{H}_T)}^m$$
$$\leq \kappa^{\epsilon/2} \mu^{\gamma-1} \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \sum_{m=0}^{2i} \mu^{\alpha m+\beta i} \cdot \mu^{-(\alpha+\delta)m} M^m \leq \kappa^{\epsilon/2} \mu^{\gamma-1} M^{4\bar{\iota}} \sum_{i=\bar{\iota}+1}^{2\bar{\iota}} \mu^{\beta i} \sum_{m=0}^{2i} \mu^{-\delta m}$$

$$\lesssim \kappa^{\epsilon/2}\mu^{\gamma-1}M^{4\bar{\iota}}\sum_{i=\bar{\iota}+1}^{2\bar{\iota}}\mu^{(\beta-2\delta)i} \lesssim \kappa^{\epsilon/2}\mu^{\gamma-1+(\beta-2\delta)(\bar{\iota}+1)}M^{4\bar{\iota}} = \kappa^{\epsilon/2}\mu^{-1-c+\delta}M^{4\bar{\iota}}.$$

Now we can repeat all these computations for the term with the Gâteaux derivative.

$$\begin{aligned}
&K_\mu * D\mathbb{1}_{0,T}F_{\kappa,\mu}\left(K_\mu * f_\mu\right)\left[K_\mu * \tilde{G}_\mu * g_\mu\right] \\
&= \sum_{i=1}^{\bar{\iota}}\sum_{m=1}^{2i} m\int_{\mathbb{M}^m} K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T}F_{\kappa,\mu}^{i,m}(z;dz_1,\ldots,dz_m)\cdot\tilde{G}_\mu * g_\mu(z_1)\prod_{j=2}^m f_\mu(z_j)dz \\
&= \sum_{i=1}^{\bar{\iota}}\sum_{m=1}^{2i} m\int_{\mathbb{M}^m} K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T}F_{0,\mu}^{i,m}(z;dz_1,\ldots,dz_m)\cdot\tilde{G}_\mu * g_\mu(z_1)\prod_{j=2}^m f_\mu(z_j)dz \\
&+ \sum_{i=1}^{\bar{\iota}}\sum_{m=1}^{2i} m\int_0^k\int_{\mathbb{M}^m} K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T}\partial_l F_{l,\mu}^{i,m}(z;dz_1,\ldots,dz_m)\cdot\tilde{G}_\mu * g_\mu(z_1)\prod_{j=2}^m f_\mu(z_j)dzdl,
\end{aligned}$$

so that

$$\begin{aligned}
&\left\|D\tilde{F}_{\kappa,\eta}^T\left(f_\eta\right)\left[\tilde{G}_\eta * g_\eta\right] - D\tilde{F}_{0,\eta}^T\left(f_\eta\right)\left[\tilde{G}_\eta * g_\eta\right]\right\|_{L^\infty(\mathbb{H}_T)} \\
&\quad\lesssim \sum_{i=1}^{\bar{\iota}}\sum_{m=1}^{2i}\int_0^\kappa \left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T}\partial_l F_{l,\mu}^{i,m}\right\|_{\mathcal{V}}\cdot\left\|\tilde{G}_\mu * g_\mu\right\|_{L^\infty(\mathbb{H}_T)}\cdot\|f_\mu\|_{L^\infty(\mathbb{H}_T)}^{m-1}dl \\
&\quad\le \sum_{i=1}^{\bar{\iota}}\sum_{m=1}^{2i}\int_0^\kappa \left\|K_\mu^{\otimes 1+m} * \mathbb{1}_{0,T}\partial_l F_{l,\mu}^{i,m}\right\|_{\mathcal{V}}\cdot\left\|\tilde{G}_\mu\right\|_{L^1(\mathbb{H}_T)}\cdot\|g_\mu\|_{L^\infty(\mathbb{H}_T)}\cdot\|f_\mu\|_{L^\infty(\mathbb{H}_T)}^{m-1}dl \\
&\quad\lesssim \sum_{i=1}^{\bar{\iota}}\sum_{m=1}^{2i}\int_0^\kappa l^{\epsilon/2-1}\mu^{\alpha m+\beta i+\gamma}\cdot\mu^{-1/2}\cdot\mu^{-c}M\cdot\mu^{-(\alpha+\delta)(m-1)}M^{m-1}dl \\
&\quad= \mu^{\gamma-1/2-c+\alpha+\delta}\sum_{i=1}^{\bar{\iota}}\sum_{m=1}^{2i}\mu^{\beta i-\delta m}M^m\int_0^\kappa l^{\epsilon/2-1}dl \\
&\quad\lesssim \kappa^{\epsilon/2}\mu^{\gamma-1/2-c+\alpha+\delta}\sum_{i=1}^{\bar{\iota}}\sum_{m=1}^{2i}\mu^{\beta i-\delta m}M^m \le \kappa^{\epsilon/2}\mu^{-1-c+\delta}M^{2\bar{\iota}}\sum_{i=1}^{\bar{\iota}}\mu^{\beta i}\sum_{m=1}^{2i}\mu^{-\delta m} \\
&\quad\lesssim \kappa^{\epsilon/2}\mu^{-1-c+\delta}M^{2\bar{\iota}}\sum_{i=1}^{\bar{\iota}}\mu^{(\beta-2\delta)i} \lesssim \kappa^{\epsilon/2}\mu^{-1-c+\delta}M^{2\bar{\iota}}.
\end{aligned}$$

This finally proves the desired estimate on the second component.

$$\begin{aligned}
&\|(S_k(f,g) - S_0\left(f,g\right))_2\|_{L^\infty(\mathbb{H}_T)} \\
&\quad\le \int_0^\mu \|P_\eta K_\mu\|_{TV}\cdot\left[\left\|\tilde{H}_{\kappa,\eta}^T\left(f_\eta\right) - \tilde{H}_{0,\eta}^T(f_\eta)\right\|_{L^\infty(\mathbb{H}_T)} + \right. \\
&\qquad\qquad\left. + \left\|D\tilde{F}_{\kappa,\eta}^T\left(f_\eta\right)\left[\tilde{G}_\eta * g_\eta\right] - D\tilde{F}_{0,\eta}^T\left(f_\eta\right)\left[\tilde{G}_\eta * g_\eta\right]\right\|_{L^\infty(\mathbb{H}_T)}\right]d\eta \\
&\quad\lesssim \int_0^\mu \left[\kappa^{\epsilon/2}\eta^{-1-c+\delta}M^{4\bar{\iota}} + \kappa^{\epsilon/2}\eta^{-1-c+\delta}M^{2\bar{\iota}}\right]d\eta
\end{aligned}$$

$$\lesssim \kappa^{\epsilon/2} \int_0^{\mu} \eta^{-1-c+\delta} M^{4\bar{\iota}} d\eta \lesssim \kappa^{\epsilon/2} \mu^{-c+\delta} M^{4\bar{\iota}}.$$

That concludes the proof of the theorem as

$$\sup_{\mu \in (0,1]} \mu^c \left\| \left(S_k(f,g) - S_0\left(f,g\right)\right)_2 \right\|_{L^\infty(\mathbb{H}_T)} \lesssim \sup_{\mu \in (0,1]} \kappa^{\epsilon/2} \mu^{\delta} M^{4\bar{\iota}} \xrightarrow{\kappa \to 0} 0.$$

□

We are ready to prove the final theorem 4.1.

The strategy consists in repeating the argument clarified in the introduction, but in reverse order, since by now, we have proved the existence of the solution of the regularised system (1.11).

We start by considering for each $\kappa \in [0,1]$, $\left(\tilde{f}_{\kappa,\bullet}, \tilde{R}_{\kappa,\bullet}\right) \in \mathscr{B}_{T,M}$ the unique fixed point of S_κ. Now, inspired by the starting computations, we define the following two

$$f_{\kappa,\mu} := K_\mu * \tilde{f}_{\kappa,\mu}, \quad R_{\kappa,\mu} := P_\mu \tilde{R}_{\kappa,\mu}.$$

Then those two solve the non-regularised system

$$\begin{cases} f_{\kappa,\mu} = -\int_\mu^1 \dot{G}_\eta * \left(F_{\kappa,\eta}(f_{\kappa,\eta}) + R_{\kappa,\eta}\right) d\eta, \\ R_{\kappa,\mu} = -\int_0^\mu \left(H_{\kappa,\eta}(f_{\kappa,\eta}) + DF_{\kappa,\eta}(f_{\kappa,\eta}) \left[\dot{G}_\eta * R_{\kappa,\eta}\right]\right) d\eta. \end{cases}$$

This proves that $R_{\kappa,\mu}$ is a function, not just a distribution.

The first equation of the last system can be simplified. In fact, by the easy computation

$$\partial_\mu \left(F_{\kappa,\mu}\left(f_{\kappa,\mu}\right) + R_{\kappa,\mu}\right) = 0,$$

we obtain

$$f_{\kappa,\mu} = G_\mu * \left(F_{\kappa,\eta}\left(f_{\kappa,\eta}\right) + R_{\kappa,\eta}\right) \tag{4.3}$$

for any $\eta \in (0,1]$.

Finally, for each $\kappa \in (0,1]$ we define

$$f_\kappa := G * \left(F_{\kappa,\eta}\left(f_{\kappa,\eta}\right) + R_{\kappa,\eta}\right).$$

We want to prove that this is the solution of the regularised Burgers equation (1.3) and that f_κ converges to some f_0 in $C^{-2\alpha-2\delta}$ for κ that goes to 0.

Fix a $\kappa \in (0,1]$. First, we will study the limit of $(f_{\kappa,\mu}, R_{\kappa,\mu})$ as μ goes to zero. Note that

$$\|f_{\kappa,\mu} - f_\kappa\|_{L^\infty(\mathbb{H}_T)} \leq \|G - G_\mu\|_{L^1(\mathbb{H}_T)} \cdot \left\|F_{\kappa,1}\left(f_{\kappa,1}\right) + R_{\kappa,1}\right\|_{L^\infty(\mathbb{H}_T)} \lesssim_\kappa \mu^{1/2} \xrightarrow{\mu \to 0} 0 \tag{4.4}$$

by proposition C.1 (C). This also proves that fixed κ, $\|f_{\kappa,\mu}\|_{L^\infty(\mathbb{H}_T)}$ is uniformly bounded in $\mu \in (0,1]$.

Moreover

$$\|R_{\kappa,\mu}\|_{L^\infty(\mathbb{H}_T)} \leq \int_0^\mu \left\|H_{\kappa,\eta}\left(f_{\kappa,\eta}\right)\right\|_{L^\infty(\mathbb{H}_T)} + \left\|DF_{\kappa,\eta}\left(f_{\kappa,\eta}\right)\left[\dot{G}_\eta * R_{\kappa,\eta}\right]\right\|_{L^\infty(\mathbb{H}_T)} d\eta$$

$$\lesssim_{\kappa,T} \int_0^\mu \eta^{-1/2} + \eta^{-1/2} \|R_{\kappa,\eta}\|_{L^\infty(\mathbb{H}_T)} \, d\eta$$
$$\lesssim \mu^{1/2} + \int_0^\mu \eta^{-1/2} \|R_{\kappa,\eta}\|_{L^\infty(\mathbb{H}_T)} \, d\eta \tag{4.5}$$

by theorem D.1 and inequality (D.1). This, together with Grönwall's lemma, gives

$$\|R_{\kappa,\mu}\|_{L^\infty(\mathbb{H}_T)} \lesssim_{\kappa,T} \mu^{1/2} \exp\left(A_{\kappa,T}\mu^{1/2}\right) \xrightarrow{\mu\to 0} 0$$

for some constant $A_{\kappa,T}$ that depends on κ and on T.

We now show that the expression $F_{\kappa,\mu}(f_{\kappa,\mu}) + R_{\kappa,\mu}$, which is constant in μ, converges to $F_\kappa(f_\kappa)$ and so it is identically equal to it. We have already shown that $R_{\kappa,\mu} \xrightarrow{\mu\to 0} 0$ and so the only thing missing is $F_{\kappa,\mu}(f_{\kappa,\mu}) \xrightarrow{\mu\to 0} F_\kappa(f_\kappa)$.

First observe that $F_\kappa(f_{\kappa,\mu}) - F_\kappa(f_\kappa) = f_{\kappa,\mu}^2 - f_\kappa^2$ and so

$$\|F_\kappa(f_{\kappa,\mu}) - F_\kappa(f_\kappa)\|_{L^\infty(\mathbb{H}_T)} = \left\|f_{\kappa,\mu}^2 - f_\kappa^2\right\|_{L^\infty(\mathbb{H}_T)} \xrightarrow{\mu\to 0} 0$$

by (4.4).

Let us prove that

$$\|F_{\kappa,\mu}(f_{\kappa,\mu}) - F_\kappa(f_{\kappa,\mu})\|_{L^\infty(\mathbb{H}_T)} \to 0$$

for μ that goes to 0.

To achieve this, we perform a computation similar to the ones in theorem 4.7.

$$\begin{aligned}
F_{\kappa,\mu}(f_{\kappa,\mu}) &= \sum_{i=0}^{\bar\iota}\sum_{m=0}^{2i}\int_{\mathbb{M}^m} F_{\kappa,\eta}^{i,m}(z;dz_1,\ldots,dz_m)\prod_{j=1}^m f_{\kappa,\mu}(z_j)dz \\
&= \sum_{i=0}^{\bar\iota}\sum_{m=0}^{2i}\int_{\mathbb{M}^m} F_{\kappa}^{i,m}(z;dz_1,\ldots,dz_m)\prod_{j=1}^m f_{\kappa,\mu}(z_j)dz \\
&+ \sum_{i=0}^{\bar\iota}\sum_{m=0}^{2i}\int_0^\mu\int_{\mathbb{M}^m} \partial_\eta F_{\kappa,\eta}^{i,m}(z;dz_1,\ldots,dz_m)\prod_{j=1}^m f_{\kappa,\mu}(z_j)dzd\eta,
\end{aligned}$$

so that

$$F_{\kappa,\mu}(f_{\kappa,\mu}) - F_\kappa(f_{\kappa,\mu}) = \sum_{i=0}^{\bar\iota}\sum_{m=0}^{2i}\int_0^\mu\int_{\mathbb{M}^m} \partial_\eta F_{\kappa,\eta}^{i,m}(z;dz_1,\ldots,dz_m)\prod_{j=1}^m f_{\kappa,\mu}(z_j)dzd\eta.$$

Taking the norm and using theorem D.1, we obtain

$$\begin{aligned}
\|F_{\kappa,\mu}(f_{\kappa,\mu}) - F_\kappa(f_{\kappa,\mu})\|_{L^\infty(\mathbb{H}_T)} &\lesssim_\kappa \sum_{i=0}^{\bar\iota}\sum_{m=0}^{2i}\int_0^\mu \|f_{\kappa,\mu}\|_{L^\infty(\mathbb{H}_T)}^m \eta^{-1/2} d\eta \\
&\lesssim_\kappa \sum_{i=0}^{\bar\iota}\sum_{m=0}^{2i}\int_0^\eta \mu^{-1/2} d\eta \lesssim \mu^{1/2} \xrightarrow{\mu\to 0} 0,
\end{aligned} \tag{4.6}$$

where we used the fact that $\|f_{\kappa,\mu}\|_{L^\infty(\mathbb{H}_T)}$ is bounded uniformly in μ.

This concludes as

$$
\begin{aligned}
\|F_{\kappa,\mu}\left(f_{\kappa,\mu}\right) - &F_{\kappa}\left(f_{\kappa}\right)\|_{L^{\infty}(\mathbb{H}_T)} \\
&\leq \|F_{\kappa,\mu}\left(f_{\kappa,\mu}\right) - F_{\kappa}\left(f_{\kappa,\mu}\right)\|_{L^{\infty}(\mathbb{H}_T)} + \|F_{\kappa}\left(f_{\kappa,\mu}\right) - F_{\kappa}\left(f_{\kappa}\right)\|_{L^{\infty}(\mathbb{H}_T)} \xrightarrow{\mu\to 0} 0
\end{aligned}
$$

and so $F_{\kappa,\mu}\left(f_{\kappa,\mu}\right) + R_{\kappa,\mu} = F_{\kappa}\left(f_{\kappa}\right)$. This finally implies

$$
f_k = G * F_{\kappa}\left(f_{\kappa}\right).
$$

We can now focus on the last part: the convergence as κ goes to 0. Heuristically, we know that $\tilde{f}_{\kappa,\mu}$ is close to $\tilde{f}_{0,\mu}$ for small κ thanks to lemma 4.6. So we want to understand how much $\tilde{f}_{\kappa,\mu}$ is close to f_{κ} to conclude. To do this, observe the following

$$
f_{\kappa} - f_{\kappa,\mu} = \left(G - G_{\mu}\right) * \left(F_{\kappa,\mu}\left(f_{\kappa,\mu}\right) + R_{\kappa,\mu}\right).
$$

This implies

$$
f_{\kappa} = K_{\mu} * \tilde{f}_{\kappa,\mu} + \left(G - G_{\mu}\right) * \left(P_{\mu}\tilde{F}^{T}_{\kappa,\mu}\left(\tilde{f}_{\kappa,\mu}\right) + P_{\mu}\tilde{R}_{\kappa,\mu}\right) \tag{4.7}
$$

for $\kappa = 0$, we let the right-hand side be the definition of the left-hand side.

To prove the desired convergence, we must study $K_{\mu} * f_{\kappa}$ (recall definition B.1). From the previous equation, this is

$$
K_{\mu} * f_{\kappa} = K_{\mu} * K_{\mu} * \tilde{f}_{\kappa,\mu} + \left(G - G_{\mu}\right) * \left(\tilde{F}^{T}_{\kappa,\mu}\left(\tilde{f}_{\kappa,\mu}\right) + \tilde{R}_{\kappa,\mu}\right).
$$

We are now ready to conclude. Observe that

$$
\begin{aligned}
\|f_{\kappa} - f_0\|_{C^{-2\alpha-2\delta}(\mathbb{H}_T)} &\leq \sup_{\mu\in(0,1]} \mu^{\alpha+\delta} \|K_{\mu} * (f_{\kappa} - f_0)\|_{L^{\infty}} \\
&\leq \sup_{\mu\in(0,1]} \mu^{\alpha+\delta} \left\|K_{\mu} * K_{\mu} * \tilde{f}_{\kappa,\mu} - K_{\mu} * K_{\mu} * \tilde{f}_{0,\mu}\right\|_{L^{\infty}(\mathbb{H}_T)} \\
&+ \sup_{\mu\in(0,1]} \mu^{\alpha+\delta} \left\|\left(G - G_{\mu}\right) * \left(\tilde{R}_{\kappa,\mu} - \tilde{R}_{0,\mu}\right)\right\|_{L^{\infty}(\mathbb{H}_T)} \\
&+ \sup_{\mu\in(0,1]} \mu^{\alpha+\delta} \left\|\left(G - G_{\mu}\right) * \left(\tilde{F}^{T}_{\kappa,\mu}\left(\tilde{f}_{\kappa,\mu}\right) - \tilde{F}^{T}_{0,\mu}\left(\tilde{f}_{0,\mu}\right)\right)\right\|_{L^{\infty}(\mathbb{H}_T)}.
\end{aligned}
$$

We bound the three terms separately.

$$
\begin{aligned}
(1) \limsup_{\kappa\to 0} \sup_{\mu\in(0,1]} \mu^{\alpha+\delta} &\left\|K_{\mu} * K_{\mu} * \tilde{f}_{\kappa,\mu} - K_{\mu} * K_{\mu} * \tilde{f}_{0,\mu}\right\|_{L^{\infty}(\mathbb{H}_T)} \\
&\leq \limsup_{\kappa\to 0} \sup_{\mu\in(0,1]} \mu^{\alpha+\delta} \|K_{\mu}\|^2_{L^1(\mathbb{H}_T)} \cdot \left\|\tilde{f}_{\kappa,\mu} - \tilde{f}_{0,\mu}\right\|_{L^{\infty}(\mathbb{H}_T)} \\
&= \limsup_{\kappa\to 0} \sup_{\mu\in(0,1]} \mu^{\alpha+\delta} \left\|\tilde{f}_{\kappa,\mu} - \tilde{f}_{0,\mu}\right\|_{L^{\infty}(\mathbb{H}_T)} \\
&\leq \limsup_{\kappa\to 0} \left\|\left|\tilde{f}_{\kappa,0} - \tilde{f}_{0,\bullet}, \tilde{R}_{\kappa,0} - \tilde{R}_{0,\bullet}\right|\right\| = 0
\end{aligned}
$$

by lemma 4.6.

$$(2)\limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{\alpha+\delta}\Big\|(G-G_\mu)*\Big(\tilde{R}_{\kappa,\mu}-\tilde{R}_{0,\mu}\Big)\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\leq \limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{\alpha+\delta}\|G-G_\mu\|_{L^1(\mathbb{H}_T)}\Big\|\tilde{R}_{\kappa,\mu}-\tilde{R}_{0,\mu}\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\lesssim \limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{\alpha+\delta+1/2}\Big\|\tilde{R}_{\kappa,\mu}-\tilde{R}_{0,\mu}\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\leq \limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{c}\Big\|\tilde{R}_{\kappa,\mu}-\tilde{R}_{0,\mu}\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\leq \limsup_{\kappa\to 0}\Big|\Big|\Big|\tilde{f}_{\kappa,0}-\tilde{f}_{0,\bullet},\tilde{R}_{\kappa,0}-\tilde{R}_{0,\bullet}\Big|\Big|\Big|=0$$

by lemma 4.6, proposition C.1 (C) and the relation $1/2+\alpha+\delta>0>c$.

To bound the last term, recall that in the proof of the Lipschitz condition of S_κ and in the proof of the convergence of S_κ to S_0 we proved relations (4.1) and (4.2). Namely

$$\Big\|\tilde{F}^T_{\kappa,\eta}\Big(\tilde{f}_{\kappa,\mu}\Big)-\tilde{F}^T_{\kappa,\eta}\Big(\tilde{f}_{0,\mu}\Big)\Big\|_{L^\infty(\mathbb{H}_T)}\lesssim \mu^\gamma M^{2\bar{\iota}-1}\Big|\Big|\Big|\tilde{f}_{\kappa,\bullet}-\tilde{f}_{0,\bullet},\tilde{R}_{\kappa,\bullet}-\tilde{R}_{0,\bullet}\Big|\Big|\Big|$$

and

$$\Big\|\tilde{F}^T_{\kappa,\mu}\Big(\tilde{f}_{0,\mu}\Big)-\tilde{F}^T_{0,\mu}(\tilde{f}_{0,\mu})\Big\|_{L^\infty(\mathbb{H}_T)}\lesssim \kappa^{\epsilon/2}\mu^\gamma M^{2\bar{\iota}}.$$

Given these inequalities, we have

$$\Big\|\tilde{F}^T_{\kappa,\mu}\Big(\tilde{f}_{\kappa,\mu}\Big)-\tilde{F}^T_{0,\mu}\Big(\tilde{f}_{0,\mu}\Big)\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\leq \Big\|\tilde{F}^T_{\kappa,\eta}\Big(\tilde{f}_{\kappa,\mu}\Big)-\tilde{F}^T_{\kappa,\eta}\Big(\tilde{f}_{0,\mu}\Big)\Big\|_{L^\infty(\mathbb{H}_T)}+\Big\|\tilde{F}^T_{\kappa,\mu}\Big(\tilde{f}_{0,\mu}\Big)-\tilde{F}^T_{0,\mu}(\tilde{f}_{0,\mu})\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\lesssim \mu^\gamma M^{2\bar{\iota}-1}\Big|\Big|\Big|\tilde{f}_{\kappa,\bullet}-\tilde{f}_{0,\bullet},\tilde{g}_{\kappa,\bullet}-\tilde{g}_{0,\bullet}\Big|\Big|\Big|+\kappa^{\epsilon/2}\mu^\gamma M^{2\bar{\iota}}.$$

Which implies the conclusion as

$$(3)\limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{\alpha+\delta}\Big\|(G-G_\mu)*\Big(\tilde{F}^T_{\kappa,\mu}\Big(\tilde{f}_{\kappa,\mu}\Big)-\tilde{F}^T_{0,\mu}\Big(\tilde{f}_{0,\mu}\Big)\Big)\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\leq \limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{\alpha+\delta}\|G-G_\mu\|_{L^1(\mathbb{H}_T)}\Big\|\tilde{F}^T_{\kappa,\mu}\Big(\tilde{f}_{\kappa,\mu}\Big)-\tilde{F}^T_{0,\mu}\Big(\tilde{f}_{0,\mu}\Big)\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\lesssim \limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{\alpha+\delta+1/2}\Big\|\tilde{F}^T_{\kappa,\mu}\Big(\tilde{f}_{\kappa,\mu}\Big)-\tilde{F}^T_{0,\mu}\Big(\tilde{f}_{0,\mu}\Big)\Big\|_{L^\infty(\mathbb{H}_T)}$$

$$\lesssim \limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{\alpha+\delta+1/2}\Big[\mu^\gamma M^{2\bar{\iota}-1}\Big|\Big|\Big|\tilde{f}_{\kappa,\bullet}-\tilde{f}_{0,\bullet},\tilde{g}_{\kappa,\bullet}-\tilde{g}_{0,\bullet}\Big|\Big|\Big|+\kappa^{\epsilon/2}\mu^\gamma M^{2\bar{\iota}}\Big]$$

$$=\limsup_{\kappa\to 0}\sup_{\mu\in(0,1]}\mu^{\delta}\Big[M^{2\bar{\iota}-1}\Big|\Big|\Big|\tilde{f}_{\kappa,\bullet}-\tilde{f}_{0,\bullet},\tilde{g}_{\kappa,\bullet}-\tilde{g}_{0,\bullet}\Big|\Big|\Big|+\kappa^{\epsilon/2}M^{2\bar{\iota}}\Big]$$

$$=\limsup_{\kappa\to 0}\Big(M^{2\bar{\iota}-1}\Big|\Big|\Big|\tilde{f}_{\kappa,\bullet}-\tilde{f}_{0,\bullet},\tilde{g}_{\kappa,\bullet}-\tilde{g}_{0,\bullet}\Big|\Big|\Big|+\kappa^{\epsilon/2}M^{2\bar{\iota}}\Big)=0,$$

where we used lemma 4.6 and proposition C.1 (C).

Note that we have proved the convergence in $C^{\bar{\alpha}}$ for $\bar{\alpha}=2\alpha+2\delta=\frac{1}{2}+4\epsilon+2\delta$. As ϵ and δ can be taken arbitrarily small, this concludes the proof of theorem 4.1.

Appendix A

Regularising kernels

This appendix presents the regularisation kernels used throughout the thesis and analyses some of their properties.

Definition A.1. *Let $\mu \in (0,1]$, then we define $\tilde{K}_\mu : \mathbb{R}^2 \to \mathbb{R}$ as*

$$\tilde{K}_\mu(t,x) = \frac{1}{2\mu^{3/2}} e^{-t/\mu} e^{-|x|/\mu^{1/2}} \mathbb{1}_{\{t \geq 0\}} \tag{A.1}$$

and $K_\mu = \tilde{K}_\mu^{*7}$. Similarly, we define $\tilde{P}_\mu = (1 + \mu\partial_t)(1 - \mu\Delta)$ and $P_\mu = \tilde{P}_\mu^7$.

The kernel $\tilde{K}_\mu$ is the fundamental solution of the differential operator $\tilde{P}_\mu$. In fact, it can be shown with a simple computation that $\bar{K}_\mu(x) := \frac{1}{2\mu^{1/2}} e^{-|x|/\mu^{1/2}}$ is the fundamental solution of $(1 - \mu\Delta)$ and $\mathring{K}_\mu(t) := \frac{1}{\mu} e^{-t/\mu} \mathbb{1}_{\{t \geq 0\}}$ is the fundamental solution of $(1 + \mu\partial_t)$.

Proposition A.2. *For each $N \geq 1$, the following estimates hold uniformly in $\mu \in (0,1]$*

$$\begin{aligned}
&(A) \quad \left\| \tilde{K}_\mu^{*N} \right\|_{L^p(\mathbb{R}^2)} \lesssim \mu^{-\frac{3}{2}\left(1 - \frac{1}{p}\right)}, \\
&(B) \quad \left\| \partial_t \tilde{K}_\mu \right\|_{TV} \lesssim \mu^{-1}, \\
&(C) \quad \left\| \partial_x \tilde{K}_\mu \right\|_{TV} \lesssim \mu^{-1/2}, \\
&(D) \quad \left\| \tilde{P}_\eta \tilde{K}_\mu \right\|_{TV} = 1 \text{ for every } \mu \geq \eta, \\
&(E) \quad \left\| \tilde{P}_\mu \partial_\mu \tilde{K}_\mu \right\|_{TV} \lesssim \mu^{-1}, \\
&(F) \quad \left\| T\tilde{K}_\mu^{*N} \right\|_{L^p(\mathbb{H})} \lesssim \mu^{-\frac{3}{2}\left(1 - \frac{1}{p}\right)},
\end{aligned}$$

where $\|\cdot\|_{TV}$ is the total variation norm of measures and T is the periodisation operator defined by $T\tilde{K}_\mu(t,x) = \sum_{y \in \mathbb{Z}} \tilde{K}_\mu(t, x+y)$

The proof of all these can be done by hand using the explicit formula (A.1). However, we point out that in [Duc21] the proofs of all the above can be found in arbitrary dimensions.

Appendix B

Besov spaces

The purpose of this chapter is to define the norm used in the thesis. First, we will define it and then compare it with a classical Besov norm.

Definition B.1. *Let $T > 0$, $\lambda \in (-1, 0]$ and $\phi \in C^\infty(\mathbb{H}_T)$. We define*

$$\|\phi\|_{C^\lambda(\mathbb{H}_T)} := \sup_{\mu \in (0,1]} \mu^{-\lambda/2} \|K_\mu * \phi\|_{L^\infty(\mathbb{H}_T)} \tag{B.1}$$

and we set $C^\lambda(\mathbb{H}_T)$ as the subset of distributions which is the closure of $C^\infty(\mathbb{H}_T)$ in the above norm.

We remark that we can define the space C^λ with a smaller λ with a similar definition if we use K_μ^{*N} instead of K_μ, for a sufficiently large N (that depends on how small λ is).

To justify the above definition, which may sound a little strange, we compare it with the standard Besov norm and show that it is contained in the latter. For the sake of simplicity, we limit ourselves to describing the case of the torus. In this case, the norm B.1 becomes

$$\|\phi\|_{C^\lambda(\mathbb{T})} := \sup_{\mu \in (0,1]} \mu^{-\lambda/2} \left\|\bar{K}_\mu^{*7} * \phi\right\|_{L^\infty(\mathbb{R})},$$

where $\bar{K}_\mu(x)$ is the fundamental solution of the operator $(1 - \mu\Delta)$ as defined in appendix A.

The following classical construction of the parabolic Besov space is taken from [GP15]. A reference for defining these spaces in the whole $\mathbb{R}^d$ is [BCD11].

We start with the following definition.

Definition B.2. *A couple of functions (ρ_{-1}, ρ_0) is called a dyadic partition of unity if they are in $C^\infty(\mathbb{R}^d, \mathbb{R})$, they are non-negative, radial, ρ_0 is supported in the ball $\{\|x\| \leq 4/3\}$, ρ_{-1} is supported in the annulus $\{3/4 \leq \|x\| \leq 8/3\}$ and such that, if we call $\rho_j(\cdot) = \rho(2^{-j}\cdot)$ for $j \geq 1$, the following hold.*

1. $\sum_{j \geq -1} \rho_j \equiv 1$,

2. *$supp(\rho_i) \cap supp(\rho_j) = \emptyset$ for each $i, j \geq -1$ such that $|i - j| \geq 2$.*

The existence of a dyadic partition of unity is a well-known result in Fourier analysis and can be found in [BCD11] (Proposition 2.10). From now on, we fix one of these partitions.

We now define the Littlewood-Paley blocks of the distribution u as

$$\Delta_j u = \mathscr{F}^{-1}\left(\rho_j \mathscr{F} u\right)$$

for any $j \geq -1$. This can be seen as

$$\Delta_j u = \mathscr{K}_j * u,$$

where $\mathscr{K}_j = \mathscr{F}^{-1}\rho_j$ and $\mathscr{F}$ is the Fourier transform.

Now we can define Besov spaces.

Definition B.3. *Let $\lambda \in \mathbb{R}$, $p, q \in [1,\infty]^2$. Then the Besov space with these 3 parameters is*

$$B^{\lambda}_{p,q}(\mathbb{T}) := \left\{ u \in \mathcal{S}' \ s.t. \ \left(\sum_{j\geq -1} \left(2^{j\lambda} \left\| \Delta_j u \right\|_{L^p}\right)^q \right)^{1/q} \right\}$$

with the obvious interpretation if $q = \infty$.
For $u \in B^{\lambda}_{p,q}(\mathbb{T})$, we set

$$\|u\|_{B^{\lambda}_{p,q}} = \left(\sum_{j\geq -1} \left(2^{j\lambda} \left\| \Delta_j u \right\|_{L^p}\right)^q \right)^{1/q}$$

with the correct interpretation if $q = \infty$.

In the case $p = q = \infty$ (the case that interests us) we write $\mathscr{C}^{\lambda}$ instead of $B^{\lambda}_{\infty,\infty}$.

We now show the desired inequality for the norms of $\mathscr{C}^{\lambda}$ and C^{λ}.

Lemma B.4. *Let $\phi \in C^{\infty}(\mathbb{T})$ and $\lambda \in (-1, 0]$. Then it holds*

$$\|\phi\|_{\mathscr{C}^{\lambda}(\mathbb{T})} \lesssim \sup_{\mu\in(0,1]} \mu^{-\lambda/2} \left\| \bar{K}^{*7}_{\mu} * \phi \right\|_{L^{\infty}(\mathbb{R})}.$$

Proof. Let $\bar{P}_{\mu} := (1 - \mu\Delta)$. Then observe that

$$\Delta_j \phi = \mathscr{K}_j * \phi = \bar{P}^7_{\mu} \mathscr{K}_j * \bar{K}^{*7}_{\mu} * \phi,$$

and that from the definition of $\mathscr{K}_j$, it holds $\mathscr{K}_j(\cdot) = 2^j \mathscr{K}_0(2^j \cdot)$. Thanks to the latter, for every $N \geq 0$ we can estimate

$$\left\| \mu^N \partial^{2N}_x \mathscr{K}_j \right\|_{L^1(\mathbb{R})} = 2^j \left\| \mu^N 2^{2Nj} (\partial^{2N}_x \mathscr{K}_0)(2^j \cdot) \right\|_{L^1(\mathbb{R})} \lesssim \mu^N 2^{2Nj}.$$

So, with $\mu = 2^{-2j}$, we obtain $\left\| \bar{P}^7_{2^{-2j}} \mathscr{K}_j \right\|_{L^1(\mathbb{R})} \lesssim 1$. This allows us to estimate the Littlewood-Paley block. In fact, with the above choice of μ, we have

$$\begin{aligned} \|\Delta_j \phi\|_{L^{\infty}(\mathbb{R})} &\leq \left\| \bar{P}^7_{2^{-2j}} \mathscr{K}_j \right\|_{L^1(\mathbb{R})} \left\| \bar{K}^{*7}_{2^{-2j}} \phi \right\|_{L^{\infty}(\mathbb{R})} \\ &\lesssim \left\| \bar{K}^{*7}_{2^{-2j}} \phi \right\|_{L^{\infty}(\mathbb{R})} \lesssim 2^{-j\lambda} \sup_{\mu\in(0,1]} \mu^{-\lambda/2} \left\| \bar{K}^{*7}_{\mu} * \phi \right\|_{L^{\infty}(\mathbb{R})}. \end{aligned}$$

Which concludes the proof. $\square$

Appendix C

Heat kernel

This appendix studies some inequalities concerning the solver of equation 1.3 and the truncated version present in equation 1.4.

First, we recall that the heat kernel $H(t,x)$ is defined as

$$H(t,x) = \frac{1}{t^{1/2}} e^{-\frac{x^2}{4t}} \mathbb{1}_{\{t>0\}} + \delta_0(x) \mathbb{1}_{\{t=0\}}.$$

Given that G is the space derivative of the heat kernel, we have

$$G(t,x) = -\frac{x}{2t^{3/2}} e^{-\frac{x^2}{4t}} \mathbb{1}_{\{t>0\}} + \delta_0'(x) \mathbb{1}_{\{t=0\}}.$$

As we will solve the equation with a zero initial condition, it is better to redefine G as

$$G(t,x) = -\frac{x}{2t^{3/2}} e^{-\frac{x^2}{4t}} \mathbb{1}_{\{t>0\}}.$$

We moreover set $\dot{G}_\mu^{(1,0)}(t,x) = t\dot{G}_\mu(t,x)$, $\dot{G}_\mu^{(0,1)}(t,x) = x\dot{G}_\mu(t,x)$ and for every $a = (a_1, a_2) \in \{(0,0),(1,0),(0,1)\}$ we set $[a] = a_1 + a_2/2$.

The following statement presents the estimates used throughout the thesis.

Proposition C.1. *For each $N \in \mathbb{N}$, $T > 0$, $\delta > 0$ and a of the above form, the following estimates hold uniformly in $\mu \in (0,1]$*

$$(A) \quad \left\| \tilde{P}^N \dot{G}_\mu^a \right\|_{L^1(\mathbb{M})} \lesssim \mu^{-1/2+[a]},$$

$$(B) \quad \left\| \tilde{P}^N \dot{G}_\mu \right\|_{L^1([0,T]\times\mathbb{R})} \lesssim T^\delta \mu^{-1/2-\delta},$$

$$(C) \quad \|G - G_\mu\|_{L^1(\mathbb{M})} \lesssim \mu^{1/2},$$

where $\tilde{P}_\mu$ has been defined in appendix A.

Proof. For the first one, it is enough to notice that $\left\| \partial_t^n \partial_x^m \dot{G}_\mu^a \right\|_{L^1(\mathbb{M})} \lesssim \mu^{-1/2+[a]-n-m/2}$ by a scaling argument and conclude by the triangular inequality. For the second one, we notice that the above argument can be refined and show that after the scaling, we have

$$\mu^{n+m/2} \left\| \partial_t^n \partial_x^m \dot{G}_\mu \right\|_{L^1([0,T]\times\mathbb{R})} \lesssim \mu^{-1/2} \int_0^{T/\mu} v^{1/2} \tilde{\chi}(v) dv$$

for a $\tilde{\chi} \in C_c^\infty((1,2))$. The latter is 0 if $T \leq \mu$ and if $T > \mu$, we have

$$\mu^{-1/2} \int_0^{T/\mu} v^{1/2} \tilde{\chi}(v) dv \lesssim \mu^{-1/2} \leq T^\delta \mu^{-1/2-\delta}.$$

Lastly, for the third, we have the following easy computation

$$\begin{aligned} \|G - G_\mu\|_{L^1(\mathbb{M})} &= \int_0^\infty \int_{\mathbb{R}} |G(t,x)| \left(1 - \chi\left(t/\mu\right)\right) dx dt \\ &\leq \int_0^{2\mu} \int_{\mathbb{R}} |G(t,x)| dx dt \lesssim \int_0^{2\mu} t^{-1/2} dt \lesssim \mu^{1/2} \end{aligned}$$

that concludes the proof. □

Appendix D

Additional estimate on the functional

This appendix should be seen as a supplement to chapter 4 and in particular to the proofs of inequalities (4.5) and (4.6).

Let us introduce a norm similar to $\mathcal{V}$, but suitable for measures.

Given $V(z; dz_1; \ldots; dz_m)$ that fixed $z \in \mathbb{H}_1$, gives a measure with finite total variation, we set

$$\|V\|_{\mathcal{V}'} := \sup_{z \in \mathbb{H}_1} \int_{\mathbb{M}^m} |V(z; dz_1, \ldots, dz_m)| \, .$$

Let us prove the following theorem.

Theorem D.1. *Let $\kappa \in (0, 1]$. Then for all $(i, m) \in \mathbb{N}^2$, the two following hold uniformly in $\mu \in (0, 1]$*

$$\begin{aligned} \left\|F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}'} &\lesssim_\kappa 1, \\ \left\|\partial_\mu F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}'} &\lesssim_\kappa \mu^{-1/2}, \end{aligned}$$

where the implicit constants depend on κ.

Note that κ is fixed in the above result. In fact, if we let κ vary, we have no hope of proving a uniform bound for the coefficients of the force (not even for the first one, which tends to the white noise as κ goes to 0). In fact, in theorem 3.3 we proved a bound uniform in κ that diverges for small μ (at least for some values of the couple (i, m)).

Proof. We will prove both by induction.

If $i = 0$, this is true as ξ_κ does not depend on μ. Let us now assume $i \geq 1$ and to know the theorem for all (i_*, m_*) such that $i_* < i$ or $i_* = i$ and $m_* > m$. Using equation (1.9), we obtain

$$\partial_\mu F^{i,m}_{\kappa,\mu} = -\sum_{l=0}^{i} \sum_{j=0}^{m} (j+1) B\left(\dot{G}_\mu, F^{l,j+1}_{\kappa,\mu}, F^{i-l,m-j}_{\kappa,\mu}\right).$$

Taking the norm, we get

$$\left\|\partial_\mu F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}'} \leq \sum_{l=0}^{i}\sum_{j=0}^{m}(j+1)\left\|B\left(\dot{G}_\mu, F^{l,j+1}_{\kappa,\mu}, F^{i-l,m-j}_{\kappa,\mu}\right)\right\|_{\mathcal{V}'}.$$

Using inequality (2.2), proposition C.1 (A) and the inductive hypothesis, we obtain

$$\begin{aligned}\left\|\partial_\mu F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}'} &\lesssim \sum_{l=0}^{i}\sum_{j=0}^{m}\left\|\dot{G}_\mu\right\|_{L^1(\mathbb{M})}\left\|F^{l,j+1}_{\kappa,\mu}\right\|_{\mathcal{V}'}\left\|F^{i-l,m-j}_{\kappa,\mu}\right\|_{\mathcal{V}'}\\ &\lesssim_\kappa \sum_{l=0}^{i}\sum_{j=0}^{m}\mu^{-1/2}\cdot 1\cdot 1 \lesssim \mu^{-1/2},\end{aligned}$$

which proves the bound on the derivative. Finally, from the relation

$$F^{i,m}_{\kappa,\mu} = F^{i,m}_{\kappa,0} + \int_0^\mu \partial_\eta F^{i,m}_{\kappa,\eta}\, d\eta,$$

and using that $F^{i,m}_{\kappa,0}$ does not depend on μ, we get

$$\begin{aligned}\left\|F^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}'} &= \left\|F^{i,m}_{\kappa,0}\right\|_{\mathcal{V}'} + \int_0^\mu \left\|\partial_\eta F^{i,m}_{\kappa,\eta}\right\|_{\mathcal{V}'} d\eta\\ &\lesssim_\kappa 1 + \int_0^\mu \eta^{-1/2} d\eta \lesssim 1 + \mu^{1/2} \lesssim 1,\end{aligned}$$

which concludes the proof. □

Similarly, we have

$$\left\|H^{i,m}_{\kappa,\mu}\right\|_{\mathcal{V}'} \leq \sum_{l=0}^{i}\sum_{j=0}^{m}(j+1)\left\|B\left(\dot{G}_\mu, F^{l,j+1}_{\kappa,\mu}, F^{i-l,m-j}_{\kappa,\mu}\right)\right\|_{\mathcal{V}'} \lesssim \sum_{l=0}^{i}\sum_{j=0}^{m}\mu^{-1/2} \lesssim \mu^{-1/2}. \tag{D.1}$$

Bibliography

[BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations.* Springer, 2011.

[BG97] Lorenzo Bertini and Giambattista Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.*, 183(3):571–607, 1997.

[CF24] Ajay Chandra and Léonard Ferdinand. A flow approach to the generalized KPZ equation. *arXiv preprint arXiv:2402.03101*, 2024.

[DGR24] Paweł Duch, Massimiliano Gubinelli, and Paolo Rinaldi. Parabolic stochastic quantisation of the fractional ϕ^4_3 model in the full subcritical regime. *arXiv preprint arXiv:2303.18112*, 2024.

[Duc21] Paweł Duch. Flow equation approach to singular stochastic PDEs. *arXiv preprint arXiv:2109.11380*, 2021.

[Duc22] Paweł Duch. Renormalization of singular elliptic stochastic PDEs using flow equation. *arXiv preprint arXiv:2201.05031*, 2022.

[Duc23] Paweł Duch. Lecture notes on flow equation approach to singular stochastic PDEs. `https://duch.faculty.wmi.amu.edu.pl/spde_flow_notes.pdf`, 2023.

[GIP15] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.

[GP15] Massimiliano Gubinelli and Nicolas Perkowski. *Lectures on singular stochastic PDEs*, volume 29 of *Ensaios Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2015.

[GP17] Massimiliano Gubinelli and Nicolas Perkowski. KPZ reloaded. *Comm. Math. Phys.*, 349(1):165–269, 2017.

[Hai14a] M. Hairer. Solving the KPZ equation. In *XVIIth International Congress on Mathematical Physics*, page 419. World Sci. Publ., Hackensack, NJ, 2014.

[Hai14b] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[KPZ86] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Physical Review Letters*, 56(9):889, 1986.

[Kup16] Antti Kupiainen. Renormalization group and stochastic PDEs. *Ann. Henri Poincaré*, 17(3):497–535, 2016.

[PT11] Giovanni Peccati and Murad S Taqqu. *Wiener Chaos: Moments, Cumulants and Diagrams: A survey with computer implementation*, volume 1. Springer Science & Business Media, 2011.