A Stochastic Control Perspective on Euclidean QFTs

Sarah-Jean Meyer

14th September 2022

Supervised by Massimiliano Gubinelli & Christian Brennecke
University of Bonn

(Euclidean-) Quantum Fields – Brief Background

- ▷ QFTs Result from combining quantum mechanical models with special relativity
- ▷ natural space: Minkowski space $\mathbb{R}^{n+1}$ with

$$\|(t, x, y, z)\|^2 = -t^2 + x^2 + y^2 + z^2.$$

- ▷ Wightman Axioms: minimal requirements for a QFT
  $\rightarrow$ Difficult to find concrete models satisfying these axioms.
- ▷ Wick-rotation: Replace $t \rightarrow it$ so that the Minkowski space $\mathbb{R}^{n+1}$ becomes a Euclidean space $\mathbb{R}^d$.
- ▷ Osterwalder-Schrader Axioms: precise conditions to swap between Minkowski and Euclidean space.

Upshot: Euclidean QFTs can (in some cases) be understood as probability measure on $\mathcal{S}'(\mathbb{R}^d)$.

EQFTs

- ▷ Probability measures on $\mathcal{S}'(\mathbb{R}^d)$ satisfying additional axioms

(i) Regularity (ii) Euclidean Invariance (iii) Reflection Positivity

- ▷ Simplest example of a EQFT:
  (massive) Gaussian free field (GFF) with covariance $(m^2 - \Delta)^{-1}$
- ▷ Formally,

$$“\mu(\mathcal{O}) = Z_\mu^{-1} \int_{\mathcal{S}'(\mathbb{R}^d)} e^{-\int_{\mathbb{R}^d}(m^2|\varphi(x)|^2 + |\nabla\varphi(x)|^2)} \mathcal{O}(\varphi) d\varphi”$$

- ▷ for $d \geq 2$: not a function.
  more precisely the GFF is a distribution of regularity $\frac{2-d}{d} - \delta$ for any $\delta > 0$ small,
  (in the sense of Besov-Hölder regularity)

Yields a **free** QFT but can be used as a starting point and reference measure for **interacting** theories.

- Probability measures on $\mathcal{S}'(\mathbb{R}^d)$ satisfying additional axioms

(i) Regularity (ii) Euclidean Invariance (iii) Reflection Positivity

- **Aim:** Construct interacting models by Gibbsian perturbations of the free theory:

$$\text{“}\nu(\mathcal{O}) = Z_\nu^{-1} \int e^{-\lambda \int_{\mathbb{R}^d} V(\varphi(x))dx} \mathcal{O}(\varphi)\mu(d\varphi)\text{”},$$

for a non-linear function $V : \mathbb{R} \to \mathbb{R}$ and μ the GFF.

This representation is ill-defined:

IR-Problem *(large scale behaviour)*. No decay in space
▷ $\int_{\mathbb{R}^d} V(\varphi(x))dx$ does not make sense.

UV-Problem *(small scale behaviour)*. $\varphi \sim \mu$ is only a distribution for $d \geq 2$
▷ $V(\varphi(x))$ cannot be defined in a pointwise manner.

Approximating the Gibbs measure

- ▷ To deal with the IR-Problem: **spacial cut-off** $\xi \in C_c^\infty(\mathbb{R}^d)$.

$$\text{replace } \int_{\mathbb{R}^d} V(\varphi(x))dx \text{ by } \int_{\mathbb{R}^d} \xi(x)V(\varphi(x))dx.$$

- ▷ To deal with the UV-Problem: **small scale cut-off** $T > 0$.
Choose μ^T with support on genuine function spaces with

$$\mu^T \to \mu. \quad (T \to \infty)$$

If V is independent of T: there are divergent contributions and $V(\varphi^T)$ with $\varphi^T \sim \mu^T$ becomes trivial in the limit.
Renormalisation for $\varphi^T \sim \mu^T$

$$\text{replace } V(\varphi^T) \text{ by } V_T(\varphi^T),$$

to compensate the divergent contributions.

Some Examples

$\rightarrow$ New objects of interest

$$\nu^{\xi,T}(d\varphi) = (Z_\nu^{\xi,T})^{-1} \exp\left(-\lambda \int_{\mathbb{R}^d} \xi(x) V_T(\varphi(x))dx\right) \mu^T(d\varphi),$$

and their limit as $T \to \infty$ and $\xi \to 1$.

$\triangleright$ Possible choices for V_T:

$\triangleright$ $d = 2, 3$:

$$V_T(\varphi) = \varphi^4 - \alpha_T \varphi^2$$

$\triangleright$ $d = 2$:

$$V_T(\varphi) = \varphi^{2p} + \sum_{\ell}^{2p-1} a_{\ell,T} \varphi^\ell, \quad \text{for any } p > 0$$

$$V_T(\varphi) = \alpha_T \cos(\beta\varphi)$$

$$V_T(\varphi) = \alpha_T \exp(\beta\varphi)$$

and linear combinations (e.g. cosh).

Plan

Goal: Construct the sine-Gordon EQFT

$$\text{“}\nu_{\mathrm{SG}}(d\varphi) = Z_{\nu_{\mathrm{SG}}}^{-1} \exp\left(-\int_{\mathbb{R}^2} \lambda \cos(\beta\varphi(x))dx\right) \mu(d\varphi)$$
$$= \widetilde{Z}_{\nu_{\mathrm{SG}}}^{-1} \exp\left(-\int_{\mathbb{R}^2} dx \lambda \cos(\beta\varphi(x)) + m^2 |\varphi(x)|^2 + |\nabla\varphi(x)|^2\right) d\varphi\text{”},$$

on $\mathcal{S}'(\mathbb{R}^2)$ for $\beta^2 < 4\pi$ as a random shift of the massive GFF.

Outline:

- ▷ The Boué-Dupuis Formula
- ▷ Decompose the Free Field
- ▷ Introduce the Stochastic Control Problem
- ▷ Stochastic Maximum Principle/ EL-equation
- ▷ BSDEs and a priori estimates
- ▷ Variational description on $\mathbb{R}^2$

Boué-Dupuis Formula

Translates the Gibbs variational problem to a stochastic control problem:

Theorem

For a bounded functional F and a Q-Brownian motion W,

$$-\log \mathbf{E}[e^{-F(W)}] = \inf_{u \in \mathbb{H}^0} \mathbf{E}\left[F(I(u) + W) + \frac{1}{2}\int_0^\infty \|u_s\|^2_{L^2(\mathbb{R}^2)} ds\right],$$

where $\mathbb{H}^0$ is the space of adapted processes, and

$$I_t(u) := \int_0^t Q_s u_s ds.$$

Extensions to more general functionals are available.

M. Boué and P. Dupuis. "A variational representation for certain functionals of Brownian motion". In: *Ann. Prob.* 26.4 (1998)

Ali Süleyman Üstünel. "Variational calculation of Laplace transforms via entropy on Wiener ...". In: *J. Funct. Anal.* 267.8 (2014)

The BD formula in the EQFT setting

Choose $\{Q_t\}_{t\geq 0}$ to be positive Hilbert-Schmidt operators s.t.

$$\int_0^\infty Q_s^2 ds = (m^2 - \Delta)^{-1}.$$

Define for a cylindrical Brownian motion B on $L^2(\mathbb{R}^2)$,

$$W_t := \int_0^t Q_s dB_s, \qquad W_\infty \sim \mu.$$

With the BD formula and $\mu^T := \mathrm{Law}(W_T)$,

$$\begin{aligned} -\log \int e^{-F(\varphi)} \mu^T(d\varphi) &= -\log \mathbf{E}[e^{-F(W_T)}] \\ &= \inf_{u \in \mathbb{H}^0} \mathbf{E}\left[F(I_T(u) + W_T) + \frac{1}{2} \int_0^\infty \|u_s\|^2_{L^2(\mathbb{R}^2)} ds \right]. \end{aligned}$$

$\triangleright$ Use this formula for the Laplace transform of $\nu^{\xi, T}$.

Variational Formula for the Laplace Transform

$$\nu^{\xi,T}(d\varphi) = Z^{-1}_{\nu^{\xi},T} \exp\left(-\lambda \int_{\mathbb{R}^2} \xi(x) V_T(\varphi(x)) dx\right) \mu^T(d\varphi),$$

$$Z_{\nu^{\xi},T} = \int \exp(-\int_{\mathbb{R}^2} \xi(x) V_T(\varphi(x)) dx) \mu^T(d\varphi).$$

Applying the BD formula with the (bounded) functional

$$V_T^{g,\xi}(\varphi) := g(\varphi) + \lambda \int_{\mathbb{R}^2} \xi(x) V_T(\varphi(x)) dx$$

the Laplace transform has the variational representation

$$-\log \int e^{-g(\varphi)} \nu^{\xi,T}(d\varphi) = -\log \left(Z^{-1}_{\nu^{\xi},T} \int e^{-V_T^{g,\xi}(\varphi)} \mu^T(d\varphi)\right)$$
$$= \inf_{u \in \mathbb{H}^0} J_T^{g,\xi}(u) - \inf_{u \in \mathbb{H}^0} J_T^{0,\xi}(u).$$

where $T \in [0, \infty)$,

$$J_T^{g,\xi}(u) = \mathbf{E}\left[V_T^{g,\xi}(X_T(u)) + \int_0^T \|u_t\|^2_{L^2} dt\right],$$

and

$$X_T(u) = W_T + \int_0^T Q_s u_s ds, \qquad W_\infty \sim \mu.$$

N. Barashkov and M. Gubinelli. "A variational method for Φ^4_3". In: *Duke Math. Jour.* 169.17 (2020)

N. Barashkov and M. Gubinelli. "On the variational method for Euclidean ...". arXiv:2112.05562. (2021)

N. Barashkov. "A stochastic control approach to Sine Gordon EQFT". arXiv:2203.06626. (2022)

Stochastic Control Problem

Control

$$X_t(u) = \int_0^t Q_s u_s ds + W_t,$$

subject to

$$\mathcal{V}_T^{g,\xi} = \inf_{u \in \mathbb{H}^0} Y_{0,T}^{g,\xi} := \inf_{u \in \mathbb{H}^0} \mathbf{E}\left[V_T^{g,\xi}(X_T(u)) + \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 ds\right].$$

Fact: The infimum is a minimum.

$\triangleright$ Introduce a variation $u_\varepsilon = u + \varepsilon \delta u$ and look for stationary controls.

(i.e. compute $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}$)

$$\nabla_\varepsilon X_t(u) = \int_0^t Q_s \delta u_s ds,$$

$$\begin{aligned}\nabla_\varepsilon Y_{0,T}^{g,\xi}(u) &= \mathbf{E}\int_0^T \nabla V_T^{g,\xi}(X_T(u))\nabla_\varepsilon X_t(u) + \int_0^T u_s \delta u_s ds \\ &= \mathbf{E}\int_0^T \mathbf{E}\left[\nabla V_T^{g,\xi}(X_s(u))Q_s + u_s \middle| \mathcal{F}_s\right]\delta u_s ds.\end{aligned}$$

$$\implies \quad u_s^* = -Q_s\,\mathbf{E}[\nabla V_T^{g,\xi}(X_T(u^*))|\mathcal{F}_s].$$

Stochastic Control Problem

Control

$$X_t(u) = \int_0^t Q_s u_s ds + W_t,$$

subject to

$$\mathcal{V}_T^{g,\xi} = \inf_{u \in \mathbb{H}^0} Y_{0,T}^{g,\xi} := \inf_{u \in \mathbb{H}^0} \mathbf{E}\left[V_T^{g,\xi}(X_T(u)) + \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 ds \right].$$

Fact: The infimum is a minimum and the optimal control satisfies

$$u_s^* = -Q_s \, \mathbf{E}[\nabla V_T^{g,\xi}(X_T(u^*)) | \mathcal{F}_s].$$

▷ The optimal dynamics are

$$X_{t,T}^{g,\xi} = -\int_0^t Q_s^2 \, \mathbf{E}\left[\nabla V_T^{g,\xi}(X_{T,T}^{g,\xi}) | \mathcal{F}_s \right] ds + W_t.$$

▷ SDE depending on the *distribution* of X with

$$\mathrm{Law}(X_{T,T}^{0,\xi}) = \nu_{\mathrm{SG}}^{\xi,T}.$$

Backward SDEs: Motivation

We derived the optimal dynamics

$$\begin{cases} X_{t,T}^{g,\xi} = -\int_0^t Q_s^2 \nabla Y_{s,T}^{g,\xi} ds + W_t, \\ \nabla Y_{t,T}^{g,\xi} = \mathbf{E}[\nabla V_T^{g,\xi}(X_{T,T}^{g,\xi}) | \mathcal{F}_t]. \end{cases}$$

Problem: Conditional expectation is inconvenient.

Solution: Martingale Representation theorem:
There is a square-integrable and adapted process ∇Z such that

$$\mathbf{E}[\nabla V_T^{g,\xi}(X_{T,T}^{g,\xi}) | \mathcal{F}_t] = E[\nabla V_T^{g,\xi}(X_{T,T}^{g,\xi})] + \int_0^t \nabla Z_{s,T} dB_s.$$

Rearranging yields

$$\mathbf{E}[\nabla V_T^{g,\xi}(X_{T,T}^{g,\xi}) | \mathcal{F}_t] = \nabla V_T^{g,\xi}(X_{T,T}^{g,\xi}) - \int_t^T \nabla Z_{s,T}^{g,\xi} dB_s.$$

In differential notation, we could write

$$d(\nabla Y_{t,T}^{g,\xi}) = \nabla Z_{t,T}^{g,\xi}, \quad Y_{T,T}^{g,\xi} = \nabla V_T(X_{T,T}^{g,\xi}).$$

- "correct" formulation for **adapted solutions** to SDEs with a **terminal condition**.
- given a *terminal condition* ξ and a *generator* f a solution is a (square-integrable) *pair* (Y, Z)

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)dt - Z_t dB_t, \ t \in [0, T], \\ Y_T = \xi. \end{cases}$$

  or equivalently

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \ t \in [0, T].$$

- Stochastic analysis for conditional expectations relying on the martingale representation theorem.
- **For us:** a priori estimates via Itô calculus

e.g. Surveys N. El Karoui, S. Peng, and M.C. Quenez. "Backward stochastic differential equations...". In: *Math Financ* 7.1 (1997) or E. Pardoux. "Backward Stochastic Differential Equations and Viscosity...". In: *Stochastic Analysis and Related Topics VI.* (1998)

Backward SDEs: a priori estimates

Given

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s,\ t \in [0, T].$$

apply Itô's formula for $|\cdot|^2$,

$$|Y_t|^2 + \int_t^T \|Z_s\|^2 ds = |\xi|^2 + \int_t^T \underbrace{2\langle Y_s, f(s, Y_s, Z_s)\rangle}_{\text{estimate in terms of } |Y|,\ \|Z\|} ds + \underbrace{\int_t^T 2\langle Y_s, Z_s dB_s\rangle}_{\text{martingale}}.$$

▷ e.g. uniform Lipschitz assumptions on f in y, z.

▷ combined with BDG inequality (provided all terms are finite) this yields estimates of the form

$$\mathbf{E}\left[\sup_t |Y_t| + \int_0^T \|Z_t\|^2 dt\right] \le C\,\mathbf{E}\,|\xi|^2 + C\,\mathbf{E}\int_0^T \|f(s,0,0)\|^2 ds.$$

Uniform estimates: Rough Outline

Rewrite the optimal dynamics in this way

$$\begin{cases} X^{g,\xi}_{t,T} = -\int_0^t Q_s^2 \nabla Y^{g,\xi}_{s,T} ds + W_t, \\ \nabla Y^{g,\xi}_{t,T} = \nabla V^{g,\xi}_T(X^{g,\xi}_{T,T}) - \int_t^T \nabla Z^{g,\xi}_s dB_s (= \mathbf{E}[\nabla V^{g,\xi}_T(X^{g,\xi}_{T,T})|\mathcal{F}_t]). \end{cases}$$

Advantage: Can be studied using stochastic analysis.

Goal: Pass to limit $\xi \to 1,\ T \to \infty$.

$\triangleright$ need uniform a priori bounds on the equation above.

Key points for the estimates:

(i) the system depends only on $\nabla V^{g,\xi}_T$; and does not involve an integral over $\mathbb{R}^2$

$$V^{0,\xi}_T(\varphi) = \lambda \int_{\mathbb{R}^2} \xi(x)\alpha_T \cos(\beta\varphi(x))dx, \quad \nabla V^{0,\xi}_T(\varphi)(x) = -\lambda\beta\alpha_T\xi(x)\sin(\beta\varphi(x)).$$

(ii) $\alpha_T \cos(W_T)$ is a martingale.

Change of variables

How to obtain bounds uniform in T?

Fact: The renormalised potential has a martingale property, that is

$$\mathbf{E}[\nabla V_T(W_T)|\mathcal{F}_t] = \nabla V_t(W_t).$$

▷ As $X_{t,T}^{g,\xi} = W_t + I_{t,T}^{g,\xi}$ we make the Ansatz

$$\mathbf{E}[\nabla V_T(X_{T,T})|\mathcal{F}_t] = \nabla V_t(X_{t,T}) + R_{t,T},$$

and try to bound $R_{t,T}$.

▷Use this in $\nabla Y_{t,T}^{g,\xi} = \mathbf{E}[\nabla V_T^{g,\xi}(X_{T,T}^{g,\xi})|\mathcal{F}_t]$,

$$R_{t,T}^{g,\xi} = \nabla g(X_{T,T}^{g,\xi}) + \underbrace{\nabla V_T^{\xi}(X_{T,T}^{g,\xi}) - \nabla V_t^{\xi}(X_{t,T}^{g,\xi})}_{\text{rewrite with Itô's formula}} - \int_t^T Z_{s,T}^{g,\xi} dB_s.$$

$$R_{t,T}^{g,\xi} = \nabla g(X_{T,T}^{g,\xi}) - \int_t^T \lambda\beta\alpha_s \nabla \sin(\beta X_{s,T}^{g,\xi}) dX_{s,T}^{g,\xi} - \int_t^T Z_{s,T}^{g,\xi} dB_s,$$

Uniform Estimates

$$R^{g,\xi}_{t,T} = \nabla g(X^{g,\xi}_{T,T}) + \int_t^T h^\xi(s, X^{g,\xi}_{s,T}, R^{g,\xi}_{s,T})ds - \int_t^T \widetilde{Z}^{g,\xi}_{s,T} dB_s,$$

with

$$h^\xi(s,x,r) = \underbrace{\beta^2\lambda\xi\alpha_s \cos(\beta x) Q_s^2 \nabla V_s^{\xi,g}(x)}_{=:\phi_s} + \underbrace{\beta^2\lambda\xi\alpha_s \cos(\beta x) Q_s^2}_{=:\gamma_s} r.$$

▷ Variation of constants type argument for the BSDE yields

$$R^{g,\xi}_{t,T} = \mathbf{E}\left[\Gamma_{t,T}\nabla g(X^{g,\xi}_{T,T}) + \int_t^T \Gamma_{t,s}\phi_s ds \Big| \mathcal{F}_t\right], \quad \Gamma_s^t = e^{\int_t^s \gamma_s ds}.$$

▷ Uniform estimates for $\beta^2 \in [0, 4\pi)$

$$\|R^{g,\xi}_{t,T}\|_{L^\infty(\mathbb{R}^2)} + \|R^{g,\xi}_{t,T}\|_{L^2(\langle x\rangle^{-\ell})} \le C_{\ell,\nabla g} + C\lambda^2\langle t\rangle^{-2\delta} \le C,$$

Dependence on T, ξ

▷ Consider the BSDEs for the differences

$$\delta_T R_t := R^{g,\xi}_{t,T_1} - R^{g,\xi}_{t,T_2}, \text{ and similarly } \delta_\xi R_t.$$

and apply the standard a priori estimates for (F)BSDEs.

▷ for "nice" functionals g, this yields

$$\mathbf{E}\left[\sup_t \|\delta_T R_t\|^2_{L^2(\langle x\rangle^{-\ell})}\right] \leq C\langle T\rangle^{-4\delta}.$$

▷ proceed analogously for the dependence ξ:

$$\mathbf{E}\left[\sup_t \|\delta_\xi R_t\|^2_{L^2(\langle x\rangle^{-\ell})}\right] \leq C\|\xi_1 - \xi_2\|^2_{L^2(\langle x\rangle^{-\ell})}.$$

▷ Convergence to some $\overline{R}^{g,\xi}$ as $T \to \infty$ and $\overline{R}^{g,\xi} \to \overline{R}^g$ as $\xi \to 1$.

▷ Convergence transfers to X and u (by a simple Gronwall argument).

Convergence of the Control

Notation for the optimal control $u^{g,\xi}_{s,T} = -Q_s \nabla Y^{g,\xi}_{s,T}$.

Proposition

(T) There is a $\overline{u}^{g,\xi}$ such that

$$\lim_{T\to\infty} \mathbf{E} \int_0^T \|u^{g,\xi}_{s,T} - \overline{u}^{\xi}_s\|^2_{L^2(\langle x\rangle^{-\ell})} ds = 0.$$

Moreover, $\overline{u}^{g,\xi}$ is optimal for the control problem at $T = \infty$.

(ξ) Similarly, there is a $\overline{u}^{g}$ such that

$$\lim_{\xi\to 1} \mathbf{E} \int_0^T \|\overline{u}^{g,\xi}_s - \overline{u}^{g}_s\|^2_{L^2(\langle x\rangle^{-\ell})} ds = 0.$$

ν_{SG} as a Random Shift of the GFF

▷ The limit $(\overline{X}^{0,\xi}, \overline{u}^{0,\xi})$ **is optimal** for the control problem, and

$$\mathrm{Law}(\overline{X}^{0,\xi}_\infty) = \mathrm{Law}(W_\infty + \mathcal{I}^\xi_\infty) = \nu^\xi_{\mathrm{SG}},$$

where $\mathcal{I}^\xi_\infty \in L^\infty(P; W^{1,\infty}(\mathbb{R}^2))$ and $\xi \in C^\infty_c(\mathbb{R}^2)$.

▷ Since $(\overline{X}^{0,\xi}, \overline{u}^{0,\xi})$ converges as $\xi \to 1$ (to a *unique* limit for $\lambda > 0$ small),

$$\nu^\xi_{\mathrm{SG}} = \mathrm{Law}(W_\infty + \mathcal{I}^\xi_\infty) \to \mathrm{Law}(W_\infty + \mathcal{I}_\infty) =: \nu_{\mathrm{SG}}$$

weakly on $H^{-\delta}(\langle x \rangle^{-\ell})$.

▷ Again $\mathcal{I}_\infty \in L^\infty(P; W^{1,\infty}(\mathbb{R}^2))$.

Summary

- **Goal:** Construct EQFTs from

$$\nu^{\xi,T}(\mathcal{O}) = Z_\nu^{-1} \int_{\mathcal{S}'(\mathbb{R}^d)} \mathcal{O}(\varphi) e^{-\lambda \int_{\mathbb{R}^d} \xi V_T(\varphi)} \mu^T(d\varphi) \text{ as } \xi \to 1,\ T \to \infty.$$

- Symmetries of the physical system: **large**- and **small-scale** problems.
- Boué-Dupuis formula $\to$ Stochastic control problem with **optimal dynamics**

$$X_{t,T}^{g,\xi} = -\int_0^t Q_s^2 \, \mathbf{E}\left[\nabla V_T^{g,\xi}(X_{T,T}^{g,\xi}) | \mathcal{F}_s\right] ds + W_t.$$

- For the sine-Gordon model, simple a priori estimates can be used to remove the cut-offs via **stochastic calculus/BSDEs**
- Variational description for the **infinite volume** EQFT the Laplace transform & characterisation as a shift of the free field

$$\nu_{\mathrm{SG}} = \mathsf{Law}(W_\infty + \mathcal{I}_\infty) = \mathsf{Law}(X_{\infty,\infty}^{0,1}).$$

Dependence on g

Is the limit $(\overline{X}, \overline{u})$ still optimal for $T = \infty, \xi = 1$?

▷ **Problem:** The cost functional for $\xi \equiv 1$ is ill-defined

$$J^{0,\xi}(u) = \mathbf{E}\left[\lambda \int_{\mathbb{R}^2} \xi(x) V_\infty(X_\infty(u))(x)dx + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 ds\right].$$

▷ Dependence of R on g is local in the sense that for **any** n

$$\mathbf{E}\left[\sup_{t\in[0,T]} \|R_{t,T}^{g,\xi} - R_{t,T}^{0,\xi}\|_{L^2(\langle x\rangle^n)}^2\right] \le C_{\nabla g,n}.$$

▷ transfers again to the optimal control $u^{g,\xi}$

$$\mathbf{E}\int_0^\infty \|u_{t,T}^{g,\xi} - u_{t,T}^{0,\xi}\|_{L^2(\langle x\rangle^n)}^2 \le C_{\nabla g,n}.$$

▷Try to pass to

$$\lim_{\xi\to 1}\inf_u J^{g,\xi}(u) - \inf_u J^{0,\xi}(u) = \lim_{\xi\to 1}\inf_u J^{g,\xi}(u) - J^{0,\xi}(\overline{u}^\xi).$$

which provides a variational problem for

$$\mathcal{W}(g) := \int e^{-g(\varphi)}\nu_{\mathsf{SG}}(d\varphi).$$

Variational Description on the Infinite Volume

Theorem

For n sufficiently large, $\lambda > 0$ small enough,

$$\mathcal{W}(g) = \lim_{\xi \to 1} \lim_{T \to \infty} \mathcal{W}^{\xi, T}(g) = \inf_{v \in \mathcal{A}(g)} \overline{J}^g(v),$$

with the cost functional

$$\overline{J}^g(v) = \mathbf{E}\left[g(X_\infty(\overline{u} + v)) + \int_{\mathbb{R}^2} (V_\infty(X_\infty(\overline{u} + v)) - V_\infty(X_\infty(\overline{u}))) + \mathcal{E}(\overline{u}, v) \right].$$

Here,

$$X_\infty(u) = I_\infty(u) + W_\infty$$

is the shifted free field.

- ▷ $\overline{u}$ is an adapted stochastic process which does not depend on g, v
- ▷ I_∞ is a linear functional, which increases regularity by 1 and does not depend on g.
- ▷ $\mathcal{E}$ is a quadratic functional, also independent of g, and
- ▷ $\mathcal{A}(g)$ contains the adapted controls v such that $\mathbf{E} \int_0^\infty \|v_s\|^2_{L^2(\langle x \rangle^n)} ds \leq C_{\nabla g, n}$.

Convergence of $\alpha_t \cos(\beta W_t) = [\![\cos(\beta W_t)]\!]$

Theorem

For any $p \geq 1$, the Wick-ordered cosine satisfies

$$\sup_{t \geq 0} \mathbf{E}\left[\| [\![\cos \beta W_t]\!] \|^p_{B_{p,p}^{-\frac{\beta^2}{4\pi}-2\delta}(\langle x \rangle^{-\ell})} \right] < \infty,$$

and converges in $L^p(P, B_{p,p}^{-\frac{\beta^2}{4\pi}-2\delta}(\langle x \rangle^{-\ell}))$ and almost surely to a limit which we denote by $[\![\cos(\beta W_\infty)]\!]$. The analogous statement holds also for the Wick-ordered sine.

J. Junnila, E. Saksman, and C. Webb. "Imaginary multiplicative chaos: Moments, regularity, and...". In: *Ann. App. Prob.* 30.5 (2020)