Axiomatic Quantum Field Theory and Stochastic Quantization

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Plan

Plan for the talk:

1. Quantum Fields as Operator-Valued Distributions
2. Wightman Axioms of Relativistic Quantum Fields
3. Wightman Reconstruction Theorem
4. Osterwalder-Schrader Axioms for Schwinger Functions and Euclidean Quantum Field Theory
5. Stochastic Quantization

Quantum Fields as Operator-Valued Distributions

Classical Fields: scalar, vector, tensor or spinor valued function of space-time

Quantum Fields: operator-valued function of space-time?

Simple math argument: free scalar Boson field, Fock space

$$\mathbb{C} \oplus L^2(\mathbb{R}^3) \oplus (L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)) \oplus \ldots = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$$

Scalar product:

$$\langle \Phi, \Psi \rangle = \bar{\Phi}_0 \, \Psi_0 + \int_{\mathbb{R}^3} \bar{\Phi}_1(x) \, \Psi_1(x) \, d^3x + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \bar{\Phi}_2(x_1, x_2) \, \Psi_2(x_1, x_2) d^3x_1 d^3x_2 + \cdots$$

Annihilation operator field $\psi(x)$:

$$(\psi(x) \, \Psi)_n(x_1, \cdots, x_n) = \sqrt{n+1} \, \Psi_{n+1}(x, x_1, \cdots, x_n)$$

Problem: the domain of creation operator $\psi^*(x)$ at the same point, as the adjoint operator of $\psi(x)$, only contains 0 vector.

Theorem 1. (Wightman 1964) *Suppose we have a quantum theory with a separable Hilbert space* $\mathcal{H}$*, and a strong continuous unitary representation of space-time translation* $\mathbb{R}^{1,3} \ni a \mapsto U(a) \in \mathcal{U}(\mathcal{H})$ *such that the spectrum of the energy momentum operator is contained in the closed forward light cone. Suppose the quantum theory has a unique vacuum vector* Ψ_0*, which is invariant under the action of space-time translation, that is*

$$U(a)\Psi_0 = \Psi_0 \, \text{for all} \, a \in \mathbb{R}^{1,3}$$

Then a map B *from a bounded open set* $\mathcal{O} \subset \mathbb{R}^{1,3}$ *to Von Neumann algebra of bounded operators on* $\mathcal{H}$*, with the following properties*

$$U(a)B(x)U(-a) = B(x+a)$$

$$[B(x), B(y)^{(*)}] = 0$$

where a *is small enough and* $(x-y)$ *is a space-like vector. Then* B *has constant value equal to a constant multiple of identity.*

Quantum Fields as Operator-Valued Distributions

Physical Viewpoint:

Bohr and Rosenfeld's (1933&1950):

measurability of electromagnetic fields in QED, only the quantities formally corresponds to the average of its classical analog over finite space-time regions are measurable

$$\frac{1}{|O|}\int_O F_{\mu\nu}(x)\, d^4 x$$

Heisenberg (1931):

used smeared fields (average over second differentiable function) to avoid the infinite fluctuation Einstein's fluctuation formula of blackbody radiation.

showed in general to measure the field in a sharply defined region, one has to use an infinite amount of energy, electromagnetic field are special cases.

Question: What differentiability and regularity conditions one should assume in order to define the smeared field?

Definition 2. (Wightman, Gårding 1965) *Suppose $\mathcal{H}$ is a Hilbert space, an operator-valued distribution is a complex linear map φ from complex-valued Schwartz function space $\mathcal{S}$ to the set of operators (bounded or unbounded) on Hilbert space $\mathcal{H}$, such that all the operators $\varphi(f), \forall f \in \mathcal{S}$ have a common dense domain D, and the map*

$$\mathcal{S} \to \mathbb{C}, f \mapsto \langle \Phi, \varphi(f)\, \Psi \rangle$$

is continuous, where $\Phi \in \mathcal{H}, \Psi \in D$ are fixed vectors.

Inspired by Schwartz's theory of distribution.

Weak continuous and temperedness only produce renormalizable theories (Bogoliubov etc.).

Different choice of test function spaces for different models, Gelfand-Shilov spaces (Jaffe 1967).

Wightman Axioms of Relativistic Quantum Fields

Minkowski space $\mathbb{R}^{1,3}$:

$\langle x_1, x_2\rangle = g_{\mu\nu} x_1^{\mu} x_2^{\nu}$ where $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$ and $g_{\mu\nu} = 0$ if $\mu \neq \nu$.

Vector types $x \in \mathbb{R}^{1,3}$

$$\begin{cases} \text{time-like,} & \text{if} \quad \langle x, x\rangle > 0 \\ \text{space-like,} & \text{if} \quad \langle x, x\rangle < 0 \\ \text{light-like,} & \text{if} \quad \langle x, x\rangle = 0 \end{cases}$$

Forward light cone: $V_+ := \{x | \langle x, x\rangle > 0, x \in \mathbb{R}^{1,3}\}$ causal future of the origin

Lorentz group: $\langle \Lambda\, x, \Lambda\, y\rangle = \langle x, y\rangle \Leftrightarrow g_{\alpha\beta}\, \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} = g_{\mu\nu}$

Restricted Lorentz group $SO^+(1,3)$: connected component of the identity

Inhomogeneous Lorentz group (Spinor group): $SL(2, \mathbb{C})$ double cover of $SO^+(1,3)$

Restricted Poincaré group P:

$$\{(a, \Lambda) | \, \Lambda \in SO^+(1,3), a \in \mathbb{R}^{1,3}\} \quad (a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2).$$

Wightman Axioms of Relativistic Quantum Fields

Born's Rule: $|\langle \Psi, \Phi \rangle|^2$

Symmetry U: maps normalized state Ψ to state $U\Psi$, such that

$$|\langle U\Psi, U\Phi \rangle|^2 = |\langle \Psi, \Phi \rangle|^2$$

Theorem 3. (Wigner 1931) *A symmetry U is an unitary or anti-unitary operator.*

Anti-unitary:

$$U(a\,\Psi + b\,\Phi) = a^*\,U(\Psi) + b^*\,U(\Phi) \text{ for } \forall a, b \in \mathbb{C}, \Psi, \Phi \in \mathcal{H};$$

$$\langle U\,\Psi, U\,\Phi \rangle = \langle \Phi, \Psi \rangle \text{ for } \forall \Psi, \Phi \in \mathcal{H}.$$

Relativity: each element $\Lambda \in \mathbb{R}^{1,3} \rtimes SO^+(1,3)$ induces a symmetry $U(\Lambda)$ (unitary)

Projective representation:

$$U(\Lambda_1)U(\Lambda_2) = e^{if(\Lambda_1,\Lambda_2)}U(\Lambda_1\Lambda_2)$$

Theorem 4. (Wigner 1939, Bargmann 1954) *Each projective representation of restricted Poincaré group* $\mathbb{R}^{1,3} \rtimes SO^+(1,3)$ *to the group* $U(\mathcal{H})$ *of unitary operators on a Hilbert space corresponds to a unique unitary representation of* $\mathbb{R}^{1,3} \rtimes SL(2,\mathbb{C})$*, this means one can change the phase factor continuously of projective representation of* $\mathbb{R}^{1,3} \rtimes SL(2,\mathbb{C})$

$$\mathbb{R}^{1,3} \rtimes SL(2,\mathbb{C}) \to \mathbb{R}^{1,3} \rtimes SO^+(1,3) \to U(\mathcal{H})$$

to get a unitary representation.

Wigner's idea:

one particle state subspace should look the same for each observer, hence a subrepresentation of Poincaré spin group.

irreducible representations corresponds to different types of the particles.

Wigner: if $m > 0$ (massive) one-to-one correspondence between

1. finite dimensional irreducible representation $D^{(s,0)}$ of $SL(2,\mathbb{C})$
2. irreducible continuous unitary representation of $\mathbb{R}^{1,3} \rtimes SL(2,\mathbb{C})$

Finite dimensional irreducible representation of $SL(2,\mathbb{C})$ are labeled by $D^{(s_+,s_-)}$ where $s_\pm = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

covariance in Wightman's axiom is motivated by transformation law in the construction of irreducible representations

1. Space of states

- States are represented as unit rays in a separable complex Hilbert space $\mathcal{H}$.

- There is a strong continuous unitary representation of the group $\mathbb{R}^{1,3} \rtimes SL(2,\mathbb{C})$.

- (**uniqueness of vacuum**) There is a unique unit ray $\{\Omega\}$ (interpreted as vacuum) such that

$$U(a,A)\Omega = \Omega$$

for any $(a,A) \in \mathbb{R}^{1,3} \rtimes SL(2,\mathbb{C})$.

- (**spectrum condition**) The generators of space-time translations (P^0, P^1, P^2, P^3), interpreted as the energy-momentum operator, has spectrum in closed forward light cone $\overline{V_+}$.

2. Observables and covariance

- A set of operator valued distributions $\{\varphi_n^{(k)} | k, n \in \mathbb{N}\}$, where k labels the type of the field which can be at most countable and n labels the components of the field which can only take finite number of values, and a dense subspace D where all the operators $\varphi_n^{(k)}(f)$ and $\varphi_n^{(k)*}(f) = \varphi_n^{(k)}(\bar{f})^*$ are defined, for all $n \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^4)$.

- The vacuum Ω is contained in D.

- The domain D is invariant under the action of $U(A, a)$, $\varphi_n^{(k)}(f)$ and $\varphi_n^{(k)*}(f)$, for all $(a, A) \in \mathbb{R}^{1,3} \rtimes SL(2, \mathbb{C})$, $n \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^4)$.

- The covariant transformation of fields operator under the action of (a, A) is given by

$$U(a, A)\varphi_n^{(k)(*)}(f)U(a, A)^{-1} = \sum_m D_{nm}^{(k)}(A^{-1})\varphi_m^{(k)(*)}((a, A)f)$$

where $D_{nm}^{(k)}(A)$ are matrices of a finite dimensional irreducible representation of the group $SL(2, \mathbb{C})$ with $\varphi_n^{(k)}$ as its components, and $(a, A)f = f(A^{-1}(x - a))$. If the representation $D(A)$ is a representation of group $\mathrm{SO}^+(1, 3)$, then the fields are called a tensor field, otherwise the fields are called spinor fields. This transformation law is linear in the test function.

- The vacuum Ω is a cyclic vector, which means the linear span D_0 of the set $\{\varphi_{i_1}^{(k_1)(*)}(f_1)\ldots\varphi_{i_m}^{(k_m)(*)}(f_m)\Omega|\, m\in\mathbb{N}, i_1,\cdots,i_m\in\mathbb{N}, f_1,\ldots,f_m\in\mathcal{S}(\mathbb{R}^4)\}$ is dense.

3. Locality or Microcausality

- For any two test functions $f, g\in\mathbb{R}^4$ whose supports consists only space-like separated points, the operators $\varphi_n^{(k)(*)}(f)$ and $\varphi_m^{(k')(*)}(g)$ satisfies

$$\varphi_n^{(k)(*)}(f)\varphi_m^{(k')(*)}(g)-\sigma(k,k')\varphi_m^{(k')(*)}(g)\varphi_n^{(k)(*)}(f)=0$$

where $\sigma(k,k')=1$ if one of k and k' is representation with integer spin, and $\sigma(k, k')=-1$ if both k and k' are representation of odd spin.

Wightman Axioms of Relativistic Quantum Fields

Comments:

1. $\sigma(k, k')$: spin-statistics theorem and Klein transformation

2. Mass operator: $(P^0)^2 - (P^1)^2 - (P^2)^2 - (P^3)^2$

A theory is said to has a **mass gap**, if there is no eigenvalue between 0 and $\Delta > 0$.

3. Rigged Hilbert space approach:

Bohm & Roberts (1966): quantum mechanics

Bogoliubov & Logunov&Todorov (1975): modified the Wightman axiom

Prigogine & Antoniou (1993): application to irreversible quantum system

4. Gauge theory:

Ferrari, Picasso & Strocchi (1974):free Maxwell's equations are inconsistent with axioms.

Wightman & Strocchi: Indefinite Hilbert space modification

Glimm&Lee (2022): axioms for quantum gauge theory

The correlation of Wightman fields, given by

$$W^{k_1^{(*)},\cdots,k_n^{(*)}}_{m_1,\cdots,m_n}(f_1,\cdots,f_n) := \left\langle \Psi_0, \varphi^{(k_1)(*)}_{m_1}(f_1)\cdots\varphi^{(k_n)(*)}_{m_n}(f_n)\,\Psi_0\right\rangle$$

for any $f_1,\cdots,f_n \in \mathcal{S}(\mathbb{R}^4)$ defines a tempered distribution on $\mathbb{R}^{4n}$ by nuclear theorem.

Proposition 5. (Hermiticity)

$$W^{k_1^{(*)},\cdots,k_n^{(*)}}_{m_1,\cdots,m_n}(f_1,\cdots,f_n) = \overline{W^{k_n^{-(*)},\cdots,k_1^{-(*)}}_{m_n,\cdots,m_1}(f_n,\cdots,f_1)}$$

where the notation $-(*)$ *means if we have index* k_i*, then we take* $k_i^{-(*)} = k_i^*$*, if we have* k_i^**, we then take* $k_i^{-(*)} = k_i$*.*

Proposition 6. (Positivity) *For any finite sequence of test functions* $\left\{ f^{k_1^{(*)},\cdots,k_i^{(*)}}_{m_1,\cdots,m_i} \,\middle|\, f^{k_1^{(*)},\cdots,k_i^{(*)}}_{m_1,\cdots,m_i} \in \mathcal{S}(\mathbb{R}^{4i}) \right\}$*, we have*

$$\sum_{i,j} \sum_{\substack{k_1'^{(*)},\cdots,k_j'^{(*)} \\ m_1',\cdots,m_j'}} \sum_{\substack{k_1^{(*)},\cdots,k_i^{(*)} \\ m_1,\cdots,m_i}} W^{k_j'^{-(*)},\cdots,k_1'^{-(*)},k_1^{(*)},\cdots,k_i^{(*)}}_{m_j',\cdots,m_1',m_1,\cdots,m_i} \left(\overline{\check{f}^{k_1'^{(*)},\cdots,k_j'^{(*)}}_{m_1',\cdots,m_j'}} \, f^{k_1^{(*)},\cdots,k_i^{(*)}}_{m_1,\cdots,m_i} \right) \geqslant 0$$

where $\check{f}^{k_1'^{(*)},\cdots,k_j'^{(*)}}_{m_1',\cdots,m_j'}$ *is the function* $f^{k_1'^{(*)},\cdots,k_j'^{(*)}}_{m_1',\cdots,m_j'}(x_j, x_{j-1}, \ldots, x_1)$.

Proof. The norm of the vector

$$\sum_i \sum_{\substack{k_1^{(*)},\cdots,k_i^{(*)} \\ m_1,\cdots,m_i}} \varphi^{(k_1)(*)}_{m_1}(f_1) \cdots \varphi^{(k_i)(*)}_{m_i}(f_i) \Psi_0$$

is non-negative. □

Proposition 7. (Covariance) *We have*

$$\sum_{n_1,\ldots,n_l} D^{(k_1)}_{m_1n_1}(A^{-1})\ldots D^{(k_l)}_{m_ln_l}(A^{-1})W^{k_1^{(*)},\cdots,k_l^{(*)}}_{n_1,\cdots,n_l}((a,A)f_1,\cdots,(a,A)f_l)$$

$$=W^{k_1^{(*)},\cdots,k_l^{(*)}}_{m_1,\cdots,m_l}(f_1,\cdots,f_l)$$

for any $(a,A)\in\mathbb{R}^{1,3}\rtimes SL(2,\mathbb{C})$.

Proposition 8. (Locality or Microcausality) *If the supports of test functions* f_l, f_{l+1} *consist only space like points, then*

$$W^{k_1^{(*)},\cdots,k_l^{(*)},k_{l+1}^{(*)},\ldots,k_n^{(*)}}_{m_1,\cdots,m_l,m_{l+1},\ldots,m_n}(f_1,\cdots,f_l,f_{l+1},\ldots,f_n)$$

$$=(-1)^{\sigma(k_l,k_{l+1})}W^{k_1^{(*)},\cdots,k_{l+1}^{(*)},k_l^{(*)},\ldots,k_n^{(*)}}_{m_1,\cdots,m_{l+1},m_l,\ldots,m_n}(f_1,\cdots,f_{l+1},f_l,\ldots,f_n)$$

for all possible indices.

Proposition 9. (Spectrum Property) *Under the change of variables*

$$\xi_1 = x_1 - x_2, \ldots, \xi_{n-1} = x_{n-1} - x_n, \xi_n = x_n$$

where $x_1, \ldots, x_n \in \mathbb{R}^4$*, each tempered distribution* $W^{k_1^{(*)}, \cdots, k_n^{(*)}}_{m_1, \cdots, m_n}$ *depends only on* $\xi_1, \ldots, \xi_{n-1}$*, that is*

$$\frac{\partial W^{k_1^{(*)}, \cdots, k_n^{(*)}}_{m_1, \cdots, m_n}}{\partial \xi_n} = 0$$

then there is a tempered distribution $M^{k_1^{(*)}, \cdots, k_n^{(*)}}_{m_1, \cdots, m_n} \in \mathcal{S}'(\mathbb{R}^{4(n-1)})$*, such that*

$$W^{k_1^{(*)}, \cdots, k_n^{(*)}}_{m_1, \cdots, m_n} = M^{k_1^{(*)}, \cdots, k_n^{(*)}}_{m_1, \cdots, m_n} \otimes 1$$

where 1 *is a constant function* 1 *on* $\mathbb{R}^4$*. Moreover, the Fourier transform* $\tilde{M}^{k_1^{(*)}, \cdots, k_n^{(*)}}_{m_1, \cdots, m_n}$ *of the tempered distribution* $M^{k_1^{(*)}, \cdots, k_n^{(*)}}_{m_1, \cdots, m_n}$ *is supported in the* $(n-1)-\mathrm{fold}$ *product of closed forward light cone* $\overline{V_+} \times \cdots \times \overline{V_+}$*.*

Proposition 10. (Cluster Property) *Suppose* $a \in \mathbb{R}^{1,3}$ *is a space-like vector, then*

$$W^{k_1^{(*)},\cdots,k_l^{(*)},k_{l+1}^{(*)},\ldots,k_n^{(*)}}_{m_1,\cdots,m_l,m_{l+1},\ldots,m_n}(f_1,\cdots,f_l,(\lambda a,I)\,f_{l+1},\ldots,(\lambda a,I)\,f_n)\xrightarrow{\lambda\to+\infty}$$

$$W^{k_1^{(*)},\cdots,k_l^{(*)}}_{m_1,\cdots,m_l}(f_1,\cdots,f_l)W^{k_{l+1}^{(*)},\cdots,k_n^{(*)}}_{m_{l+1},\cdots,m_n}(f_{l+1},\cdots,f_n)$$

for all possible indices and test functions.

Proof. In general one can show that

$$\lim_{\lambda\to\infty}\langle\Phi,U(\lambda a,I)\,\Psi\rangle=\langle\Phi,\Psi_0\rangle\langle\Psi_0,\Psi\rangle$$ □

Wightman Reconstruction Theorem

Wightman distributions a collection of tempered distributions $\left\{W^{k_1^{(*)},\cdots,k_n^{(*)}}_{m_1,\cdots,m_n} \in \mathcal{S}'(\mathbb{R}^{4n})\middle| n \in \mathbb{N}, 0 \leqslant m_i \leqslant 2s(k_i)+1 < +\infty, 1 \leqslant i \leqslant n\right\}$ and $W^{[0]}$ without any index is assumes to be 1, where $s(k_i)$ is understood to be the spin described by $k_i - \mathrm{th}$ field, with following properties:

1. **(Hermiticity)**
2. **(Positivity)**
3. **(Covariance)**
4. **(Locality or Microcausality)**
5. **(Cluster Property)**
6. **(Spectrum Property)**

Theorem 11. (Wightman Reconstruction Theorem) *For a given set of Wightman distributions satisfying axioms 1 to 6, there exists a unique Wightman quantum field theory up to unitary equivalence.*

Proof: Sketch

Vector space H: sequence $f=(f_0,f_1,\ldots)$ where $f_i\in\mathcal{S}(\mathbb{R}^{4i})$ with only a finite number of nonzero components.

pre vacuum vector: $\Psi_0=(1,0,0,\ldots)$

Inner product:

$$\langle f,g\rangle := \sum_{i,j=0}^{\infty} W_{i+j}(\bar{\bar{f}}_i\otimes g_j)$$

It is skew symmetric by hermiticity and non-negative definite by positivity.

Wightman Reconstruction Theorem

Representation of Poincaré group:

$$U(a,\Lambda)f = (f_0, (a,\Lambda)f_1, (a,\Lambda)f_2, \dots)$$

then by covariance assumption, the skew linear form is preserved, and clearly the vacuum is an invariant vector.

Field: for $h \in \mathcal{S}$, the operator $\varphi(h)$ is

$$\varphi(h)f = (0, h \otimes f_0, h \otimes f_1, \dots)$$

Define $H_0 := \{f \in H \,|\, \langle f, f\rangle = 0\}$

complete the space H/H_0 to get the physical Hilbert space $\mathcal{H}$

Verify other axioms...

Transition probability:

$$\langle \varphi_f | U(t_1 - t_0) | \varphi_i \rangle, U(t_1 - t_0) = e^{-i(t_1 - t_0)\frac{H}{\hbar}}$$

Feynman path integral:

$$\text{probability theory on states} \rightarrow \text{probability theory on histories}$$

$$\langle \varphi_f | U(t_1 - t_0) | \varphi_i \rangle = \langle \varphi_f \left| e^{-i(t_1 - t_0)\frac{H}{\hbar}} \right| \varphi_i \rangle = \int_{\substack{\varphi(t_0) = \varphi_i \\ \varphi(t_1) = \varphi_f}} e^{\frac{i}{\hbar} S[\varphi]} \mathcal{D}[\varphi]$$

where $S[\varphi]$ is the classical action functional.

The expectation of time ordered product operators (operators in Heisenberg picture)

$$\begin{aligned}
&\int_{\substack{\varphi(t_0)=\varphi_i \\ \varphi(t_{n+1})=\varphi_f}} \varphi(t_n)\cdots\varphi(t_1)e^{\frac{i}{\hbar}S[\varphi]}\mathcal{D}[\varphi] \\
=\ &\int d\varphi_n\ldots\int d\varphi_1\langle\varphi_f\,|U(t_{n+1}-t_n)|\varphi_n\rangle\varphi_n\langle\varphi_n|\cdots|\varphi_1\rangle\varphi_1\langle\varphi_1|U(t_1-t_0)|\varphi_i\rangle \\
=\ &\langle\varphi_f\,|U(t_{n+1})\Phi(t_n)\cdots\Phi(t_1)U(-t_0)|\varphi_i\rangle \\
=\ &\langle t=0,\varphi_f\,|\Phi(t_n)\cdots\Phi(t_1)|\,0,\varphi_i\rangle
\end{aligned}$$

where $t_{n+1}>t_n>t_{n-1}>\cdots>t_1>t_0$ and Φ is the quantum analog of φ, usually given by canonical quantization. The state $|\,t=0,\varphi_i\rangle$ means the time zero state which evolves to labeled by configuration φ_i in time t_0.

Usually the Hamiltonian is positive $H \geqslant 0$: $\langle \psi | H | \psi \rangle \geqslant 0$ for all possible state $| \psi \rangle$

Wick rotation: Replace $t \geqslant 0$ in $e^{-it\frac{H}{\hbar}}$ by $-i\tau$ for $\tau \geqslant 0$, then

$$e^{-it\frac{H}{\hbar}} \to e^{-\tau\frac{H}{\hbar}}$$

is bounded, and the path integral formula turns into

$$\langle \varphi_f | U_E(\tau_1 - \tau_0) | \varphi_i \rangle = \langle \varphi_f \left| e^{-(\tau_1 - \tau_0)\frac{H}{\hbar}} \right| \varphi_i \rangle = \int_{\substack{\varphi(\tau_0) = \varphi_i \\ \varphi(\tau_1) = \varphi_f}} e^{-\frac{1}{\hbar} S_E[\varphi]} \mathcal{D}[\varphi]$$

where $S_E[\varphi]$ is the Euclidean action, which is positive usually. For expectations with $\tau_{n+1} > \tau_n > \tau_{n-1} > \cdots > \tau_1 > \tau_0$, we have

$$\langle \tau = 0, \varphi_f | \Phi(\tau_n) \cdots \Phi(\tau_1) | \tau = 0, \varphi_i \rangle = \int_{\substack{\varphi(\tau_0) = \varphi_i \\ \varphi(\tau_{n+1}) = \varphi_f}} \varphi(\tau_n) \cdots \varphi(\tau_1) e^{-\frac{1}{\hbar} S_E[\varphi]} \mathcal{D}[\varphi]$$

$$\text{Euclidean path integral measure} \leftrightarrow \text{Boltzmann distribution}$$

$$\text{Euclidean time}\,\tau \leftrightarrow \text{temperature}\,T \qquad \tau = \frac{\hbar}{kT}$$

According to quantum statistical mechanics, if we assume the spectrum of H is discrete for convenience, say $E_0 < E_1 < \cdots < E_n < \cdots$, the ensemble average at temperature T of an operator A should be given by

$$\langle A \rangle = \frac{\mathrm{Tr}\left(e^{-\frac{1}{kT}H} A\right)}{\mathrm{Tr}\left(e^{-\frac{1}{kT}H}\right)} = \frac{\sum_{i=0}^{\infty} e^{-\frac{1}{kT}E_i} \langle i | A | i \rangle}{\sum_{i=0}^{\infty} e^{-\frac{1}{kT}E_i}}$$

where k is the Boltzmann constant. Zero temperature limit $T \to 0$, we have

$$\frac{\sum_{i=0}^{\infty} e^{-\frac{1}{kT}E_i} \langle i | A | i \rangle}{\sum_{i=0}^{\infty} e^{-\frac{1}{kT}E_i}} \xrightarrow{T \to 0} \langle 0 | A | 0 \rangle$$

and this limit corresponds to the limit $\tau \to +\infty$. Thus the equation

$$\langle \Phi(\tau_n) \cdots \Phi(\tau_1) \rangle = \frac{\int d\varphi' \int_{\substack{\varphi(\tau_0)=\varphi_i \\ \varphi(\tau_{n+1})=\varphi_f \\ \varphi(0)=\varphi'}} \varphi(\tau_n) \cdots \varphi(\tau_1) e^{-\frac{1}{\hbar} S_E[\varphi]} \mathcal{D}[\varphi] d\varphi'}{\int d\varphi' \int_{\substack{\varphi(\tau_0)=\varphi_i \\ \varphi(\tau_{n+1})=\varphi_f \\ \varphi(0)=\varphi'}} e^{-\frac{1}{\hbar} S_E[\varphi]} \mathcal{D}[\varphi] d\varphi'}$$

with the limit $\tau_0 \to -\infty, \tau_{n+1} \to +\infty$, one has

$$\langle 0 | \Phi(\tau_n) \cdots \Phi(\tau_1) | 0 \rangle = \frac{\int \varphi(\tau_n) \cdots \varphi(\tau_1) e^{-\frac{1}{\hbar} S_E[\varphi]} \mathcal{D}'[\varphi]}{\int e^{-\frac{1}{\hbar} S_E[\varphi]} \mathcal{D}'[\varphi]}$$

where the path integral measure is over the space of all configurations with some decay property (one can also pose the periodic condition on Euclidean time, and then study the limit that the period goes to infinity).

Construction of Schwinger functions: single scalar Boson field

Step 1: use spectral property

Theorem 12. *There are holomorphic functions* $W_n(z_1,\ldots,z_n)$ *and* $M_{n-1}(\tilde{z}_1,\ldots,\tilde{z}_{n-1})$*, where* $z_i = (z_i^0, z_i^1, z_i^2, z_i^3)$ *and denote* $\tilde{z}_j = x_j - i\, y_j$*, such that*

$$W_n(z_1,\cdots,z_n) = M_{n-1}(z_1 - z_2,\ldots,z_{n-1} - z_n)$$

defined on the **tube** $T_{n-1} = \{-\mathrm{Im}\,(z_i - z_{i+1}) \in V_+ | \, i = 1,\cdots,n-1\}$*, and polynomially bounded, such that the boundary value* $M_{n-1}(\tilde{z}_1,\ldots,\tilde{z}_{n-1})$ *is the distribution* M_{n-1}*, i.e*

$$\lim_{y_j \to 0} M_{n-1}(x_1 - i\, y_1,\cdots,x_{n-1} - i\, y_{n-1}) = M_{n-1}$$

in the sense of tempered distribution.

Step 2: use covariance

Complex Lorentz group $L(\mathbb{C})$: connected component of the identity of the group of complex matrices that preserve the complex bilinear form

$$\langle z_1, z_2\rangle = z_1^0 z_2^0 - z_1^1 z_2^1 - z_1^2 z_2^2 - z_1^3 z_2^3$$

on the space $\mathbb{C}^{1,3}$. A point in the space $\mathbb{C}^{1,3}$ is called a **Euclidean point** if it as the form $(-ix^0, x^1, x^2, x^3)$ where $x^0, x^1, x^2, x^3 \in \mathbb{R}$.

Since W_n transforms as

$$W_n(x_1, \cdots, x_n) = W_n(\Lambda\, x_1, \cdots, \Lambda\, x_n), \text{for all}\, \Lambda \in SO^+(1,3)$$

by the uniqueness of the analytic continuation we have

$$W_n(z_1, \cdots, z_n) = W_n(\Lambda\, z_1, \cdots, \Lambda\, z_n)$$

and

$$M_{n-1}(\tilde{z}_1, \cdots, \tilde{z}_{n-1}) = M_{n-1}(\Lambda\, \tilde{z}_1, \cdots, \Lambda\, \tilde{z}_{n-1})$$

but we can see here that the Lorentz transformations preserve the tube T_{n-1}, now one can use this identity to extend the action of Lorentz group to complex Lorentz group. Then one can define M_{n-1} on the so called **extended tube**

$$T^e_{n-1} := \bigcup_{\Lambda \in L(\mathbb{C})} \Lambda T_{n-1}$$

and this extension is single valued.

Osterwalder-Schrader Axioms

Step 3: use locality

For any permutation σ

$$W_n(x_1, \ldots, x_n) = W_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

Define the value of $W_n(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ by $W_n(z_1, \cdots, z_n)$, and hence one can define M_{n-1} when $W_n(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ is define.

This analytic continuation is well-defined and also single-valued.

The intersection of Euclidean points E^n and W_n's holomorphic domain is

$$\mathbb{R}^{4n}_{\neq} := \{(x_1, x_2, \cdots, x_n) | x_i \in \mathbb{R}^4, x_i \neq x_j \text{ if } i \neq j\} \subset E^n$$

Schwinger function S_n: restriction of W_n on $\mathbb{R}^{4n}_{\neq}$

note that it is polynomially bounded, hence also a tempered distribution.

Osterwalder-Schrader Axioms

OS axioms: Schwinger distribution is a collection of tempered distributions $\{S_n \in \mathcal{S}'(\mathbb{R}^{4n}_{\neq}) | n \in \mathbb{N}\}$ and $S_0 := 1$, with following properties:

1. (Linear Growth) There exists an integer $s \in \mathbb{N}$, and a sequence $\{\sigma_n\}$ of positive numbers, such that $\sigma_n \leqslant C(n!)^{C'}$ for some constants C, C' independent of n, and

$$|S_n(f)| \leqslant \sigma_n \|f\|_{n \times s}$$

for all $f \in \mathbb{R}^{4n}_{\neq}$ and $n \in \mathbb{N}$.

2. (Euclidean Invariance) for any $(a, A) \in \mathbb{R}^4 \rtimes \mathrm{SO}(4)$

$$S_n((a, A) f_1, \cdots, (a, A) f_n)$$

$$= S_n(f_1, \cdots, f_n)$$

3. (Reflection Positivity) For any finite sequence of test functions f_i

$$\sum_{i,j} S_{i+j}(\overline{\Theta \check{f}_j} \otimes f_i) \geqslant 0.$$

where $\Theta f(x_1, \ldots, x_n) = f(\Theta x_1, \ldots, \Theta x_n)$ and $\Theta(x_0, x_1, x_2, x_3) = (-x_0, x_1, x_2, x_3)$.

4. (Symmetry)

$$S_n(f_1, \cdots, f_l, f_{l+1}, \ldots, f_n)$$

$$= S_n(f_1, \cdots, f_{l+1}, f_l, \ldots, f_n)$$

5. (Cluster Property) Suppose $a \in \mathbb{R}^4$ is a non-zero vector of the form $(0, a_1, a_2, a_3)$

$$S_{m+n}(f_m, (\lambda a, I) f_n) \xrightarrow{\lambda \to +\infty} S_m(f_m) S_n(f_n)$$

Osterwalder-Schrader Axioms

Why reflection positivity?

Reflection positivity correponds to positivity of Wightman distributions

Conjugate transformation of Minkowski Hermitian scalar field

$$\varphi(t,x)=e^{itP^0-ix_1P^1-ix_2P^2-ix_3P^3}\varphi(0,0)e^{-itP^0+ix_1P^1+ix_2P^2+ix_3P^3}$$

and applyWick rotation $t=-i\tau$ we have

$$\varphi(\tau,x)=e^{\tau P^0-ix_1P^1-ix_2P^2-ix_3P^3}\varphi(0,0)e^{-\tau P^0+ix_1P^1+ix_2P^2+ix_3P^3}$$

then we have

$$\varphi^*(\tau,x)=\varphi(-\tau,x)$$

for any Euclidean time τ, note that conjugate transformation of a Hermitian scalar Boson field in Euclidean theory is different from Minkowski theory.

Osterwalder-Schrader Axioms

Theorem 13. (Osterwalder-Schrader Reconstruction) *There exist a unique Wightman quantum field theory whose Schwinger functions agree with the given set with properties listed above.*

Remarks:

1. This works for general spinor or tensor fields.

2. Different similar axioms, Jaffe and Glimm (2012), Fröhlich (1974), probabilities on distributions. They imply Wightman axioms, but not equivalent to.

Equivalence can be find in Bogolubov etc. (2012) with different condition corresponding to the spectrum condition (a different topology on test functions).

3. OS axioms can be used to construct Euclidean fields that are operator-valued distributions, which is also called the reconstruction theorem.

Stochastic Quantization

Question: How to construct a qft model satisfies OS-axioms?

Stochastic Quantization: construct Eucldiean path integral measure and verify OS-axioms.

Idea: Think $e^{-S_E[\varphi]}\mathcal{D}'[\varphi]$ as the Boltzmann distribution in equilibrium statistical mechanics and use Langevin dynamics to construct a hypothetical non-equilibrium process converging to this equilibrium.

$$\partial_t\varphi(t,x) = -\left(\frac{\delta S_E[\varphi]}{\delta\varphi}\right)|_{\varphi=\varphi(t,x)} + \xi$$

or equivalently

$$\partial_t\varphi(t,x) = -\frac{\delta\hat{S}_E[\varphi]}{\delta\varphi} + \xi, \hat{S}_E[\varphi] = \int dt\, S_E[\varphi]$$

where δ is the functional derivative, t is fictitious time, and ξ is the space-time white noise.

Since the white noise is delta correlated in time direction, then the solution process if exists, should be a Markov process.

Compute invaraint measure:

Fokker-Planck equation? equation of the probability distribution $P(\varphi, t)$

Then the Fokker-Planck equation is given by

$$\partial_t P(\varphi, t) = \int dx \frac{\delta}{\delta \varphi} \left[\frac{\delta \hat{S}_E}{\delta \varphi} + \frac{\delta}{\delta \varphi} \right] P(\varphi, t)$$

and it is clear $e^{-S_E[\varphi]}$ is a stationary solution.

Remark:

1. Proposed by Parisi and Wu (1981), different from Nelson's stochastic mechanics (1966), and De Broglie–Bohm (1952), where they used the real time. The SQ method is based on an hypothetical process depending on a fictitious time. And one can use different Langevin equations to describe the Euclidean path integral measure. There are cases (Chern-Simons) where the stochastic quantization method does not work (Ferrari, Huffel 1991).

2. The stochastic quantization equation requires us to study the solution theory for stochastic partial differential equations, establish the meaning of the equation, the existence of local solutions, and most importantly the long time existence of the solution, since we need to take $t \to \infty$ to get the equilibrium state.

3. We need to verify that the limiting measure should satisfy the Osterwalder-Schrader axioms or its some kind of modifications, this is not easy. It has been shown by Jaffe (2015), that reflection positivity is not satisfied in the finite time non-equilibrium state of the solution for free scalar field.

Thanks!