Topics in Non-commutative Probability

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Introduction

Non-commutative probability is an area of mathematics which tries to understand non-commutative algebras inspired by classical probability theory. In this framework, non-commutative random variables are defined as abstract elements of non-commutative algebras. Such a theory can emerge as a non-commutative generalization of classical probability theory.

The fundamental notion in classical probability theory is that of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a set, $\mathcal{F}$ is a σ-algebra on Ω and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. Starting with these data, it is possible to build commutative algebras of random variables $X: \Omega \to \mathbb{C}$ (e.g. $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C})$). In probability and statistics, a lot of times is of interest the computation or the estimation of the expected value $\mathbb{E}[X]$, in order to obtain useful information for the random variable $X \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C})$. Thus, the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is the building block which allows us to define random variables $X \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C})$ and their expected values $\mathbb{E}[X]$. A first step in order to pass from classical probability theory to non-commutative probability is to ignore the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and to consider the algebra of random variables (that we are interested in to study) and their expectations, as being the foundational objects. This leads to a more general formalism of "algebraic probability" where the algebra of random variables is replaced by an abstract unital algebra $\mathcal{A}$ over $\mathbb{C}$ and the random variables can be identified with elements $a \in \mathcal{A}$. Moreover, the expectation $\mathbb{E}$ is replaced by a linear functional $\phi: \mathcal{A} \to \mathbb{C}$, such that $\phi(1_{\mathcal{A}}) = 1$, where $1_{\mathcal{A}}$ represents the unit of the algebra $\mathcal{A}$. Of course the condition $\phi(1_{\mathcal{A}}) = 1$ is proportional to the fact $\mathbb{E}[1_{L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C})}] = 1$. Thus, in this algebraic framework our main players are the elements $a \in \mathcal{A}$ that we want to observe and the functional $\phi: \mathcal{A} \to \mathbb{C}$ which can be thought as the experimental mechanism which correspnds to every observable $a \in \mathcal{A}$ a value $\phi(a) \in \mathbb{C}$, which provides useful information about a. For $a_1, \ldots, a_n \in \mathcal{A}$, the expressions $\phi(a_1 \ldots a_n)$ can be thought as the joint moments of the random variables $a_1, \ldots, a_n \in \mathcal{A}$. In the context of non-commutative probability theory, the algebra $\mathcal{A}$ is not commutative and the elements $a \in \mathcal{A}$ are called non-commutative random variables. This more general formalism encompasses quantum mechanics, since the random variables $a \in \mathcal{A}$ can take the rolle of physical observables and the expectation $\phi(a)$ can be thought as the expected value of a on a given quantum state [...].

Free probability is a mathematical theory that studies non-commutative random variables. It was introduced by Dan Voiculescu in the 1980's. This branch of mathematics can be described by the exact formula

$$\text{free probability} = \text{non-commutative probability} + \text{free independence}.$$

In the context of non-commutative probability, free independence is one of the most celebrated notions of stochastic independence, which comes to replace the classical notion of independence. Free probability theory is extremely useful in order to study large random matrices. Dan Voiculescu proved that independent Gaussian matrices behave as free random variables in a non-commutative probability space, as their size goes to infinity [...]. Except of it's strong connections with random matrices, free probability has relations with other fields of mathematics, such as combinatorics, operator algebras and representations of symmetric groups. It's relations with combinatorics (especially non-crossing partitions) leads to a lot of comparisons of free probability with classical probability theory, where the difference between the two is very concrete and stable. This fact leads to "free analogues" of some of the most crucial notions of classical probability theory, such as Gaussian distribution, Poisson distribution and Brownian motion. The goal of this Master Thesis is to present some of these notions and to provide interpolations between classical and free probability.

The first part of this Master Thesis aims to present some aspects of free probability theory, concentrated on their combinatorial structure. In the first section we present the basics of free probability theory. In subsections 1.1 and 1.2 we start by defining the notions of non-commutative probability space, non-commutative random variables and that of free independence. We do so in mainly following the book [...]. In subsection 1.3 we concentrate on the computation of joint moments of free random variables. Our goal is to better understand the definition of free independence and to see whether this new notion of independence can be related, in some sense, with the classical one. We also introduce the notion of non-crossing partition, which will be crucial for the further development of the theory. Our main reference for this subsection is [...]. Then, mainly based on [...], in subsection 1.4 we formulate the free analogues of the central limit theorem and of the Poisson limit theorem. These lead to the determination of the free analogues of the Gaussian variables and of the Poisson variables. At the end of the first section, in subsection 1.5, we give an example of "free Gaussian variables" in some non-commutative probability space. In order to do so, we introduce the notion of Dyck paths. This example was taken by the book [...].

In section 2, we present the basics of the theory of free cumulants, which was introduced by Speicher in proportion to the theory of classical cumulants of Rota. We start by introducing some useful combinatorial properties about the lattice of non-crossing partitions. One of them is the Möbious inversion formula which will be very important for the determination of free cumulants. Then, we continue with the definition of free cumulants. The definition of free cumulants shows the analogy with classical cumulants quite clearly, in the sense that compared to the latter, the lattice of all partitions of the set $\{1, \ldots, n\}$ is replaced by the lattice of non-crossing partitions of that set. Our final goal is to show a characterization of the notion of free independence via free cumulants. Similarly to classical probability theory, in the context of free probability we have that non-commutative random variables are freely independent if and only if certain cumulants vanish. In this section we mainly follow [...].

In section 3 we turn to non-commutative stochastic processes. More precisely, in the context of free probability we focus on non-commutative processes with stationary and independent increments. In this framework, for a non-commutative process $(c_t)_{t\geq 0}$ in some non-commutative probability space, we demand the non-commutative random variables $c_{t_1} - c_{t_0}, c_{t_2} - c_{t_1}, \ldots, c_{t_m} - c_{t_{m-1}}$ to be freely independent, for all $m \in \mathbb{N}$ and for all $0 \leq t_0 < t_1 < \cdots < t_m$. We give the definition of free Levy processes and we examine the combinatorics of their non-commutative distribution. The special examples of free Levy processes that we are interested in are the free Brownian motion and the free Poisson process. We prove the free analogue of Wick formula that allows us to determine the non-commutative distribution of the free Brownian motion. Whereas the moments $\mathbb{E}[X_{t_1} \ldots X_{t_{2n}}]$ of a classical Brownian motion $(\Omega, \mathcal{F}, \mathbb{P}, (X_t)_{t\geq 0})$ are related to the second moments $\mathbb{E}[X_{t_i} X_{t_j}]$ via a sum on the whole set of 2-partitions of $\{1, \ldots, 2n\}$, the corresponding moments of a free Brownian motion can be computed with the help of non-crossing partitions. Finally, we realize the free Brownian motion and the free Poisson process as processes on the full Fock space of $L^2(\mathbb{R})$. Our main reference for this section are [...] and [...].

Section 4 corresponds to the work of Bożejko and Speicher about the generalized Brownian motion [...,...]. Inspired by the results of section 3 about free Brownian motion, we present a non-commutative stochastic process which gives an interpolation between classical and free Brownian motion. In order to do so, we present an interpolation between the canonical commutation relations (C.C.R.) and canonical anticommutation relations (C.A.R.). The C.C.R. are fundamental relations which are algebraically defined by

$$a_i^* a_j - a_j a_i^* = \delta_{i,j} 1.$$

These relations describe bosons in quantum mechanics. On the other hand, the C.A.R. are defined by

$$a_i^* a_j + a_j a_i^* = \delta_{i,j} 1$$

and they descibe fermions. In order to provide an interpolation between C.C.R. and C.A.R., we are interested in the generalized commutation relations, which are defined by

$$a_i^* a_j - \mu\, a_j a_i^* = \delta_{i,j} 1,$$

where $-1 \leq \mu \leq 1$. These relations were also proposed by Greenberg [...], in order to describe particles with statistics intermediate between Bose and Fermi statistics. Inspired by the fact that the bosonic Fock space gives a natural realization of the C.C.R. and the fermionic Fock space gives a realization of the C.A.R., we introduce the μ-Fock space ($-1 \leq \mu \leq 1$) which gives an interpolation between the bosonic Fock space and fermionic Fock space. It also includes the full Fock space (for $\mu = 0$). In order to define the μ-Fock space of a Hilbert space H, we consider the completion of the set of finite linear combinations of product vectors, with respect to a scalar product $\langle \cdot, \cdot \rangle_\mu$. This scalar product differs from the usual scalar product of $\bigoplus_{n \geq 0} H^{\otimes n}$. In this direction, the main problem is the positive definiteness of $\langle \cdot, \cdot \rangle_\mu$. For this reason, we introduce the notion of a positive definite kernel, mainly following in [...]. We define the corresponding creation and annihilation operators on the μ-Fock space and we show that they satisfy the generalized commutation relation. This is an analogous situation with the well known bosonic and fermionic relations. Then, we give an example of a generalized Brownian motion, which is formulated by considering the sum of creation and annihilation operators on the μ-Fock space. This comes in proportion to the examples of non-commutative Brownian motions, which are given by [...].

Finally, in section 5 we continue our investigation of the generalized Brownian motion, which was presented in section 4. We show that this non-commutative Brownian motion which was motivated by the fact that the moments of classical Brownian motion can be calculated by 2-partitions, can actually emerge via a central limit theorem. More precisely, we examine the Gaussian distribution corresponding to this process and we show that it can be derived from a non-commutative central limit theorem. In order to do so, we use a stochastic interploation. Taking into account this non-commutative central limit theorem, we generalize it to an invariance principle, which leads to the generalized Brownian motion of section 4. The formulation and proof of this non-commutative central limit theorem is due to Speicher [...]. Moreover, in order to determine this non-commutative analogue of Gaussian distribution, we introduce the μ-analogues of the Hermite polynomials, mainly following in [...] and [...]. As a consequence, this leads to the determination of the probability measure which characterizes the non-commutative distribution of this non-commutative Gaussian variable. Inspired by the definition of μ-Hermite polynomials, we present the μ-analogues of Charlier-Poisson polynomials. Given this sequence of polynomials, it arises an orthogonalizing probability measure on $\mathbb{R}$ which is an μ-analogue of the Poisson distribution. We realize μ-Poisson variables on the μ-Fock space. This is done in full proportion to the corresponding result of [...], which gives non-commutative realizations of Poisson variables in the bosonic Fock space. Our main reference for μ-Poisson variables is [...].

1 Combinatorics of Free Probability

Free probability is a non-commutative probability theory, in which the notion of independence of classical probability theory is replaced by that of free independence. It was introduced by D.Voiculescu around 1985, in order to study some problems in the context of von Neumann algebras, but some years later, around 1991, he realized that a concrete probabilistic model of free random variables is afforded by large independent random matrices. Since then, free probability became the natural framework to study random matrices as their size go to infinity. While the relation of free probability with random matrices gave rise to important results about operator algebras and random matrices, we shall not discuss these relations here. Free Probability Theory has a combinatorial description, which was developed by Speicher around 1994, based on free cumulants. This combinatorial description shows the analogy with classical probability theory quite clearly, since compared to the later, in free probability the lattice of all partitions of a finite set is replaced by the lattice of non-crossing partitions.

1.1 Non-commutative Probability Spaces

First, we start by introducing the notions of non-commutative probability space and non-commutative random variables. As we already mentioned, non-commutative probability can be thought as a kind of generalization of classical probability from a merely algebraic point of view.

Definition 1.1. *We say that a pair $(\mathcal{A}, \phi)$ is a non-commutative probability space, if the following conditions hold:*

1. *$\mathcal{A}$ is a unital (associative) algebra over $\mathbb{C}$*
2. *ϕ is a linear functional $\phi: \mathcal{A} \to \mathbb{C}$ such that $\phi(1) = 1$.*

An element $a \in \mathcal{A}$ is called non-commutative random variable.

By abuse of notation, we will denote with 1 the unit of a unital algebra $\mathcal{A}$ over $\mathbb{C}$. An additional property that can be given to ϕ is that

$$\phi(ab) = \phi(ba), \quad \text{for all } a, b \in \mathcal{A}.$$

Then we say that the non-commutative probability space $(\mathcal{A}, \phi)$ is tracial.

The previous definition can be extended to the case where the algebra $\mathcal{A}$ is endowed with more structure. More precisely, if $(\mathcal{A}, \phi)$ is a non-commutative probability space where $\mathcal{A}$ is a $*$-algebra and ϕ is a positive functional (i.e. $\phi(aa^*) \geq 0$ for all $a \in \mathcal{A}$), then we say that the pair $(\mathcal{A}, \phi)$ is a $*$-probability space. In the framework of a $*$-probability space, for a non-commutative random variable $a \in \mathcal{A}$ we have that

- a is said normal if $a^* a = a a^*$
- a is said self-adjoint if $a^* = a$.

Note that if $(\mathcal{A}, \phi)$ is a $*$-probability space, then every non-commutative random variable $a \in \mathcal{A}$ can be written in the form $a = x + \mathrm{i}y$ where $x, y \in \mathcal{A}$ are self-adjoint variables.

Definition 1.2. *Let $(\mathcal{A}, \phi)$ be a $*$-probability space. The linear functional ϕ is said faithful, if for $a \in \mathcal{A}$ we have*

$$\phi(aa^*) = 0 \quad \textit{if and only if} \quad a = 0.$$

Remark 1.3. For a $*$-probability space $(\mathcal{A}, \phi)$ using the positivity of ϕ, it is easy to note that the functional ϕ is self-adjoint, which means that

$$\phi(a^*) = \overline{\phi(a)}, \quad \text{for all } a \in \mathcal{A}.$$

Example 1.4. We give some examples of $*$-probability spaces.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space. We define the algebra

$$L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C}) := \bigcap_{1 \le p < \infty} L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C}),$$

   of classical random variables with finite moments of any order. Then we have that the pair $(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C}), \mathbb{E})$ is a $*$-probability space if we consider the $*$-operation as the conjugation of complex functions.

2. For $d \in \mathbb{N}$, we consider $M_d(\mathbb{C})$ the algebra of $d \times d$ complex matrices. Then $M_d(\mathbb{C})$ is a $*$-algebra on $\mathbb{C}$, if we consider the $*$-operation $*\colon M_d(\mathbb{C}) \to M_d(\mathbb{C})$, where for $A = (a_{i,j}) \in M_d(\mathbb{C})$ we define $A^* = (a^*_{i,j})$ such that $a^*_{i,j} := \overline{a_{j,i}}$ for all $i, j \in \{1, \ldots, d\}$. Moreover, if we define the trace $\mathrm{tr}\colon M_d(\mathbb{C}) \to \mathbb{C}$ by

$$\mathrm{tr}(A) := \sum_{i=1}^{d} a_{i,i}, \quad \text{for all } A = (a_{i,j}) \in M_d(\mathbb{C}),$$

   then the pair $\left(M_d(\mathbb{C}), \frac{1}{d} \cdot \mathrm{tr}\right)$ is a $*$-probability space.

3. Let $(\mathcal{A}, \phi)$ and $(\mathcal{B}, \psi)$ be two $*$-probability spaces. Then the vector space tensor product $\mathcal{A} \otimes \mathcal{B}$ is turned into a $*$-algebra by setting

$$(a \otimes b)(\tilde{a} \otimes \tilde{b}) := a\tilde{a} \otimes b\tilde{b}$$

   and

$$(a \otimes b)^* := a^* \otimes b^*,$$

   for all $a, \tilde{a} \in \mathcal{A}$ and $b, \tilde{b} \in \mathcal{B}$. Moreover, if we define the functional $\phi \otimes \psi\colon \mathcal{A} \otimes \mathcal{B} \to \mathbb{C}$ by

$$(\phi \otimes \psi)(a \otimes b) := \phi(a)\psi(b), \quad \text{for all } (a, b) \in \mathcal{A} \times \mathcal{B},$$

   then $(\mathcal{A} \otimes \mathcal{B}, \phi \otimes \psi)$ is a $*$-probability space.

For a $*$-probability space $(\mathcal{A},\phi)$, given a family of non-commutative random variables $\{a_i\}_{i\in I}$ we can define the set of moments of $\{a_i\}_{i\in I}$ as

$$\left\{\phi\left(a_{i(1)}^{k(1)}\dots a_{i(r)}^{k(r)}\right) \mid r\in\mathbb{N},\ (i(1),\dots,i(r))\in I^r,\ (k(1),\dots,k(r))\in\{1,*\}^r\right\}.$$

C^*-algebras provide an appropriate environment where non-commutative probabilistic ideas can be developed. For a C^*-algebra $\mathcal{A}$ and a state (i.e. positive linear functional) ϕ we have that the moments of a normal element $a\in\mathcal{A}$ can be characterized by a probability measure μ on $\mathbb{C}$, in the sense that there exist a probability measure μ on $\mathbb{C}$ (where it's support is contained in the spectrum of a), such that

$$\phi(a^k(a^*)^l)=\int_{\mathbb{C}} z^k\bar{z}^l\mu(dz),\quad \text{for every } k,l\in\mathbb{N}.$$

The existence of such a probability measure emerges from the functional calculus for a and Riesz's theorem. Note that if the element $a\in\mathcal{A}$ is self-adjoint, then we have

$$\begin{aligned}\int_{\mathbb{C}}|z-\bar{z}|^2\mu(dz) &= \int_{\mathbb{C}}(z-\bar{z})\overline{(z-\bar{z})}\mu(dz)\\ &= 2\int_{\mathbb{C}} z\bar{z}\mu(dz)-\int_{\mathbb{C}} z^2\mu(dz)-\int_{\mathbb{C}}\bar{z}^2\mu(dz)\\ &= 2\phi(aa^*)-\phi(a^2)-\phi((a^*)^2)\\ &= 0.\end{aligned}$$

Therefore, using that the function $\mathbb{C}\ni z\mapsto|z-\bar{z}|^2$ is continuous and non-negative, we deduce that $\operatorname{supp}(\mu)\subseteq\{z\in\mathbb{C}\mid z=\bar{z}\}=\mathbb{R}$. Hence the measure μ can be seen as a probability measure on $\mathbb{R}$ and we will have that

$$\phi(a^k)=\int_{\mathbb{R}} x^k\mu(dx),\quad \text{for all } k\in\mathbb{N}.$$

So in this case we see that the moments of the non-commutative random variable $a\in\mathcal{A}$ can be identified with the moments of a classical random variable.

Definition 1.5. *A C^*-probability space is a $*$-probability space $(\mathcal{A},\phi)$ where $\mathcal{A}$ is a unital C^*-algebra.*

Remark 1.6. For every C^*-probability space $(\mathcal{A},\phi)$ we have that the linear functional ϕ is bounded.

Example 1.7. Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space. We consider the C^*-algebra $B(H)$ of bounded linear operators on H (where the $*$-operation is given by taking the adjoint operator). Moreover, for a vector $e\in H$ such that $\langle e,e\rangle=1$, we define the linear functional $\rho\colon B(H)\to\mathbb{C}$ determined by

$$\rho(T)=\langle e,Te\rangle,\quad \text{for all } T\in B(H).$$

Then ρ is a state and $(B(H),\rho)$ is a C^*-probability space.

1.2 The Notion of Freenes

We now present the notion of free independence which is one of the most celebrated notions of stochastic independence in the context of non-commutative probability theory. We start by defining freenes and by giving some examples in order to better understand whether this new notion of independence is related with the classical one.

Definition 1.8. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and I be an index set. A family of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$ is called freely independent in $(\mathcal{A}, \phi)$ if $\phi(a_1 \ldots a_k) = 0$ whenever*

1. *$k \in \mathbb{N}$,*
2. *$i(j) \in I$ for all $j = 1, \ldots, k$,*
3. *$a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \ldots, k$,*
4. *neighboring elements in $a_1 \ldots a_k$ are from different subalgebras, i.e. $i(j) \neq i(j+1)$ for all $j = 1, \ldots, k-1$,*
5. *$\phi(a_j) = 0$ for all $j = 1, \ldots, k$.*

Remark 1.9. A family of non-commutative random variables $(a_i)_{i\in I}$ in some non-commutative probability space $(\mathcal{A}, \phi)$ is said free if the unital subalgebras that each of a_i generate are free. In the context of $*$-probability spaces, we say that the family $(a_i)_{i\in I}$ is free if the unital $*$-subalgebras that each of a_i generate are free.

The concept of free independence has a probabilistic flavour. It is easy to note that if a is a non-commutative random variable, then we have that $a, 1$ are free.

Example 1.10. Let $\mathcal{A}_1$, $\mathcal{A}_2$ be free unital subalgebras in $(\mathcal{A}, \phi)$. For an arbitrary variable $c \in \mathcal{A}$ we define the zero mean variable $c^0 := c - \phi(c)1$.

1. If $a_1, a_2 \in \mathcal{A}_1$ and $b_1, b_2 \in \mathcal{A}_2$, we have

$$0 = \phi(a_1^0 b_1^0 a_2^0 b_2^0) = \phi[(a_1 - \phi(a_1)1)(b_1 - \phi(b_1)1)(a_2 - \phi(a_2)1)(b_2 - \phi(b_2)1)]$$

   which implies,

$$\phi(a_1 b_1 a_2 b_2) = \phi(a_1 a_2)\phi(b_1)\phi(b_2) + \phi(a_1)\phi(a_2)\phi(b_1 b_2) - \phi(a_1)\phi(b_1)\phi(a_2)\phi(b_2).$$

2. If $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$, we have

$$0 = \phi(a^0 b^0) = \phi[(a - \phi(a)1)(b - \phi(b)1)] = \phi(ab) - \phi(a)\phi(b).$$

   Hence,

$$\phi(ab) = \phi(a)\phi(b).$$

The previous example shows that independence and free independence are quite different. In fact, freenes is a really non-commutative notion of independence, which does not have particular interest in the classical case, where $\mathcal{A} = L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C})$ and $\phi(X) = \int X(\omega)\mathbb{P}(d\omega)$, for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 1.11. *Let $(\mathcal{A}, \phi)$ be a $*$-probability space and ϕ be a faithful functional. Assume that $x, y \in \mathcal{A}$ are self-adjoint and free random variables such that $x\,y = y\,x$. Then we have that,*

$$x = \phi(x)1 \ \ \textit{or} \ \ y = \phi(y)1.$$

Therefore, at least one of them must be a constant.

Proof. By the previous example we have

$$\phi(x^2)\phi(y^2) = \phi(x^2y^2) = \phi(x\,y\,x\,y) = \phi(x^2)\phi(y)^2 + \phi(x)^2\phi(y^2) - \phi(x)^2\phi(y)^2,$$

where in the second equality we used that x, y commute.

Hence, it follows,

$$0 = \phi(x^2)\phi(y^2) + \phi(x)^2\phi(y)^2 - \phi(x^2)\phi(y)^2 - \phi(x)^2\phi(y^2) = [\phi(x^2) - \phi(x)^2][\phi(y^2) - \phi(y)^2].$$

Therefore, if $\phi(x^2) - \phi(x)^2 = 0$ we have,

$$0 = \phi(x^2) - \phi(x)^2 = \phi[(x - \phi(x)1)(x - \phi(x)1)^*],$$

where in the second equality we used that x is self-adjoint and $\phi(a) \in \mathbb{R}$ for every self-adjoint random variable $a \in \mathcal{A}$. Thus, because ϕ is faithful the claim holds. $\square$

For the time being, we have not seen any example of free random variables. In classical probability theory we can create independent random variables by forming products of probability spaces. Analogously, in free probability we can create free random variables by forming free products of non-commutative probability spaces.

Definition 1.12. *Let $\{(\mathcal{A}_i, \phi_i)\}_{i \in I}$ be a family of C^*-probability spaces. Furthermore, let a unital C^*-algebra $\hat{\mathcal{A}}$ and a state $\hat{\phi}$ on $\hat{\mathcal{A}}$ be given. Then, the C^*-probability space $(\hat{\mathcal{A}}, \hat{\phi})$ is called the reduced free product of the $\{(\mathcal{A}_i, \phi_i)\}_{i \in I}$, if the following hold:*

1. *For all $i \in I$, there exist unital $*$-homomorphisms $j_i: \mathcal{A}_i \to \hat{\mathcal{A}}$, such that $\hat{\mathcal{A}}$ is generated by the union of $j_i(\mathcal{A}_i)$, $i \in I$ as a C^*-algebra.*

2. *$\hat{\phi} \circ j_i = \phi_i$ for all $i \in I$.*

3. *For $k \in \mathbb{N}$, if $i(1), \ldots, i(k) \in I$ such that $i(m) \neq i(m+1)$ for all $m = 1, \ldots, k-1$ and $a_j \in \mathcal{A}_{i(j)}$ with $\phi_{i(j)}(a_{i(j)}) = 0$ for all $j = 1, \ldots, k$, then we have,*

$$\hat{\phi}(j_{i(1)}(a_1) \ldots j_{i(k)}(a_k)) = 0.$$

4. The GNS representation of $(\hat{\mathcal{A}}, \hat{\phi})$ is faithful.

Voiculescu proved that for every family of C^*-probability spaces $\{(\mathcal{A}_i, \phi_i)\}_{i\in I}$ the reduced free product exists and it is unique up to isomorphism. The proof can be found in [...]. We will denote the reduced free product of $\{(\mathcal{A}_i, \phi_i)\}_{i\in I}$ by $*_{i\in I}(\mathcal{A}_i, \phi_i)$.

1.3 Joint Moments of Free Random Variables

Similarly with classical probability theory, the concept of freenes gives a rule in order to compute joint moments of free random variables from the moments of the individual random variables. Some simple properties of this computation procedure are given below.

Lemma 1.13. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space with free unital subalgebras $(\mathcal{A}_i)_{i\in I}$ and let non-commutative random variables $a_1, \dots, a_k \in \mathcal{A}$, such that $a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \dots, k$, where $i(1), \dots i(k) \in I$. We assume the following:*

1. *There exists a $m \in \{1, \dots, k\}$ such that $i(m) \neq i(j)$ for all $j = 1, \dots, k$.*
2. *$\phi(a_m) = 0$.*

Then $\phi(a_1 \dots a_k) = 0$.

Proof. We assume that $i(j) \neq i(j+1)$ for every $j = 1, \dots, k-1$. Otherwise, we consider the product of neighbouring random variables that belong to the same subalgebra, as a single random variable. We will prove the claim by induction. Obviously for $k = 1$ the claim holds and we assume that it also holds for all $l < k$. From the freenes condition we have

$$\phi(a_1 \dots a_k) = \phi[(a_1^0 + \phi(a_1)1) \dots (a_k^0 + \phi(a_k)1)] = \sum_{\pi, \sigma, n} \phi(a_{\pi(1)}) \dots \phi(a_{\pi(n)}) \phi(a_{\sigma(1)}^0 \dots a_{\sigma(k-n)}^0)$$

where we take the sum over all $n = 1, \dots, k$ (we exclude the case $n = 0$ because $i(j) \neq i(j+1)$ for all $j = 1, \dots, k-1$ and the subalgebras $(\mathcal{A}_i)_{i\in I}$ are free) and all $\{\{\pi(1) < \dots < \pi(n)\}, \{\sigma(1) < \dots < \sigma(k-n)\}\}$ partitions of $\{1, \dots, k\}$. For such a partition $\{\{\pi(1) < \dots < \pi(n)\}, \{\sigma(1) < \dots < \sigma(k-n)\}\}$ of $\{1, \dots, k\}$, we assume that $m \in \{\pi(1) < \dots < \pi(n)\}$. Then $\phi(a_{\pi(1)}) \dots \phi(a_{\pi(n)}) = 0$ because the non-commutative random variable a_m has zero mean. Otherwise, if $m \in \{\sigma(1) < \dots < \sigma(k-n)\}$, and we consider the product of neighbouring random variables that belong to the same subalgebra as a single random variable, then $a_{\sigma(1)}^0 \dots a_{\sigma(k-n)}^0$ can be writtten in the form $b_1 \dots b_l$ where $l < k$ and $b_1, \dots, b_l \in \mathcal{A}$ are non-commutative random variables that satisfy the conditions of the lemma. Hence, our induction hypothesis guarantees that $\phi(a_{\sigma(1)}^0 \dots a_{\sigma(k-n)}^0) = 0$. Therefore, the claim holds. □

Corollary 1.14. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space with free unital subalgebras $(\mathcal{A}_i)_{i\in I}$ and let non-commutative random variables $a_1, \dots, a_k \in \mathcal{A}$ such that $a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \dots, k$, where $i(1), \dots, i(k) \in I$. Assume that there exists a $m \in \{1, \dots, k\}$ such that $i(m) \neq i(j)$ for all $j = 1, \dots, k$. Then we have,*

$$\phi(a_1 \dots a_k) = \phi(a_m) \phi(a_1 \dots a_{m-1} a_{m+1} \dots a_k).$$

Proof. By the previous lemma it arises that,

$$\phi(a_1\ldots a_k)=\phi(a_1\ldots a_{m-1}(a_m^0+\phi(a_m)1)a_{m+1}\ldots a_k)=\phi(a_m)\phi(a_1\ldots a_{m-1}a_{m+1}\ldots a_k). \quad \square$$

We consider $a_1....a_k$ to be a product of non-commutative random variables, which satisfy the conditions of the previous corollary. Also, let's assume that a_m is the unique random variable among $a_1,...,a_k$ that belongs to the subalgebra $\mathcal{A}_{i(m)}$, i.e. $i(m)\neq i(l)$ for all $l\in\{1,..., k\}\backslash\{m\}$. Hence in order to compute $\phi(a_1...a_k)$, we have to compute $\phi(a_1...a_{m-1}a_{m+1}...a_k)$. If every time that we remove some random variable from the product $a_1\ldots a_k$ we have that the new product is a product of non-commutative random variables that satisfy the conditions of the previous corollary (for example, for $a_1\ldots a_{m-1}a_{m+1}\ldots a_k$ this could be done if $i(m-1)=i(m+1)\neq i(l)$ for all $l\in\{1,\ldots,k\}\backslash\{m-1,m+1\}$, and in that case the random variable $a_{m-1}a_{m+1}$ could be removed), then we will get that

$$\phi(a_1\ldots a_k)=\phi\left(\overrightarrow{\prod_{j:i(j)=k_1}} a_j\right)\ldots\phi\left(\overrightarrow{\prod_{j:i(j)=k_n}} a_j\right),$$

where we assume that $\{i(1),\ldots,i(k)\}=\{k_1,\ldots,k_n\}$, $k_i\neq k_j$ for all $i\neq j$ and $\overrightarrow{\prod}$ denotes the product of factors in the same order as they appear in the product $a_1\ldots a_k$.

Thus, in that case the joint moments of free random variables can be computed as in classical probability theory. In order to better understand what kind of products satisfy these conditions, we introduce the notion of non-crossing partitions, which will be crucial for the further development of the theory.

We will denote the set of all partitions of a finite totally ordered set S by $P(S)$. For a partition $\pi=\{V_1,\ldots,V_r\}$ of the set S, the elements $V_1,\ldots,V_r\subseteq S$ are called blocks of π. For two elements $p,q\in S$ we write,

$$p\sim_\pi q \quad \text{if and only if} \quad p,q \quad \text{are in the same block of } \pi.$$

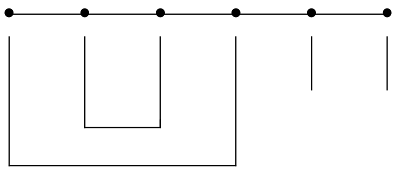
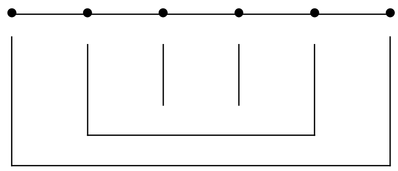
Moreover, we denote by $P_2(S)$ the set of all 2-partitions of S, i.e. $\pi\in P_2(S)$ if each block of π is an ordered set containing exactly two elements. Of course, when the set S has an odd number of elements, then we have $P_2(S)=\emptyset$.

Definition 1.15. *Let S be a finite totally ordered set and $\pi=\{V_1,...,V_r\}\in P(S)$ a partition with blocks $V_i\subseteq S$.*

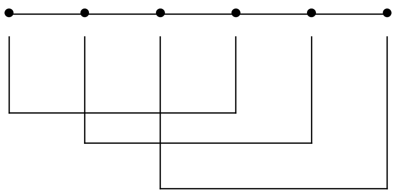
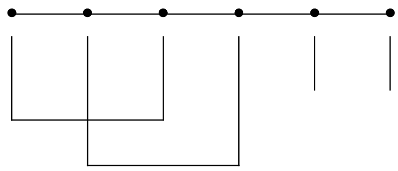
1. *The partition π is called crossing if there exist $p_1,q_1,p_2,q_2\in S$ such that*
   - $p_1<q_1<p_2<q_2$,
   - $p_1\sim_\pi p_2$, $q_1\sim_\pi q_2$ *and* $p_1\nsim_\pi q_1$.
2. *If π is not a crossing partition, then it will be called non-crossing and we denote by $\mathrm{NC}(S)$ the set of all non-crossing partitions of S.*

We can depict crossing and non-crossing partitions in the following way: For a $\pi\in P(S)$, if we build bridges in order to connect the points of S that belong to the same block, then these bridges will not cross if and only if $\pi\in\mathrm{NC}(S)$.

Example 1.16. The partitions {(1,4),(2,3),(5),(6))} and {(1,6),(2,5),(3),(4)} are non-crossing partitions of {1,2,3,4,5,6} and they correpond to the following pictures.

On the other hand, the partitions $\{(1,4),(2,5),(3,6)\}$ and $\{(1,3),(2,4),(5),(6)\}$ are crossing partitions. The respective pictures are

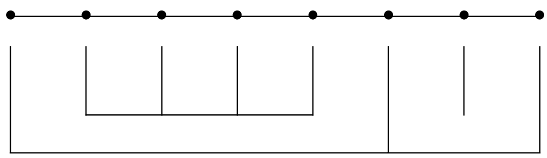



For a totally ordered set S we will freely identify $P(S)$ with $P(1,2,\ldots,\#S)$, since only the order of S matters. It is easy to note that every non-crossing partition π of the finite set $\{1,\ldots,n\}$ contains an interval block, i.e. a block $V\in\pi$ of the form

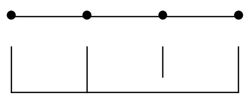
$$V=\{k,k+1,\ldots,k+r\}\subseteq\{1,\ldots,n\},$$

and that $\pi\backslash\{V\}$ is a non-crossing partition of $\{1,2,\ldots,n-\#V\}$.

For example, π={(1,6,8),(2,3,4,5),(7)} is a non-crossing partition of {1,2,3,4,5,6,7,8}



and $\pi\backslash\{(2,3,4,5)\}=\{(1,2,4),(3)\}$ is a non-crossing partition of {1,2,3,4}



Returning to the discussion about the computation rule of the joint moments of free random variables, let's assume that $(\mathcal{A},\phi)$ is a non-commutative probability space with free unital subalgebras $(\mathcal{A}_i)_{i\in I}$ and non-commutative random variables $a_1,\ldots,a_k\in\mathcal{A}$ such that $a_j\in\mathcal{A}_{i(j)}$ for all $j=1,\ldots,k$, where $i(1),\ldots,i(k)\in I$. Then, by the previous corollary we see that the criterion for deciding if the calculation rule for the moments $\phi(a_1\ldots a_k)$ is the same as in the classical case, is whether for two random variables a_n,a_m we have that they belong to the same subalgebra or not, i.e. for which $n,m\in\{1,\ldots,k\}$ it is true that $i(n)=i(m)$. This is the reason that leads to the following definition.

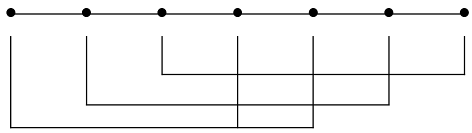
Definition 1.17. *Let I be a set, $k \in \mathbb{N}$ and $i = (i(1), \ldots, i(k))$ be a k-tuple of elements of I. For such an i we define its kernel $\ker(i) \in P(1, \ldots, k)$, by demanding for all $p, q \in \{1, \ldots, k\}$:*

$$p \sim_{\ker(i)} q \quad \text{if and only if} \quad i(p) = i(q).$$

Example 1.18. For $i = (10, 3, 4, 10, 10, 3, 4)$ we have that its kernel is given by

$$\ker(i) = \{\{1, 4, 5\}, \{2, 6\}, \{3, 7\}\} \in P(1, \ldots, 7)$$

which corresponds to the following picture



Note that for $j = (9, 3, 7, 9, 9, 3, 7)$ we have $\ker(i)$=$\ker(j)$.

Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, with unital subalgebras $(\mathcal{A}_i)_{i \in I}$ and non-commuatative random variables $a_1, \ldots, a_k \in \mathcal{A}$ such that $a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \ldots, k$ where $i(1), \ldots, i(k) \in I$. Then, taking into account the previous corollary in order to obtain that the calculation rule for the joint moments $\phi(a_1 \ldots a_k)$ is the same as in the classical case, it is necessary every time that we remove an element from the product $a_1 \ldots a_k$ to be valid that in the new product there are some neighboring random variables that belong to the same subalgebra and there is not other random variable, except of them, which belongs to this subalgebra. It is easy to note, that this is true if and only if for $i = (i(1), \ldots, i(k))$ we have that $\ker(i) \in \mathrm{NC}(1, \ldots, k)$. Hence, we see that non-crossing partitions appear when we want to deduce some properties of freenes with probabilistic flavour.

We present some lemmas, in order to better understand the calculation rule for the joint moments of free random variables, when their product do not correspond in the case that we described above.

Lemma 1.19. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, with free unital subalgebras $(\mathcal{A}_i)_{i \in I}$ and random variables $a_1, \ldots, a_k \in \mathcal{A}$ such that $a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \ldots, k$, where $i(1), \ldots, i(k) \in I$. We consider $s := \#\{i(1), \ldots, i(k)\}$. Then $\phi(a_1 \ldots a_k)$ can be written as a sum of products of elementary moments (i.e. moments of random variables that belong to the same subalgebra) of the $a_1, \ldots, a_k$, where each summand contains at least s factors.*

Proof. We prove the claim by induction. For $k = 1, 2$ the claim holds and we also assume that it holds for all $l < k$. It suffices to assume that $i(j) \neq i(j+1)$ for all $j = 1, \ldots, k-1$, because otherwise the induction hypothesis guarantees the claim. Similarly with the proof of Lemma 1.13 we consider the factorization,

$$\phi(a_1 \ldots a_k) = \sum_{\pi, \sigma, n} \phi(a_{\pi(1)}) \ldots \phi(a_{\pi(n)}) \phi(a^0_{\sigma(1)} \ldots a^0_{\sigma(k-n)})$$

where the sum runs over all $n=1,\ldots,k$ and over all $\{\{\pi(1)<\cdots<\pi(n)\},\{\sigma(1)<\cdots<\sigma(k-n)\}\}$ partitions of $\{1,\ldots,k\}$. Hence, if $\{\{\pi(1)<\cdots<\pi(n)\},\{\sigma(1)<\cdots<\sigma(k-n)\}\}$ is a partition of $\{1,\ldots,k\}$, because $\#\{i(1),\ldots,i(k)\}=s$, we will have that

$$s-n\leq\#\{i(\sigma(1)),\ldots,i(\sigma(k-n))\}.$$

Therefore, by the induction hypothesis we deduce that $\phi(a^0_{\sigma(1)}\ldots a^0_{\sigma(k-n)})$ can be written as a sum of products of elementary moments of the $a^0_{\sigma(j)}$, where each product contains at least $s-n$ factors. But the elementary moments of the $a^0_{\sigma(j)}$ are sums of products of elementary moments of the a_j and consequently, $\phi(a^0_{\sigma(1)}\ldots a^0_{\sigma(k-n)})$ is a sum of products of elementary moments of the a_j, where each product contains at least $s-n$ factors. Taking into account that $\phi(a^0_{\sigma(1)}\ldots a^0_{\sigma(k-n)})$ is multiplied by $\phi(a_{\pi(1)})\ldots\phi(a_{\pi(n)})$, it emerges that the claim holds. ☐

In the case where $\ker(i)\in \mathrm{NC}(1,\ldots,k)$, we saw that $\phi(a_1\ldots a_k)$ is the product of s elementary moments. Using the previous lemma we deduce the following result for the case where $\ker(i)$ is a crossing partition.

Lemma 1.20. *Let $(\mathcal{A},\phi)$ be a non-commutative probability space, with free unital subalgebras $(\mathcal{A}_i)_{i\in I}$ and random variables $a_1,\ldots,a_k\in\mathcal{A}$ such that $a_j\in\mathcal{A}_{i(j)}$ for all $j=1,\ldots,k$, where $i(1),\ldots,i(k)\in I$. We also assume that* $\ker(i)$ *is a crossing partition of $\{1,\ldots,k\}$ and we consider $s:=\#\{i(1),\ldots,i(k)\}$. Then $\phi(a_1\ldots a_k)$ can be written as a sum of elementary moments of the a_j, where each summand contains at least $s+1$ factors.*

Proof. We can assume that $i(j)\neq i(j+1)$ for all $j=1,\ldots,k-1$. Otherwise, we just consider the product of neighbouring random variables that belong to the same subalgebra as a single random variable. Also, by Corollary 1.14 it suffices to assume that each $i(j)$ occures at least twice. We consider again the factorization,

$$\phi(a_1\ldots a_k)=\sum_{\pi,\sigma,n}\phi(a_{\pi(1)})\ldots\phi(a_{\pi(n)})\phi(a^0_{\sigma(1)}\ldots a^0_{\sigma(k-n)}).$$

Let $\{\{\pi(1)<\cdots<\pi(n)\},\{\sigma(1)<\cdots<\sigma(k-n)\}\}$ a partition of $\{1,\ldots,k\}$. If we have that $\#\{i(\pi(1)),\ldots,i(\pi(n))\}=n$, then $\{i(\pi(1)),\ldots,i(\pi(n))\}\subseteq\{i(\sigma(1)),\ldots,i(\sigma(k-n))\}$ becauce each $i(j)$ occures at least twice by assumption. Hence, $s=\#\{i(\sigma(1)),\ldots,i(\sigma(k-n))\}$. By Lemma 1.19 it emerges that the moments $\phi(a^0_{\sigma(1)}\ldots a^0_{\sigma(k-n)})$ can be written as a sum of products of elementary moments of the $a^0_{\sigma(j)}$ where each summand contains at least $\#\{i(\sigma(1)),\ldots,i(\sigma(k-n))\}=s$ factors. But, since the elementary moments of the $a^0_{\sigma(j)}$ can be written as a sum of products of elementary moments of the a_j, we see that $\phi(a_{\pi(1)})\ldots\phi(a_{\pi(n)})\phi(a^0_{\sigma(1)}\ldots a^0_{\sigma(k-n)})$ is a sum of products of elementary moments of the a_j, where each summand contains at least $s+1$ factors. On the other hand, if we have $\#\{i(\pi(1)),\ldots,i(\pi(n))\}<n$, i.e. there exist $m,j\in\{1,\ldots,n\}$ such that $i(\pi(m))=i(\pi(j))$, then

$$s\leq\#\{i(\pi(1)),\ldots,i(\pi(n))\}+\#\{i(\sigma(1)),\ldots,i(\sigma(k-n))\}\leq n-1+\#\{i(\sigma(1)),\ldots,i(\sigma(k-n))\}.$$

Therefore, we will have $\#\{i(\sigma(1)),\ldots,i(\sigma(k-n))\} \geq s-n+1$. Taking into account Lemma 1.19, the moments $\phi(a^0_{\sigma(1)}\ldots a^0_{\sigma(k-n)})$ can be written as a sum of elementary moments of the $a^0_{\sigma(j)}$ where each summand contains at least $s-n+1$ factors. Hence, since the elementary moments of the $a^0_{\sigma(j)}$ can be written as a sum of products of elementary moments of the a_j, we deduce that $\phi(a_{\pi(1)})\ldots\phi(a_{\pi(n)})\phi(a^0_{\sigma(1)}\ldots a^0_{\sigma(k-n)})$ is a sum of products of elementary moments of the a_j, where each summand contains at least $s+1$ factors. Therefore, the claim has been proven. □

1.4 Free Central Limit Theorem

In this subsection, we want to investigate if some standard notions and results of classical probability can be generalized to the non-commutative setting. Our first attempt will be to try to understand if there exists some analogue of the Gaussian distribution, in the context of free probability. One of the reasons that makes Gaussian distribution important and usefull is the fact that arises by the central limit theorem. The free central limit theorem will be the main criterion which will make us decide which is the analog of Gaussian distribution in the non-commutative case. Hence we will show that the normalized sum of free random variables converges (in some sense) to a specific probability distribution.

Before we formulate and prove our limit theorem, we pause in order to introduce some notation. Given a unital algebra $\mathcal{A}$ over $\mathbb{C}$ and $\mathcal{X} \subseteq \mathcal{A}$, we denote by $\mathrm{alg}(\mathcal{X})$ the subalgebra generated by $\mathcal{X}$.

Theorem 1.21. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. For $n \in \mathbb{N}$ fixed, we consider n non-commutative random variables $a^1_{M,N}, \ldots, a^n_{M,N} \in \mathcal{A}$, where $M, N \in \mathbb{N}$. We assume that for every $N \in \mathbb{N}$, the unital subalgebras $\{\mathrm{alg}(1, a^1_{M,N}, \ldots, a^n_{M,N})\}_{M\in\mathbb{N}}$ are free and the non-commutative random variables $\{(a^1_{M,N}, \ldots, a^n_{M,N})\}_{M\in\mathbb{N}}$ have the same joint distribution, which means that for $P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$ arbitrary we have*

$$\phi[P(a^1_{M,N}, \ldots, a^n_{M,N})] = \phi[P(a^1_{K,N}, \ldots, a^n_{K,N})],$$

for all $K, M \in \mathbb{N}$. For $P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$ we define,

$$\phi[P(a^1_N, \ldots, a^n_N)] := \phi[P(a^1_{M,N}, \ldots, a^n_{M,N})].$$

If for all $k \in \mathbb{N}$ and $i(1), \ldots, i(k) \in \{1, \ldots, n\}$ the limit

$$Q(i(1), \ldots, i(k)) := \lim_{N\to\infty} N \cdot \phi\big(a^{i(1)}_N \ldots a^{i(k)}_N\big)$$

exists, then we have for the sums

$$S^m_N := a^m_{1,N} + \cdots + a^m_{N,N}$$

for all $k \in \mathbb{N}$ and $i(1), \ldots, i(k) \in \{1, \ldots, n\}$:

$$\lim_{N\to\infty} \phi\big(S^{i(1)}_N \ldots S^{i(k)}_N\big) = \sum_{p=1}^{k} \sum_{\{V_1,\ldots,V_p\}\in \mathrm{NC}(1,\ldots,k)} Q(V_1)\ldots Q(V_p),$$

where for $V = \{v_1 < \cdots < v_l\} \subseteq \{1, \ldots, k\}$, $Q(V)$ stands for $Q(i(v_1), \ldots, i(v_l))$.

Proof. We consider $k\in\mathbb{N}$ and $i(1),\ldots,i(k)\in\{1,\ldots,n\}$ to be fixed. Then, we have,

$$\begin{aligned}\phi\big(S_N^{i(1)}\ldots S_N^{i(k)}\big) &= \phi\big[\big(a_{1,N}^{i(1)}+\cdots+a_{N,N}^{i(1)}\big)\ldots\big(a_{1,N}^{i(k)}+\cdots+a_{N,N}^{i(k)}\big)\big]\\ &= \sum_{j(1),\ldots,j(k)=1}^{N}\phi\big(a_{j(1),N}^{i(1)}\ldots a_{j(k),N}^{i(k)}\big)\\ &= \sum_{p=1}^{k}\sum_{\{V_1,\ldots,V_p\}\in P(1,\ldots,k)}\sum_{\substack{j(1),\ldots,j(k)=1\\ \ker(j)=V}}^{N}\phi\big(a_{j(1),N}^{i(1)}\ldots a_{j(k),N}^{i(k)}\big).\end{aligned}$$

Let two k-tuples $j=(j(1),\ldots,j(k))$ and $l=(l(1),\ldots,l(k))$ such that $\ker(j)=\ker(l)$, i.e. $j(a)=j(b)$ if and only if $l(a)=l(b)$. Then, we have,

$$\phi\big(a_{j(1),N}^{i(1)}\ldots a_{j(k),N}^{i(k)}\big)=\phi\big(a_{l(1),N}^{i(1)}\ldots a_{l(k),N}^{i(k)}\big).$$

This is true because for all $r\in\mathbb{N}$ and for all $P_1,\ldots,P_r\in\mathbb{C}\langle X_1,\ldots,X_n\rangle$ we have,

$$\phi[P_1(a_{j(1),N}^1,\ldots,a_{j(1),N}^n)\ldots P_r(a_{j(r),N}^1,\ldots,a_{j(r),N}^n)]$$

$$=\phi[P_1(a_{l(1),N}^1,\ldots,a_{l(1),N}^n)\ldots.P_r(a_{l(r),N}^1,\ldots a_{l(r),N}^n)].$$

Ineed for $r=1$ the above equality holds because the non-commutative random variables $\{(a_{M,N}^1,\ldots,a_{M,N}^n)\}_{\mathrm{M}\in\mathbb{N}}$ have the same joint distribution. Also, because the unital subalgebras $\{\mathrm{alg}(1,a_{M,N}^1,\ldots,a_{M,N}^n)\}_{M\in\mathbb{N}}$ are freely independent, by induction we can see that the above equality holds for all $r\in\mathbb{N}$.

For $V=\{V_1,\ldots,V_p\}\in P(1,\ldots,k)$, the number of k-tuples $j=(j(1),\ldots,j(k))$ such that $j(1),\ldots,j(k)\in\{1,\ldots,N\}$ and $\ker(j)=V$, is $A_{p,N}:=N(N-1)\ldots(N-p+1)$. For such a j, because the expressions $\phi\big(a_{j(1),N}^{i(1)}\ldots a_{j(k),N}^{i(k)}\big)$ do not depend on j, they will be denoted by $\phi(V_1,\ldots,V_p;N)$. Therefore, we have,

$$\phi(S_N^{i(1)}\ldots.S_N^{i(k)})=\sum_{p=1}^{k}A_{p,N}\sum_{\{V_1,\ldots,V_p\}\in\mathcal{P}(1,\ldots,k)}\phi(V_1,\ldots,V_p;N).$$

We consider a crossing partition $V=\{V_1,\ldots,V_p\}\in P(1,\ldots,k)$ and $j=(j(1),\ldots,j(k))$ such that $\ker(j)=V$. By definition, $\phi(V_1,\ldots,V_p;N)=\phi\big(a_{j(1),N}^{i(1)}\ldots a_{j(k),N}^{i(k)}\big)$ and since $\ker(j)$ is crossing, by Lemma 1.20, we can write $\phi(V_1,\ldots,V_p;N)$ as a sum of products of elementary moments of the $a_{j(1),N}^{i(1)},\ldots,a_{j(k),N}^{i(k)}$, where each product contains at least $p+1$ factors. The existence of the limits $Q(i(1),\ldots,i(r))$, implies that,

$$\lim_{N\to\infty}\phi(a_N^{i(1)}\ldots a_N^{i(r)})=0,$$

for all $r\in\mathbb{N}$ and all $i(1),\ldots,i(r)\in\{1,\ldots,n\}$. Therefore, for every crossing partition $V\in P(1,\ldots,k)$, we deduce that,

$$\lim_{N\to\infty}A_{p,N}\cdot\phi(V_1,\ldots,V_p;N)=0.$$

On the other hand, if $V = \{V_1, \ldots, V_p\}$ is a non-crossing partition, then using Corollary 1.14, we have $\phi(V_1, \ldots, V_p; N) = \phi(V_1; N) \ldots \phi(V_p; N)$, which implies,

$$\lim_{N\to\infty} A_{p,N} \cdot \phi(V_1, \ldots, V_p; N) = Q(V_1) \ldots Q(V_p).$$

Therefore, the asserttion holds. □

In the previous theorem, a lot of conditions about the non-commutative random variables were necessary in order to prove the limit theorem. These conditions are satisfied, if we adopt the following setting: We consider $(\mathcal{A}, \phi)$ to be a C*-probability space and for each $i \in \mathbb{N}$ let $(\mathcal{A}_i, \phi_i) := (\mathcal{A}, \phi)$. Let also $(\hat{\mathcal{A}}, \hat{\phi}) = *_{i=1}^{\infty}(\mathcal{A}_i, \phi_i)$ to be the reduced free product of the C*-probability spaces $(\mathcal{A}_i, \phi_i)$, with the canonical embeddings $j_i \colon \mathcal{A}_i \to \hat{\mathcal{A}}$. Given $n \in \mathbb{N}$ fixed, for each $N \in \mathbb{N}$ we consider n random variables $b_N^1, \ldots, b_N^n \in (\mathcal{A}, \phi)$. Then, if we define $a_{M,N}^k := j_M(b_N^k)$, it is clear that for $N \in \mathbb{N}$ fixed, the non-commutative random variables $a_{M,N}^1, \ldots, a_{M,N}^n \in (\hat{\mathcal{A}}, \hat{\phi})$ have the same joint distributions for all $M \in \mathbb{N}$, because the linear maps $j_i \colon \mathcal{A}_i \to \hat{\mathcal{A}}_i$ are $*$-homomorphisms, and $\hat{\phi} \circ j_i = \phi$ for all $i \in \mathbb{N}$. The free independence of the unital subalgebras $\{\mathrm{alg}(1, a_{M,N}^1, \ldots, a_{M,N}^n)\}_{M \in \mathbb{N}}$ is also satisfied by the definition of the reduced free product.

Theorem 1.22. (free central limit theorem) *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. For $n \in \mathbb{N}$ fixed, we consider non-commutative random variables $a_M^1, \ldots,$ $a_M^n \in \mathcal{A}$, where $M \in \mathbb{N}$. We assume that the unital subalgebras $\{\mathrm{alg}(1, a_M^1, \ldots, a_M^n)\}_{M \in \mathbb{N}}$ are free and the non-commutative random variables $\{(a_M^1, \ldots, a_M^n)\}_{M \in \mathbb{N}}$ have the same joint distribution, which means that*

$$\phi[P(a_M^1, \ldots, a_M^n)] = \phi[P(a_K^1, \ldots, a_K^n)],$$

for all $P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$ and for all $M, K \in \mathbb{N}$. For $P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$ we define,

$$\phi[P(a^1, \ldots, a^n)] := \phi[P(a_M^1, \ldots, a_M^n)].$$

We also assume that $\phi(a^m) = 0$ for all $m \in \{1, \ldots, n\}$. Then, if we consider the sums

$$S_N^m := \frac{a_1^m + \cdots + a_N^m}{\sqrt{N}},$$

for all $k \in \mathbb{N}$ and $i(1), \ldots, i(k) \in \{1, \ldots, n\}$ we have:

$$\lim_{N\to\infty} \phi\big(S_N^{i(1)} \ldots S_N^{i(k)}\big) = \begin{cases} 0, & k \text{ odd} \\ \displaystyle\sum_{\substack{\{(e_1,z_1), \ldots, (e_r,z_r)\} \\ \in \mathrm{NC}(1,\ldots,k)}} \phi(a^{i(e_1)} a^{i(z_1)}) \ldots \phi(a^{i(e_r)} a^{i(z_r)}), & k = 2r. \end{cases}$$

Proof. For $m \in \{1, \ldots, n\}$ and $M, N \in \mathbb{N}$, we consider $a_{M,N}^m := \frac{1}{\sqrt{N}} a_M^m$. We want to show that the non-commutative random variables $a_{M,N}^k$ satisfy the assumptions of Theorem 1.21 . Hence, it suffices to show that for all $k \in \mathbb{N}$ and for all $i(1), \ldots, i(k) \in \{1, \ldots, n\}$ we have,

$$\lim_{N\to\infty} N \phi(a_{M,N}^{i(1)} \ldots a_{M,N}^{i(k)}) = 0.$$

For $k=1$ we have,

$$\lim_{N\to\infty} N\phi(a_{M,N}^{i(1)}) = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \cdot \phi(a^{i(1)}) = 0.$$

For $k=2$ we have,

$$\lim_{n\to\infty} N\phi(a_{M,N}^{i(1)} \cdot a_{M,N}^{i(2)}) = \lim_{N\to\infty} \phi(a^{i(1)}a^{i(2)}) = \phi(a^{i(1)}a^{i(2)}).$$

For $k \geq 3$ we see that,

$$\lim_{N\to\infty} N\phi(a_{M,N}^{i(1)} \dots a_{M,N}^{i(k)}) = \lim_{N\to\infty} \frac{1}{N^{\frac{k}{2}-1}} \cdot \phi(a^{i(1)} \dots a^{i(k)}) = 0.$$

Therefore, by Theorem 1.21 the assertion holds. □

Following the setting of the previous theorem, for $n=1$ we have a sequence $(a_M)_{M\in\mathbb{N}}$ of free random variables such that $\phi(a_M^k) = \phi(a_N^k)$ for all $k, M, N \in \mathbb{N}$ (identically distributed) and $\phi(a_M) = 0$ for all $M \in \mathbb{N}$ (zero mean). Hence, under the assumption $\phi(a_M^2) = \sigma^2 > 0$ for all $M \in \mathbb{N}$, by Theorem 1.22 is obtained that if

$$S_N = \frac{a_1 + \cdots + a_N}{\sqrt{N}}$$

is the normalized sum of the free random variables, then

$$\lim_{N\to\infty} \phi[(S_N)^k] = \begin{cases} 0, & \text{for } k \text{ odd} \\ \#\mathrm{NC}_2(1, \dots, k) \cdot \sigma^k, & \text{for } k \text{ even.} \end{cases}$$

We recall that in the classical case, where $\mathcal{A} = L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and $\phi(X) = \mathbb{E}[X]$, if $(X_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with zero mean and variance σ^2, then by the classical central limit theorem, for the normalized sum

$$S_N = \frac{X_1 + \cdots + X_N}{\sqrt{N}}$$

we have

$$\lim_{N\to\infty} \mathbb{E}[(S_N)^k] = \int_{\mathbb{R}} t^k \gamma(\mathrm{d}t) = \begin{cases} 0, & \text{for } k \text{ odd} \\ \#\mathcal{P}_2(1, \dots, k) \cdot \sigma^k, & \text{for } k \text{ even} \end{cases}$$

where γ is the Gaussian measuse on $\mathbb{R}$ with mean 0 and variance σ^2.

Hence, we see another case where partitions of a finite set have to be replaced by non-crossing partitions in order to obtain some results in the context of free probability theory, which have a classical probabilistic flavour.

Definition 1.23. *Let $\{(\mathcal{A}_n, \phi_n)\}_{n\in\mathbb{N}}$ as well as $(\mathcal{A}, \phi)$ be non-commutative probability spaces. We consider $(b_n)_{n\in\mathbb{N}}$ to be a sequence of non-commutative random variables with $b_n \in \mathcal{A}_n$ for every $n \in \mathbb{N}$ and let $b \in \mathcal{A}$. We say that b_n converges in distribution to b, if and only if,*

$$\lim_{n\to\infty} \phi_n(b_n^k) = \phi(b^k), \quad \textit{for all } k \in \mathbb{N}.$$

Convergence in distribution will be denoted by $b_n \overset{\text{distr}}{\to} b$.

According to the previous definition, the classical central limit theorem shows that in the classical case, where $(\mathcal{A}, \phi) = (L^{\infty -}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}), \mathbb{E})$, the random variable S_N converges in distribution to some Gaussian random variable $X \in L^{\infty -}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. It is easy to note that the free central limit theorem gives a similar outcome. This is due to the fact that the number of non-crossing pairings of a finite set can be determined by a well know recurrence relation.

Lemma 1.24. *Let* $m \in \mathbb{N}$ *and* $C_m := \#\mathrm{NC}_2(1, \dots, 2m)$ *be the number of non-crossing pairings of the finite set* $\{1, \dots, 2m\}$. *We also define* $C_0 := 1$. *Then, for the sequence* $(C_m)_{m \in \mathbb{N}}$ *we have for all* $m \in \mathbb{N}$,

$$C_m = \sum_{k=1}^{m} C_{k-1} C_{m-k}.$$

Proof. For $m \in \mathbb{N}$, we consider $\pi = \{(e_1, z_1), \dots, (e_m, z_m)\} \in \mathrm{NC}_2(1, \dots, 2m)$, where $e_i < z_i$ for all $i = 1, \dots, m$ and $1 = e_1 < \cdots < e_m$. Because π is a non-crossing partition, z_1 must be an even number. We consider $j = 1, \dots, m$ such that $e_j \in \{2, \dots, z_1 - 1\}$. Then, since π is non-crossing, we must have $e_1 < e_j < z_j \leq z_1 - 1$. Therefore, we will have that the restriction of π to $\{2, \dots, z_1 - 1\}$ gives a non-crossing partition of $\{2, \dots, z_1 - 1\}$. In the same way, the fact that π is non-crossing implies that if $z_1 < z_j$ for some $j = 1, \dots, m$, then $z_1 < e_j < z_j$. Hence, the restriction of π to $\{z_1 + 1, \dots, 2m\}$ leads to the construction of a non-crossing partition of the finite set $\{z_1 + 1, \dots, 2m\}$. The previous observations show that for some non-crossing pair partition π of $\{1, \dots, 2m\}$, with z_1 fixed, the procedure of making the pairs $(e_2, z_2), \dots, (e_m, z_m)$ is equivalent to the construction of a non-crossing pair partition of $\{2, \dots, z_1 - 1\}$ and of a non-crossing pair partition of $\{z_1 + 1, \dots, 2m\}$. Hence, we see that for an even number $2l \in \{2, \dots, 2m\}$,

$$\#\{\{(e_1, z_1), \dots, (e_m, z_m)\} \in \mathrm{NC}_2(1, \dots, 2m) \mid z_1 = 2l\} = \#\mathrm{NC}_2(2, \dots, l-1) \cdot \#\mathrm{NC}_2(l+1, \dots, 2m)$$

$$= C_{\frac{2l-2}{2}} \cdot C_{\frac{2m-2l}{2}} = C_{l-1} \cdot C_{m-l}.$$

Therefore, considering all the different possible values that z_1 may have, we deduce that the claim holds. ☐

The unique solution of the recursion equation of Lemma 1.24 is given by the sequence $(C_m)_{m \in \mathbb{N}}$, where

$$C_m = \frac{1}{m+1} \binom{2m}{m},$$

is the the mth Catalan number.

Catalan numbers occur in various counting problems and for every $m \in \mathbb{N}$ they have the following integral representations

$$C_m = \frac{1}{2\pi} \int_{-2}^{2} t^{2m} \sqrt{4 - t^2}\, dt.$$

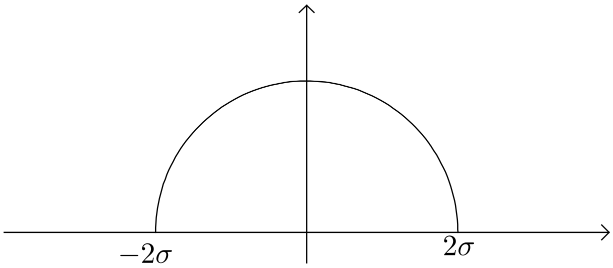
Therefore, we conclude that the free central limit theorem implies that

$$\lim_{n\to\infty} \phi[(S_N)^k] = \int_{\mathbb{R}} t^k s(dt),$$

where the probability measure $s(dx) = f(x)dx$, with density function,

$$f(x) = \frac{1}{2\pi\sigma^2}\mathbf{1}_{[-2\sigma,2\sigma]}(x)\sqrt{4\sigma^2 - x^2}$$

is called semicircle distribution.



Definition 1.25. *Let $(\mathcal{A}, \phi)$ be a $*$-probability space and $\sigma > 0$. A self-adjoint random variable $a \in \mathcal{A}$ is called semicircular variable of variance σ^2, if it's moments have the following form*

$$\phi(s^k) = \begin{cases} 0, & \text{for } k \text{ odd}, \\ \sigma^{2m} C_m, & \text{for } k = 2m \text{ for some } m \in \mathbb{N}. \end{cases}$$

A semicircular variable will be called standard or normalized if $\sigma^2 = 1$. The conclusion of the free central limit theorem is that the sum S_N converges in distribution to a semicircular variable. In free probability, the semicircle distribution is the analogue of the Gaussian distribution, since for a semicircular variable $a \in (\mathcal{A}, \phi)$ of variance $\sigma^2 > 0$, we have

$$\phi(a^k) = \frac{1}{2\pi\sigma^2}\int_{-2\sigma}^{2\sigma} t^k \sqrt{4\sigma^2 - t^2}dt, \quad \text{for all } k \in \mathbb{N}.$$

Theorem 1.26. (Poisson limit theorem) *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. For $M, N \in \mathbb{N}$ we consider non-commutative random variables $a_{M,N} \in \mathcal{A}$, such that for every $N \in \mathbb{N}$ the unital subalgebras $\{\mathrm{alg}(1, a_{M,N})\}_{M\in\mathbb{N}}$ are freely independent and the non-commutative random variables $\{a_{M,N}\}_{M\in\mathbb{N}}$ have the same distribution, which means that for every $k, M, K \in \mathbb{N}$ we have*

$$\phi[(a_{M,N})^k] = \phi[(a_{K,N})^k].$$

For $k, N \in \mathbb{N}$ we define $\phi[(a_N)^k] := \phi[(a_{M,N})^k]$. We also assume that there exist λ such that for every $k \in \mathbb{N}$

$$\lim_{N\to\infty} N\phi[(a_N)^k] = \lambda.$$

Then we have for the sums

$$S_N := a_{1,N} + \cdots + a_{N,N}$$

for all $k \in \mathbb{N}$:

$$\lim_{N\to\infty} \phi[(S_N)^k] = \sum_{p=1}^{k} \sum_{\{V_1,\ldots,V_p\}\in \mathrm{NC}(1,\ldots,k)} \lambda^p.$$

Proof. Since for every $k \in \mathbb{N}$ we have

$$\lim_{N\to\infty} N\phi[(a_N)^k] = \lambda,$$

the claim holds by Theorem 1.21 . □

We recall, that in the classical case the analogous theorem gives a limit theorem for the Poisson distribution. Due to this analogy, it is reasonable to consider that the sequence $(S_N)_{N\in\mathbb{N}}$ of Theorem 1.26, converges in distribution to the free analogue of a Poisson random variable. This is the reason that leads to the following defition.

Definition 1.27. *Let* $(\mathcal{A}, \phi)$ *be a* $*$*-probability space and* $\lambda > 0$*. A self-adjoint random variable* $a \in \mathcal{A}$ *is called free Poisson variable with parameter* λ*, if it's moments have the following form*

$$\phi(a^k) = \sum_{p=1}^{k} \sum_{\{V_1,\ldots,V_p\}\in \mathrm{NC}(1,\ldots,k)} \lambda^p, \quad \textit{for every } k \in \mathbb{N}.$$

As we expected, in terms of their moments, the difference between Poisson random variables and free Poisson variables comes by replacing partitions with non-crossing partitions.

1.5 Semicircular variables

In the previous subsection we identified the free analogues of the Gaussian and Poisson distribution. However, we do not know yet if non-commutative realizations of these distributions exist. In this subsection we present an example of semicircular variables on some non-commutative probability space. Examples of free Poisson variables will be presented later. Throughout this subesction we consider a $*$-probability space $(\mathcal{A}, \phi)$ and a non-commutative random variable $a \in \mathcal{A}$ such that the following conditions hold,

- $a^* a = 1 \neq a a^*$.
- a gerenates $\mathcal{A}$ as a $*$-algebra.
- the elements $\{a^m (a^*)^n \mid m, n \geq 0\}$ are linearly independent.
- the linear functional $\phi\colon \mathcal{A} \to \mathbb{C}$ satisfies the following equation

$$\phi(a^m (a^*)^n) = \begin{cases} 1, & \text{if } m = n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Later on, we will present specific examples of non-commutative probability spaces $(\mathcal{A}, \phi)$ where there exist a random variable $a \in \mathcal{A}$ such that the above conditions are satisfied. By the previous conditions, it is easy to see that $\mathcal{A} = \text{span}\{(a)^m (a^*)^n \mid m, n \geq 0\}$ because, due to the first assumption, we have that the family $\{a^m (a^*)^n \mid m, n \geq 0\}$ is closed under multiplication. Obviously, it is also closed under $*$-operation. Hence it's linear span is a $*$-subalgebra of $\mathcal{A}$ which contains a. Since a generates $\mathcal{A}$ as a $*$-algebra, we must have that the two $*$-algebras are equal.

Such a $*$-algebra it is easy to be studied since it is closely related with the Toeplitz algebra. We recall that on the Hilbert space $l^2 := l^2(\mathbb{N} \cup \{0\}; \mathbb{C})$ there is a standard orthonormal basis $(e_n)_{n \geq 0}$, where,

$$e_n = (0, \ldots, 0, 1, 0, 0, \ldots),$$

with the 1 appearing on the nth component.

We consider the one-sided shift operator $S \in B(l^2)$, determined by the relation

$$S e_n = e_{n+1}, \quad \text{for all } n \geq 0. \tag{1.1}$$

It is straightforward to verify that the adjoint operator S^* is determined by

$$S^* e_0 = 0 \quad \text{and} \quad S^* e_n = e_{n-1}, \quad \text{for all } n \geq 1 \tag{1.2}$$

and the condition $S^* S = 1_{B(l^2)} \neq S S^*$ is satisfied. The Toeplitz algebra is the C^*-algebra generated by the one-sided shift operator $S \in B(l^2)$.

Since $\mathcal{A} = \text{span}\{a^m (a^*)^n \mid m, n \geq 0\}$, we can define a linear map $\pi \colon \mathcal{A} \to B(l^2)$ such that

$$\pi(a^m (a^*)^n) = S^m (S^*)^n, \quad \text{for all } m, n \geq 0$$

and extend by linearity. Note that since the elements $\{a^m (a^*)^n \mid m, n \geq 0\}$ are linearly independent, we have that the map π is well defined. Moreover, it is easily verified that π is a unital $*$-homomorphism.

Proposition 1.28. *The elements* $\{S^m (S^*)^n \mid m, n \geq 0\}$ *are linearly independent.*

Proof. Let $T \in \text{span}\{S^m (S^*)^n \mid m, n \geq 0\}$ such that $T = 0$. Then the operator T can be written as a finite sum of operators $a\, S^m (S^*)^n$, where $a \in \mathbb{C}$ and $m, n \geq 0$. We consider $a_1 S^{m_1} (S^*)^n, \ldots, a_k S^{m_k} (S^*)^n$ to be the summands of T such that n is minimal and $m_i \neq m_j$ for all $i \neq j$. Then we have

$$\begin{aligned} a_1 S^{m_1} (S^*)^n e_n + \cdots + a_k S^{m_k} (S^*)^n e_n &= a_1 S^{m_1} e_0 + \cdots + a_k S^{m_k} e_0 \\ &= a_1 e_{m_1} + \cdots + a_k e_{m_k} \end{aligned}$$

and $S^M (S^*)^N e_n = 0$ for all $M \in \mathbb{N}$ and $N > n$.

Therefore, the fact $T e_n = 0$ implies that

$$a_1 e_{m_1} + \cdots + a_k e_{m_k} = 0.$$

Since $m_i \neq m_j$ for all $i \neq j$, we deduce that $a_1 = \cdots = a_k = 0$.

Continuing in this way, we see that if $a\, S^m (S^*)^n$ is a summand of T, then $a = 0$. $\square$

Because the elements $\{S^m(S^*)^n \,|\, m,n \geq 0\}$ are linearly independent, the $*$-homomorphism π is injective. Therefore, the $*$-algebra $\mathcal{A}$ can be identified with an algebra of bounded operators on l^2. This identification allows us to compute moments of random variables in $\mathcal{A}$ with respect to ϕ, by computing moments of random variables on $B(l^2)$, with respect to a suitable state. Indeed, we consider the vacuum state $\phi_0: B(l^2) \to \mathbb{C}$, determined by

$$\phi_0(T) = \langle e_o, Te_0\rangle_{l^2}, \quad \text{for all } T \in B(l^2).$$

Then, by definition we have

$$\phi_0(S^m(S^*)^n) = \langle (S^*)^m e_0, (S^*)^n e_0\rangle_{l^2} = \begin{cases} 1, & \text{for } m=n=0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \phi(a^m(a^*)^n).$$

Therefore, the triple (l^2, π, e_0) is a representation of $(\mathcal{A}, \phi)$, which means that

$$\phi(x) = \langle e_0, \pi(x)e_0\rangle_{l^2}, \quad \text{for all } x \in \mathcal{A}. \tag{1.3}$$

We remind, that our purpose is to give non-commutative realizations of semicircular random variables. In our example, the semicircular random variable will be an element of the algebra $\mathcal{A}$. Hence, from now on, we will focus on the computation of the moments $\phi(a^{\varepsilon(1)} \dots a^{\varepsilon(k)})$, where $k \in \mathbb{N}$ and $\{\varepsilon(1), \dots, \varepsilon(k)\} \in \{1, *\}$. Up to now, we saw that we can reduce the study of the moments $\phi(a^{\varepsilon(1)} \dots a^{\varepsilon(k)})$, to the study of the corresponding moments $\phi_0(S^{\varepsilon(1)} \dots S^{\varepsilon(k)})$. In order to understand how the k-tuple $(\varepsilon(1), \dots, \varepsilon(k))$ affects the value of $\phi_0(S^{\varepsilon(1)} \dots S^{\varepsilon(k)})$, we introduce the notion of Dyck paths.

Definition 1.29. *We define* NE-SE paths*, (where* NE *stands for North-East and* SE *stands for South-East) to be paths in $\mathbb{Z}^2$ which starts at the origin and makes finitely many steps either of the form $(1,1)$ (North-East steps) or of the form $(1,-1)$ (South-East steps).*

Remark 1.30. For $k \in \mathbb{N}$, there is a bijection between the set of NE-SE paths with k steps and the set $\{-1,+1\}^k$. More precisely, we can identify a NE-SE path with k steps, with a k-tuple $(\lambda(1), \dots, \lambda(k)) \in \{-1,+1\}^k$, where for $j \in \{1, \dots, k\}$, $\lambda(j) = 1$ means that the j-th step was a NE step, while $\lambda(j) = -1$ means that the j-th step was a SE step.

We will distinguish some NE-SE paths, called Dyck paths.

Definition 1.31. *A* Dyck path *is a* NE-SE path *which stays above the x-axis and ends on the x-axis. This happens if the path visits only points of the form $(i,j) \in \mathbb{Z}^2$ with $j \geq 0$ and the last point that it visits is of the form $(k,0) \in \mathbb{Z}^2$, where $k \in \mathbb{N}$ is the number of steps.*

Remark 1.32. For $k \in \mathbb{N}$, we consider the identification of NE-SE paths with k steps and $\{-1,1\}^k$. Then it is obvious that a NE-SE path $(\lambda(1), \dots, \lambda(k)) \in \{-1,1\}^k$ is a Dyck path if and only if

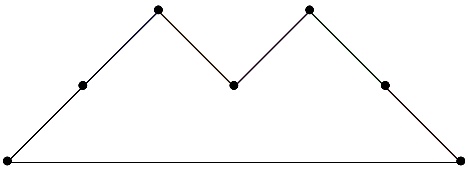
$$\lambda(1) + \cdots + \lambda(j) \geq 0, \text{ for all } j \in \{1, \dots, k\} \quad \text{and} \quad \lambda(1) + \cdots + \lambda(k) = 0. \tag{1.4}$$

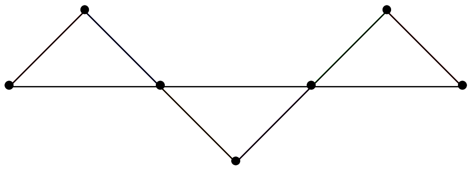
The inequalities in (1.4) guarantee that the path never goes strictly below the x-axis, while the equality in (1.4) guarantees that the path ends on the x-axis. Also, the same equality implies that

$$\begin{aligned} k &= \#\{j=1,\dots,k \mid \lambda(j)=1\} + \#\{j=1,\dots,k \mid \lambda(j)=-1\} \\ &= 2\,\#\{j=1,\dots,k \mid \lambda(j)=1\}. \end{aligned}$$

Therefore, a Dyck path makes even number of steps.

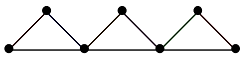
Example 1.33. The path $(1,1,-1,1,-1,-1) \in \{\pm 1\}^6$ is a Dyck path with 6 steps, but the NE-SE path $(1,-1,-1,1,1,-1) \in \{\pm 1\}^6$ is not a Dyck path because at the third step it goes strictly below the x-axis. We can visualize the previous paths in the following way



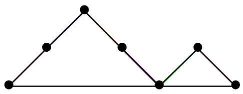


There are five Dyck paths with 6 steps:

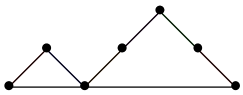
$(1,-1,1,-1,1,-1)$



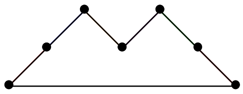
$(1,1,-1,-1,1,-1)$



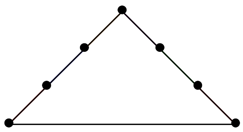
$(1,-1,1,1,-1,-1)$



$(1,1,-1,1,-1,-1)$



$(1,1,1,-1,-1,-1)$



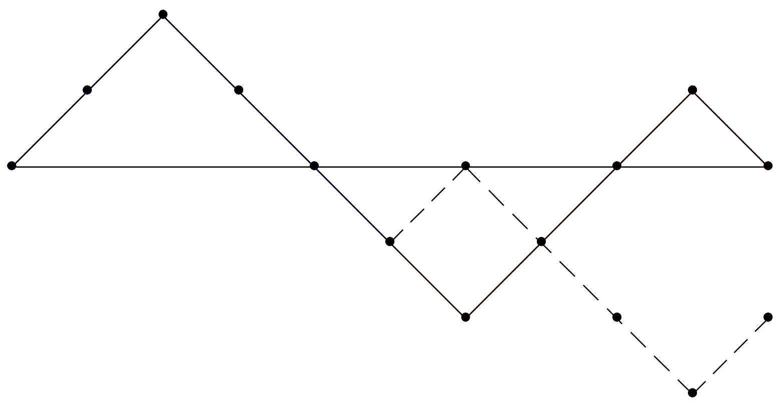
We recall that for an integer $n \in \mathbb{N}$, we defined the nth Catalan number

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

Proposition 1.34. *For all $p \in \mathbb{N}$, the number of Dyck paths with $2p$ steps is equal to the pth Catalan number.*

Proof. Let $p \in \mathbb{N}$. In order to compute the number of Dyck paths with $2p$ steps, we will compute the number of NE-SE paths with $2p$ steps and the number of NE-SE paths with $2p$ steps, which are not Dyck paths. We consider a NE-SE path which makes u NE steps, and v SE steps. Then, $u+v>0$, and the ending point of the path must be $(u+v, u-v) \in \mathbb{Z}^2$. Note that for a point $(m,n) \in \mathbb{Z}^2$, if there exist $u, v \in \mathbb{N} \cup \{0\}$ such that $u+v>0$, $m=u+v$ and $n=u-v$, then we have $m>0$, $|n| \leq m$ and the integers m, n have the same parity. On the other hand, for $m, n \in \mathbb{Z}$ such that $m>0$, $|n| \leq m$ and m, n have the same parity, there exist unique $u = \frac{1}{2}(m+n) \in \mathbb{N} \cup \{0\}$ and $v = \frac{1}{2}(m-n) \in \mathbb{N} \cup \{0\}$, such that $u+v>0$, $m=u+v$ and $n=u-v$. Therefore, we deduce that a pair $(m,n) \in \mathbb{Z}^2$ is an ending point of a NE-SE path, if and only if, $m>0$, $|n| \leq m$ and m, n have the same parity. In that case the path makes $\frac{1}{2}(m+n)$ NE-steps and $\frac{1}{2}(m-n)$ SE steps. Therefore, identifying the NE-SE paths that make m steps with $\{-1,1\}^m$, we see that the number of NE-SE paths with ending point $(m,n) \in \mathbb{Z}^2$ is equal to $\binom{m}{(m+n)/2}$. Therefore, the number of NE-SE paths arriving at $(2p, 0)$ is $\binom{2p}{p}$.

Now, we want to compute the number of NE-SE paths arriving at $(2p,0)$ which are not Dyck paths. Let γ be a NE-SE path arriving at $(2p,0)$ which is not a Dyck path. Since the ending point of γ is on x-axis, we must have that after some steps γ goes strictly below the x-axis. Let $j \in \{1, \ldots, 2p-1\}$ be the number of steps that γ needs to do in order to go under the x-axis for the first time. Of course, this automatically means that after the jth step the path γ will be in the position $(j,-1)$. Therefore, we can split γ in two paths γ_1 and γ_2, where γ_1 goes from $(0,0)$ to $(j,-1)$, and γ_2 goes from $(j,-1)$ to $(2p,0)$. In that case, we will write $\gamma = \gamma_1 \vee \gamma_2$. We consider γ_3 to be the reflection of γ_2 in the horizontal line of the equation $y=-1$. This automatically means that γ_3 is a path from $(j,-1)$ to $(2p,-2)$. We define $F(\gamma) := \gamma_1 \vee \gamma_3$ which is a NE-SE path arriving at $(2p,-2)$. In order to better understand the construction of the path $F(\gamma)$, we give a concrete example: Let $p=5$ and $\gamma = (1,1,-1,-1,-1,-1,1,1,1,-1)$. In that case, $j=5$, γ_1 is a path from $(0,0)$ to $(5,-1)$, γ_2 is a path from $(5,-1)$ to $(10,0)$, and γ_3 is a path from $(5,-1)$ to $(10,-2)$.



Hence, we have constructed a map F from the set of NE-SE paths ending at $(2p,0)$ and which are not Dyck paths, to the set of NE-SE paths ending at $(2p,-2)$. From the definition of F, it is clear that F is injective. Moreover, the map F is surjective. Indeed, let δ be a NE-SE path ending at $(2p,-2)$. Let $i\in\{1,\ldots,2p-1\}$ be the number of steps that δ needs to do in order to go under the x-axis for the first time. Like before, we can write $\delta=\delta_1\vee\delta_2$, where δ_1 is a path from $(0,0)$ to $(i,-1)$ and δ_2 is a path from $(i,-1)$ to $(2p,-2)$. Therefore, if δ_3 is the reflection of δ_2 in the horizontal line $y=-1$, then we have $F(\delta_1\vee\delta_3)=\delta$.

As a consequence, the number of NE-SE paths ending at $(2p,0)$ and which are not Dyck paths is equal to the number of NE-SE paths ending at $(2p,-2)$, which is $\binom{2p}{p-1}$. Finally, we deduce that the number of Dyck paths with $2p$ steps is

$$\binom{2p}{p}-\binom{2p}{p-1}=C_p\,. \qquad \square$$

We now return to our discussion about semicircular variables. We recall that we consider the non-commutative probability space $(\mathcal{A},\phi)$ and $a\in\mathcal{A}$ introduced at the beginning of the current subsection. Dyck paths will give us a rule in order to compute the moments $\phi(a^{\varepsilon(1)}\ldots a^{\varepsilon(k)})$, where $k\in\mathbb{N}$ and $\varepsilon(1),\ldots,\varepsilon(k)\in\{1,*\}$.

Proposition 1.35. *Let $k\in\mathbb{N}$ and $\varepsilon(1),\ldots,\varepsilon(k)\in\{1,*\}$. For all $j\in\{1,\ldots,k\}$ we define*

$$\lambda(j):=\begin{cases}1 & \text{if } \varepsilon(j)=*\\ -1 & \text{if } \varepsilon(j)=1.\end{cases} \tag{1.5}$$

Let γ be the NE-SE path *which corresponds to the k-tuple $(\lambda(1),\ldots,\lambda(k))\in\{-1,1\}^k$. Then we have that*

$$\phi(a^{\varepsilon(1)}\ldots a^{\varepsilon(k)})=\begin{cases}1 & \textit{if } \gamma \textit{ is a } \text{Dyck path}\\ 0 & \text{otherwise}\,.\end{cases} \tag{1.6}$$

Proof. From the relation (1.3) we will have that

$$\phi(a^{\varepsilon(1)}\ldots a^{\varepsilon(k)})=\langle e_0,S^{\varepsilon(1)}\ldots S^{\varepsilon(k)}e_0\rangle_{l^2}=\langle (S^{\varepsilon(k)})^*\ldots(S^{\varepsilon(1)})^*e_0,e_0\rangle_{l^2}.$$

Using the relations (1.1) and (1.2) we can see that the vector $(S^{\varepsilon(k)})^*\ldots(S^{\varepsilon(1)})^*e_0$ is equal either to the zero-vector, or to an element of the orthonormal basis $\{e_n|\, n\geq 0\}$ of l^2. In fact, by induction on $j=1,\ldots,k$, we have that

$$(S^{\varepsilon(j)})^*\ldots(S^{\varepsilon(1)})^*e_0=\begin{cases}e_{\lambda(1)+\cdots+\lambda(j)} & \text{if } \lambda(1)+\cdots+\lambda(m)\geq 0, \text{for all } 1\leq m\leq j\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, we obtain

$$\phi(a^{\varepsilon(1)}\ldots a^{\varepsilon(k)})=\begin{cases}\langle e_0,e_{\lambda(1)+\cdots+\lambda(k)}\rangle_{l^2} & \text{if } \lambda(1)+\cdots+\lambda(j)\geq 0, \text{for all } 1\leq j\leq k\\ 0 & \text{otherwise.}\end{cases}$$

Since the vectors $\{e_n\}_{n\geq 0}$ are orthonormal we will have that $\langle e_0, e_{\lambda(1)+\cdots+\lambda(k)}\rangle_{l^2}=1$, if and only if $\lambda(1)+\cdots+\lambda(k)=0$, or equivalently, $\phi(a^{\varepsilon(1)}\ldots a^{\varepsilon(k)})=1$, if and only if γ is a Dyck path. Finally, we deduce that

$$\phi(a^{\varepsilon(1)},\ldots,a^{\varepsilon(k)})=\begin{cases}1, & \text{if } \gamma \text{ is a Dyck path}\\ 0, & \text{otherwise.}\end{cases}\qquad\square$$

The previous proposition provides a rule in order to decide whether the moments $\phi(a^{\varepsilon(1)}\ldots a^{\varepsilon(k)})$ are equal either to one, or to zero. Using this rule we will conclude that the self-adjoint random variable $a+a^*\in(\mathcal{A},\phi)$ is semicircular.

Corollary 1.36. *Let k be a positive integer. For the self-adjoint random variable $a+a^*\in(\mathcal{A},\phi)$ we have*

$$\phi[(a+a^*)^k]=\begin{cases}0, & \text{for } k \text{ odd}\\ C_p, & \text{for } k=2p.\end{cases}$$

Proof. We have that

$$\begin{aligned}\phi[(a+a^*)^k] &= \phi\left(\sum_{\varepsilon(1),\ldots,\varepsilon(k)\in\{1,*\}} a^{\varepsilon(1)}\ldots a^{\varepsilon(k)}\right)\\ &= \sum_{\varepsilon(1),\ldots,\varepsilon(k)\in\{1,*\}} \phi(a^{\varepsilon(1)}\ldots a^{\varepsilon(k)}).\end{aligned}$$

Using the relation (1.5) we have that the set of k-tuples $(\varepsilon(1),\ldots,\varepsilon(k))\in\{1,*\}^k$ is identified with $\{-1,1\}^k$. Moreover, taking into account the relation (1.6) and the identification between the NE-SE paths with k steps and $\{-1,1\}^k$, we deduce that

$$\phi[(a+a^*)^k]=\sum_{\text{Dyck paths with } k \text{ steps}} 1=\begin{cases}0, & \text{for } k \text{ odd}\\ C_{k/2}, & \text{for } k \text{ even.}\end{cases}\qquad\square$$

Remark 1.37. Up to now, we have seen only one example of a $*$-probability space such that the conditions that we made at the beginning of the current subsection are satisfied. We refer to the case where a is the one-sided shift operator on l^2, and ϕ is the vacuum state on $B(l^2)$. Later on we will consider the case, where a is the creation operator on the full Fock space $\mathcal{F}(H)$ of some Hilbert space H, and ϕ is the vacuum state on $B(\mathcal{F}(H))$.

The previous corollary shows that the operator $S+S^*\in(B(l^2),\phi_0)$ is a standard semicircular variable. Similarly, we can construct an unbounded operator, defined on a dense subset of l^2, which is a non-commutative standard Gaussian variable. To be more precise, we define the linear operators $S_1, S_1^*\colon \text{span}\{e_n\,|\,n\geq 0\}\to\text{span}\{e_n\,|\,n\geq 0\}$ by

$$S_1^* e_0 := 0, \tag{1.7}$$

$$S_1^* e_n := \sqrt{n}\, e_{n-1}, \quad \text{for all } n\geq 1$$

and

$$S_1 e_n := \sqrt{n+1}\, e_{n+1}, \quad \text{for all } n \geq 0.$$

It is easy to verify that the operators S_1, S_1^* satisfy the relations

$$\langle S_1^* \xi, \eta \rangle_{l^2} = \langle \xi, S_1 \eta \rangle_{l^2}, \quad \text{for all } \xi, \eta \in \operatorname{span}\{e_n \mid n \geq 0\} \tag{1.8}$$

and

$$S_1^* S_1 - S_1 S_1^* = 1. \tag{1.9}$$

It has been noticed [...] that the relations (1.7), (1.8) and (1.9) imply that the non-commutative random variable $S_1 + S_1^*$ is a standard Gaussian variable, which means that

$$\phi_0[(S_1 + S_1^*)^k] = \langle e_0, (S_1 + S_1^*)^k e_0 \rangle_{l^2} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^k \exp\left(-\frac{t^2}{2}\right) dt, \quad \text{for all } k \in \mathbb{N}.$$

Notice that the operators S_1, S_1^* do not differ a lot from the operators S, S^* respectively. For this reason, later on we will present an example of a non-commutative random variable in $(B(l^2), \phi_0)$, which gives an interpolation between standard Gaussian variables and standard semicircular variables.

2 Free Cumulants

In this section, we introduce the notion of free cumulants which is one of the main tools for dealing with free independence. Free cumulants were introduced by R. Speicher around 1994. Cumulants are quantities related to some combinatorial notion of connectivity and some probabilistic notion of independence. Similarly with the classical case, we can obtain an equivalent characterization of free independence via free cumulants.

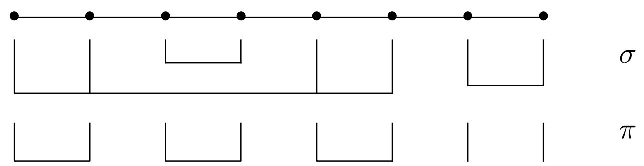
2.1 Basic combinatorics

In order to formulate and present the basic results about free cumulants, we first introduce some useful combinatorial properties about the lattice of non-crossing partitions.

Definition 2.1. *For $n \in \mathbb{N}$ and $\pi, \sigma \in \mathrm{NC}(1, \ldots, n)$, we write $\pi \leq \sigma$ if and only if, for every block of π, there exists (a unique) block of σ which contains it.*

Note that the pair $(\mathrm{NC}(1, \ldots, n), \leq)$ is a partially ordered set (poset).

Example 2.2. Consider the non-crossing partitions $\sigma = \{\{1, 2, 5, 6\}, \{3, 4\}, \{7, 8\}\}$, and $\pi = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7\}, \{8\}\}$. Then it is clear that $\pi \leq \sigma$.



For $n \in \mathbb{N}$, the maximal element of $\mathrm{NC}(1, \ldots, n)$ is denoted by $1_n := \{\{1, 2, \ldots,, n\}\}$ and the minimal element of $\mathrm{NC}(1, \ldots, n)$ by $0_n := \{\{1\}, \{2\}, \ldots, \{n\}\}$.

Definition 2.3. *Let P be a finite poset and define $P^{(2)} := \{(\pi, \sigma) \in P \times P \mid \pi \leq \sigma\}$. For functions $F, G\colon P^{(2)} \to \mathbb{C}$ we define their convolution $F * G\colon P^{(2)} \to \mathbb{C}$ by demanding*

$$(F * G)(\pi, \sigma) := \sum_{\substack{\pi \leq \tau \leq \sigma \\ \tau \in P}} F(\pi, \tau)\, G(\tau, \sigma), \quad \textit{for all } (\pi, \sigma) \in P^{(2)}. \tag{2.1}$$

Similarly, given two functions $f\colon P \to \mathbb{C}$, and $G\colon P^{(2)} \to \mathbb{C}$, we define $G * f$, $f * G\colon P \to \mathbb{C}$ by requiring

$$(f * G)(\sigma) := \sum_{\substack{\tau \leq \sigma \\ \tau \in P}} f(\tau) G(\tau, \sigma), \quad \textit{for all } \sigma \in P$$

and

$$(G * f)(\sigma) := \sum_{\substack{\sigma \leq \tau \\ \tau \in P}} G(\sigma, \tau) f(\tau), \quad \textit{for all } \sigma \in P.$$

Remark 2.4. Let P be a finite poset.

1. Assume that P has a unique minimal element 0 (e.g. $P=\mathrm{NC}(1,\ldots,n)$). Then, a one-variable function $f\colon P\to\mathbb{C}$ can be seen as the restriction of some two-variables function $F\colon P^{(2)}\to\mathbb{C}$ with $f(\sigma)=F(0,\sigma)$ for all $\sigma\in P$.

2. The convolution $*$ is associative, whichs means that for all $F,G,H\colon P^{(2)}\to\mathbb{C}$, we have
$$(F*G)*H=F*(G*H).$$

3. Consider the function $\delta\colon P^{(2)}\to\mathbb{C}$, defined by
$$\delta(\pi,\sigma):=\begin{cases}1, & \text{if} \quad \pi=\sigma\\ \\ 0, & \text{if} \quad \pi<\sigma\end{cases} \qquad \text{for all } (\pi,\sigma)\in P^{(2)}.$$
Note that δ is the unit of the $*$ operation, namely for all $F\colon P^{(2)}\to\mathbb{C}$, we have
$$F*\delta=\delta*F.$$
Therefore, the set of all functions $F\colon P^{(2)}\to\mathbb{C}$ equipped with pointwise defined addition and with the convolution $*$ as multiplication, is a unital (associative) algebra over $\mathbb{C}$.

For a finite poset P, we also define the function $\zeta\colon P^{(2)}\to\mathbb{C}$ by requiring $\zeta(\pi,\sigma):=1$ for all $(\pi,\sigma)\in P^{(2)}$.

In classical probability theory, moments of classical random variables can be written as a sum over partitions of classical cumulants. We will see that a similar formula holds in free probability except that partitions have to be non-crossing. In order to formulate this relation between moments of non-commutative random variables and free cumulants, we introduce the notion of Mobious function.

Proposition 2.5. *Let P be a finite poset. It's ζ function is invertible, namely there exists a function $\mu\colon P^{(2)}\to\mathbb{C}$, called Mobious function, such that*
$$\mu*\zeta=\delta=\zeta*\mu. \tag{2.2}$$

Proof. For a function $\mu\colon P^{(2)}\to\mathbb{C}$ that satisfies the first equality of (2.2), we must have for all $(\pi,\sigma)\in P^{(2)}$,
$$\sum_{\substack{\pi\leq\tau\leq\sigma\\ \tau\in P}}\mu(\pi,\tau)=\begin{cases}1, & \text{if } \pi=\sigma\\ \\ 0, & \text{if } \pi<\sigma.\end{cases} \tag{2.3}$$

Since P is finite (2.3) can be solved recursively by defining $\mu(\pi,\pi):=1$ for all $\pi\in P$ and
$$\mu(\pi,\sigma):=-\sum_{\substack{\pi\leq\tau\leq\sigma\\ \tau\in P}}\mu(\pi,\tau), \quad \text{for all } (\pi,\sigma)\in P^{(2)}.$$

Therefore, there exists a function $\mu: P^{(2)} \to \mathbb{C}$ such that $\mu * \zeta = \delta$. However, it is not sure that for this function $\mu: P^{(2)} \to \mathbb{C}$, the relation $\zeta * \mu = \delta$ is satisfied.

We make the following observation: We assume that $P = \{p_1, \ldots, p_m\}$ and we consider three functions $F, G, H: P^{(2)} \to \mathbb{C}$. Then, defining,

$$F(p_i, p_j) := 0, \quad G(p_i, p_j) := 0 \quad \text{and } H(p_i, p_j) := 0,$$

for all $i, j \in \{1, \ldots, m\}$ such that $p_i \nleq p_j$, we have constructed three $m \times m$ matrices $\hat{F}$, $\hat{G}$, $\hat{H}$, with entries $\hat{F}_{i,j} := F(p_i, p_j)$, $\hat{G}_{i,j} := G(p_i, p_j)$ and $\hat{H}_{i,j} := H(p_i, p_j)$ for all $i, j = 1, \ldots, m$. By the relation (2.1), it is clear that $F * G = H$, if and only if $\hat{F} \cdot \hat{G} = \hat{H}$. Note that $\hat{\delta}$ is the identity matrix. Then, the relation $\hat{\mu} \cdot \hat{\zeta} = \hat{\delta}$ implies that $\hat{\zeta} \cdot \hat{\mu} = \hat{\delta}$. As a consequence, we have that $\mu * \zeta = \delta = \zeta * \mu$. □

Corollary 2.6. *Let P be a finite poset. Then, for any $f, g: P \to \mathbb{C}$ we have*

$$f = g * \zeta \quad \textit{if and only if} \quad g = f * \mu.$$

Proof. We assume that $P = \{p_1, \ldots, p_m\}$. For $f, g: P^{(2)} \to \mathbb{C}$, we consider the $1 \times m$ matrices

$$\hat{f} := (\ f(p_1) \ \ \ldots \ \ f(p_m)\ ) \quad \text{and} \quad \hat{g} := (\ g(p_1) \ \ \ldots \ \ g(p_m)\ ).$$

Then, since the matrix $\hat{\zeta}$ is the inverse of $\hat{\mu}$, we have that

$$f = g * \zeta \Leftrightarrow \hat{f} = \hat{g} \cdot \hat{\zeta} \Leftrightarrow \hat{g} = \hat{f} \cdot \hat{\mu} \Leftrightarrow g = f * \mu. \qquad \square$$

From now on we will consider our finite poset to be the set $\mathrm{NC}(1, \ldots, n)$, for some $n \in \mathbb{N}$. Before we give the definition of free cumulants, we need to racall a key property of non-crossing partitions.

Proposition 2.7. *For each $n \in \mathbb{N}$, the poset $\mathrm{NC}(1, \ldots, n)$ is a lattice. This means that:*

1. *Let $\{\pi_i\}_{i=1}^k$ be a finite family of non-crossing partitions of $\{1, \ldots, n\}$. Then there exist a greatest lower bound of $\{\pi_i\}_{i=1}^k$ denoted by $\wedge_{i=1}^k \pi_i$.*

2. *Let $\{\pi_i\}_{i=1}^k$ be a finite family of non-crossing partitions of $\{1, \ldots, n\}$. Then there exist a least upper bound of $\{\pi_i\}_{i=1}^k$ denoted by $\vee_{i=1}^k \pi_i$.*

Proof. Let $k, n \in \mathbb{N}$ and $\{\pi_i\}_{i=1}^k$ be a family of non-crossing partitions of $\{1, \ldots, n\}$.

1. We define the partition $\wedge_{i=1}^k \pi_i$ in the following way: For $p, q \in \{1, \ldots, n\}$we demand,

$$p \sim_{\wedge_{i=1}^k \pi_i} q \quad \text{if and only if} \quad p \sim_{\pi_j} q \quad \text{for all } j = 1, \ldots, k.$$

   It is a routine to check that $\wedge_{i=1}^k \pi_i$ is a non-crossing partition. By definition, for every $j \in \{1, \ldots, k\}$ we have

$$\wedge_{i=1}^k \pi_i \leq \pi_j.$$

It remains to show that the partition $\wedge_{i=1}^{k}\pi_i$ is a greatest lower bound of $\{\pi_i\}_{i=1}^{k}$. Let τ be a non-crossing partition of $\{1,\ldots,n\}$ such that $\tau \leq \pi_j$ for all $j=1,\ldots,k$. Then, for $p,q \in \{1,\ldots,n\}$ such that $p \sim_\tau q$, we have $p \sim_{\pi_j} q$ for all $j=1,\ldots,k$, or equivalently $p \sim_{\wedge_{i=1}^{k}\pi_i} q$. Therefore, we deduce that $\tau \leq \wedge_{i=1}^{k}\pi_i$.

2. The family $\{\tau \in \mathrm{NC}(1,\ldots,n) \mid \pi_j \leq \tau \text{ for all } j=1,\ldots,k\}$ is non-empty since it contains 1_n. It is straightforward to verify that the partition

$$\vee_{i=1}^{k}\pi_i := \wedge\{\tau \in \mathrm{NC}(1,\ldots,n) \mid \pi_j \leq \tau \text{ for all } j=1,\ldots,k\},$$

satisfies all the desired properties. □

Example 2.8. For $\pi = \{\{1\},\{2,3,4\}\}$ and $\sigma = \{\{1,4\},\{2,3\}\}$ we have

$$\pi \wedge \sigma = \{\{1\},\{2,3\},\{4\}\} \text{ and } \pi \vee \sigma = 1_4.$$

2.2 Definition of free cumulants

Before we define free cumulants, we pause to introduce some notation. Define $\mathrm{NC} := \cup_{n\geq 1}\mathrm{NC}(1,\ldots,n)$. Suppose we are given a sequence of multilinear functionals $\{\rho_n\colon \mathcal{A}^n \to \mathbb{C}\}_{n\in\mathbb{N}}$ on a fixed complex algebra $\mathcal{A}$. Then we extend $\{\rho_n\colon \mathcal{A}^n \to \mathbb{C}\}_{n\in\mathbb{N}}$ to a family $\{\rho_\pi\colon \mathcal{A}^n \to \mathbb{C}\}_{\pi\in\mathrm{NC}}$ in the following way: Let $n \in \mathbb{N}$ and let π be a non-crossing partition of $\{1,\ldots,n\}$. We define the functional $\rho_\pi\colon \mathcal{A}^n \to \mathbb{C}$ by requiring, for all $(a_1,\ldots,a_n) \in \mathcal{A}^n$,

$$\rho_\pi(a_1,\ldots,a_n) := \prod_{V\in\pi} \rho_{\#V}(a_1,\ldots,a_n|V),$$

where for every block $V = \{l_1 < \cdots < l_m\} \in \pi$,

$$\rho_{\#V}(a_1,\ldots,a_n|V) := \rho_m(a_{l_1},\ldots,a_{l_m}).$$

It is straightforward to see that for all $\pi \in \mathrm{NC}$, $\rho_\pi\colon \mathcal{A}^n \to \mathbb{C}$ is a multilinear functional. The family $(\rho_\pi)_{\pi\in\mathrm{NC}}$ is called the multiplicative family of functionals on NC determined by $(\rho_n)_{n\in\mathbb{N}}$.

Example 2.9. Let $\mathcal{A}$ be a complex algebra and let $\{\rho_n\colon \mathcal{A}^n \to \mathbb{C}\}_{n\in\mathbb{N}}$ be a sequence of multilinear functionlas on $\mathcal{A}$.

1. Then, for the multiplicative family of functionals $\{\rho_\pi\colon \mathcal{A}^n \to \mathbb{C}\}_{n\in\mathbb{N}}$, we have for all $n \in \mathbb{N}$ and for all $(a_1,\ldots,a_n) \in \mathcal{A}^n$,

$$\rho_{1_n}(a_1,\ldots,a_n) = \rho_n(a_1,\ldots,a_n) \tag{2.4}$$

and

$$\rho_{0_n}(a_1,\ldots,a_n) = \rho_1(a_1)\ldots\rho_1(a_n).$$

The relation (2.4) justifies thinking of $(\rho_\pi)_{\pi\in\mathrm{NC}}$ as "extending" the family $(\rho_n)_{n\in\mathbb{N}}$.

2. For $\pi = \{\{1,4\},\{2,3\},\{5\}\} \in \mathrm{NC}(1,\ldots,5)$ and $a_1,\ldots,a_5 \in \mathcal{A}$, we have

$$\rho_\pi(a_1,\ldots,a_5) = \rho_2(a_1,a_4)\rho_2(a_2,a_3)\rho_1(a_5).$$

Definition 2.10. *Let $(\mathcal{A},\phi)$ be a non-commutative probability space. Then, we define, for every $n\in\mathbb{N}$, the multilinear functionals $\phi_n: \mathcal{A}^n \to \mathbb{C}$ by*

$$\phi_n(a_1,\ldots,a_n) := \phi(a_1\ldots a_n) \quad \textit{for all } (a_1,\ldots,a_n)\in\mathcal{A}^n$$

and consider $(\phi_\pi)_{\pi\in\mathrm{NC}}$ to be the multiplicative family of functionals on NC *determined by $(\phi_n)_{n\in\mathbb{N}}$.*

The corresponding free cumulants $\kappa := (\kappa_\pi)_{\pi\in\mathrm{NC}}$ are for each $n\in\mathbb{N}$,$\pi\in\mathrm{NC}(1,\ldots,n)$, multilinear functionals

$$\begin{array}{ccc} \mathcal{A}^n & \xrightarrow{\kappa_\pi} & \mathbb{C} \\ (a_1,\ldots,a_n) & \longmapsto & \kappa_\pi(a_1,\ldots,a_n) \end{array}$$

which are defined by $\kappa := \phi * \mu$, which means, by

$$\kappa_\sigma(a_1,\ldots,a_n) := \sum_{\substack{\pi\le\sigma \\ \pi\in\mathrm{NC}(1,\ldots,n)}} \phi_\pi(a_1,\ldots,a_n)\mu(\pi,\sigma),$$

for all $n\in\mathbb{N}, \sigma\in\mathrm{NC}(1,\ldots,n)$ and $(a_1,\ldots,a_n)\in\mathcal{A}^n$.

Proposition 2.11. *Let $(\mathcal{A},\phi)$ be a non-commutative probability space with free cumulants $(\kappa_\pi)_{\pi\in\mathrm{NC}}$ and consider the multilinear functionals $\kappa_n := \kappa_{1_n}$ $(n\in\mathbb{N})$. Then, the family $(\kappa_\pi)_{\pi\in\mathrm{NC}}$ is multiplicative, determined by the family $(\kappa_n)_{n\in\mathbb{N}}$.*

Proof. See in [...]. □

Remark 2.12. Let $(\mathcal{A},\phi)$ be a non-commutative probability space with free cumulants $(\kappa_\pi)_{\pi\in\mathrm{NC}}$. By Corollary 2.6, the definition $\kappa := \phi*\mu$ is equivalent to demanding $\phi = \kappa * \zeta$. Then, our moment-cumulant formula can be written

$$\phi_\sigma(a_1,\ldots,a_n) = \sum_{\substack{\pi\le\sigma \\ \pi\in\mathrm{NC}(1,\ldots,n)}} \kappa_\pi(a_1,\ldots,a_n),$$

for all $n\in\mathbb{N}$, $\sigma\in\mathrm{NC}(1,\ldots,n)$ and $(a_1,\ldots,a_n)\in\mathcal{A}^n$. More precisely, the free cumulants are determined by the fact that $(\kappa_\pi)_{\pi\in\mathrm{NC}}$ is a multiplicative family of functionals and that, for all $n\in\mathbb{N}$, and $(a_1,\ldots,a_n)\in\mathcal{A}^n$

$$\phi(a_1\ldots a_n) = \sum_{\pi\in\mathrm{NC}(1,\ldots,n)} \kappa_\pi(a_1,\ldots,a_n). \tag{2.5}$$

We give some examples in order to better understand how the free cumulants can be comptuted using the relation (2.5).

Example 2.13. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space with free cumulants $(\kappa_\pi)_{\pi\in\mathrm{NC}}$. Using the relation (2.5) and the statement that $\pi\mapsto\kappa_\pi$ is a multiplicative family of functionals, we make the following computations:

1. Since $\mathrm{NC}(1)=\{\{1\}\}$, for $a_1\in\mathcal{A}$ we have

$$\phi(a_1)=\phi_1(a_1)=\kappa_1(a_1).$$

2. Let a_1, a_2 be in $\mathcal{A}$. Taking into account that the elements of $\mathrm{NC}(1,2)$ are the partitions $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$, it follows that

$$\begin{aligned}\phi(a_1\,a_2) &= \phi_2(a_1, a_2)\\ &= \kappa_2(a_1, a_2)+\kappa_1(a_1)\kappa_1(a_2)\\ &= \kappa_2(a_1, a_2)+\phi(a_1)\phi(a_2).\end{aligned}$$

Hence, we deduce that $\kappa_2(a_1, a_2)=\phi(a_1\,a_2)-\phi(a_1)\phi(a_2)$.

3. Let $s\in\mathcal{A}$ be a standard semicircular variable. Then, for all $n\in\mathbb{N}$,

$$\phi(s^n)=\begin{cases}0, & \text{for n odd}\\ \#\mathrm{NC}_2(1,\dots,n), & \text{for n even.}\end{cases}$$

For every $n\in\mathbb{N}$, we consider

$$\kappa_n(s,\dots,s)=\begin{cases}1, & \text{if } n=2\\ 0, & \text{otherwise.}\end{cases}$$

Let $(\kappa_\pi(s,\dots,s))_{\pi\in\mathrm{NC}}$ be the family of multiplicative functionals on NC, determined by $(\kappa_n(s,\dots,s))_{n\in\mathbb{N}}$, which means that for every $n\in\mathbb{N}$ and $\pi\in\mathrm{NC}(1,\dots,n)$,

$$\kappa_\pi(s,\dots,s)=\prod_{V\in\pi}\kappa_{\#V}(s,\dots,s\,|\,V)=\begin{cases}1, & \text{for } \pi\in\mathrm{NC}_2(1,\dots,n)\\ 0, & \text{otherwise.}\end{cases}$$

As a consequence, for every $n\in\mathbb{N}$, we have

$$\phi(s^n)=\sum_{\pi\in\mathrm{NC}(1,\dots,n)}\kappa_\pi(s,\dots,s),$$

and by Remark 2.12, the expressions $\kappa_\pi(s,\dots,s)$ must be the free cumulants of s.

4. Let $a\in\mathcal{A}$ be a free Poisson variable of parameter $\lambda>0$ and define

$$\kappa_n(a,\dots a):=\lambda \qquad (n\in\mathbb{N}).$$

In the same way, let $(\kappa_\pi(a,\ldots,a))_{\pi\in\mathrm{NC}}$ be the family of multiplicative functionals on NC, determined by $(\kappa_n(a,\ldots,a))_{n\in\mathbb{N}}$. Then, it follows that for all $n\in\mathbb{N}$,

$$\sum_{\pi\in\mathrm{NC}(1,\ldots,n)}\kappa_\pi(a,\ldots,a)=\sum_{\pi\in\mathrm{NC}(1,\ldots,n)}\prod_{V\in\pi}\lambda=\sum_{\pi\in\mathrm{NC}(1,\ldots,n)}\lambda^{\#\pi}=\phi(a^n)$$

and by Remark 2.12, the expressions $\kappa_\pi(a,\ldots,a)$ must be the free cumulants of a.

Let $(\mathcal{A},\phi)$ be a non-commutative probability space with free cumulants $(\kappa_\pi)_{\pi\in\mathrm{NC}}$. As mentioned earlier, for every $n\in\mathbb{N}$ and $\pi\in\mathrm{NC}(1,\ldots,n)$, the functionals $\kappa_\pi\colon\mathcal{A}^n\to\mathbb{C}$ are multilinear. We are interested in to understand how the algebra structure of $\mathcal{A}$, affects the cumulants. Note that the moment functionals are "associative", e.g. for all $a_1,a_2,a_3\in\mathcal{A}$,

$$\phi_2(a_1a_2,a_3)=\phi[(a_1a_2)a_3]=\phi[a_1(a_2\,a_3)]=\phi(a_1,a_2\,a_3).$$

Since, the functionals ϕ and $(\kappa_\pi)_{\pi\in\mathrm{NC}}$ are related through (2.5), it is natural to ask if κ_2 has a similar behavior with respect to the multiplicative structure of $\mathcal{A}$.

Before we consider this question, we start by introducing some notation: Let $(\mathcal{A},\phi)$ be a non-commutative probability space and let $m,n\in\mathbb{N}$ such that $m\le n$. Then, for $1\le i(1)<\cdots<i(m-1)<i(m)=n$ and $a_1,\ldots,a_n\in\mathcal{A}$, we define

$$\begin{aligned}A_1 &:= a_1\ldots a_{i(1)}\\ A_2 &:= a_{i(1)+1}\ldots a_{i(2)}\\ &\vdots\\ A_m &:= a_{i(m-1)+1}\ldots a_n.\end{aligned}$$

Therefore we group the elements $a_1,\ldots,a_n\in\mathcal{A}$, by taking into account their order in the product $a_1\ldots a_n$.

We want to relate the free cumulants of $(a_1,\ldots,a_n)$ and $(A_1,\ldots,A_m)$. First, we have to understand for which partitions $\sigma\in\mathrm{NC}(1,\ldots,n)$ and $\pi\in\mathrm{NC}(1,\ldots,m)$, we may have a relation between $\kappa_\sigma(a_1,\ldots,a_n)$ and $\kappa_\pi(A_1,\ldots,A_m)$.

For $\pi\in\mathrm{NC}(1,\ldots,m)$, we define a partition $\hat{\pi}\in\mathrm{NC}(1,\ldots,n)$, in the following way: Define $i(0):=0$. For $j,k\in\{1,\ldots,n\}$, we require $j\sim_{\hat{\pi}}k$ if and only if there exist $p,q\in\{1,\ldots,m\}$ such that

- $j\in\{i(p-1)+1,\ldots,i(p)\}$, i.e. a_j is a factor in A_p,
- $k\in\{i(q-1)+1,\ldots,i(q)\}$, i.e. a_k is a factor in A_q,
- $p\sim_\pi q$.

Using that $\pi\in\mathrm{NC}(1,\ldots,m)$, it is easy to observe that the new partition $\hat{\pi}$ is a non-crossing partition of $\{1,\ldots,n\}$. Also, we have $\hat{1}_m=1_n$, but is not true in general that $\hat{0}_m=0_n$.

Given $n,m\in\mathbb{N}$ such that $n<m$ and $a_1,\ldots,a_n\in\mathcal{A}$, the construction of the variables $A_1,\ldots,A_m\in\mathcal{A}$ is equivalent to the construction of an interval partition $(\hat{0}_m)$ of $\{1,\ldots,n\}$, with m blocks. For every $\pi\in\mathrm{NC}(1,\ldots,m)$, it is easy to see that,

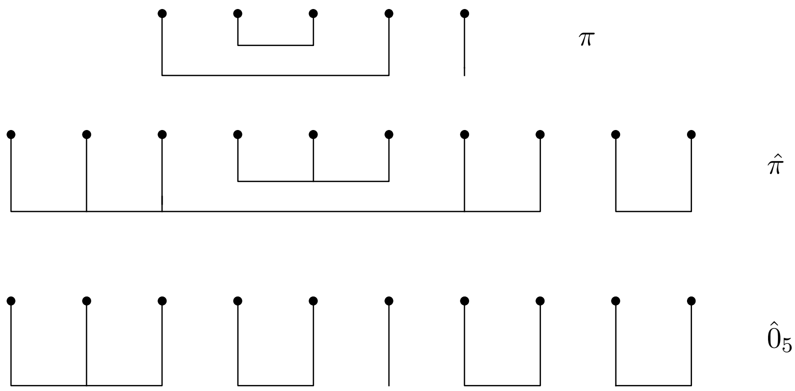
$$\phi_\pi(A_1,\ldots,A_m)=\phi_{\hat{\pi}}(a_1,\ldots,a_n). \tag{2.6}$$

We give an example in order to illustrate the action of the map $\hat{}: \mathrm{NC}(1,\ldots,m) \to \mathrm{NC}(1,\ldots,n)$.

Example 2.14. Let $m=5$, $n=10$, $\pi = \{\{1,4\},\{2,3\},\{5\}\}$ and for $a_1,\ldots,a_{10} \in \mathcal{A}$, define,

$$A_1 := a_1\, a_2\, a_3, \quad A_2 := a_4\, a_5, \quad A_3 := a_6, \quad A_4 := a_7\, a_8, \quad A_5 := a_9\, a_{10}.$$

Then $\hat{\pi} := \{\{1,2,3,7,8\},\{4,5,6\},\{9,10\}\}$, and $\hat{0}_5 = \{\{1,2,3\},\{4,5\},\{6\},\{7,8\},\{9,10\}\}$.



Remark 2.15. We want to mention some key properties of the map $\hat{}: \mathrm{NC}(1,\ldots,m) \to \mathrm{NC}(1,\ldots,n)$.

1. The map $\mathrm{NC}(1,\ldots,m) \ni \tau \mapsto \hat{\tau} \in \mathrm{NC}(1,\ldots,n)$ is injective and it's image is

$$\widehat{\mathrm{NC}(1,\ldots,m)} = \{\tau \in \mathrm{NC}(1,\ldots,n) \,|\, \hat{0}_m \leq \tau\}.$$

   More generally, for $\pi,\sigma \in \mathrm{NC}(1,\ldots,m)^{(2)}$ we have

$$\widehat{\{\tau \in \mathrm{NC}(1,\ldots,m) \,|\, \pi \leq \tau \leq \sigma\}} = \{\tau \in \mathrm{NC}(1,\ldots,n) \,|\, \hat{\pi} \leq \tau \leq \hat{\sigma}\}. \tag{2.7}$$

   Note also that $\pi \leq \sigma$ for $\pi,\sigma \in \mathrm{NC}(1,\ldots,m)$ implies $\hat{\pi} \leq \hat{\sigma}$.

2. We have

$$\mu(\pi,\sigma) = \mu(\hat{\pi},\hat{\sigma}), \quad \text{for all } (\pi,\sigma) \in \mathrm{NC}(1,\ldots,m)^{(2)}. \tag{2.8}$$

   Indeed, we define the function $\tilde{\mu}: \mathrm{NC}(1,\ldots,m)^{(2)} \to \mathbb{C}$ by demanding,

$$\tilde{\mu}(\pi,\sigma) := \mu(\hat{\pi},\hat{\sigma}), \quad \text{for all } (\pi,\sigma) \in \mathrm{NC}(1,\ldots,m)^{(2)}.$$

   Then, for $(\pi,\sigma) \in \mathrm{NC}(1,\ldots,m)^{(2)}$, we have

$$\begin{aligned}
(\zeta * \tilde{\mu})(\pi,\sigma) &= \sum_{\substack{\pi \leq \tau \leq \sigma \\ \tau \in \mathrm{NC}(1,\ldots,m)}} \mu(\hat{\tau},\hat{\sigma}) \\
&= \sum_{\substack{\hat{\pi} \leq \tau \leq \hat{\sigma} \\ \tau \in \mathrm{NC}(1,\ldots,n)}} \mu(\tau,\hat{\sigma}) \\
&= (\zeta * \mu)(\hat{\pi},\hat{\sigma}) = \delta(\hat{\pi},\hat{\sigma}) = \delta(\pi,\sigma).
\end{aligned}$$

Similarly, we can prove $\tilde{\mu} * \zeta = \delta$ and the relation (2.8) holds.

Proposition 2.16. *Consider a non-commutative probability space $(\mathcal{A}, \phi)$ and let $(\kappa_\pi)_{\pi \in \mathrm{NC}}$ be the corresponding free cumulants. Let m, n be positive integers, such that $m < n$ and let $i(0), \ldots, i(m) \in \{1, \ldots, n\}$, such that $0 = i(0) < 1 \leq i(1) < i(2) < \cdots < i(m) = n$. For arbitrary non-commutative random variables $a_1, \ldots, a_n \in \mathcal{A}$, define*

$$A_k := a_{i(k-1)+1} \ldots a_{i(k)}, \quad \textit{for all } k = 1, \ldots, m.$$

Then, for all $\sigma \in \mathrm{NC}(1, \ldots, m)$ we have

$$\kappa_\sigma(A_1, \ldots, A_m) = \sum_{\substack{\pi \vee \hat{0}_m = \hat{\sigma} \\ \pi \in \mathrm{NC}(1, \ldots, n)}} \kappa_\pi(a_1, \ldots, a_n)\,.$$

Proof. Using the relations (2.6),(2.7) and (2.8) we have

$$\begin{aligned}
\kappa_\sigma(A_1, \ldots, A_m) &= \sum_{\substack{\tau \leq \sigma \\ \tau \in \mathrm{NC}(1, \ldots, m)}} \phi_\tau(A_1, \ldots, A_m)\mu(\tau, \sigma) \\
&= \sum_{\substack{\tau \leq \sigma \\ \tau \in \mathrm{NC}(1, \ldots, m)}} \phi_{\hat{\tau}}(a_1, \ldots, a_n)\mu(\hat{\tau}, \hat{\sigma}) \\
&= \sum_{\substack{\hat{0}_m \leq \omega \leq \hat{\sigma} \\ \omega \in \mathrm{NC}(1, \ldots, n)}} \phi_\omega(a_1, \ldots, a_n)\mu(\omega, \hat{\sigma}) \\
&= \sum_{\substack{\hat{0}_m \leq \omega \leq \hat{\sigma} \\ \omega \in \mathrm{NC}(1, \ldots, n)}} \sum_{\substack{\pi \leq \omega \\ \pi \in \mathrm{NC}(1, \ldots, n)}} \kappa_\pi(a_1, \ldots, a_n)\mu(\omega, \hat{\sigma}).
\end{aligned}$$

Since

$$\{(\omega, \pi) \in \mathrm{NC}(1, \ldots, n)^2 |\, \hat{0}_m \leq \omega \leq \hat{\sigma}, \pi \leq \omega\} = \{(\omega, \pi) \in \mathrm{NC}(1, \ldots, n)^2 |\, \pi \leq \hat{\sigma}, \hat{0}_m \vee \pi \leq \omega \leq \hat{\sigma}\}$$

we have

$$\begin{aligned}
\kappa_\sigma(A_1, \ldots, A_m) &= \sum_{\substack{\pi \leq \hat{\sigma} \\ \pi \in \mathrm{NC}(1, \ldots, n)}} \sum_{\substack{\hat{0}_m \vee \pi \leq \omega \leq \hat{\sigma} \\ \omega \in \mathrm{NC}(1, \ldots, n)}} \kappa_\pi(a_1, \ldots, a_n)\mu(\omega, \hat{\sigma}) \\
&= \sum_{\substack{\pi \leq \hat{\sigma} \\ \pi \in \mathrm{NC}(1, \ldots, n)}} \left(\sum_{\substack{\hat{0}_m \vee \pi \leq \omega \leq \hat{\sigma} \\ \omega \in \mathrm{NC}(1, \ldots, n)}} \mu(\omega, \hat{\sigma}) \right) \kappa_\pi(a_1, \ldots, a_n) \\
&= \sum_{\substack{\pi \leq \hat{\sigma} \\ \pi \in \mathrm{NC}(1, \ldots, n)}} \delta(\hat{0}_m \vee \pi, \hat{\sigma})\kappa_\pi(a_1, \ldots, a_n).
\end{aligned}$$

By the definition of δ function we conclude that the partitions π which satisfy the relation $\hat{0}_m \vee \pi \lneqq \hat{\sigma}$, do not contribute to the sum. Hence,

$$\kappa_\sigma(A_1, \ldots, A_m) = \sum_{\substack{\pi \vee \hat{0}_m = \hat{\sigma} \\ \pi \in \mathrm{NC}(1, \ldots, n)}} \kappa_\pi(a_1, \ldots, a_n). \qquad \square$$

2.3 Free cumulants and Free independence

We now turn to the description of the notion of free independence in terms of free cumulants. We have mentioned several times that compared to classical probability, in free probability the lattice of all partitions of a set is replaced by the lattice of non-crossing partitions. In that spirit, we recall that in the theory of classical cumulants, which was developed by Rota around 1964, the classical cumulants are defined in an analogous way, as a multiplicative family of functionals on $P := \cup_{n \geq 1} P(1, \ldots, n)$. Moreover, in classical probability theory we have a characterization of the notion of classical independence via cumulants. Roughly speaking, classical random variables are independent if and only if certain cumulants vanish. An analogous result holds in the case where cumulants and independent random variables are replaced by free cumulants and free random variables.

Proposition 2.17. *Consider a non-commutative probability space $(\mathcal{A}, \phi)$ with free cumulants $(\kappa_\pi)_{\pi \in \mathrm{NC}}$. For $n \geq 2$ and $a_1, \ldots, a_n \in \mathcal{A}$, we have $\kappa_n(a_1, \ldots, a_n) = 0$, if there exist at least one $i \in \{1, \ldots, n\}$, such that $a_i = 1$.*

Proof. For simplicity, we consider the case $i = n$. Therefore, we have to prove that $\kappa_n(a_1, \ldots, a_{n-1}, 1) = 0$, for $n \geq 2$. We give a proof by induction over n.

For $n = 2$, we have shown that

$$\kappa_2(a_1, 1) = \phi(a_1 \cdot 1) - \phi(a_1)\phi(1) = 0.$$

Now, we assume that for all $k < n$ the assertion holds. Then, using that $\phi = \kappa * \zeta$, we have

$$\begin{aligned} \kappa_n(a_1, \ldots, a_{n-1}, 1) + \sum_{\substack{\pi < 1_n \\ \pi \in \mathrm{NC}(1, \ldots, n)}} \kappa_\pi(a_1, \ldots, a_{n-1}, 1) &= \sum_{\pi \in \mathrm{NC}(1, \ldots, n)} \kappa_\pi(a_1, \ldots, a_{n-1}, 1) \\ &= \phi(a_1 \ldots a_{n-1} \cdot 1) \\ &= \phi(a_1 \ldots a_{n-1}). \end{aligned}$$

Taking into account that the family $(\kappa_\pi)_{\pi \in \mathrm{NC}}$ is multiplicative and the induction hypothesis, we see that for a non-crossing partition $\pi < 1_n$ we may have $\kappa_\pi(a_1, \ldots, a_{n-1}, 1) \neq 0$ only if the set $\{n\}$ is a block of π. Therefore, in order to compute the sum on the left hand side, it suffices to consider partitions $\pi < 1_n$, such that $\pi = \sigma \cup \{\{n\}\}$, for some $\sigma \in \mathrm{NC}(1, \ldots, n-1)$. For such a partition, we have

$$\kappa_\pi(a_1, \ldots, a_{n-1}, 1) = \kappa_\sigma(a_1, \ldots, a_{n-1})\kappa_1(1) = \kappa_\sigma(a_1, \ldots, a_{n-1}).$$

Therefore, using again the relation $\phi = \kappa * \zeta$, we have

$$\begin{aligned} \kappa_n(a_1, \ldots, a_{n-1}, 1) &= \phi(a_1 \ldots a_{n-1}) - \sum_{\sigma \in \mathrm{NC}(1, \ldots, n-1)} \kappa_\sigma(a_1, \ldots, a_{n-1}) \\ &= \phi(a_1 \ldots a_{n-1}) - \phi(a_1 \ldots a_{n-1}) \\ &= 0 \end{aligned}$$

and the assertion holds. □

Finally, we are ready to prove the main theorem of this subsection which gives a characterization of free independence through free cumulants. The previous proposition is a special case of the next theorem and it mainly holds because the random variable $1 \in \mathcal{A}$ is free from every random variable in $\mathcal{A}$.

Theorem 2.18. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, with free cumulants $(\kappa_\pi)_{\pi \in \mathrm{NC}}$ and let $(\mathcal{A}_i)_{i \in I}$ be a family of unital subalgebras of $\mathcal{A}$. Then, the subalgebras $(\mathcal{A}_i)_{i \in I}$ are independent in $(\mathcal{A}, \phi)$ if and only if for all $n \geq 2$ and for all $a_j \in \mathcal{A}_{i(j)}$ with $i(1), \ldots, i(n) \in I$, we have that $\kappa_n(a_1, \ldots, a_n) = 0$ whenever there exist $l, k \in \{1, \ldots, n\}$ such that $i(l) \neq i(k)$.*

Proof. We assume that the free cumulants $(\kappa_n)_{n \geq 2}$ vanish when they are evaluated at elements of different algebras. We want to prove that the subalgebras $(\mathcal{A}_i)_{i \in I}$ are free. Let n be a positive integer and let $i(1), \ldots, i(n) \in I$ such that $i(j) \neq i(j+1)$ for every $j = 1, \ldots, n-1$. We consider non-commutative random variables $a_1, \ldots, a_n \in \mathcal{A}$ such that $a_j \in \mathcal{A}_{i(j)}$ and $\phi(a_j) = 0$ for all $j = 1, \ldots, n$. We want to prove that $\phi(a_1 \ldots a_n) = 0$. Using the relation $\phi = \kappa * \zeta$ and the fact that the family $(\kappa_\pi)_{\pi \in \mathrm{NC}}$ is multiplicative, we have

$$\begin{aligned} \phi(a_1 \ldots a_n) &= \sum_{\pi \in \mathrm{NC}(1, \ldots, n)} \kappa_\pi(a_1, \ldots, a_n) \\ &= \sum_{\pi \in \mathrm{NC}(1, \ldots, n)} \prod_{V \in \pi} \kappa_{\#V}(a_1, \ldots, a_n | V). \end{aligned}$$

As it has been mentioned, if $\pi \in \mathrm{NC}(1, \ldots, n)$, then it will have a block V of the form $V = \{l, l+1, \ldots, l+p\} \subseteq \{1, \ldots, n\}$. For such a block V, if $p = 0$, then

$$\kappa_{\#V}(a_1, \ldots, a_n | V) = \kappa_1(a_l) = \phi(a_l) = 0,$$

because it is assumed that the non-commutative random variables $a_1, \ldots, a_n \in \mathcal{A}$ have zero mean. Otherwise, for $p > 0$ we will have $\#V = p + 1 \geq 2$, and

$$\kappa_{\#V}(a_1, \ldots, a_n | V) = \kappa_{p+1}(a_l, \ldots, a_{l+p}) = 0,$$

by our assumption of vanishing of cumulants, because $a_l \in \mathcal{A}_{i(l)}$, $a_{l+1} \in \mathcal{A}_{i(l+1)}$ and $i(l) \neq i(l+1)$. As a consequence, we deduce that $\phi(a_1 \ldots a_n) = 0$, and the subalgebras $(\mathcal{A}_i)_{i \in I}$ are free.

We continue with the proof of the inverse direction. Let $(\mathcal{A}_i)_{i \in I}$ be a family of unital subalgebras of $\mathcal{A}$ such that $(\mathcal{A}_i)_{i \in I}$ are freely independent in $(\mathcal{A}, \phi)$. For $n \geq 2$, indices $i(1), \ldots, i(n) \in I$ such that $\#\{i(1), \ldots, i(n)\} \geq 2$, and $a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \ldots, n$, we have to prove that $\kappa_n(a_1, \ldots, a_n) = 0$. Assume first that neighboring elements in $a_1 \ldots a_n$ are from different subalgebras, i.e. that $i(j) \neq i(j+1)$ for all $j = 1, \ldots, n-1$. Also, note that by the previous proposition we have

$$\kappa_n(a_1, \ldots, a_n) = \kappa_n(a_1^0, \ldots, a_n^0).$$

Therefore, it suffices to show that $\kappa_n(a_1^0, \ldots, a_n^0) = 0$. By definition, we have

$$\begin{aligned} \kappa_n(a_1^0, \ldots, a_n^0) &= \sum_{\pi \in \mathrm{NC}(1, \ldots, n)} \phi_\pi(a_1^0, \ldots, a_n^0) \mu(\pi, 1_n) \\ &= \sum_{\pi \in \mathrm{NC}(1, \ldots, n)} \mu(\pi, 1_n) \prod_{V \in \pi} \phi_{\#V}(a_1^0, \ldots, a_n^0 | V). \end{aligned}$$

As before, for $\pi \in \mathrm{NC}(1,\ldots,n)$, choosing an interval partition $V \in \pi$, we have $\phi_{\#V}(a_1^0,\ldots, a_n^0|V) = 0$. Indeed, this is true if $\#V = 1$ because the non-commutative random variables $a_1^0,\ldots,a_n^0 \in \mathcal{A}$ have zero mean, but it is also true if $\#V > 1$ because the subalgebras $(\mathcal{A}_i)_{i\in I}$ are freely independent in $(\mathcal{A}, \phi)$. Hence we have $\kappa_n(a_1^0,\ldots,a_n^0) = 0$.

Now, we treat the more general case where $\{i(1),\ldots,i(n)\}$ is a set of more than one element, but not necessarily $i(j) \neq i(j+1)$ for all $j = 1,\ldots,n-1$. We proceed by induction on $n \geq 2$. Obviously, for $n = 2$ the assertion holds and we also assume that it holds for all $2 \leq r < n$. After combining neighbouring elements from the same subalgebra, we have that $a_1 \ldots a_n = A_1 \ldots A_m$, where $A_j \in \mathcal{A}_{l(j)}$ for all $j = 1,\ldots,m$ and $l(j) \neq l(j+1)$ for all $j = 1,\ldots, m-1$. Since, $\#\{i(1),\ldots,i(n)\} \geq 2$, we have that $m \geq 2$. Thus, from the above we have $\kappa_m(A_1,\ldots,A_m) = 0$. Moreover, by Proposition 2.16 it follows that

$$\begin{aligned} 0 = \kappa_m(A_1,\ldots,A_m) &= \sum_{\substack{\pi \vee \hat{0}_m = 1_n \\ \pi \in \mathrm{NC}(1,\ldots,n)}} \kappa_\pi(a_1,\ldots,a_n) \\ &= \kappa_{1_n}(a_1,\ldots,a_n) + \sum_{\substack{\pi \vee \hat{0}_m = 1_n \text{ and } \pi \neq 1_n \\ \pi \in \mathrm{NC}(1,\ldots,n)}} \kappa_\pi(a_1,\ldots,a_n) \\ &= \kappa_n(a_1,\ldots,a_n) + \sum_{\substack{\pi \vee \hat{0}_m = 1_n \text{ and } \pi \neq 1_n \\ \pi \in \mathrm{NC}(1,\ldots,n)}} \kappa_\pi(a_1,\ldots,a_n). \end{aligned}$$

We will show that the sum on the right hand side is equal to zero. Let $\pi \in \mathrm{NC}(1,\ldots,n)$ such that $\pi \neq 1_n$ and $\pi \vee \hat{0}_m = 1_n$. For a block $V = \{p_1 < \cdots < p_r\} \in \pi$, we have $r < n$ because $\pi \neq 1_n$. Assume that $\#\{i(p_1),\ldots,i(p_r)\} \geq 2$. Then, our induction hypothesis guarantees that

$$\kappa_{\#V}(a_1,\ldots,a_n|V) = \kappa_r(a_{p_1},\ldots,a_{p_r}) = 0,$$

which implies that $\kappa_\pi(a_1,\ldots,a_n) = 0$. Hence, for such a $\pi \in \mathrm{NC}(1,\ldots,n)$, we have $\kappa_\pi(a_1,\ldots, a_n) \neq 0$, only if $i(p) = i(q)$ for every $p, q \in \{1,\ldots,n\}$ such that $p \sim_\pi q$. Therefore, it follows that

$$\sum_{\substack{\pi \vee \hat{0}_m = 1_n \text{ and } \pi \neq 1_n \\ \pi \in \mathrm{NC}(1,\ldots,n)}} \kappa_\pi(a_1,\ldots,a_n) = \sum_{\substack{\pi \vee \hat{0}_m = 1_n,\ \pi \neq 1_n \text{ and } \ker(i) \geq \pi \\ \pi \in \mathrm{NC}(1,\ldots,n)}} \kappa_\pi(a_1,\ldots,a_n).$$

Note that $\ker(i) \geq \hat{0}_m$. Hence, for $\pi \in \mathrm{NC}(1,\ldots,n)$ such that $\pi \vee \hat{0}_m = 1_n$, $\pi \neq 1_n$ and $\ker(i) \geq \pi$, we have $\ker(i) = 1_n$, saying that $\#\{i(1),\ldots,i(n)\} = 1$. Thus, by contradiction we deduce that

$$\kappa_n(a_1,\ldots,a_n) = \kappa_m(A_1,\ldots,A_m) = 0. \qquad \square$$

3 Non-commutative Stochastic Processes

In this section we focus on non-commutative stochastic processes. In the context of classical probability, stochastic processes with stationary and independent increments, form one of the best studied and important classes of stochastic processes. The understanding of their structure was crucial for many developments in classical probability theory. We will present the natural non-commutative analogues of some of the most celebrated examples (e.g. Brownian motion and Poisson process) of such stochastic processes, in the context of free probability theory. Therefore, in our case we are interested in stochastic processes with “free increments”.

3.1 Definition and Combinatorics of Free Levy Process

We recall that in the classical case, a stochastic process $(X_t)_{t\geq 0}$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{C}^n$, is called an independent increment process, if for any $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the random variables X_{t_1}, $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$, are mutually independent. Also, the process $(X_t)_{t\geq 0}$ is called a stationary increments process, if for every $s < t$, the random variable $X_t - X_s$ is equal in distribution to X_{t-s} .

Now, we start by defining such a stochastic process, in the context of free probability and studying the behaviour of it's moments.

In the following, let $\mathcal{R}$ be the ring generated by all semiclosed intervals $I \subseteq \mathbb{R}$, of the form $I = [s, t)$, for $s < t$. Also, we denote by λ, the Lebesgue measure on $\mathbb{R}$.

Definition 3.1. *A n-dimensional free Levy process $(\mathcal{C}, \rho, (c_I^1, \ldots, c_I^n)_{I\in\mathcal{R}})$ consists of*

1. *a unital C^*-algebra $\mathcal{C}$,*
2. *a state ρ on $\mathcal{C}$,*
3. *a finitely additive mapping $\mathcal{R} \to \mathcal{C}^n$, $I \mapsto (c_I^1, \ldots, c_I^n)$,*

such that if $\mathcal{C}_I := C^*(1, c_I^1, \ldots, c_I^n)$ is the C^*-algebra generated by $1, c_I^1, \ldots, c_I^n \in \mathcal{C}$, then the following conditions are satisfied:

1. *The C^*-subalgebras $\mathcal{C}_{I_1}, \ldots, \mathcal{C}_{I_r}$ are freely independent in $(\mathcal{C}, \rho)$ for all $r \in \mathbb{N}$ and for all $I_1, \ldots, I_r \in \mathcal{R}$ disjoint.*
2. *The joint distribution of the non-commutative random variables $c_I^1,\ldots,c_I^n$ with respect to ρ depends only on $\lambda(I)$. This means that for every $I, J \in \mathcal{R}$ such that $\lambda(I) = \lambda(J)$, we have for all $r \in \mathbb{N}$ and for all $k(1), \ldots, k(r) \in \{1, \ldots, n\}$,*

$$\rho\big(\hat{c}_I^{k(1)} \ldots \hat{c}_I^{k(r)}\big) = \rho(\hat{c}_J^{k(1)} \ldots \hat{c}_J^{k(r)}),$$

where $\hat{c}$ stands for $c \in \mathcal{C}$ or it's adjoint $c^* \in \mathcal{C}$.

Remark 3.2. In order to compare our definition of free white noise with the analogous definition in the classical case, we can think of a 1-dimensional free white noise, as

$$X_t := c_{[0,t)}, \quad \text{for every } t \geq 0.$$

Then, for $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the fact that the map $\mathcal{R} \to \mathcal{C}$, $I \mapsto c_I$ is finitely additive, implies that for every $k = 2, \ldots, n$,

$$X_{t_k} - X_{t_{k-1}} = c_{[0,t_k)} - c_{[0,t_{k-1})} = c_{[t_{k-1},t_k)}.$$

Therefore, since the C^*-subalgebras $\mathcal{C}_{[0,t_1)}, \mathcal{C}_{[t_1,t_2)}, \ldots, \mathcal{C}_{[t_{n-1},t_n)}$ are freely independent in $(\mathcal{C}, \rho)$, this implies that the non-commutative random variables $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are free.

Finally, since the distribution of $c_I \in \mathcal{C}$, depends only on $\lambda(I)$ and $X_t - X_s = c_{[s,t)}$, $X_{t-s} = c_{[0,t-s)}$, for $s < t$, we deduce that the non-commutative random variables $X_t - X_s$, X_{t-s} have the same distribution, with respect to ρ.

In the next theorem, we see that for certain n-dimensional free Levy processes, it suffices to know their moments at time $t_0 = 1$, in order to determine their moments at every time $t \in \mathbb{R}_{>0}$.

Theorem 3.3. *Consider a n-dimensional free Levy process $(\mathcal{C}, \rho, (c_I^1, \ldots, c_I^n)_{I \in \mathcal{R}})$, and define $c_t^k := c_{[0,t)}^k$ for all $k = 1, \ldots, n$. Assume that for all $r \in \mathbb{N}$ and all $k(1), \ldots, k(r) \in \{1, \ldots, n\}$, we have*

$$\lim_{t \to 0} \rho\big(c_t^{k(1)} \ldots c_t^{k(r)}\big) = 0.$$

Then, for every $t \geq 0$, there exist a map $Q_t: \cup_{m \geq 1} \{1, \ldots, n\}^m \to \mathbb{C}$, called generator, such that for all $r \in \mathbb{N}$, and all $k(1), \ldots, k(r) \in \{1, \ldots, n\}$ we have

$$\rho\big(c_t^{k(1)} \ldots c_t^{k(r)}\big) = \sum_{p=1}^{r} \sum_{\{V_1, \ldots, V_p\} \in \mathrm{NC}(1, \ldots, r)} Q_t(V_1) \ldots Q_t(V_p),$$

where we define $Q_t(V) := Q_t(k(v_1), \ldots, k(v_m))$, for $V = \{v_1 < \cdots < v_m\} \subseteq \{1, \ldots, r\}$. Moreover, for every $t \geq 0$, we have

$$Q_t = t \cdot Q_1.$$

Proof. Let r be a positive integer and let $k(1), \ldots, k(r) \in \{1, \ldots, n\}$. By induction, we see that the continuity of the map $t \mapsto \rho(c_t^{k(1)} \ldots c_t^{k(r)})$ at $t_0 = 0$, implies that the map is continuous at every t. We will only treat the cases $r = 1, 2$, because it saves us a lot of indices and illustrates the procedure sufficiently. For $r = 1$, and $t > t_0$, we have

$$\rho(c_t^{k(1)}) = \rho(c_{t_0}^{k(1)}) + \rho(c_{[t_0,t)}^{k(1)}) = \rho(c_{t_0}^{k(1)}) + \rho(c_{t-t_0}^{k(1)}),$$

which implies that $\rho(c_t^{k(1)}) \to \rho(c_{t_0}^{k(1)})$, as $t \downarrow t_0$. For $r = 2$, we have

$$\begin{aligned}
\rho\big(c_t^{k(1)} c_t^{k(2)}\big) &= \rho\big[\big(c_{t_0}^{k(1)} + c_{[t_0,t)}^{k(1)}\big)\big(c_{t_0}^{k(2)} + c_{[t_0,t)}^{k(2)}\big)\big] \\
&= \rho\big(c_{t_0}^{k(1)} c_{t_0}^{k(2)}\big) + \rho\big(c_{t_0}^{k(1)}\big)\rho\big(c_{[t_0,t)}^{k(2)}\big) + \rho\big(c_{[t_0,t)}^{k(1)}\big)\rho\big(c_{t_0}^{k(2)}\big) + \rho\big(c_{[t_0,t)}^{k(1)} c_{[t_0,t)}^{k(2)}\big) \\
&= \rho\big(c_{t_0}^{k(1)} c_{t_0}^{k(2)}\big) + \rho\big(c_{t_0}^{k(1)}\big)\rho\big(c_{t-t_0}^{k(2)}\big) + \rho\big(c_{t-t_0}^{k(1)}\big)\rho\big(c_{t_0}^{k(2)}\big) + \rho\big(c_{t-t_0}^{k(1)} c_{t-t_0}^{k(2)}\big)
\end{aligned}$$

where in the second equality we used that the C^*-algebras $\mathcal{C}_{[0,t_0)}, \mathcal{C}_{[t_0,t)}$ are freely independent and in the third equality we used that the joint distribution of $c_I^1, \ldots, c_I^n$, depends only on $\lambda(I)$. Therefore, we have that $\rho(c_t^{k(1)} c_t^{k(2)}) \to \rho(c_{t_0}^{k(1)} c_{t_0}^{k(2)})$, as $t \downarrow t_0$, and similar arguments for $t < t_0$, show that

$$\lim_{t \downarrow t_0} \rho(c_t^{k(1)} \ldots c_t^{k(r)}) = \rho(c_{t_0}^{k(1)} \ldots c_{t_0}^{k(r)}) = \lim_{t \uparrow t_0} \rho(c_t^{k(1)} \ldots c_t^{k(r)}).$$

The fact that the map $I \mapsto (c_I^1, \ldots, c_I^n)$ is finitely additive, implies that for all $k \in \mathbb{N}$ and $t \geq 0$, we have

$$c_t^k = \sum_{M=0}^{N-1} c_{I_M}^k \quad \text{where} \quad I_M = [Mt/N, (M+1)\, t/N)\,. \tag{3.1}$$

Since, $(\mathcal{C}, \rho, (c_I^1, \ldots, c_I^n)_{I \in \mathcal{R}})$ is a n-dimensional free Levy process, by the relation (3.1) it follows that c_t^k, can be written as a sum S_N^k of Theorem 1.21 if we identify $c_{I_M}^k$ with $a_{M,N}^k$. Since, for every $M = 0, \ldots, N-1$,

$$\rho\big(c_{I_M}^{k(1)} \ldots c_{I_M}^{k(r)}\big) = \rho\big(c_{t/N}^{k(1)} \ldots c_{t/N}^{k(r)}\big),$$

from Theorem 1.21 follows that if the limit

$$Q_t(k(1), \ldots, k(r)) := \lim_{N \to \infty} N \cdot \rho(c_{t/N}^{k(1)} \ldots c_{t/N}^{k(r)}), \tag{3.2}$$

exists, then the first assertion holds. We will prove the existence of the limit stated in (3.2), by induction on $r \in \mathbb{N}$. Again, in order to avoid extremely lengthy computations, we only treat the cases $r = 1$ and $r = 2$, since they illustrate the procedure sufficiently. For $r = 1$ we have

$$\rho\big(c_t^{k(1)}\big) = \sum_{M=0}^{N-1} \rho\big(c_{I_M}^{k(1)}\big) = N\rho\big(c_{t/N}^{k(1)}\big),$$

which implies that $Q_t(k(1)) = \rho(c_t^{k(1)})$. For $r = 2$, we have

$$\begin{aligned}
\rho\big(c_t^{k(1)} c_t^{k(2)}\big) &= \sum_{l,m=0}^{N-1} \rho\big(c_{[l \cdot t/N, (l+1) \cdot t/N)}^{k(1)}\, c_{[m \cdot t/N, (m+1) \cdot t/N)}^{k(2)}\big) \\
&= \sum_{\substack{l,m=0 \\ l=m}}^{N-1} \rho\big(c_{t/N}^{k(1)}\, c_{t/N}^{k(2)}\big) + \sum_{\substack{l,m=0 \\ l \neq m}}^{N-1} \rho\big(c_{t/N}^{k(1)}\big) \rho\big(c_{t/N}^{k(2)}\big) \\
&= N\rho\big(c_{t/N}^{k(1)}\, c_{t/N}^{k(2)}\big) + N(N-1)\, \rho\big(c_{t/N}^{k(1)}\big) \rho\big(c_{t/N}^{k(2)}\big)\,.
\end{aligned}$$

Thus, $Q_t(k(1), k(2)) = \rho\big(c_t^{k(1)} c_t^{k(2)}\big) - Q_t(k(1)) Q_t(k(2))\,.$

In order to finish the proof it remains to show that $Q_t = t \cdot Q_1$, for every $t \geq 0$. Let $r \in \mathbb{N}$, and $k(1), \ldots, k(r) \in \{1, \ldots, n\}$. Since the function $t \mapsto \rho\big(c_t^{k(1)} \ldots c_t^{k(r)}\big)$ is continuous, it is easy to see that the function $t \mapsto Q_t(k(1), \ldots, k(r))$ is also continuous. Therefore, it suffices to show that $Q_{t+s} = Q_t + Q_s$, for every $t, s \geq 0$. As above, we only treat the cases $r = 1$ and $r = 2$ since they indicate sufficiently why the cases $r > 3$ hold. For $r = 1$, we have

$$\begin{aligned}
Q_{t+s}(k(1)) &= \lim_{N \to \infty} N\rho\big(c_{(t+s)/N}^{k(1)}\big) \\
&= \lim_{N \to \infty} N\rho\big(c_{t/N}^{k(1)}\big) + \lim_{N \to \infty} N\rho\big(c_{[t/N, (t+s)/N)}^{k(1)}\big) \\
&= Q_t(k(1)) + \lim_{N \to \infty} N\rho\big(c_{s/N}^{k(1)}\big) \\
&= Q_t(k(1)) + Q_s(k(1)).
\end{aligned}$$

For the case $r=2$, we have

$$Q_{t+s}(k(1),k(2)) = \lim_{N\to\infty} N\rho\big[\big(c^{k(1)}_{t/N}+c^{k(1)}_{[t/N,(t+s)/N)}\big)\big(c^{k(2)}_{t/N}+c^{k(2)}_{[t/N,(t+s)/N)}\big)\big]$$

$$= Q_t(k(1),k(2)) + \lim_{n\to\infty} N\left[\rho\big(c^{k(1)}_{t/N}\big)\rho\big(c^{k(2)}_{s/N}\big)+\rho\big(c^{k(1)}_{s/N}\big)\rho\big(c^{k(2)}_{t/N}\big)\right] + Q_s(k(1),k(2))$$

$$= Q_t(k(1),k(2)) + Q_s(k(1),k(2)). \qquad \square$$

The special examples of free Levy process that we are interested in are the free analogues of Brownian motion and Poisson process.

Definition 3.4. *Let $(\mathcal{C},\rho,(c_I)_{I\in\mathcal{R}})$ be a 1-dimensional free Levy process.*

1. *The triple $(\mathcal{C},\rho,(c_I)_{I\in\mathcal{R}})$ is called a free Brownian motion if for every $I\in\mathcal{R}$ with $\lambda(I)>0$, the element $c_I\in\mathcal{C}$ is a semicircular variable of variance $\lambda(I)$.*

2. *The triple $(\mathcal{C},\rho,(c_I)_{I\in\mathcal{R}})$ is called a free Poisson process if for every $I\in\mathcal{R}$ with $\lambda(I)>0$, the element $c_I\in\mathcal{C}$ is a free Poisson variable with parameter $\lambda(I)$.*

Remark 3.5. Let $(\mathcal{C},\rho,(c_I)_{I\in\mathcal{R}})$ be a free Brownian motion. For $0<s<t$ we have

$$\rho(c_t\, c_s) = \rho(c_s\, c_s) + \rho(c_{[s,t)})\rho(c_s) = s$$

and

$$\rho(c_s\, c_t) = \rho(c_s\, c_s) + \rho(c_s)\rho(c_{[s,t)}) = s.$$

Therefore, similarly to the classical case, for $s,t>0$ the "covariance" of c_s and c_t is

$$\rho(c_t\, c_s) = s\wedge t = \rho(c_s\, c_t),$$

where $s\wedge t$ stands for the minimum of s,t .

We recall, that for a classical Brownian motion $(\Omega,\mathcal{F},\mathbb{P},(X_t)_{t\geq 0})$, we can compute it's moments $\mathbb{E}[X_{t_1}\dots X_{t_n}]$ explicitly, using it's Gaussian structure. More precisely, from Wick's formula we have that for every $t_1,\dots,t_n\geq 0$,

$$\mathbb{E}[X_{t_1}\dots X_{t_n}] = \begin{cases} 0, & \text{for } n \text{ odd} \\ \sum\limits_{\pi\in P_2(1,\dots,n)} \prod\limits_{\{i,j\}\in\pi} t_i\wedge t_j, & \text{for } n \text{ even.} \end{cases}$$

A similar formula holds for the moments $\rho(c_{t_1}\dots c_{t_n})$ of free Brownian motion except that partitions have to be non-crossing.

Theorem 3.6. (free Wick formula) *Let $(\mathcal{C}, \rho, (c_I)_{I \in \mathcal{R}})$ be a free Brownian motion. For every $n \in \mathbb{N}$, and $t_1, \ldots, t_n \geq 0$, we have*

$$\rho(c_{t_1} \ldots c_{t_n}) = \begin{cases} 0, & \text{for } n \text{ odd} \\ \sum_{\pi \in \mathrm{NC}_2(1,\ldots,n)} \prod_{\{i,j\} \in \pi} t_i \wedge t_j, & \text{for } n \text{ even.} \end{cases} \tag{3.3}$$

Proof. Let n be a positive integer, let $t_1, \ldots, t_n > 0$ and consider $0 < s_1 < \cdots < s_m$, such that $\{t_1, \ldots, t_n\} = \{s_1, \ldots, s_m\}$. We will compute the moments $\rho(c_{t_1} \ldots c_{t_n})$ using the relation $\rho = \kappa * \zeta$, where $\kappa = (\kappa_\pi)_{\pi \in \mathrm{NC}}$ are the free cumulants of $(\mathcal{C}, \rho)$. We have that

$$\rho(c_{t_1} \ldots c_{t_n}) = \sum_{\pi \in \mathrm{NC}(1,\ldots,n)} \kappa_\pi(c_{t_1}, \ldots, c_{t_n}).$$

Since the map $I \mapsto c_I$ is finitely additive, it follows that the non-commutative random variables $c_{s_1}, \ldots, c_{s_m}$ can be written in the form

$$c_{s_1} = X_1, \quad c_{s_2} = X_1 + X_2, \quad \cdots \quad, c_{s_m} = X_1 + X_2 + \cdots + X_m, \tag{3.4}$$

where $X_1, \ldots, X_m \in \mathcal{C}$, are free semicircular variables. Let $\pi \in \mathrm{NC}(1, \ldots, n)$, and assume that there exists $V \in \pi$, such that $\# V = 1$. Using that $(\kappa_\pi)_{\pi \in \mathrm{NC}}$ is a multiplicative family of functionals and $\kappa_1(c_{t_i}) = \rho(c_{t_i}) = 0$ for every $i = 1, \ldots, n$, we deduce that $\kappa_\pi(c_{t_1}, \ldots, c_{t_n}) = 0$. Now, let $\pi \in \mathrm{NC}(1, \ldots, n)$ such that there exist $V \in \pi$ with $\# V \geq 3$. Using the equalities stated in (3.4) and the fact that the cumulants are multilinear, we see that $\kappa_{\# V}(c_{t_1}, \ldots, c_{t_n} \mid V)$ can be written as a sum of free cumulants $\kappa_{\# V}(X_{i(1)}, \ldots, X_{i(\# V)})$, where $i(1), \ldots, i(\# V) \in \{1, \ldots, m\}$. We claim that $\kappa_l(X_{i(1)}, \ldots, X_{i(l)}) = 0$ for all $l \geq 3$ and all $i(1), \ldots, i(l) \in \{1, \ldots, m\}$. Indeed, let $l \geq 3$ and $i(1), \ldots, i(l) \in \{1, \ldots, m\}$. If there exist $a, b \in \{1, \ldots, l\}$ such that $i(a) \neq i(b)$, then it follows that $\kappa_l(X_{i(1)}, \ldots, X_{i(l)}) = 0$, because $l \geq 3$ and the random variables $X_{i(a)}, X_{i(b)}$ are freely independent. On the other hand, if $i(1) = i(2) = \cdots = i(l)$, then we have,

$$\begin{aligned} \kappa_l(X_{i(1)}, \ldots, X_{i(l)}) &= \kappa_l(X_{i(1)}, \ldots, X_{i(1)}) \\ &= [\rho(X_{i(1)}^2)]^{l/2} \kappa_l\left(\frac{X_{i(1)}}{\sqrt{\rho(X_{i(1)}^2)}}, \ldots, \frac{X_{i(1)}}{\sqrt{\rho(X_{i(1)}^2)}} \right) \\ &= 0, \end{aligned}$$

because the element $[\rho(X_{i(1)}^2)]^{-1/2} X_{i(1)} \in \mathcal{C}$ is a standard semicircular variable. Therefore the claim holds.

From the above, we deduce that $\kappa_\pi(c_{t_1}, \ldots, c_{t_n}) \neq 0$, only if $\pi \in \mathrm{NC}_2(1, \ldots, n)$. Then, we have $\rho(c_{t_1} \ldots c_{t_n}) = 0$ for n odd and

$$\begin{aligned} \rho(c_{t_1} \ldots c_{t_n}) &= \sum_{\pi \in \mathrm{NC}_2(1,\ldots,n)} \kappa_\pi(c_{t_1}, \ldots, c_{t_n}) \\ &= \sum_{\pi \in \mathrm{NC}_2(1,\ldots,n)} \prod_{\{i,j\} \in \pi} \kappa_2(c_{t_i}, c_{t_j}) \\ &= \sum_{\pi \in \mathrm{NC}_2(1,\ldots,n)} \prod_{\{i,j\} \in \pi} t_i \wedge t_j, \qquad \text{for } n \text{ even.} \end{aligned}$$

Hence, the relation (3.3) has been proven. □

3.2 Free Levy Processes on the Full Fock Space

In this section, we realize free Brownian motion and free Poisson process as processes on the full Fock space of $L^2(\mathbb{R})$. In that case, our C^*-algebra $\mathcal{A}$ will be the algebra of bounded operators on the full Fock space of $L^2(\mathbb{R})$, and our state ρ will be the corresponding vacuum state.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. The full Fock space of H is the Hilbert space

$$\mathcal{F}(H) := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}$$

with scalar product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle = \delta_{n,m} \langle f_1, g_1 \rangle \cdots \langle f_n, g_n \rangle$$

$$\langle \Omega, f_1 \otimes \cdots \otimes f_n \rangle = 0$$

$$\langle \Omega, \Omega \rangle = 1,$$

where $n, m \in \mathbb{N}$, and $f_1, \ldots, f_n, g_1, \ldots, g_m \in H$. By $\Omega = (1, 0, 0, \ldots)$ we denote the vacuum and by $\mathcal{F}_{\mathrm{lin}}$ the set of finite linear combinations of product vectors.

On the C^*-algebra $B(\mathcal{F}(H))$ of bounded operators on $\mathcal{F}(H)$, we consider the vacuum state $\rho: B(\mathcal{F}(H)) \to \mathbb{C}$, which is given by

$$\rho(X) = \langle \Omega, X\Omega \rangle \quad \text{for every } X \in B(\mathcal{F}(H)).$$

For $f \in H$, we define the left annihilation operator $l^-(f)$ and the left creation operator $l^+(f)$ by

$$l^-(f)\Omega := 0,$$

$$l^-(f) f_1 \otimes \cdots \otimes f_n := \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n$$

and

$$l^+(f)\Omega := f,$$

$$l^+(f) := f \otimes f_1 \otimes \cdots \otimes f_n,$$

where $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in H$.

Furthermore, given $T \in B(H)$, we define the gauge operator $p(T)$ by

$$p(T)\Omega := 0$$

$$p(T) f_1 \otimes \cdots \otimes f_n := (T(f_1)) \otimes f_2 \otimes \cdots \otimes f_n,$$

where $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in H$.

Creation operators, annihilation operators and gauge operators are extended by linearity to $\mathcal{F}_{\mathrm{lin}}$. Therefore, the above operators have been defined on the dense subset $\mathcal{F}_{\mathrm{lin}}$ of the full Fock space $\mathcal{F}(H)$. However, the operators $l^-(f), l^+(f), p(T)$ can be extended on the full Fock space.

Proposition 3.7. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then, for $f \in H$ and $T \in B(H)$, we have that the operators $l^+(f), l^-(f), p(T)$ are bounded and have the norms*

$$\|l^+(f)\| = \|l^-(f)\| = \|f\|,$$

$$\|p(T)\| = \|T\|.$$

Also, we have $(l^+(f))^* = l^-(f)$, which means that the operators $l^-(f)$ and $l^+(f)$ are mutually adjoint.

Proof. Let $f \in H$ and $\eta, \xi \in \mathcal{F}_{\text{lin}}$. Then by the definition of $l^+(f)$, $l^-(f)$, it is easy to note that

$$l^-(f)l^+(f)\eta = \langle f, f\rangle \eta \qquad \text{and} \qquad \langle l^+(f)\xi, \eta\rangle = \langle \xi, l^-(f)\eta\rangle.$$

Therefore, we have that

$$\langle l^+(f)\eta, l^+(f)\eta\rangle = \langle \eta, l^-(f)l^+(f)\eta\rangle = \langle f, f\rangle\langle \eta, \eta\rangle$$

and we deduce that the operator $l^+(f)$ is bounded, with norm $\|l^+(f)\| = \|f\|$. Since $\langle l^+(f)\xi, \eta\rangle = \langle \xi, l^-(f)\eta\rangle$ for every $\xi, \eta \in \mathcal{F}_{\text{lin}}$, we deduce that $l^-(f) = (l^+(f))^*$ and consequently we have $\|l^-(f)\| = \|l^+(f)\| = \|f\|$.

It remains to show that $\|p(T)\| = \|T\|$. Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in H$. Then, we consider the $n \times n$ Hermitian matrices $A = (a_{i,j}) \in M_n(\mathbb{C})$ and $B = (b_{i,j}) \in M_n(\mathbb{C})$, with entries

$$a_{i,j} = \langle T(x_i), T(x_j)\rangle \quad \text{and} \quad b_{i,j} = \langle x_i, x_j\rangle, \quad \text{for every } i, j \in \{1, \dots, n\}.$$

Then, for $\lambda \in \mathbb{C}^{n\times 1}$, we have

$$\begin{aligned} \lambda^* A \lambda &= \sum_{i,j=1}^{n} \bar{\lambda_i}\lambda_j \langle T(x_i), T(x_j)\rangle \\ &= \left\langle T\left(\sum_{i=1}^{n} \lambda_i x_i\right), T\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right\rangle \\ &\leq \|T\|^2 \left\langle \sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i x_i \right\rangle. \end{aligned}$$

Since

$$\lambda^* B \lambda = \left\langle \sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i x_i \right\rangle,$$

we deduce that $\lambda^*(\|T\|^2 B - A)\lambda \geq 0$, which implies that $\|T\|^2 B - A \geq 0$. Therefore, there exists a matrix $C \in M_n(\mathbb{C})$, such that $\|T\|^2 B - A = CC^*$. Without loss of generality, we assume that $\|T\| = 1$. Let $d \in \mathbb{N}$ and $x_i^j \in H$, where $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, d\}$. From the above (if we identify x_i with x_1^i) we have that there exist $c_{i,j} \in \mathbb{C}$, such that for every $i, j \in \{1, \dots, n\}$,

$$\langle x_1^i, x_1^j\rangle = \langle T(x_1^i), T(x_1^j)\rangle + \sum_{k=1}^{n} c_{i,k}\overline{c_{j,k}}\,. \tag{3.5}$$

Then, multiplying both sides of the relation (3.5) by $\langle x_2^i \otimes \cdots \otimes x_n^i, x_2^j \otimes \cdots \otimes x_n^j \rangle$, we have

$$\langle x_1^i \otimes \cdots \otimes x_n^i, x_1^j \otimes \cdots \otimes x_n^j \rangle =$$

$$= \langle T(x_1^i) \otimes x_2^i \otimes \cdots \otimes x_n^i, T(x_1^j) \otimes x_2^j \otimes \cdots \otimes x_n^j \rangle + \sum_{k=1}^{n} c_{i,k} \overline{c_{j,k}} \langle x_2^i \otimes \cdots \otimes x_n^i, x_2^j \otimes \cdots \otimes x_n^j \rangle.$$

Therefore, it follows that

$$\left\| \sum_{i=1}^{d} x_1^i \otimes \cdots \otimes x_n^i \right\|^2 = \sum_{i,j=1}^{d} \langle x_1^i \otimes \cdots \otimes x_n^i, x_1^j \otimes \cdots \otimes x_n^j \rangle$$

$$= \left\| \sum_{i=1}^{d} T(x_1^i) \otimes x_2^i \otimes \cdots \otimes x_n^i \right\|^2 + \sum_{k=1}^{n} \left\| \sum_{i=1}^{d} (\overline{c_{i,k}}\, x_2^i) \otimes x_3^i \otimes \cdots \otimes x_n^i \right\|^2.$$

As a consequence, we deduce that

$$\left\| p(T) \left(\sum_{i=1}^{d} x_1^i \otimes \cdots \otimes x_n^i \right) \right\| \le \left\| \sum_{i=1}^{d} x_1^i \otimes \cdots \otimes x_n^i \right\|,$$

or equivalently $\|p(T)\| \le 1$. Since,

$$1 = \|T\| = \sup_{\|f\|=1} \|T(f)\| = \sup_{\|f\|=1} \|p(T) f\| \le \|p(T)\|,$$

we have that $\|p(T)\| = 1$.

For $T \in B(H)$ arbitrary, since $p(T) = \|T\|\, p(\|T\|^{-1} T)$, we deduce that $p(T) = \|T\|$. □

In addition, for $\lambda, \mu \in \mathbb{C}$, $f, g \in H$, and $T, T_1, T_2 \in B(H)$, we have the following properties:

$$l^+(\lambda f + \mu g) = \lambda\, l^+(f) + \mu\, l^+(g) \tag{3.6}$$

$$l^-(f) l^+(g) = \langle f, g \rangle 1 \tag{3.7}$$

$$(l^-(f))^* = l^+(f) \tag{3.8}$$

$$p(T^*) = p(T)^*$$

$$p(\lambda T_1 + \mu T_2) = \lambda\, p(T_1) + \mu\, p(T_2)$$

$$p(T_1 T_2) = p(T_1)\, p(T_2)$$

$$l^-(f)\, p(T) = l^-(T^* f)$$

$$p(T)\, l^+(f) = l^+(Tf)$$

$$l^-(f) p(T) l^+(g) = \langle f, T(g) \rangle 1.$$

As we already mentioned, for $f,g\in H$, the operator $l^-(f)l^+(g)$ is trivial, due to the relation (3.7). On the other hand, the operator $l^+(f)l^-(g)$ can be expressed as a gauge operator. More precisely, if we define $T_{f,g}\in B(H)$ by demanding $T_{f,g}(h):=\langle g,h\rangle f$ for every $h\in H$, then $l^+(f)l^-(g)=p(T_{f,g})$.

For $f\in H$, taking into account the relations (3.7) and (3.8), we see that the unital $*$-subalgebra of $B(\mathcal{F}(H))$ generated by $l^+(f)$ is equal to $\operatorname{span}\{(l^+(f))^m(l^-(f))^n\,|\,m,n\geq 0\}$. Furthermore, by the definition of $l^+(f)$ and $l^-(f)$, we have that

$$\rho((l^+(f))^m(l^-(f))^n)=\langle\Omega,(l^+(f))^m(l^-(f))^n\Omega\rangle=\begin{cases}1, & m=n=0\\ 0, & \text{otherwise}.\end{cases}$$

Therefore, we see that if the elements $\{(l^+(f))^m(l^-(f))^n\,|\,m,n\geq 0\}$ are linearly independent and $\langle f,f\rangle=1$, then the triple $(l^+(f),\operatorname{span}\{(l^+(f))^m(l^-(f))^n|m,n\geq 0\},\rho)$ can be identified with the triple $(a,\mathcal{A},\phi)$ that we studied in subsection 1.5. In that case we have that the operator $l^+(f)+l^-(f)\in B(\mathcal{F}(H))$ is a standard semicircular variable with respect to ρ.

Proposition 3.8. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space and let f be a non-zero vector of H. Then, the operator $l^+(f)+l^-(f)\in B(\mathcal{F}(H))$ is a semicircular variable of variance $\|f\|^2$, with respect to ρ.*

Proof. First, we consider $f\in H$, with $\|f\|=1$. As we mentioned above, the assertion holds if the elements $\{(l^+(f))^m(l^-(f))^n|\ m,n\geq 0\}$ are linearly independent. Let $T\in\operatorname{span}\{(l^+(f))^m(l^-(f))^n\,|\,m,n\geq 0\}$, such that $T=0$. Then, the operator T can be written as a finite sum of operators $a\,(l^+(f))^m(l^-(f))^n$, where $a\in\mathbb{C}$ and $m,n\geq 0$. We consider $a_1(l^+(f))^{m_1}(l^-(f))^n,\dots,a_k(l^+(f))^{m_k}(l^-(f))^n$, to be the summands of T such that n is minimal and $m_i\neq m_j$ for all $i\neq j$. Then we have,

$$\begin{aligned}a_1(l^+(f))^{m_1}(l^-(f))^nf^{\otimes n}+\cdots+a_k(l^+(f))^{m_k}(l^-(f))^nf^{\otimes n}\\ =a_1\langle f,f\rangle^nf^{\otimes m_1}+\cdots+a_k\langle f,f\rangle^nf^{\otimes m_k}\end{aligned}$$

and $(l^+(f))^M(l^-(f))^Nf^{\otimes n}=0$ for all $M\in\mathbb{N}$ and $N>n$.

Therefore, the fact $Tf^{\otimes n}=0$ implies that,

$$a_1f^{\otimes m_1}+\cdots+a_kf^{\otimes m_k}=0$$

and because the vectors $f^{\otimes m_1},\dots,f^{\otimes m_k}\in\mathcal{F}(H)$ are linearly independent, we deduce that $a_1=a_2=\cdots=a_k=0$.

Continuing in this way, we see that if $a\,(l^+(f))^m(l^-(f))^n$ is a summand of T, then $a=0$. Therefore, the assertion holds for $f\in H$ with $\|f\|=1$.

For a non-zero vector $f\in H$, since the operator $l^+(\|f\|^{-1}f)+l^-(\|f\|^{-1}f)$ is a standard semicircular variable, we have that the operator $l^+(f)+l^-(f)=\|f\|(l^-(\|f\|^{-1}f)+l^+(\|f\|^{-1}f))$ is a semicircular variable of variance $\|f\|^2$. □

As we saw in Example 2.13 semicircular variables and free Poisson variables can be characterized by their free cumulants. The free cumulants of creation, annihilation and gauge operators on a full Fock space, can be computed explicitly.

Proposition 3.9. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and consider the C^*-probability space $(B(F(H)), \rho)$. Then, for $f, g \in H$ and $T \in B(H)$, the cumulants of the non-commutative random variables $l^+(f), l^-(g), p(T) \in B(F(H))$ are of the following form. We have*

$$\kappa_n(l^-(f), p(T_1), \ldots, p(T_{n-2}), l^+(g)) = \langle f, T_1 \ldots T_{n-2}\, g\rangle,$$

for all $n \geq 2$, $f, g \in H$ and $T_1, \ldots, T_{n-2} \in B(H)$. All the other free cumulants of the form $\kappa_n(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in \{l^+(f) \mid f \in H\} \cup \{l^-(g) \mid g \in H\} \cup \{p(T) \mid T \in B(H)\}$, are equal to zero.

Proof. See in [...] □

Remark 3.10. Let $f \in H$ with $\|f\| = 1$ and $\lambda > 0$. We consider the operator

$$x(f, \lambda) = l^+(f) l^-(f) + \sqrt{\lambda}(l^+(f) + l^-(f)) + \lambda \cdot 1 = p(T_{f,f}) + \sqrt{\lambda}(l^+(f) + l^-(f)) + \sqrt{\lambda} \cdot 1.$$

Note that $\kappa_1(x(f, \lambda)) = \rho(x(f, \lambda)) = \lambda$ and

$$\kappa_2(x(f, \lambda), x(f, \lambda)) = \lambda\, \rho(l^-(f) l^+(f)) = \lambda.$$

By Proposition 3.9 we also have

$$\begin{aligned}\kappa_n(x(f,\lambda), \ldots, x(f,\lambda)) &= \kappa_n(\sqrt{\lambda} l^-(f), p(T_{f,f}), \ldots, p(T_{f,f}), \sqrt{\lambda} l^+(f)) \\ &= \lambda\, \langle f, (T_{f,f})^{n-2} f\rangle \\ &= \lambda\, \langle f, f\rangle \\ &= \lambda.\end{aligned}$$

Therefore, we deduce that the operator $l^+(f) l^-(f) + \sqrt{\lambda}(l^+(f) + l^-(f)) + \lambda \cdot 1 \in B(F(H))$ is a free Poisson variable of parameter λ. Since $\phi_0(S^m (S^*)^n) = \rho((l^+(f))^m (l^-(f))^n)$ for all $m, n \geq 0$, we also have that the operator $S S^* + \sqrt{\lambda}(S + S^*) + \lambda \cdot 1 \in B(l^2(\mathbb{N} \cup \{0\}))$ is a free Poisson variable of parameter λ.

Notice that up to now we have not seen any example of freely independent subalgebras on some non-commutative probability space. As we will see in the next proposition, the creation and annihilation operators on the full Fock space are connected with the notion of free independence because the orthogonality of vectors translates into free independence of the corresponding creation and annihilation operators. Actually, we can even say more.

Proposition 3.11. *We consider the C^*-probability space $(B(\mathcal{F}(H)), \rho)$, where $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space. We also consider $H_1, \ldots, H_k$ to be linear subspaces of H and we assume that for every $i, j \in \{1, \ldots, k\}$ such that $i \neq j$, the subspaces $H_i, H_j \subseteq H$ are orthogonal. For every $i \in \{1, \ldots, k\}$, let $\mathcal{A}_i$ be the unital C^*-subalgebra of $B(\mathcal{F}(H))$ generated by the elements $\{l^-(f) \mid f \in H_i\} \cup \{p(T) \mid T \in B(H), T(H_i) \subseteq H_i$ and T vanishes on H_j for every $j \neq i\}$, then the C^*-subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are freely independent in $(B(\mathcal{F}(H)), \rho)$.*

Proof. For simplicity, we will only prove that the C^*-subalgebras generated by the elements $\{l^-(f) \mid f \in H_i\}$ are freely independent. For every $i = 1, \ldots, k$, let $\mathcal{A}_i$ be the C^*-subalgebra generated by $\{l^-(f) \mid f \in H_i\}$ and let $\mathcal{B}_i$ be the $*$-subalgebra generated by $\{l^-(f) \mid f \in H_i\}$. Note that since ρ is continuous, the subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are freely independent if and only if the subalgebras $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are freely independent. Therefore, we will show that $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are freely independent. Using the relations (3.6) and (3.7) we see that for every $i = 1, \ldots, k$, the elements of $\mathcal{B}_i$ can be written in the form:

$$T = a\,1 + \sum_{j=1}^{p} l^+(f_{j,1}) \ldots l^+(f_{j,n(j)}) l^-(g_{j,1}) \ldots l^-(g_{j,m(j)}),$$

where $a \in \mathbb{C}$, $p \in \mathbb{N}$ and for every $j \in \{1, \ldots, p\}$ we have $(m(j), n(j)) \neq (0,0)$ and $f_{j,1}, \ldots, f_{j,n(j)}, g_{j,1}, \ldots,, g_{j,m(j)} \in H_i$. In order to apply the definition of free independence, we need to know which elements $T \in \mathcal{B}_i$ have zero mean with respect to ρ, i.e. $\rho(T) = 0$. For $j = 1, \ldots, p$, using that $(m(j), n(j)) \neq (0,0)$, we have

$$\begin{aligned}\rho(l^+(f_{j,1}) \ldots l^+(f_{j,n(j)}) l^-(g_{j,1}) \ldots l^-(g_{j,m(j)})) \\ = \langle l^-(f_{j,n(j)}) \ldots l^-(f_{j,1})\Omega, l^-(g_{j,1}) \ldots l^-(g_{j,m(j)})\Omega \rangle = 0,\end{aligned}$$

because $l^-(f)\Omega = 0$, for every $f \in H$. Therefore, due to the linearity of ρ, we have $\rho(T) = 0$ if and only if $a = 0$. As a consequence, for every $i = 1, \ldots, k$, if we denote $\mathcal{B}_i^0 := \{T \in \mathcal{B}_i \mid \rho(T) = 0\}$, then we have

$$\mathcal{B}_i^0 = \operatorname{span}\{l^+(f_1) \ldots l^+(f_n) l^-(g_1) \ldots l^-(g_m) \mid (m,n) \neq (0,0) \text{ and } f_1, \ldots, f_n, g_1, \ldots, g_m \in H_i\}.$$

Now, we are ready to check (using the definition of free independence) if the $*$-subalgebras $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are freely independent. In order to do so, we consider $l \in \mathbb{N}$ and $k_1, \ldots, k_l \in \{1, \ldots, k\}$ such that $k_j \neq k_{j+1}$ for every $j \in \{1, \ldots, l-1\}$. We also consider $T_1, \ldots, T_l \in B(\mathcal{F}(H))$, such that $T_j \in \mathcal{B}_{k_j}^0$ for every $j = 1, \ldots, l$. Our goal is to show that $\rho(T_1 \ldots T_l) = 0$.

Using the linearity of ρ, for every $j = 1,..., l$ it suffices to consider operatots T_j of the form

$$T_j = l^+(f_{j,1}) \ldots l^+(f_{j,n(j)}) l^-(g_{j,1}) \ldots l^-(g_{j,m(j)}),$$

where $(m(j), n(j)) \neq (0,0)$ and $f_{j,1}, \ldots, f_{j,n(j)}, g_{j,1}, \ldots, g_{j,m(j)} \in H_{k_j}$. It is easy to note that in certain cases the orthogonality of $H_i, H_j \subseteq H$, for $i \neq j$, in combination with the relation (3.7) implies that $T_1 \ldots T_l = 0$. Indeed, we assume that there exists $j \in \{1, \ldots, l-1\}$ such that $m(j) \neq 0$ and $n(j+1) \neq 0$. By definition, the operator $T_1 \ldots T_l$ is a product of creation and annihilation operators. But, due to our assumption, the operator $l^-(g_{j,m(j)}) l^+(f_{j+1,1})$ is a factor of the product. Using that $g_{j,m(j)} \in H_{k_j}$, $f_{j+1,1} \in H_{k_{j+1}}$ and $k_j \neq k_{j+1}$, we have that $l^-(g_{j,m(j)}) l^+(f_{j+1,1}) = \langle g_{j,m(j)}, f_{j+1,1} \rangle 1 = 0$. Therefore, we deduce that $T_1 \ldots T_l = 0$, which implies that $\rho(T_1 \ldots T_l) = 0$.

On the other hand, let's assume that for every $j \in \{1,..., l-1\}$ we have $m(j)\, n(j+1) = 0$. By contradiction, we will show that $\rho(T_1 \ldots T_l) = 0$. If $\rho(T_1 \ldots T_l) \neq 0$, then since $l^+(f) = (l^-(f))^*$ and $l^-(f)\Omega = 0$ for every $f \in H$, we must have $n(1) = 0$. Then, because $(m(1), n(1)) \neq (0,0)$ and $m(1)\, n(2) = 0$, we must have $m(1) > 0$ and $n(2) = 0$. But, taking into account that $(m(2), n(2)) \neq (0,0)$ and $m(2)\, n(3) = 0$, we must have $m(2) > 0$ and $n(3) = 0$. Continuing in that way, we must have

$$m(j) > 0 \quad \text{and} \quad n(j) = 0, \quad \text{for every } j \in \{1, \ldots, l\}.$$

Then

$$T_1 \dots T_l = l^-(g_{1,1}) \dots l^-(g_{1,m(1)}) \dots l^-(g_{l,1}) \dots l^-(g_{l,m(l)})$$

and $\rho(T_1 \dots T_l) = 0$, which cannot be true by our assumption. Therefore, by contradiction we deduce that the claim holds. □

We recall that our goal is to realize free Brownian motion and free Poisson process as processes on the full Fock space of $L^2(\mathbb{R})$. From now on we consider $H = L^2(\mathbb{R})$. For $h \in L^\infty(\mathbb{R})$, we define the multiplication operator $T_h: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $T_h(f) := hf$, for every $f \in L^2(\mathbb{R})$. We also define $p(h) := p(T_h)$.

For every $I \in \mathcal{R}$, let us denote by $\mathbf{1}_I$ the indicator function of I. By Proposition 3.8 we have that the operator $l^+(\mathbf{1}_I) + l^-(\mathbf{1}_I)$ is a semicircular variable of variance $\lambda(I)$, for every $I \in \mathcal{R}$ with $\lambda(I) > 0$. Moreover, for $I \in \mathcal{R}$ let $\mathcal{C}_I$ be the unital C^*-algebra generated by $l^+(\mathbf{1}_I) + l^-(\mathbf{1}_I) \in B(F(L^2(\mathbb{R})))$. Then, by Proposition 3.11 we have that the C^*-subalgebras $\mathcal{C}_{I_1}, \dots, \mathcal{C}_{I_r}$ are freely independent in $(B(F(L^2(\mathbb{R}))), \rho)$ for all $r \in \mathbb{N}$ and disjoint $I_1, \dots, I_r \in \mathcal{R}$. Now we can summarize our result.

Theorem 3.12. *The triple $(B(\mathcal{F}(L^2(\mathbb{R}))), \rho, (l^+(\mathbf{1}_I) + l^-(\mathbf{1}_I))_{I \in \mathcal{R}})$ is a free Brownian motion.*

Now, we turn our attention to the realization of free Poisson process. For every $I \in \mathcal{R}$ we consider the self-adjoint operator $p(\mathbf{1}_I) + l^-(\mathbf{1}_I) + l^+(\mathbf{1}_I) + \lambda(I)\, 1 \in B(F(L^2(\mathbb{R})))$. It is easy to note that the distribution of this operator with respect to ρ depends only on $\lambda(I)$. Given $I_1, \dots, I_r \in \mathcal{R}$ disjoint, we consider the subspaces of $L^2(\mathbb{R})$

$$H_i = \{f \in L^2(\mathbb{R}) \mid f(x) = 0, \text{ for every } x \notin I_i\}, \quad \text{for every } i \in \{1, \dots, r\}.$$

Obviously, $\mathbf{1}_{I_i} \in H_i$ for every $i \in \{1, \dots, r\}$ and the subspaces $H_1, \dots, H_r$ are orthogonal. Then, for $i, j \in \{1, \dots, r\}$ such that $i \neq j$, by definition it is clear that $T_{\mathbf{1}_{I_i}}(H_i) \subseteq H_i$, and $T_{\mathbf{1}_{I_i}}(f) = 0$, for every $f \in H_j$. For $I \in \mathcal{R}$ let $\tilde{\mathcal{C}}_I$ be the unital C^*-algebra generated by $p(\mathbf{1}_I) + l^-(\mathbf{1}_I) + l^+(\mathbf{1}_I) + \lambda(I)\, 1 \in B(F(L^2(\mathbb{R})))$. Then, by Proposition 3.11 we have that the C^*-subalgebras $\tilde{\mathcal{C}}_{I_1}, \dots, \tilde{\mathcal{C}}_{I_r}$ are freely independent in $(B(F(L^2(\mathbb{R}))), \rho)$.

Theorem 3.13. *The triple $(B(F(L^2(\mathbb{R}))), \rho, (p(\mathbf{1}_I) + l^-(\mathbf{1}_I) + l^+(\mathbf{1}_I) + \lambda(I)\, 1)_{I \in \mathcal{R}})$ is a free Poisson process.*

Proof. Let $c_t := p(\mathbf{1}_{[0,t)}) + l^-(\mathbf{1}_{[0,t)}) + l^+(\mathbf{1}_{[0,t)}) + t\, 1$, where $t > 0$. In order to prove the assertion, it suffices to show that for every $r \in \mathbb{N}$ and $t > 0$, we have

$$\rho[(c_t)^r] = \sum_{p=1}^{r} \sum_{\{V_1, \dots, V_p\} \in \mathrm{NC}(1, \dots, r)} t^p.$$

Using that

$$c_t = \sum_{M=0}^{N-1} c_{I_M} \quad \text{where} \quad I_M = [Mt/N, (M+1)\, t/N),$$

we see c_t can be written as a sum S_N of Theorem 1.26, if we identify c_{I_M} with $a_{M,N}$. Therefore, by Theorem 1.26 the assertion holds, if for every $r \in \mathbb{N}$ we have

$$\lim_{N\to\infty} N\rho[(c_{t/N})^r] = t. \tag{3.9}$$

Because $\rho(c_{t/N}) = t/N$ and $\rho(c_{t/N}c_{t/N}) = (t/N)^2 + t/N$, we see that the relation (3.9) is satisfied for $r = 1, 2$. For $r \geq 3$, we have that $\rho[(c_{t/N})^r]$ can be written as a sum of joint moments of the operators $l^+(\mathbf{1}_{[0,t/N)})$, $l^-(\mathbf{1}_{[0,t/N)})$, $p(\mathbf{1}_{[0,t/N)})$ and $\frac{t}{N}1$. Using that $T_{\mathbf{1}_I}^2 = T_{\mathbf{1}_I}$ for every $I \in \mathcal{R}$, we see that

$$\rho[l^-(\mathbf{1}_{[0,t/N)})[p(\mathbf{1}_{[0,t/N)})]^{r-2}l^+(\mathbf{1}_{[0,t/N)})] = \rho[l^-(\mathbf{1}_{[0,t/N)})p(\mathbf{1}_{[0,t/N)})l^+(\mathbf{1}_{[0,t/N)})] = t/N.$$

Moreover, it is easy to note that the other summands are equal to 0 or $(t/N)^n$, for some $n \geq 2$. For these reasons, the claim holds. □

4 The generalized Brownian motion

In this section we present an example of a non-commutative stochastic process which provides an interpolation between fermionic, free and bosonic Brownian motion. This non-commutative process gives an example of a generalized Brownian motion. The notion of generalized Brownian motion was introduced by Bożejko and Speicher around 1990, who established the existence of such a non-commutative stochastic process.

4.1 Motivation

In the previous sections we presented some of the basic results of free probability theory (free central limt theorem, free cumulants, free Levy processes) and we concentrated on their combinatorial structure. As we stressed out multiple times, our goal was to show that free probability is related to the lattice of non-crossing partitions of the finite set $\{1,\dots,n\}$ in the same way in which classical probability is related to the lattice of all partitions of that set. A concrete example that we examined was the Wick formula. More precisely for a classical Brownian motion $(\Omega,\mathcal{F},\mathbb{P},(W_t)_{t\geq 0})$, for every $t_1,\dots,t_{2n+1}>0$ it's moments are determined by the relations

$$\mathbb{E}(W_{t_1}\cdots W_{t_{2n}})=\sum_{\pi\in P_2(1,\dots,2n)}\prod_{\{i,j\}\in\pi}t_i\wedge t_j \quad\text{and}\quad \mathbb{E}(W_{t_1}\cdots W_{t_{2n+1}})=0. \tag{4.1}$$

On the other hand, for a free Brownian motion $(\mathcal{C},\rho,(c_I)_{I\in\mathcal{R}})$, for every $t_1,\dots,t_{2n+1}>0$ it's moments are determined by the relations

$$\rho(c_{[0,t_1)}\cdots c_{[0,t_{2n})})=\sum_{\pi\in \mathrm{NC}_2(1,\dots,2n)}\prod_{\{i,j\}\in\pi}t_i\wedge t_j \quad\text{and}\quad \rho(c_{[0,t_1)}\cdots c_{[0,t_{2n+1})})=0. \tag{4.2}$$

Using this analogy, we will present an example of a non-commutative stochastic process which depends on a parameter $\mu\in[-1,1]$ and gives an interpolation between classical and free Brownian motion, in the sense that in terms of it's moments, for $\mu=0$ the non-commutative stochastic process coincides with the free Brownian motion, while for $\mu\to 1$ we obtain the dynamics of the classical Brownian motion.

We want to introduce our framework for the construction of such a non-commutative stochastic process. In the previous section, we introduced non-commutative stochastic processes $(c_I)_{I\in\mathcal{R}}$, on C^*-probability spaces $(\mathcal{C},\rho)$. Now, for our purpose we have to replace the C^*-algebraic framework with a $*$-algebraic framework. The main reason is that we are interested in $(\mu=1)$ self-adjoint random variables a on some $*$-probability space $(\mathcal{C},\rho)$, where their non-commutative distribution is determined by a Gaussian measure γ on $\mathbb{R}$ (with mean 0). This means that for every $k\in\mathbb{N}$,

$$\rho(a^k)=\int_{\mathbb{R}}t^k\gamma(dt).$$

Therefore, if $\mathcal{C}$ is a C^*-algebra and ρ is a state, using that $|\rho(a^k)|\leq\|a^k\|\leq\|a\|^k$, we will have for every $k\in\mathbb{N}$

$$\int_{\mathbb{R}}t^k\gamma(dt)\leq\|a\|^k,$$

which is impossible for a Gaussian measure γ on $\mathbb{R}$ (with mean 0). Therefore, we are interested in (n-dimensional) non-commutative stochastic processes $(\mathcal{C}, \rho, ((c_I^1, c_I^{1*}),\ldots,(c_I^n, c_I^{n*}))_{I\in\mathcal{R}})$, where $\mathcal{C}$ is a unital $*$-algebra, ρ is a state and for every $I_1, I_2 \in \mathcal{R}$ disjoint we have

$$c^i_{I_1\cup I_2} = c^i_{I_1} + c^i_{I_2} \quad \text{for all } i = 1,\ldots,n.$$

Of course, the element c_I^{i*} denotes the adjoint of c_I^i. For $I\in\mathcal{R}$, we consider $\mathcal{C}_I\subseteq\mathcal{C}$ to be the unital $*$-subalgebra generated by $c_I^1,\ldots,c_I^n$.

Such an n-dimensional stochastic process will be a generalized Brownian motion if for $I_1,\ldots,I_r\in\mathcal{R}$ disjoint we have some notion of independence for the subalgebras $\mathcal{C}_{I_1},\ldots,\mathcal{C}_{I_r}$ (independent increments). We will use the following notion of independence which was given by Kümmerer [...]. This allows us to obtain a calculation rule for certain joint moments with respect to ρ. More precisely, we demand that pyramidally ordered products factorize, i.e.

$$\rho(a_1\ldots a_r\, b_r\ldots b_1) = \rho(a_1\, b_1)\ldots\rho(a_r\, b_r),$$

if $a_i, b_i\in\mathcal{C}_{I_i}$ and $I_1<\cdots<I_r$, where $J_1<J_2$ means that for all $t_1\in J_1$ and $t_2\in J_2$ we have $t_1<t_2$.

Example 4.1. Let $a\in\mathcal{C}_{I_1}$ and $b\in\mathcal{C}_{I_2}$ with $I_1<I_2$. Then, for the products $a\,a\,b\,b = a\,a\cdot 1\cdot b\,b\cdot 1$ and $a\,b\,b\,a\in\mathcal{C}$ we have

$$\rho(aabb) = \rho(abba) = \rho(aa)\rho(bb)\,.$$

Note that we do not have any rule in order to compute $\rho(abab)$.

We will now give the definition of generalized Brownian motion which is due to Bożejko and Speicher [...].

Definition 4.2. (Bożejko-Speicher 1991) *Let $(\mathcal{C},\rho)$ be a $*$-probability space and let $(c_I^1,\ldots,c_I^n)_{I\in\mathcal{R}}$ be a family of non-commutative random variables such that the map $\mathcal{R}\to\mathcal{C}^n$, $I\mapsto(c_I^1,\ldots,c_I^n)$ is finitely additive. Then, the triple $(\mathcal{C},\rho,((c_I^1,c_I^{1*}),\ldots,(c_I^n,c_I^{n*}))_{I\in\mathcal{R}})$ is called an n-dimensional generalized Brownian motion if,*

1. *pyramidally ordered moments factorize (independent increments).*
2. *If $\hat{c}$ stands for c or c^* and $I+t := \{s+t \mid s\in I\}$, then the moments $\rho\big(\hat{c}^{k(1)}_{I_1+t}\cdots\hat{c}^{k(r)}_{I_r+t}\big)$ are independent of $t\in\mathbb{R}$ for all $r\in\mathbb{N}$, $k(1),\ldots,k(r)\in\{1,\ldots,n\}$ and $I_1,\ldots,I_r\in\mathcal{R}$ (stationarity).*
3. *For every $r\in\mathbb{N}$, $k(1),\ldots,k(r)\in\{1,\ldots,n\}$ and $I\in\mathcal{R}$ we have*

$$\rho\big(\hat{c}_I^{k(1)}\cdots\hat{c}_I^{k(r)}\big) = \begin{cases} 0, & r \text{ odd} \\ \lambda(I)^{r/2}\rho\big(\hat{c}^{k(1)}_{[0,1]}\cdots\hat{c}^{k(r)}_{[0,1]}\big), & r \text{ even}\end{cases}$$

(Gaussianity of the distribution).

In the case that we have an 1-dimensional generalized Brownian motion $(\mathcal{C}, \rho, (c_I, c_I^*)_{I\in\mathcal{R}})$, in contrast with the classical or free case, we do not assume that c_I has a specific non-commutative distribution with respect to ρ. This is because our goal is to construct, for every $\mu \in [-1,1]$, an 1-dimensional generalized Brownian motion $(\mathcal{C}_\mu, \rho_\mu, (c_I, c_I^*)_{I\in\mathcal{R}})$ which gives an interpolation between classical and free Brownian motion, as far as moments are concerned. Therefore, for different $\mu \in [-1,1]$ the values $\rho_\mu\big(\hat{c}_I^{k(1)} \ldots \hat{c}_I^{k(r)}\big)$ should differ. In order to motivate the construction of the example that we will give in the next subsection, we recall a few facts about the non-commutative realization of the classical Brownian motion on the bosonic Fock space of $L^2(\mathbb{R})$.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. For $h_1, \ldots, h_n \in H$ we define the symmetric tensor product

$$h_1 \circ \cdots \circ h_n := \frac{1}{n!} \sum_{\sigma \in S_n} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(n)},$$

which is the orthogonal projection of $h_1 \otimes \cdots \otimes h_n \in H^{\otimes n}$ to the subspace of symmetric tensors.

The closed subspace of $H^{\otimes n}$ generated by $h_1 \circ \cdots \circ h_n$ is denoted by $H^{\circ n}$ and it is called the n-fold symmetric tensor product of H. Then, the bosonic Fock space of H is the Hilbert space

$$\mathcal{F}_s(H) := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} H^{\circ n}$$

with scalar product

$$\langle f_1 \circ \cdots \circ f_n, g_1 \circ \cdots \circ g_m \rangle_\circ = \delta_{n,m} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \langle f_i, g_{\sigma(i)} \rangle$$

$$\langle \Omega, f_1 \circ \cdots \circ f_n \rangle_\circ = 0$$

$$\langle \Omega, \Omega \rangle_\circ = 1,$$

where $n, m \in \mathbb{N}$ and $f_1, \ldots, f_n, g_1, \ldots, g_m \in H$.

For $f \in H$, we define the bosonic annihilation operator $a^-(f)$ and the bosonic creation operator $a^+(f)$ by

$$a^-(f)\Omega := 0,$$

$$a^-(f) f_1 \circ \cdots \circ f_n := \sum_{i=1}^{n} \langle f, f_i \rangle f_1 \circ \cdots \circ \check{f}_i \circ \cdots \circ f_n$$

and

$$a^+(f)\Omega := f,$$

$$a^+(f) f_1 \circ \cdots \circ f_n := f \circ f_1 \circ \cdots \circ f_n.$$

The symbol $\check{f}_i$ means that f_i has to be deleted in the product. The above operators extend to the set of finite linear combinations of symmetric product vectors. Moreover, it is easy to note that the operators $a^-(f)$ and $a^+(g)$ satisfy the C.C.R..

For $H = L^2(\mathbb{R})$, if we consider $a_t := a^-(\mathbf{1}_{[0,t)}) + a^+(\mathbf{1}_{[0,t)})$, then for $t_1, \ldots, t_n \geq 0$ the joint moments of $a_{t_1}, \ldots, a_{t_n}$ with respect to the vacuum $\rho_\circ(\cdot) := \langle \Omega, \cdot \Omega \rangle_\circ$ are given by the formula

$$\rho_\circ(a_{t_1} \ldots a_{t_n}) = \begin{cases} 0, & \text{for } n \text{ odd} \\ \sum_{\pi \in P_2(1,\ldots,n)} \prod_{\{i,j\} \in \pi} t_i \wedge t_j, & \text{for } n \text{ even.} \end{cases} \tag{4.3}$$

Therefore, in terms of it's moments, $(a_t)_{t \geq 0}$ is a non-commutative realization of the classical Brownian motion.

An anti-commuting analogue of Brownian motion is obtained by replacing the bosonic Fock space of $L^2(\mathbb{R})$ by the fermionic Fock space of $L^2(\mathbb{R})$. To be more precise, let H be a Hilbert space. For $f_1, \ldots, f_n \in H$ we define the antisymmetric tensor product

$$f_1 \wedge \cdots \wedge f_n := \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon_\sigma f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},$$

where ε_σ is the signature of the permutation $\sigma \in S_n$. The closed subspace of $H^{\otimes n}$ generated by $f_1 \wedge \cdots \wedge f_n$ is denoted by $H^{\wedge n}$ and it is called the n-fold antisymmetric tensor product of H. Then, the fermionic Fock space of H is the Hilbert space

$$\mathcal{F}_a(H) := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} H^{\wedge n}$$

with scalar product

$$\langle f_1 \wedge \cdots \wedge f_n, g_1 \wedge \cdots \wedge g_m \rangle_\wedge = \delta_{n,m} \det[(\langle f_i, g_j \rangle)_{1 \leq i,j \leq n}]$$

$$\langle \Omega, f_1 \wedge \cdots \wedge f_n \rangle_\wedge = 0$$

$$\langle \Omega, \Omega \rangle_\wedge = 1,$$

where $n, m \in \mathbb{N}$ and $f_1, \ldots, f_n, g_1, \ldots, g_m \in H$.

For $f \in H$, we define the fermionic annihilation operator $b^-(f)$ and the fermionic creation operator $b^+(f)$ by

$$b^-(f)\Omega := 0,$$

$$b^-(f) f_1 \wedge \cdots \wedge f_n := \sum_{i=1}^{n} (-1)^i \langle f, f_i \rangle f_1 \wedge \cdots \wedge \check{f}_i \wedge \cdots \wedge f_n$$

and

$$b^+(f)\Omega := f,$$

$$b^+(f) f_1 \wedge \cdots \wedge f_n := f \wedge f_1 \wedge \cdots \wedge f_n .$$

The operators $b^+(f)$ and $b^-(f)$ are bounded and they extend to the space $\mathcal{F}_a(H)$. Furthermore, it is easy to note that the operators $b^+(f)$ and $b^-(g)$ satisfy the C.A.R.. Therefore, for $H = L^2(\mathbb{R})$, the non-commutative process $(b^+(\mathbf{1}_{[0,t)}) + b^-(\mathbf{1}_{[0,t)}))_{t \geq 0}$ can be considered as an anti-commuting analogue of Brownian motion. Putting $b_t := b^+(\mathbf{1}_{[0,t)}) + b^-(\mathbf{1}_{[0,t)})$ for all $t \geq 0$, note that the joint moments of $b_{t_1}, \ldots, b_{t_n}$ with respect to the vacuum state $\rho_\wedge(\cdot) := \langle \Omega, \cdot \Omega \rangle_\wedge$ are not equal to the left hand side of (4.3). Thus, the connection with classical Brownian motion is now only formal.

In the context of non-commutative probability theory, except of free independence, there are other notions of stochastic independence, which lead to Brownian motions $(a_t)_{t\geq 0}$ (bosonic Brownian motion) and $(b_t)_{t\geq 0}$ (fermionic Brownian motion).

Definition 4.3. *Let $(\mathcal{A}, \phi)$ be $*$-probability space. A family of sub-$*$-algebras $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of the $*$-algebra $\mathcal{A}$ is independent in the sense of the Bose independence if the algebras $\mathcal{A}_k$ commute with each other (i.e. $a_k a_l = a_l a_k$ if $a_k \in \mathcal{A}_k, a_l \in \mathcal{A}_l$ and $k \neq l$) and*

$$\phi(a_1 \dots a_m) = \phi(a_1) \dots \phi(a_m),$$

whenever $a_i \in \mathcal{A}_{k_i}$ and $i \neq j$ implies $k_i \neq k_j$.

We omit the definition of Fermi independence [...]. As we saw in Proposition 3.11, the free independence appears in the full Fock space. Similarly, we can prove that the Bose independence appears in the bosonic Fock space and the Fermi independence appears in the fermionic Fock space.

In the following, we will construct generalized Brownian motions which are interpolations between fermionic, free and bosonic Brownian motion. We recall that the proof of (4.3) is based on the relations

$$\langle a^+(f)\eta, \xi\rangle_\circ = \langle \eta, a^-(f)\xi\rangle_\circ, \quad \text{for every } \xi, \eta \text{ in the domain of } a^+(f), a^-(f)$$

and

$$a^-(f)a^+(g) - a^+(g)a^-(f) = \langle f, g\rangle 1, \quad a^-(f)\Omega = 0.$$

On the other hand, for the creation and annihilation operators on the full Fock space of H, the relations (3.7) and (3.8) were crucial in order to prove that the self-adjoint operator $l^+(f) + l^-(f)$ ($f \in H$ with $f \neq 0$) is a semicircular variable. These observations will be our main motivation in order to define the μ-Fock space and the corresponding creation and annihilation operators.

4.2 The μ-Fock space

Our goal is to construct generalized Brownian motions which are interpolations between fermionic, free and bosonic Brownian motion. In this context, taking into account our observations in the previous subsection, our aim is to consider operators $c^-(f), c^+(g)$ and a vacuum vector Ω such that $c^-(f)\Omega = 0$ for every $f \in L^2(\mathbb{R})$ and

$$c^-(f)c^+(g) - \mu c^+(g)c^-(f) = \langle f, g\rangle 1 \quad \text{for all } f, g \in L^2(\mathbb{R}) \tag{4.4}$$

with $-1 \leq \mu \leq 1$. We refer to the relations (4.4) as generalized commutation relations.

Our main goal in this section is to show that there exist operators on some Hilbert space and a corresponding vacuum vector in this Hilbert space which fulfill the above relations. In this direction, let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and consider the full Fock space of H $\mathcal{F}(H)$. As before, by $\Omega = (1, 0, 0, \dots.)$ we denote the vacuum vector and by $\mathcal{F}_{\text{lin}}$ the set of finite linear combinations of product vectors. For each $f \in H$ we define the μ-creation operator $c^+(f)$ and the μ-annihilation operator $c^-(f)$ by

$$c^+(f)\Omega := f,$$

$$c^+(f) f_1 \otimes \cdots \otimes f_n := f \otimes f_1 \otimes \cdots \otimes f_n$$

and

$$c^-(f)\Omega := 0,$$

$$c^-(f) f_1 \otimes \cdots \otimes f_n := \sum_{k=1}^{n} \mu^{k-1} \langle f, f_k \rangle f_1 \otimes \cdots \otimes \check{f}_k \otimes \cdots \otimes f_n .$$

The operators $c^+(f), c^-(f)$ are extended by linearity to $\mathcal{F}_{\text{lin}}$. Note that for $\mu = 0$ we have that $c^+(f) = l^+(f)$ and $c^-(f) = l^-(f)$, for every $f \in H$.

Lemma 4.4. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. For every $f, g \in H$, the operators $c^-(f)$, $c^+(g)$ satisfy on $\mathcal{F}_{\text{lin}}$ the relation*

$$c^-(f) c^+(g) - \mu\, c^+(g) c^-(f) = \langle f, g \rangle 1. \tag{4.5}$$

Proof. For $f, g, f_1, \ldots, f_n \in H$, using the definition of $c^+(g)$ we have

$$c^-(f) c^+(g) f_1 \otimes \cdots \otimes f_n = c^-(f) g \otimes f_1 \otimes \cdots \otimes f_n.$$

But, by definition,

$$c^-(f) g \otimes f_1 \otimes \cdots \otimes f_n = \langle f, g \rangle f_1 \otimes \cdots \otimes f_n + \mu\, g \otimes [c^-(f) f_1 \otimes \cdots \otimes f_n].$$

Therefore, we deduce that

$$c^-(f) c^+(g) f_1 \otimes \cdots \otimes f_n = [\langle f, g \rangle 1 + \mu\, c^+(g) c^-(f)] f_1 \otimes \cdots \otimes f_n$$

and by the linearity of $c^-(f), c^+(g)$, the claim holds. □

Therefore, we have found operators on $\mathcal{F}_{\text{lin}}$ that satisfy the generalized commutation relations. It remains to find a suitable scalar product $\langle \cdot, \cdot \rangle_\mu$ such that for every $f \in H$ and $\xi, \eta \in \mathcal{F}_{\text{lin}}$ the relation $\langle c^+(f)\xi, \eta \rangle_\mu = \langle \xi, c^-(f)\eta \rangle_\mu$ will be satisfied.

We define the symmetric bilinear form $\langle \cdot, \cdot \rangle_\mu$ on $\mathcal{F}_{\text{lin}}$ which is determined by

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_\mu := 0 \quad \text{for } n \neq m$$

and otherwise recursively by

$$\begin{aligned} \langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_n \rangle_\mu &:= \langle f_2 \otimes \cdots \otimes f_n, c^-(f_1) g_1 \otimes \cdots \otimes g_n \rangle_\mu \\ &= \sum_{k=1}^{n} \mu^{k-1} \langle f_1, g_k \rangle \langle f_2 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes \check{g}_k \otimes \cdots \otimes g_n \rangle_\mu . \end{aligned}$$

Lemma 4.5. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. For every $f \in H$ and $\xi, \eta \in \mathcal{F}_{\text{lin}}$ we have*

$$\langle c^+(f)\xi, \eta \rangle_\mu = \langle \xi, c^-(f)\eta \rangle_\mu . \tag{4.6}$$

Proof. For $n, m \in \mathbb{N}$ and $f, f_1, \ldots, f_n, g_1, \ldots, g_m \in H$ by definition we have

$$\langle c^+(f) f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_\mu = \langle f_1 \otimes \cdots \otimes f_n, c^-(f) g_1 \otimes \cdots \otimes g_{n+1} \rangle_\mu,$$

if $m=n+1$, and otherwise

$$\langle c^+(f) f_1\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes g_m\rangle_\mu = 0 = \langle f_1\otimes\cdots\otimes f_n, c^-(f) g_1\otimes\cdots\otimes g_m\rangle_\mu.$$

Therefore, by linearity the assertion holds. □

As a consequence, we have for every $f_1,\ldots,f_n,g_1,\ldots,g_n\in H$,

$$\begin{aligned}\langle f_1\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes g_n\rangle_\mu &= \langle c^+(f_1)\ldots c^+(f_n)\Omega, g_1\otimes\cdots\otimes g_n\rangle_\mu\\ &= \langle \Omega, c^-(f_n)\ldots c^-(f_1) g_1\otimes\cdots\otimes g_n\rangle.\end{aligned}$$

In order to prove that the bilinear form $\langle\cdot,\cdot\rangle_\mu$ is a scalar product, the hardest part is to prove that it is positive definite. For this purpose, for $\mu\in[-1,1]$ we want to consider a map $P_\mu\colon\mathcal{F}_{\text{lin}}\to\mathcal{F}_{\text{lin}}$, such that $\langle\xi,\eta\rangle_\mu=\langle\xi,P_\mu\eta\rangle$ for evey $\xi,\eta\in\mathcal{F}_{\text{lin}}$, where we denote with $\langle\cdot,\cdot\rangle$ the usual scalar product on the full Fock space of H. In that case, we are interested in whether P_μ is a (strictly) positive operator. We consider

$$P_\mu=\bigoplus_{n=0}^{\infty} P_\mu^{(n)} \quad \text{with} \quad P_\mu^{(n)}\colon H^{\otimes n}\to H^{\otimes n}.$$

In order to define $P_\mu^{(n)}$, we consider for every $n\in\mathbb{N}$ and $\pi\in S_n$, the operator $U_\pi\colon H^{\otimes n}\to H^{\otimes n}$ such that

$$U_\pi f_1\otimes\cdots\otimes f_n := f_{\pi(1)}\otimes\cdots\otimes f_{\pi(n)} \quad \text{for all } f_1,\ldots,f_n\in H.$$

For $\pi\in S_n$, we define $i(\pi)$ to be the number of inversions of π, which means that

$$i(\pi):=\#\{(i,j)\in\{1,\ldots,n\}^2 \mid i<j \text{ and } \pi(i)>\pi(j)\}.$$

Then, we define

$$P_\mu^{(n)} := \sum_{\pi\in S_n}\mu^{i(\pi)}\,U_\pi.$$

Since $i(\pi^{-1})=i(\pi)$ and $(U_\pi)^*=U_{\pi^{-1}}$ for all $\pi\in S_n$, we have

$$\big(P_\mu^{(n)}\big)^* = \sum_{\pi\in S_n}\mu^{i(\pi)}\,U_{\pi^{-1}} = P_\mu^{(n)}, \quad \text{for all } n\in\mathbb{N}.$$

Lemma 4.6. *Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space. For all $\xi,\eta\in\mathcal{F}_{\text{lin}}$, we have*

$$\langle\xi,\eta\rangle_\mu=\langle\xi,P_\mu\eta\rangle.$$

Proof. By the definition of $\langle\cdot,\cdot\rangle_\mu$ and due to linearity, it suffices to show that for every $n\in\mathbb{N}$ and $f_1,\ldots,f_n,g_1,\ldots,g_n\in H$ we have

$$\begin{aligned}\langle f_1\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes g_n\rangle_\mu &= \big\langle f_1\otimes\cdots\otimes f_n, P_\mu^{(n)} g_1\otimes\cdots\otimes g_n\big\rangle\\ &= \sum_{\pi\in S_n}\mu^{i(\pi)}\langle f_1, g_{\pi(1)}\rangle\ldots\langle f_n, g_{\pi(n)}\rangle.\end{aligned}$$

This can be proved by induction on n. For $n=1$ we have nothing to prove and we assume that the claim holds for $n-1$. We consider $S_{n-1}^{(k)}$ to be the set of all bijections from $\{2,\ldots, n\}$ to $\{1,\ldots,\check{k},\ldots,n\}$. Then, for every $\pi\in S_n$ there exist unique $k\in\{1,\ldots,n\}$ and $\sigma\in S_{n-1}^{(k)}$ such that $\pi(1)=k$ and $\pi(l)=\sigma(l)$ for all $l=2,\ldots,n$. Similarly, we can define for $\sigma\in S_{n-1}^{(k)}$ the number of inversions, i.e. the number of $(i,j)\in\{2,\ldots,n\}^2$, such that $i<j$ and $\sigma(i)>\sigma(j)$. Therefore, for such $\pi\in S_n$, $k\in\mathbb{N}$ and $\sigma\in S_{n-1}^{(k)}$, we have

$$\begin{aligned} i(\pi) &= i(\sigma)+\#\{j=2,\ldots,n\,|\,\pi(1)>\pi(j)\} \\ &= i(\sigma)+\#\{j=2,\ldots,n\,|\,k>\pi(j)\} \\ &= i(\sigma)+k-1. \end{aligned}$$

By the definition of $\langle\cdot,\cdot\rangle_\mu$ we have,

$$\langle f_1\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes g_n\rangle_\mu=\sum_{k=1}^{n}\mu^{k-1}\langle f_1,g_k\rangle\langle f_2\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes\check{g}_k\otimes\cdots\otimes g_n\rangle_\mu$$

and for $k=1,\ldots,n$ our induction hypothesis guarantees that

$$\langle f_2\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes\check{g}_k\otimes\cdots\otimes g_n\rangle_\mu=\sum_{\sigma\in S_{n-1}^{(k)}}\mu^{i(\sigma)}\langle f_2,g_{\sigma(2)}\rangle\ldots\langle f_n,g_{\sigma(n)}\rangle.$$

Therefore, taking into acount that

$$i(\pi)=i(\sigma)+k-1 \quad\text{and}\quad \sum_{\pi\in S_n}=\sum_{k=1}^{n}\sum_{\sigma\in S_{n-1}^{(k)}},$$

we deduce that

$$\langle f_1\otimes\cdots\otimes f_n, g_1\otimes\cdots\otimes g_n\rangle_\mu=\sum_{\pi\in S_n}\mu^{i(\pi)}\langle f_1,g_{\pi(1)}\rangle\cdots\langle f_n,g_{\pi(n)}\rangle. \qquad\square$$

Remark 4.7. It is obvious that for every $n\in\mathbb{N}$ the operator $P_\mu^{(n)}\colon H^{\otimes n}\to H^{\otimes n}$ is bounded (as a finite sum of bounded operators). On the other hand, for $f\in H$ such that $\langle f,f\rangle=1$, we have that

$$\langle f^{\otimes n}, f^{\otimes n}\rangle_\mu=(1+\mu+\cdots+\mu^{n-1})\langle f^{\otimes(n-1)}, f^{\otimes(n-1)}\rangle_\mu.$$

Therefore, for every $n\in\mathbb{N}$, we deduce that

$$\langle f^{\otimes n}, P_\mu f^{\otimes n}\rangle=\langle f^{\otimes n}, f^{\otimes n}\rangle_\mu=\prod_{i=0}^{n-1}(1+\mu+\cdots+\mu^i)$$

and the operator P_μ is unbounded for $\mu>0$.

4.3 Positive definite kernels

Before we prove that the map P_μ is positive, we pause in order to introduce the notion of positive definite kernel which appears in (non-commutative) probability theory [...] and other areas of Mathematics. A positive definite kernel is a generalization of a positive definite function.

Definition 4.8. *Let X be an arbitrary set. A complex valued function $K: X \times X \to \mathbb{C}$ is called a Hermitian kernel if $K(x,y) = \overline{K(y,x)}$ for all $x, y \in X$. A Hermitian kernel K is called positive definite kernel on X if, for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $a_1, \ldots, a_n \in \mathbb{C}$, we have*

$$\sum_{i,j=1}^{n} a_i \bar{a_j} K(x_i, x_j) \geq 0. \tag{4.7}$$

A Hermitian kernel K is said to be strictly positive definite if for every $n \in \mathbb{N}, x_1, \ldots, x_n \in X$ and $a_1, \ldots, a_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} a_i \bar{a_j} K(x_i, x_j) = 0 \quad \textit{if and only if} \quad a_1 = \cdots = a_n = 0.$$

Remark 4.9. Let X be any set and let $\{K_m\}_{m \in \mathbb{N}}$ be a family of positive definite kernels on X. Given $a_1, \ldots, a_n \geq 0$, the map $\sum_{i=1}^{n} a_i K_i$ is a positive definite kernel on X. Moreover, if the sequence $\{K_m\}_{m \in \mathbb{N}}$ converges pointwise, we will also have that $\lim_{m \to \infty} K_m$ is a positive definite kernel on X.

Example 4.10. Let X be any set, let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and consider a map $\nu: X \to H$. Then, we define the map $K: X \times X \to \mathbb{C}$ by requiring, for all $(x,y) \in X \times X$, $K(x,y) := \langle \nu(x), \nu(y) \rangle$. Note that K is a positive definite kernel on X since, for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $a_1, \ldots, a_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{n} a_i \bar{a_j} K(x_i, x_j) = \left\| \sum_{i=1}^{n} a_i \nu(x_i) \right\|^2 \geq 0.$$

Conversely, we have that every positive definite kernel can be written in this form.

Theorem 4.11. *Let X be an arbitrary set and let $K: X \times X \to \mathbb{C}$ be a positive definite kernel on X. Then, there exists a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and a map $\nu: X \to H$ such that, for every $x, y \in X$,*

$$K(x,y) = \langle \nu(x), \nu(y) \rangle.$$

Proof. Let n be a positive integer and let $x_1, \ldots, x_n \in X$. Since K is a positive definite kernel on X, by definition we have that the Hermitian matrix $(K(x_i, x_j))_{i,j=1,\ldots,n}$ is positive definite. Therefore, there exists a Gaussian measure $\mu_{x_1,\ldots,x_n}$ on $\mathbb{C}^n$ with zero mean and covariance matrix $(K(x_i, x_j))_{i,j=1,\ldots,n}$. It is clear that the family of probability measures $\mu_{x_1,\ldots,x_n}$ is consistent. We consider $\mathcal{X}$ to be the set of all functions from X to $\mathbb{C}$ and $\mathfrak{A}$ to be the σ-algebra generated by the projections $\Pi_y: \mathcal{X} \to \mathbb{C}$, $f \mapsto f(y)$. Then, by Kolmogorov's extension theorem we have that there exists a probability measure μ on $(\mathcal{X}, \mathfrak{A})$ such that, for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $A \subseteq \mathbb{C}^n$ Borel subset we have

$$\mu(\{f \in \mathcal{X} \mid (f(x_1), \ldots, f(x_n)) \in A\}) = \mu_{x_1,\ldots,x_n}(A).$$

We consider the Hilbert space $H = L^2(\mu)$ and for every $x \in X$ we define $\nu(x): \mathcal{X} \to \mathbb{C}$, by demanding, for all $f \in \mathcal{X}$, $\nu(x)(f) := f(x)$. Then $\nu(x) \in L^2(\mu)$ for every $x \in X$, because

$$\int_{\mathcal{X}} |f(x)|^2 \mu(df) = \int_{\mathbb{C}} |z|^2 \mu_x(dz) = K(x,x).$$

Therefore, for $x, y \in X$ we deduce that

$$\langle \nu(x), \nu(y) \rangle_{L^2(\mu)} = \int_{\mathcal{X}} f(x)\overline{f(y)}\mu(df) = \int_{\mathbb{C}\times\mathbb{C}} z\,\bar{w}\mu_{x,y}(dzdw) = K(x,y)\,. \qquad \square$$

Corollary 4.12. *Let X be an arbitrary set and let K, L be two complex valued functions on $X \times X$. If the maps $K, L: X \times X \to \mathbb{C}$ are (strictly) positive definite kernels on X, then the pointwise product $K \cdot L: X \times X \to \mathbb{C}$ is a (strictly) positive definite kernel on X.*

Proof. First, we assume that the maps $K, L: X \times X \to \mathbb{C}$ are positive definite kernels on X. By the previous theorem there exist Hilbert spaces $(H_1, \langle\cdot,\cdot\rangle_1)$, $(H_2, \langle\cdot,\cdot\rangle_2)$ and maps $\nu_1: X \to H_1$, $\nu_2: X \to H_2$ such that for every $x, y \in X$ the relations

$$K(x,y) = \langle \nu_1(x), \nu_1(y)\rangle_1 \quad \text{and} \quad L(x,y) = \langle \nu_2(x), \nu_2(y)\rangle_2,$$

hold. Then, for $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in X$ we have

$$\begin{aligned} \sum_{i,j=1}^{n} a_i \bar{a}_j (K \cdot L)(x_i, x_j) &= \sum_{i,j=1}^{n} a_i \bar{a}_j \langle \nu_1(x_i), \nu_1(x_j)\rangle_1 \langle \nu_2(x_i), \nu_2(x_j)\rangle_2 \\ &= \left\| \sum_{i=1}^{n} a_i\, \nu_1(x_i) \otimes \nu_2(x_i) \right\|^2 \\ &\geq 0, \end{aligned}$$

and we deduce that $K \cdot L: X \times X \to \mathbb{C}$ is a positive definite kernel.

Now, we assume that the maps K, L are strictly positive definite kernels on X. Let, n be a positive integer and let $x_1, \ldots, x_n \in X$. Without loss of generality, we assume that $\nu_1(x_i) \neq 0$ for all $i = 1, \ldots, n$ and $\nu_2(x_i) \neq 0$ for all $i = 1, \ldots, n$. Since K is a strictly positive kernel on X, we have that the elements $\nu_1(x_1), \ldots, \nu_1(x_n) \in H_1$ are linearly independent. Similarly, we also have that the elements $\nu_2(x_1), \ldots, \nu_2(x_n) \in H_2$ are linearly independent. Then, it is easy to note that the vectors $\{\nu_1(x_i) \otimes \nu_2(x_j) \mid i, j = 1, \ldots, n\} \subseteq H_1 \otimes H_2$ are linearly independent, which implies that $K \cdot L$ is a strictly positive definite kernel on X. $\square$

Now, we are ready to tackle the problem of the positive definiteness of $\langle\cdot,\cdot\rangle_\mu$.

Proposition 4.13. *Let $(H, \langle\cdot,\cdot\rangle)$ be a complex separable Hilbert space. The operator P_μ: $\mathcal{F}_{\text{lin}} \to \mathcal{F}_{\text{lin}}$ is positive for all $\mu \in [-1,1]$ and strictly positive for all $\mu \in (-1,1)$.*

Proof. It suffices to show that the corresponding assertions hold for the operators $P_\mu^{(n)}$, for every $n \in \mathbb{N}$. For $\mu \in [-1,1]$, we will first prove that the map $K_\mu: S_n \times S_n \to \mathbb{C}$, $(\sigma, \pi) \mapsto \mu^{i(\pi^{-1}\sigma)}$ is a positive definite kernel on S_n, i.e. for any $r: S_n \to \mathbb{C}$ we have

$$\sum_{\pi,\sigma \in S_n} r(\sigma)\overline{r(\pi)}\mu^{i(\pi^{-1}\sigma)} \geq 0\,.$$

In order to do so we have to find a nice formula for $\mu^{i(\pi^{-1}\sigma)}$.

We define

$$\begin{aligned}\Phi &:= \{(i,j)\in\{1,\dots,n\}^2 \,|\, i\neq j\},\\ \Phi^+ &:= \{(i,j)\in\Phi \,|\, i<j\}.\end{aligned}$$

For $\pi\in S_n$ and $A\subseteq\Phi$, we also define

$$\pi(A) := \{(\pi(i),\pi(j)) \,|\, (i,j)\in A\}\subseteq\Phi.$$

Since the map $A\ni(i,j)\mapsto(\pi(i),\pi(j))\in\pi(A)$ is a bijection, we have that $\#A=\#\pi(A)$. Moreover, since the elements of the set $\pi(\Phi^+)\backslash\Phi^+$ are of the form $(\pi(i),\pi(j))$ where $i<j$ and $\pi(i)>\pi(j)$, taking into account that $i(\pi)=\#\{i,j=1,\dots,n\,|\,i<j,\pi(i)>\pi(j)\}$, we see that the relation $i(\pi)=\#\pi(\Phi^+)\backslash\Phi^+$ holds. Using the fact that the map

$$\begin{aligned}\{(i,j)\in\{1,\dots,n\}^2\,|\,i<j,\pi(i)>\pi(j)\} &\longrightarrow \{(i,j)\in\{1,\dots,n\}^2\,|\,i<j,\pi^{-1}(i)>\pi^{-1}(j)\}\\ (i,j) &\longmapsto (\pi(j),\pi(i))\end{aligned}$$

is a bijection, we deduce that $i(\pi)=i(\pi^{-1})$ and the map K is a Hermitian kernel on S_n. From the above we have that $i(\pi)=i(\pi^{-1})=\#\pi^{-1}(\Phi^+)\backslash\Phi^+$. But, taking into account that the map

$$\begin{aligned}\pi^{-1}(\Phi^+)\backslash\Phi^+ &\longrightarrow \Phi^+\backslash\pi(\Phi^+)\\ (\pi^{-1}(i),\pi^{-1}(j)) &\longmapsto (i,j)\end{aligned}$$

is a bijection, we have $\#\pi^{-1}(\Phi^+)\backslash\Phi^+=\#\Phi^+\backslash\pi(\Phi^+)$, which implies that

$$2i(\pi)=i(\pi)+i(\pi^{-1})=\#\pi(\Phi^+)\backslash\Phi^+ + \#\Phi^+\backslash\pi(\Phi^+)=\#\pi(\Phi^+)\triangle\Phi^+,$$

where for two sets A,B we denote by $A\triangle B$ the symmetric difference $(A\backslash B)\cup(B\backslash A)$ of A and B. For every $\pi,\sigma\in S_n$, using that the map

$$\begin{aligned}\{(i,j)\in\{1,\dots,n\}^2\,|\,i<j,\ \pi^{-1}\sigma(i)>\pi^{-1}\sigma(j)\} &\longrightarrow \sigma(\Phi^+)\backslash\pi(\Phi^+)\\ (i,j) &\longmapsto (\sigma(i),\sigma(j))\end{aligned}$$

is a bijection, we deduce that

$$2i(\pi^{-1}\sigma)=i(\pi^{-1}\sigma)+i(\sigma^{-1}\pi)=\#\sigma(\Phi^+)\backslash\pi(\Phi^+)+\#\pi(\Phi^+)\backslash\sigma(\Phi^+)=\pi(\Phi^+)\triangle\sigma(\Phi^+)\,.$$

But, for $A,B\subseteq\Phi$ we can write

$$\#A\triangle B=\sum_{x\in\Phi}|\mathbf{1}_A(x)-\mathbf{1}_B(x)|=\sum_{x\in\Phi}|\mathbf{1}_A(x)-\mathbf{1}_B(x)|^2.$$

In order to show that K_μ $(-1\leq\mu\leq1)$ is a positive definite kernel on S_n, we will consider two separate cases. First, we consider the case $0<\mu\leq1$. Then, we can write $\mu=e^{-\lambda}$, for some $\lambda\geq0$. Hence, we have

$$\begin{aligned}\mu^{i(\pi^{-1}\sigma)} &= \exp(-\lambda i(\pi^{-1}\sigma))\\ &= \exp\Big(-\frac{\lambda}{2}\cdot\#\sigma(\Phi^+)\triangle\pi(\Phi^+)\Big)\\ &= \prod_{x\in\Phi}\exp\Big(-\frac{\lambda}{2}|\mathbf{1}_{\sigma(\Phi^+)}(x)-\mathbf{1}_{\pi(\Phi^+)}(x)|^2\Big).\end{aligned}$$

Therefore, taking into account that the product of positive definite kernels is a positive definite kernel, we see that in order to prove that K_μ is a positive definite kernel on S_n, it suffices to show that, for every $x \in \Phi$, the maps $S_n \times S_n \ni (\sigma, \pi) \mapsto \exp\Big(-\frac{\lambda}{2}|\mathbf{1}_{\sigma(\Phi^+)}(x) - \mathbf{1}_{\pi(\Phi^+)}(x)|^2\Big)$ are positive definite kernels on S_n, i.e.

$$\sum_{\pi,\sigma\in S_n} \exp\Big(-\frac{\lambda}{2}|\mathbf{1}_{\sigma(\Phi^+)}(x) - \mathbf{1}_{\pi(\Phi^+)}(x)|^2\Big) r(\sigma)\overline{r(\pi)} \geq 0.$$

Let $x \in \Phi$. Note that defining $y_0 := 0$, $y_1 := 1$ and

$$r(y_0) := \sum_{\substack{\sigma\in S_n \\ x\notin\sigma(\Phi^+)}} r(\sigma), \qquad r(y_1) := \sum_{\substack{\sigma\in S_n \\ x\in\sigma(\Phi^+)}} r(\sigma),$$

we have

$$\begin{aligned}\sum_{\pi,\sigma\in S_n} \exp\Big(-\frac{\lambda}{2}|\mathbf{1}_{\sigma(\Phi^+)}(x) - \mathbf{1}_{\pi(\Phi^+)}(x)|^2\Big) r(\sigma)\overline{r(\pi)} &= \sum_{i,j=0}^{1} \exp\Big(-\frac{\lambda}{2}|y_i - y_j|^2\Big) r(y_i)\overline{r(y_j)} \\ &\geq 0,\end{aligned}$$

because the function $\mathbb{R} \ni x \mapsto \exp\Big(-\frac{\lambda}{2}\cdot x^2\Big)$ is the Fourier transform of a Gaussian measure on $\mathbb{R}$, which means that is a positive definite function, by Bochner's theorem. Hence, the map K_μ is a positive definite kernel on S_n, for every $0 < \mu \leq 1$.

For the case $-1 \leq \mu < 0$, we first notice that K_{-1} is a positive definite kernel on S_n. Indeed, using that the signature function $S_n \ni \pi \mapsto (-1)^{i(\pi)}$ is a character on S_n, we see that

$$\sum_{\pi,\sigma\in S_n} (-1)^{i(\pi^{-1}\sigma)} r(\sigma)\overline{r(\pi)} = \sum_{\pi,\sigma\in S_n} (-1)^{i(\pi)}(-1)^{i(\sigma)} r(\sigma)\overline{r(\pi)} = \Big|\sum_{\pi\in S_n} (-1)^{i(\pi)} r(\pi)\Big|^2 \geq 0.$$

For $-1 \leq \mu < 0$ we have $K_\mu = K_{-1}\cdot K_{-\mu}$. Therefore, using that the product of positive definite kernels is again a positive definite kernel, we deduce that K_μ is a positive definite kernel on S_n.

For $\mu = 0$, since $K_0(\sigma, \pi) = 1$ for $\sigma = \pi$ and $K_0(\sigma, \pi) = 0$ for $\sigma \neq \pi$, it is obvious that K_0 is a positive definite kernel on S_n.

Now, we are ready to show that for every $n \in \mathbb{N}$, $P_\mu^{(n)}$ is a positive operator, i.e. we have $\langle \eta, P_\mu^{(n)}\eta\rangle \geq 0$, for all $\eta \in H^{\otimes n}$. We consider $\{e_i\}$ to be a CONS of $H^{\otimes n}$. Then, we have

$$\begin{aligned}\langle \eta, P_\mu^{(n)}\eta\rangle &= \sum_{\pi\in S_n} \mu^{i(\pi)}\langle \eta, U_\pi\eta\rangle \\ &= \frac{1}{n!}\sum_{\pi,\sigma\in S_n} \mu^{i(\pi^{-1}\sigma)}\langle \eta, U_{\pi^{-1}\sigma}\eta\rangle \\ &= \frac{1}{n!}\sum_{\pi,\sigma\in S_n} \mu^{i(\pi^{-1}\sigma)}\langle U_\pi\eta, U_\sigma\eta\rangle \\ &= \frac{1}{n!}\sum_{\pi,\sigma\in S_n}\sum_{\{e_i\}} \mu^{i(\pi^{-1}\sigma)}\langle U_\pi\eta, e_i\rangle\langle e_i, U_\sigma\eta\rangle \\ &= \frac{1}{n!}\sum_{\{e_i\}}\Big\{\sum_{\pi,\sigma\in S_n} \mu^{i(\pi^{-1}\sigma)}\overline{\langle e_i, U_\pi\eta\rangle}\langle e_i, U_\sigma\eta\rangle\Big\} \\ &\geq 0,\end{aligned}$$

where the third equality holds because for every $\pi, \sigma \in S_n$, $U_\pi U_\sigma = U_{\pi\sigma}$ and $U_{\pi^{-1}} = (U_\pi)^*$, the fourth equality holds because $\{e_i\}$ is a CONS of $H^{\otimes n}$ and the last inequality holds because K_μ is a positive definite kernel on S_n.

In order to conclude the proof it remains to show that for every $\mu \in (-1,1)$ and $n \in \mathbb{N}$, the operator $P_\mu^{(n)}$ is strictly positive. Note that it is sufficient to show that K_μ is a strictly positive definite kernel on S_n. We will only consider the case $\mu \in (0,1)$ since the case $\mu \in (-1,0)$ can be proved in a completely similar manner. The case $\mu = 0$ is trivial.

We assume that there exist $\mu \in (0,1)$ such that K_μ is not a strictly positive definite kernel on S_n. Then, since $K_\mu = K_{\sqrt{\mu}} \cdot K_{\sqrt{\mu}}$ and the product of strictly positive definite kernels is a strictly positive definite kernel, we deduce that $K_{\sqrt{\mu}}$ is not a strictly positive definite kernel on S_n. Because $\sqrt{\mu} \neq \mu$, we get in this way infinitely many positive definite kernels K_{μ_i}, $i \in \mathbb{N}$, which are not strictly positive definite kernels. But the fact that K_μ is not a strictly positive definite kernel, implies that $\det(A) = 0$, where $A := (\mu^{i(\pi^{-1}\sigma)})_{\pi,\sigma \in S_n}$. Since $\det(A)$ is a non-constant polynomial in μ we deduce that the polynomial equation $\det(A) = 0$ has finitely many solutions and this fact leads to a contradiction. Therefore, for every $\mu \in (0,1)$ we have that K_μ is a strictly positive definite kernel on S_n. □

From now on, we will denote by $\mathcal{F}_\mu(H)$ the completion of $\mathcal{F}_{\text{lin}}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_\mu$. In the cases $\mu = 1$ and $\mu = -1$ we first have to divide the kernel of P_μ, thus leading to the bosonic and fermionic Fock space, respectively.

4.4 Interpolation between fermionic, free and bosonic Brownian motions

Lemma 4.14. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. For every $\mu \in [-1,1)$ the operator $c^+(f)$ on $\mathcal{F}_\mu(H)$ is bounded and has the norm*

$$\|c^+(f)\|_\mu = \frac{1}{\sqrt{1-\mu}} \|f\| \quad \text{for } \mu \in [0,1), \tag{4.8}$$

$$\|c^+(f)\|_\mu = \|f\| \quad \text{for } \mu \in [-1,0]. \tag{4.9}$$

Proof. We will first treat the case $\mu \in [-1,0]$. Using the generalized commutation relations stated in (4.5), for $f \in H$ and $\eta \in \mathcal{F}_{\text{lin}}$, we have

$$\begin{aligned} \langle c^+(f)\eta, c^+(f)\eta \rangle_\mu &= \langle \eta, c^-(f)c^+(f)\eta \rangle_\mu \\ &= \langle f, f \rangle \langle \eta, \eta \rangle_\mu + \mu \langle c^-(f)\eta, c^-(f)\eta \rangle_\mu \\ &\leq \langle f, f \rangle \langle \eta, \eta \rangle_\mu, \end{aligned}$$

because $\mu \leq 0$. Therefore, we have $\|c^+(f)\|_\mu \leq \|f\|$. Using that $\|f\| = \|c^+(f)\Omega\|_\mu \leq \|c^+(f)\|_\mu$, we deduce that the relation (4.8) holds.

Now, we turn to the case $\mu \in [0,1)$. For every $i < n$, we consider $\pi_i \in S_n$ to be the transpositions of the symmetric group, i.e. π_i interchanges i and $i+1$ and keeps all the other elements fixed. We recall that the symmetric group S_n is generated by the transpositions $\pi_1, \ldots, \pi_{n-1}$. Moreover, the maps $\pi_1, \ldots, \pi_{n-1}$ satisfy the relations:

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, \quad \text{for all } i = 1, \ldots, n-2$$

$$\pi_i \pi_j = \pi_j \pi_i, \quad \text{for all } i, j = 1, \ldots, n-1 \text{ with } |i-j| \geq 2,$$

$$\pi_i = \pi_i^{-1}, \quad \text{for all } i = 1, \ldots, n-1.$$

Using the above relations, we can easily see that each permutation π of the symmetric group can be written uniquely in the form $\pi_{k(1)}\pi_{k(1)+1}\cdots\pi_{k(1)+r(1)}\cdots\pi_{k(i)}\pi_{k(i)+1}\cdots\pi_{k(i)+r(i)}$ with $i\geq 0$, $r(i)\geq 0$ and $k(1)>k(2)>\cdots>k(i)$. Furthermore, for such a permutation its number of inversions is equal to $(r(1)+1)+(r(2)+1)+\cdots+(r(i)+1)$. From these facts, it follows that the bounded operator $P_\mu^{(n+1)}$ can be written in the form:

$$P_\mu^{(n+1)}=\big(1\otimes P_\mu^{(n)}\big)(1+\mu U_{\pi_1}+\mu^2 U_{\pi_1}U_{\pi_2}+\cdots+\mu^n U_{\pi_1}U_{\pi_2}\ldots U_{\pi_n}).$$

Since, $U_\pi U_\sigma=U_{\pi\sigma}$ and $(U_\pi)^*=U_{\pi^{-1}}$ for all $\pi,\sigma\in S_n$, it follows that,

$$P_\mu^{(n+1)}P_\mu^{(n+1)}=P_\mu^{(n+1)}\big(P_\mu^{(n+1)}\big)^*$$

$$=\big(1\otimes P_\mu^{(n)}\big)(1+\mu\, U_{\pi_1}+\cdots+\mu^n U_{\pi_1\ldots\pi_n})(1+\mu\, U_{\pi_1}+\cdots+\mu^n U_{\pi_n\ldots\pi_1})\big(1\otimes P_\mu^{(n)}\big)^*.$$

Therefore, using that $U_\pi\leq\|U_\pi\|1=1$ for all $\pi\in S_n$, we have,

$$(1+\mu\, U_{\pi_1}+\cdots+\mu^n U_{\pi_1\ldots\pi_n})(1+\mu\, U_{\pi_1}+\cdots+\mu^n U_{\pi_n\ldots\pi_1})$$

$$\leq(1+\mu\cdot 1+\cdots+\mu^n\cdot 1)(1+\mu\cdot 1+\cdots+\mu^n\cdot 1).$$

This implies

$$P_\mu^{(n+1)}P_\mu^{(n+1)}\leq\big(1\otimes P_\mu^{(n)}\big)(1+\mu\cdot 1+\cdots+\mu^n\cdot 1)(1+\mu\cdot 1+\cdots+\mu^n\cdot 1)\big(1\otimes P_\mu^{(n)}\big).$$

Finally, we deduce that

$$P_\mu^{(n+1)}=\sqrt{P_\mu^{(n+1)}P_\mu^{(n+1)}}$$

$$\leq\sqrt{\big(1\otimes P_\mu^{(n)}\big)(1+\mu\cdot 1+\cdots+\mu^n\cdot 1)(1+\mu\cdot 1+\cdots+\mu^n\cdot 1)\big(1\otimes P_\mu^{(n)}\big)}$$

$$=(1+\mu+\cdots+\mu^n)1\otimes P_\mu^{(n)}\leq\frac{1}{1-\mu}1\otimes P_\mu^{(n)}.$$

Now, for $f\in H$ and $\eta\in H^{\otimes n}$, we get

$$\begin{aligned}\langle f\otimes\eta, f\otimes\eta\rangle_\mu &= \big\langle f\otimes\eta, P_\mu^{(n+1)}f\otimes\eta\big\rangle\\ &\leq \frac{1}{1-\mu}\big\langle f\otimes\eta,\big(1\otimes P_\mu^{(n)}\big)(f\otimes\eta)\big\rangle\\ &= \frac{1}{1-\mu}\langle f,f\rangle\big\langle\eta,P_\mu^{(n)}\eta\big\rangle.\end{aligned}$$

This shows that $\|c^+(f)\|\leq(1-\mu)^{-1/2}\|f\|$. That the norm of $c^+(f)$ is equal to $(1-\mu)^{-1/2}\|f\|$ can be seen from $c^+(f)f^{\otimes n}=f^{\otimes(n+1)}$ and

$$\langle f^{\otimes(n+1)},f^{\otimes(n+1)}\rangle_\mu=(1+\mu+\cdots+\mu^n)\langle f,f\rangle\langle f^{\otimes n},f^{\otimes n}\rangle_\mu. \qquad\square$$

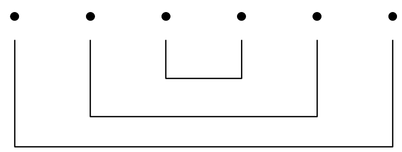
The previous lemma implies that for every $f\in H$ and $\mu\in[-1,1)$, the operators $c^+(f)$, $c^-(f)$ can be extended on $\mathcal{F}_\mu(H)$. Moreover, the relations (4.6) and (4.5) hold on $\mathcal{F}_\mu(H)$. On the other hand, for $\mu=1$ our operators are unbounded and they can be defined only on the dense domain $\mathcal{F}_{\text{lin}}$ of $\mathcal{F}_\mu(H)$.

Now, we are ready to consider our concrete example of a generalized Brownian motion. For this purpose, we choose $H = L^2(\mathbb{R})$ and fix a $\mu \in [-1, 1]$. We consider $\mathcal{C}_\mu$ to be the unital $*$-algebra generated by all $c^-(\mathbf{1}_I)$ for $I \in \mathcal{R}$ and $\rho_\mu(\cdot) := \langle \Omega, \cdot \Omega \rangle_\mu$ to be the vacuum state on $\mathcal{C}_\mu$. We claim that the triple $(\mathcal{C}_\mu, \rho_\mu, (c^-(\mathbf{1}_I), c^+(\mathbf{1}_I))_{I \in \mathcal{R}})$ is an 1-dimensional generalized Brownian motion. By induction, it is easy to note the factorizing of pyramidally ordered moments. Therefore, in order to prove the claim it remains to show the stationarity and the Gaussianity of the corresponding distribution. This is a consequence of the fact that all (joint) moments of our operators, with respect to ρ_μ, are determined in a specific way by the second (joint) moments. In order to describe this connection between the moments we have to define the number of inversions $i(V)$ of a 2-partition V. The number of inversions counts the number of crossing points that we have if we build bridges that connect the points which belong to the same block of the 2-partition V. More precisely, for r even and a 2-partition $V = \{(e_1, z_1), \ldots, (e_{r/2}, z_{r/2})\} \in P_2(1, \ldots, r)$, where $e_i < e_j$ for all $i < j$ and $e_i < z_i$ for all $i = 1, \ldots, r/2$, the number of inversions is defined as

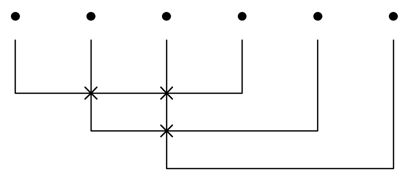
$$i(V) := \#\{(i, j) \in \{1, \ldots, r/2\}^2 \mid e_i < e_j < z_i < z_j\}.$$

Example 4.15. For the partitions $V_1 = \{(1, 6), (2, 5), (3, 4)\}$, $V_2 = \{(1, 4), (2, 5), (3, 6)\}$ and $V_3 = \{(1, 3), (2, 4), (5, 6)\}$ we have $i(V_1) = 0$, $i(V_2) = 3$, $i(V_3) = 1$ and the above partitions can be depicted in the following way

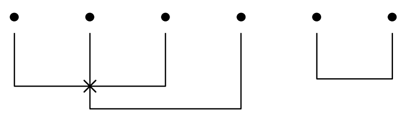
V_1



V_2



V_3



Remark 4.16. Let V be a 2-partition of the set $\{1, \ldots, r\}$. Since $i(V)$ counts the number of crossing points, we have that $i(V) = 0$ if and only if V is a non-crossing partition.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. For $f \in H$, let $c^{-1}(f) := c^-(f)$ and $c^1(f) := c^+(f)$. Our first task is to compute expressions of the form $\rho_\mu(c^{k(1)}(f_1) \ldots c^{k(r)}(f_r))$, where $r \in \mathbb{N}$ and $k(1), \ldots, k(r) \in \{-1, 1\}$. For $k(1), \ldots, k(r) \in \{-1, 1\}$, we define

$$\begin{aligned}
\sigma_r &= k(r), \\
\sigma_{r-1} &= k(r-1) + k(r), \\
&\vdots \\
\sigma_2 &= k(2) + \cdots + k(r), \\
\sigma_1 &= k(1) + k(2) + \cdots + k(r).
\end{aligned}$$

A straightforward induction shows that if $\sigma_r, \sigma_{r-1}, \ldots, \sigma_l \geq 0$, then $c^{k(l)}(f_l) \ldots c^{k(r)}(f_r)\Omega \in H^{\otimes \sigma_l}$ and otherwise $c^{k(l)}(f_l) \ldots c^{k(r)}(f_r)\Omega = 0$. Therefore, by the definition of ρ_μ we deduce that $\rho_\mu(c^{k(1)}(f_1) \ldots c^{k(r)}(f_r)) \neq 0$ only if $(k(r), k(r-1), \ldots, k(1)) \in \{-1, 1\}^r$ is a Dyck path. Hence, we have $\rho_\mu(c^{k(1)}(f_1) \ldots c^{k(r)}(f_r)) = 0$, for r odd.

Proposition 4.17. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. For $n\in\mathbb{N}$, $k(1),\dots,k(2n)\in\{-1,1\}$ and $f_1,\dots,f_{2n}\in H$ we have*

$$\rho_\mu(c^{k(1)}(f_1)\dots c^{k(2n)}(f_{2n}))=\sum_{\substack{V=\{(e_1,z_1),\dots,(e_n,z_n)\}\\ \in P_2(1,\dots,2n)}}\mu^{i(V)}\prod_{i=1}^{n}\rho_\mu(c^{k(e_i)}(f_{e_i})\,c^{k(z_i)}(f_{z_i}))\,. \tag{4.10}$$

Proof. First, we show that the above formula is valid for products of the form $c^-(f_1)\dots c^-(f_m)c^+(f_{m+1})c^+(f_{m+2})\dots c^+(f_{2n})$. For $m\neq n$, taking into account that $\rho_\mu(c^{k(1)}(f_1)\dots c^{k(2n)}(f_{2n}))\neq 0$ only if $(k(2n),k(2n-1),\dots,k(1))\in\{-1,1\}^{2n}$ is a Dyck path and $\rho_\mu(c^k(f)c^l(g))\neq 0$ only if $(k,l)=(-1,1)$, we see that both sides of the above formula vanish. Now, we assume that $m=n$, i.e. $k(1)=\dots=k(n)=-1$ and $k(n+1)=\dots=k(2n)=1$. We have

$$\begin{aligned}\rho_\mu(c^{k(1)}(f_1)\dots c^{k(2n)}(f_{2n})) &= \langle c^+(f_n)\dots c^+(f_1)\Omega, c^+(f_{n+1})\dots c^+(f_{2n})\Omega\rangle_\mu\\ &= \langle f_n\otimes\dots\otimes f_1, f_{n+1}\otimes\dots\otimes f_{2n}\rangle_\mu\\ &= \left\langle f_n\otimes\dots\otimes f_1, P_\mu^{(n)} f_{n+1}\otimes\dots\otimes f_{2n}\right\rangle\\ &= \sum_{\pi\in S_n}\langle f_n, f_{n+\pi(1)}\rangle\cdots\langle f_1, f_{n+\pi(n)}\rangle\mu^{i(\pi)}.\end{aligned}$$

In order to compute the left hand side of (4.10), we see that for $V=\{(e_1,z_1),\dots,(e_n,z_n)\}$ we have $\rho_\mu(c^{k(e_1)}(f_{e_1})\,c^{k(z_1)}(f_{z_1}))\dots\rho_\mu(c^{k(e_n)}(f_{e_n})\,c^{k(z_n)}(f_{z_n}))\neq 0$ only if $k(e_1)=\dots=k(e_n)=-1$ and $k(z_1)=\dots=k(z_n)=1$, i.e. only if $e_i=i$ for all $i=1,\dots,n$ and $z_1,\dots,z_n\in\{n+1,\dots,2n\}$. Then, we have

$$\sum_{\substack{V=\{(e_1,z_1),\dots,(e_n,z_n)\}\\ \in P_2(1,\dots,2n)}}\mu^{i(V)}\prod_{i=1}^{n}\rho_\mu(c^{k(e_i)}(f_{e_i})c^{k(z_i)}(f_{z_i}))=\sum_{\substack{V=\{(1,z_1),\dots,(n,z_n)\}\\ \in P_2(1,\dots,2n)}}\mu^{i(V)}\prod_{i=1}^{n}\langle f_i, f_{z_i}\rangle.$$

But, every partition $V=\{(1,z_1),\dots,(n,z_n)\}\in P_2(1,\dots,2n)$ corresponds to a unique $\pi\in S_n$, by defining $\pi(i)=z_{n+1-i}-n$, for all $i=1,\dots,n$. Then, for these π, V using that me map

$$\begin{aligned}\{(i,j)\in\{1,\dots,n\}^2\,|\,i<j,\pi(i)>\pi(j)\} &\longrightarrow \{(i,j)\in\{1,\dots,n\}^2\,|\,i<j<z_i<z_j\}\\ (i,j) &\longmapsto (n+1-j, n+1-i)\end{aligned}$$

is a bijection, we deduce that $i(\pi)=i(V)$. Therefore, we have

$$\begin{aligned}\sum_{\substack{V=\{(1,z_1),\dots,(n,z_n)\}\\ \in P_2(1,\dots,2n)}}\mu^{i(V)}\prod_{i=1}^{n}\langle f_i, f_{z_i}\rangle &= \sum_{\pi\in S_n}\mu^{i(\pi)}\prod_{i=1}^{n}\langle f_i, f_{n+\pi(n+1-i)}\rangle\\ &= \sum_{\pi\in S_n}\langle f_n, f_{n+\pi(1)}\rangle\dots\langle f_1, f_{n+\pi(n)}\rangle\mu^{i(\pi)}\\ &= \rho_\mu(c^{k(1)}(f_1)\dots c^{k(2n)}(f_{2n}))\,.\end{aligned}$$

Hence, the formula is valid for products of the form $c^-(f_1)\dots c^-(f_m)c^+(f_{m+1})\dots c^+(f_{2n})$.

The assertion has been proven because it is not hard to note that both sides of the formula stated in (4.10) change in the same way if we replace in $c^{k(1)}(f_1)\dots c^{k(2n)}(f_{2n})$ a factor $c^-(f)c^+(g)$ by $c^+(g)c^-(f)$, using the generalized commutation relations. □

If we return to the case $H=L^2(\mathbb{R})$, for every $I\in\mathcal{R}$ we denote $c_I^{-1}:=c^{-1}(\mathbf{1}_I)$, $c_I^1:=c^1(\mathbf{1}_I)$. The Gaussianity and the stationarity of the distribution of the process $(\mathcal{C}_\mu, \rho_\mu, (c_I^{-1}, c_I^1)_{I\in\mathcal{R}})$ emerges from the previous proposition, and the relation

$$\begin{pmatrix} \rho_\mu(c_I^{-1}c_J^{-1}) & \rho_\mu(c_I^{-1}c_J^1) \\ \rho_\mu(c_I^1c_J^{-1}) & \rho_\mu(c_I^1c_J^1) \end{pmatrix} = \begin{pmatrix} 0 & \lambda(I\cap J) \\ 0 & 0 \end{pmatrix}.$$

Therefore, for every $\mu\in[-1,1]$ the triple $(\mathcal{C}_\mu, \rho_\mu, (c_I^{-1}, c_I^1)_{I\in\mathcal{R}})$ is an 1-dimensional generalized Brownian motion. The importance of this specific example of a generalized Brownian motion comes from the fact that it gives an interpolation between fermionic, bosonic and free Brownian motion. More precisely, for $t_1,\ldots,t_{2n+1}\geq 0$, we have

$$\rho_\mu[(c_{[0,t_1)}^{-1}+c_{[0,t_1)}^1)\ldots(c_{[0,t_{2n})}^{-1}+c_{[0,t_{2n})}^1)] = \sum_{k(1),\ldots,k(2n)\in\{-1,1\}} \rho_\mu\big(c_{[0,t_1)}^{k(1)}\ldots c_{[0,t_{2n})}^{k(2n)}\big)$$

and using the relation (4.10) and the fact $\rho_\mu(c^k(f)c^l(g))\neq 0$ only if $(k,l)=(-1,1)$, we have

$$\begin{aligned} \rho_\mu[(c_{[0,t_1)}^{-1}+c_{[0,t_1)}^1)\ldots(c_{[0,t_{2n})}^{-1}+c_{[0,t_{2n})}^1)] &= \sum_{\substack{V=\{(e_1,z_1),\ldots,(e_n,z_n)\}\\ \in P_2(1,\ldots,2n)}} \mu^{i(V)} \prod_{i=1}^n \rho_\mu\big(c_{[0,t_{e_i})}^{-1}\, c_{[0,t_{z_i})}^1\big) \\ &= \sum_{V\in P_2(1,\ldots,2n)} \mu^{i(V)} \prod_{\{i,j\}\in V} t_i\wedge t_j \end{aligned}$$

and

$$\rho_\mu[(c_{[0,t_1)}^{-1}+c_{[0,t_1)}^1)\ldots(c_{[0,t_{2n+1})}^{-1}+c_{[0,t_{2n+1})}^1)] = 0.$$

Therefore, the joint moments $\rho_\mu[(c_{[0,t_1)}^{-1}+c_{[0,t_1)}^1)\ldots(c_{[0,t_r)}^{-1}+c_{[0,t_r)}^1)]$ coincide with the corresponding joint moments of a fermionic, free and bosonic Brownian motion, for the cases $\mu=-1$, $\mu=0$ and $\mu=1$, respectively.

4.5 Another representation of the generalized commutation relations

In the previous subsection, we saw that the generalized commutation relations

$$c^-(f)c^+(g) - \mu\, c^+(g)c^-(f) = \langle f, g\rangle 1 \quad (f, g\in L^2(\mathbb{R})),$$

were crucial in order to construct an example of a generalized Brownian motion, which gives an interpolation between fermionic, free and bosonic Brownian motion. For $f\in L^2(\mathbb{R})$ fixed, in order to construct the operators $c^+(f)$, $c^-(f)$, we considered $c^+(f)=l^+(f)$ and we defined $c^-(f)$ in such a way that the generalized commutation relations are satisfied. By the definition of $c^-(f)$, we have $c^-(f)=l^-(f)$ for $\mu=0$. Now, we will give another example of a pair of operators $(d^-(f), d^+(g))$ that satisfy the generalized commutation relations. For $f\in L^2(\mathbb{R})$ fixed, in order to define $d^+(f)$, $d^-(f)$, we will do the inverse procedure than the one that we did in order to define $c^+(f)$ and $c^-(f)$. Namely, we consider $d^-(f)=l^-(f)$ and we will define $d^+(f)$ in such a way that the generalized commutation relations are satisfied. For $\mu=0$, we will have $d^+(f)=l^+(f)$.

Let $(H, \langle\cdot,\cdot\rangle)$ be a (complex) Hilbert space. For each $f\in H$, we define the operators $d^+(f), d^-(f)$ by

$$d^-(f) := l^-(f) \tag{4.11}$$

and

$$d^+(f)\Omega := f,$$

$$\begin{aligned} d^+(f) f_1 \otimes \cdots \otimes f_n &:= f \otimes f_1 \otimes \cdots \otimes f_n + \mu f_1 \otimes f \otimes f_2 \otimes \cdots \otimes f_n + \cdots + \\ &\quad + \mu^{n-1} f_1 \otimes \cdots \otimes f_{n-1} \otimes f \otimes f_n + \mu^n f_1 \otimes \cdots \otimes f_n \otimes f. \end{aligned}$$

The operator $d^+(f)$ is extended by linearity to $\mathcal{F}_{\text{lin}}$. Note that, for $\mu = 0$, we have that $d^+(f) = l^+(f)$.

Lemma 4.18. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. The operators $d^-(f), d^+(g)$ on $\mathcal{F}_{\text{lin}}$ fulfill, for all $f, g \in H$, the relations*

$$d^-(f) d^+(g) - \mu\, d^+(g) d^-(f) = \langle f, g \rangle 1.$$

Proof. Let n be a positive integer and let $f_1, \ldots, f_{n+1}, g, g_1, \ldots, g_n \in H$. Then, for every $k \in \{1, \ldots, n\}$ we have,

$$\begin{aligned} &\langle f_1 \otimes \cdots \otimes f_{n+1}, g_1 \otimes \cdots \otimes g_{k-1} \otimes g \otimes g_k \otimes \cdots \otimes g_n \rangle \\ &\qquad = \langle \langle g, f_k \rangle f_1 \otimes \cdots \otimes \check{f_k} \otimes \cdots \otimes f_{n+1}, g_1 \otimes \cdots \otimes g_n \rangle. \end{aligned}$$

Then, if we multiply both sides by μ^{k-1} and take the sum over all $k = 1, \ldots, n+1$, we have

$$\langle f_1 \otimes \cdots \otimes f_{n+1}, d^+(g) g_1 \otimes \cdots \otimes g_n \rangle = \langle c^-(g) f_1 \otimes \cdots \otimes f_{n+1}, g_1 \otimes \cdots \otimes g_n \rangle,$$

which implies that

$$\langle \xi, d^+(g)\eta \rangle = \langle c^-(g)\xi, \eta \rangle, \quad \text{for all } \xi, \eta \in \mathcal{F}_{\text{lin}}. \tag{4.12}$$

Then, for $\xi, \eta \in \mathcal{F}_{\text{lin}}$ and $f, g \in H$ we have

$$\begin{aligned} \langle \xi, d^-(f) d^+(g) \eta \rangle &= \langle l^+(f)\xi, d^+(g)\eta \rangle \\ &= \langle c^-(g) c^+(f) \xi, \eta \rangle \\ &= \langle \mu\, c^+(f) c^-(g) \xi + \langle g, f \rangle \xi, \eta \rangle \\ &= \langle \xi, \mu\, d^+(g) d^-(f) \eta + \langle f, g \rangle \eta \rangle, \end{aligned}$$

where we have used that $d^-(f) = l^-(f)$, $c^+(f) = l^+(f)$ and $l^+(f) = (l^-(f))^*$. Hence, the assertion holds. □

Let $\mu \in (-1, 1)$ be fixed in the following. In order to show that the operators $d^+(f)$, $d^-(f)$ behave exactly as the operators $c^+(f), c^-(f)$, we have to introduce a scalar product which makes $d^+(f)$ and $d^-(f)$ adjoints of one another. To work in this direction, we choose a scalar product $\langle \cdot, \cdot \rangle^\mu$ using P_μ^{-1}. More precisely, we define $\langle \xi, \eta \rangle^\mu := \langle \xi, P_\mu^{-1} \eta \rangle$, for every $\xi, \eta \in \mathcal{F}_{\text{lin}}$. Note that P_μ^{-1} exists, i.e. $P_\mu^{-1}(\mathcal{F}_{\text{lin}}) = \mathcal{F}_{\text{lin}}$.

Lemma 4.19. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. For all $f \in H$ and $\xi, \eta \in \mathcal{F}_{\text{lin}}$ we have*

$$\langle \eta, d^+(f)\xi \rangle^\mu = \langle d^-(f)\eta, \xi \rangle^\mu.$$

Proof. Using the relation (4.12), for all $\xi, \eta \in \mathcal{F}_{\text{lin}}$ we have

$$\begin{aligned}\langle \xi, P_\mu c^+(f)\eta\rangle = \langle \xi, c^+(f)\eta\rangle_\mu &= \langle c^-(f)\xi, \eta\rangle_\mu \\ &= \langle c^-(f)\xi, P_\mu \eta\rangle \\ &= \langle \xi, d^+(f) P_\mu \eta\rangle.\end{aligned}$$

Therefore, we have $P_\mu c^+(f) = d^+(f) P_\mu$, which implies $c^+(f) P_\mu^{-1} = P_\mu^{-1} d^+(f)$. Hence, by the definition of $\langle \cdot, \cdot\rangle^\mu$ we deduce that

$$\begin{aligned}\langle \eta, d^+(f)\xi\rangle^\mu &= \langle \eta, P_\mu^{-1} d^+(f)\xi\rangle \\ &= \langle \eta, c^+(f) P_\mu^{-1}\xi\rangle \\ &= \langle d^-(f)\eta, P_\mu^{-1}\xi\rangle \\ &= \langle d^{-1}(f)\eta, \xi\rangle^\mu,\end{aligned}$$

where in the third equality we used that $c^+(f) = l^+(f)$ and $(l^+(f))^* = l^-(f) = d^-(f)$. □

We now denote by $\mathcal{F}^\mu(H)$ the completion of $\mathcal{F}_{\text{lin}}$ with respect to $\langle \cdot, \cdot\rangle^\mu$.

Lemma 4.20. *Let $(H, \langle \cdot, \cdot\rangle)$ be a Hilbert space. For $f \in H$, the operator $d^+(f)$ is bounded on $\mathcal{F}^\mu(H)$ and has the same norm as $c^+(f)$ on $\mathcal{F}_\mu(H)$.*

Proof. First, for $\eta \in \mathcal{F}_{\text{lin}}$ and $\xi = P_\mu^{-1}\eta$ we see that

$$\langle \xi, \xi\rangle_\mu = \langle \xi, P_\mu \xi\rangle = \langle P_\mu^{-1}\eta, \eta\rangle = \langle \eta, \eta\rangle^\mu.$$

Furthermore, using that $\eta = P_\mu \xi$ we have

$$\begin{aligned}\langle d^+(f)\eta, d^+(f)\eta\rangle^\mu &= \langle d^+(f) P_\mu \xi, P_\mu^{-1} d^+(f) P_\mu \xi\rangle \\ &= \langle P_\mu c^+(f)\xi, P_\mu^{-1} P_\mu c^+(f)\xi\rangle \\ &= \langle P_\mu c^+(f)\xi, c^+(f)\xi\rangle \\ &= \langle c^+(f)\xi, c^+(f)\xi\rangle_\mu,\end{aligned}$$

where, in the second equality we used that $d^+(f) P_\mu = P_\mu c^+(f)$. Therefore, we deduce that the claim holds. □

The previous lemma implies that Lemmata 4.18 and 4.19, which were proved only on the dense domain $\mathcal{F}_{\text{lin}}$, remain valid also on $\mathcal{F}^\mu(H)$.

Given a Hilbert space $(H, \langle \cdot, \cdot\rangle)$, we consider the vacuum state $\rho^\mu(\cdot) := \langle \Omega, \cdot\Omega\rangle^\mu$ on $B(\mathcal{F}^\mu(H))$. Let $d^1(f) := d^+(f)$, $d^{-1}(f) := d^-(f)$, for every $f \in H$. Then, the joint moments of the non-commutative random variables $d^{-1}(f)$ with respect to ρ^μ are equal to the corresponding joint moments of the non-commutative random variables $c^{-1}(f)$ with respect to ρ_μ.

Proposition 4.21. *Let $(H, \langle \cdot, \cdot\rangle)$ be a Hilbert space. For every $r \in \mathbb{N}$, $k(1), \ldots, k(r) \in \{-1, 1\}$ and $f_1, \ldots, f_r \in H$, we have*

$$\rho^\mu\big(d^{k(1)}(f_1) \ldots d^{k(r)}(f_r)\big) = \rho_\mu(c^{k(1)}(f_1) \ldots c^{k(r)}(f_r)). \tag{4.13}$$

Proof. It suffices to prove the weaker condition: for all $n, m \in \mathbb{N}$ and $g_1, \dots, g_m, h_1, \dots, h_n \in H$ we have

$$\rho^\mu(d^{-1}(g_1)\dots d^{-1}(g_m)d^1(h_1)\dots d^1(h_n)) = \rho_\mu(c^{-1}(g_1)\dots c^{-1}(g_m)c^1(h_1)\dots c^1(h_n)).$$

Indeed, this is so because using the generalized commutation relations and the linearity of ρ^μ, ρ_μ we see that both sides of the formula stated in (4.13) change in the same way if we replace in $d^{k(1)}(f_1)\dots d^{k(r)}(f_r)$ a factor $d^{-1}(f)d^1(g)$ by $d^1(g)d^{-1}(f)$ and respectively if we replace in $c^{k(1)}(f_1)\dots c^{k(r)}(f_r)$ the factor $c^{-1}(f)c^1(g)$ by $c^1(g)c^{-1}(f)$. Therefore, we have

$$\begin{aligned}\rho^\mu(d^{-1}(g_1)\dots d^{-1}(g_m)d^1(h_1)\dots d^1(h_n)) &= \langle d^1(g_m)\dots d^1(g_1)\Omega, d^1(h_1)\dots d^1(h_n)\Omega\rangle^\mu \\ &= \langle d^1(g_m)\dots d^1(g_1)\Omega, P_\mu^{-1}d^1(h_1)\dots d^1(h_n)\Omega\rangle.\end{aligned}$$

But, since $d^+(f)P_\mu = P_\mu c^+(f)$, for every $f \in H$, we have $d^1(g_m) = P_\mu c^1(g_m)P_\mu^{-1}$, which implies

$$d^1(g_m)\dots d^1(g_1)\Omega = P_\mu c^1(g_m)P_\mu^{-1}d^1(g_{m-1})\dots d^1(g_1)\Omega.$$

Using that $P_\mu^{-1}d^+(f) = c^+(f)P_\mu^{-1}$ for every $f \in H$ and $P_\mu\Omega = \Omega$, we have

$$d^1(g_m)\dots d^1(g_1)\Omega = P_\mu c^1(g_m)\dots c^1(g_1)\Omega$$

and similarly

$$d^1(h_1)\dots d^1(h_n)\Omega = P_\mu c^1(h_1)\dots c^1(h_n)\Omega.$$

Therefore, we deduce that

$$\begin{aligned}\rho^\mu(d^{-1}(g_1)\dots d^{-1}(g_m)d^1(h_1)\dots d^1(h_n)) &= \langle P_\mu c^1(g_m)\dots c^1(g_1)\Omega, c^1(h_1)\dots c^1(h_n)\Omega\rangle \\ &= \langle c^1(g_m)\dots c^1(g_1)\Omega, P_\mu c^1(h_1)\dots c^1(h_n)\Omega\rangle \\ &= \langle c^1(g_m)\dots c^1(g_1)\Omega, c^1(h_1)\dots c^1(h_n)\Omega\rangle_\mu \\ &= \rho_\mu(c^{-1}(g_1)\dots c^{-1}(g_m)c^1(h_1)\dots c^1(h_n))\end{aligned}$$

and the assertion holds. □

Taking into account that, for $\mu \in [-1, 1]$, the triple $(\mathcal{C}_\mu, \rho_\mu, (c_I^{-1}, c_I)_{I\in\mathcal{R}})$ is a generalized Brownian motion, by the previous proposition we deduce that, for $\mu \in (-1, 1)$, the triple $(\mathcal{D}_\mu, \rho^\mu, (d_I^{-1}, d_I^1)_{I\in\mathcal{R}})$ is a generalized Brownian motion, where we consider $\mathcal{D}_\mu$ to be the unital $*$-algebra generated by all d_I^{-1}, for $I \in \mathcal{R}$.

5 A non-commutative central limit theorem

In this section we continue the investigation of the non-commutative stochastic process $(\mathcal{C}_\mu, \rho_\mu, (c_I^{-1}, c_I^1)_{I\in\mathcal{R}})$. We will show that the 1-dimensional generalized Brownian motion $(\mathcal{C}_\mu, \rho_\mu, (c_I^{-1}, c_I^1)_{I\in\mathcal{R}})$ arises via some non-commutative central limit theorem relying on the notion of Kümmerer independence [...]. In order to determine the Gaussian distribution corresponding to this process we introduce the μ-Hermite polynomials. In full proportion with the above, we also introduce the μ-Poisson distribution and the μ-analogues of Charlier-Poisson polynomials.

5.1 Overview

In the previous section we introduced generalized Brownian motion and we gave an example of an 1-dimensional generalized Brownian motion which gives an interpolation between fermionic, free and bosonic Brownian motion. More precisely, for the non-commutative stochastic process $(\mathcal{C}_\mu, \rho_\mu, (c^+(\mathbf{1}_I)+c^-(\mathbf{1}_I))_{I\in\mathcal{R}})$ we have seen that the interpolation is given by the relations

$$\rho_\mu[(c^+(\mathbf{1}_{[0,t_1)})+c^-(\mathbf{1}_{[0,t_1)}))\dots(c^+(\mathbf{1}_{[0,t_{2n})})+c^-(\mathbf{1}_{[0,t_{2n})}))]=\sum_{V\in P_2(1,\dots,2n)}\mu^{i(V)}\prod_{\{i,j\}\in V} t_i\wedge t_j$$

and

$$\rho_\mu[(c^+(\mathbf{1}_{[0,t_1)})+c^-(\mathbf{1}_{[0,t_1)}))\dots(c^+(\mathbf{1}_{[0,t_{2n+1})})+c^-(\mathbf{1}_{[0,t_{2n+1})}))]=0,$$

for every $\mu\in[-1,1]$ and for every $t_1,\dots,t_{2n+1}\geq 0$.

In this section we want to understand if the Gaussian distribution, which correspods to this process, can be emerged from a central limit theorem and whether this can be generalized to an invariance principle yielding the process itself. By Gaussian distribution corresponding to this process, we mean the probability measure ν_μ on $\mathbb{R}$ which determines the non-commutative distribution of the self-adjoint variable $c^+(f)+c^-(f)$ for $\|f\|=1$ ($f\in L^2(\mathbb{R})$), in the sense that

$$\rho_\mu[(c^+(f)+c^-(f))^k]=\int_{\mathbb{R}} t^k\,\nu_\mu(dt),$$

for every $k\in\mathbb{N}$. We have seen that the moments of $c^-(f), c^+(f)$ with respect to ρ_μ depend only on $\|f\|$ and μ. Therefore the measure ν_μ does not depend on $f\in L^2(\mathbb{R})$. Of course, the cases $\mu=0$ and $\mu=1$ correspond to the semicircle and normal distribution respectively. For $\mu\in(-1,1)$, since $c^+(f)+c^-(f)$ is a bounded and self-adjoint operator on $\mathcal{F}_\mu(L^2(\mathbb{R}))$, the existence of such a probability measure ν_μ can be derived from the functional calculus for $c^+(f)+c^-(f)$ and Riesz's theorem. For the fermionic case, $\mu=-1$, it is not hard to note that the corresponding measure ν_{-1} is given by

$$\nu_{-1}(dx)=\frac{1}{2}(\delta_{-1}(dx)+\delta_1(dx)).$$

Since our goal is to obtain ν_μ via a central limit theorem, we are interested in finding appropriate $*$-probability spaces $(\mathcal{A},\phi)$ and non-commutative random variables $a_i\in\mathcal{A}$ for every $i\in\mathbb{N}$, such that for the sum

$$S_N=\frac{a_1+\dots+a_N}{\sqrt{N}},$$

we have that $S_N \in (\mathcal{A}, \phi)$ converges in distribution to $c^-(f) \in (\mathcal{C}_\mu, \rho_\mu)$ as $N \to \infty$. By convergence in distribution we mean that all moments of S_N, S_N^* with respect to ϕ converge to the corresponding moments of $c^-(f), c^+(f)$ with respect to ρ_μ. In that case, for every $k \in \mathbb{N}$, we will have

$$\lim_{N\to\infty} \phi[(S_N + S_N^*)^k] = \rho_\mu[(c^-(f) + c^+(f))^k] = \int_{\mathbb{R}} t^k \, \nu_\mu(dt).$$

As usual, in order to formulate and prove the non-commutative central limit theorem, the non-commutative random variables a_i will be considered in some sense as independent and as identically distributed. Form now on, we shall call a sequence $(a_i)_{i\in\mathbb{N}}$ of non-commutative random variables on some $*$-probability space $(\mathcal{A}, \phi)$ independent with respect to ϕ, if ϕ of naturally ordered products factorizes in the following sense: If $\mathcal{A}_i$ is the unital $*$-algebra generated by a_i, then we demand

$$\phi(\tilde{a}_{i(1)} \dots \tilde{a}_{i(r)}) = \phi(\tilde{a}_{i(1)}) \dots \phi(\tilde{a}_{i(r)}),$$

for every $r \in \mathbb{N}$, $\tilde{a}_{i(k)} \in \mathcal{A}_{i(k)}$ and $i(1) < \dots < i(r)$.

This notion of independence was introduced by Kümmerer and it is called Kümmerer independence. The important examples of Kümmerer independence are free independence, Bose independence and Fermi independence.

5.2 Central limit theorem

We shall now formulate and prove the non-commutative central limit theorem. In order to do so, we consider the following framework: Let $(\mathcal{A}, \phi)$ be a $*$-probability space and let $(a_i)_{i\in\mathbb{N}}$ be a sequence of non-commutative random variables such that ϕ of naturally ordered products factorizes. We define $a_i^{-1} := a_i$ and $a_i^1 := a_i^*$, for every $i \in \mathbb{N}$. For $i \neq j$ and $k, l \in \{-1, 1\}$ we assume that the non-commutative random variables a_i^k and a_j^l commute or anticommute, meaning that

$$a_i^k a_j^l = s(i, j) a_j^l a_i^k, \tag{5.1}$$

where $s(i, j) \in \{-1, 1\}$. For concreteness we may also define $s(i, i) := 0$. Note that by definition $s(i, j) = s(j, i)$, for every $i, j \in \mathbb{N}$ and for $i \neq j$ the relation (5.1) implies that if a_i^1 and a_j^1 commute (or anticommute) then the variables a_i^{-1} and a_j^1 will also commute (or anticommute).

To formulate the central limit theorem some assumptions have to be done for the infinite symmetric matrix $\boldsymbol{s} = (s(i, j))_{i,j=1}^{\infty}$. Before we present these assumptions let us introduce some notation: For an arbitrary partition $V = \{V_1, \dots, V_n\} \in P_2(1, \dots, 2n)$ we will write $V_i = (e_i, z_i)$ with $e_i < z_i$, for all $i = 1, \dots, n$. We can also assume $e_1 < \dots < e_n$. For such a partition, we define the set of inversions of V by

$$I(V) := \{(i, j) \in \{1, \dots, n\}^2 \mid e_i < e_j < z_i < z_j\}.$$

We recall that we denote the number of inversions of V by $i(V) = \#I(V)$ and we have $I(V) = \emptyset$ if and only if V is a non-crossing partition.

Our assumption on $\boldsymbol{s}$ will be the convergence of specific sums where each of them corresponds to a 2-set partition. More precisely for $V \in P_2(1,\dots,2n)\backslash \mathrm{NC}_2(1,\dots,2n)$ we assume the existence of

$$t(V) := \lim_{N\to\infty} \frac{1}{N^n} \sum_{\substack{i(e_1),\dots,i(e_n)=1 \\ i(e_j)\neq i(e_m) \text{ for } j\neq m}}^{N} \prod_{(k,l)\in I(V)} s(i(e_k), i(e_l)). \tag{5.2}$$

For $V \in \mathrm{NC}_2(1,\dots,2n)$ (i.e. $I(V)=\emptyset$) we define $t(V):=1$.

If $s(i,j)=1$ for every $i,j\in\mathbb{N}$, the limit exists and we have

$$t(V) = \lim_{N\to\infty} \frac{N(N-1)\dots(N-n+1)}{N^n} = 1$$

Similarly $t(V)=(-1)^{i(V)}$ if $s(i,j)=-1$ for every $i,j\in\mathbb{N}$.

Example 5.1. For the partitions $V=\{(1,3),(2,4),(5,6)\}$ and $W=\{(1,4),(2,5),(3,6)\}$, we have $I(V)=\{(1,2)\}$ and $I(W)=\{(1,2),(1,3),(2,3)\}$. This implies that

$$t(V) = \lim_{N\to\infty} \frac{1}{N^3} \sum_{\substack{i(1),i(2),i(3)=1 \\ i(1)\neq i(2)\neq i(3)\neq i(1)}}^{N} s(i(1),i(2)) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{i(1),i(2)=1 \\ i(1)\neq i(2)}}^{N} s(i(1),i(2))$$

and

$$t(W) = \lim_{N\to\infty} \frac{1}{N^3} \sum_{\substack{i(1),i(2),i(3)=1 \\ i(1)\neq i(2)\neq i(3)\neq i(1)}}^{N} s(i(1),i(2))s(i(1),i(3))s(i(2),i(3)).$$

Now we are ready to prove the non-commutative central limit theorem under the assumption of the convergence of $t(V)$ for all $n\in\mathbb{N}$ and for all $V\in P_2(1,\dots,2n)$. Later on, we shall see that under some probabilistic assumptions about $\boldsymbol{s}$ we can compute $t(V)$ for almost all infinite symmetric matrices $\boldsymbol{s}$. The computation of $t(V)$ will show that the measure ν_μ (where $\mu\in[-1,1]$ will be specified from our probabilistic assumptions) arises from the non-commutative central limit theorem as a non-commutative analogue of the Gaussian distribution.

Theorem 5.2. (non-commutative central limit theorem) *Let (A,ϕ) be a $*$-probability space and let $(a_i)_{i\in\mathbb{N}}$ be a sequence of non-commutative random variables such that ϕ of naturally ordered products factorizes. We consider, for every $i\in\mathbb{N}$, $a_i^{-1}:=a_i$ and $a_i^1:=a_i^*$ and we assume, for every $i\in\mathbb{N}$ and $k,l\in\{-1,1\}$, that $\phi(a_i^k)=0$ and the covariance $\phi(a_i^k a_i^l)$ is independent of i. We define $\phi(a^k a^l):=\phi(a_i^k a_i^l)$. Assume further that for all $i,j\in\mathbb{N}$ with $i\neq j$ and $k,l\in\{-1,1\}$,*

$$a_i^k a_j^l = s(i,j)\, a_j^l a_i^k, \quad \text{with} \quad s(i,j)\in\{-1,1\}, \tag{5.3}$$

and that for all $n\in\mathbb{N}$ and $V\in P_2(1,\dots,2n)$ the limit $t(V)$ exists. Then, if we consider the sums

$$S_N^{-1} := S_N = \frac{a_1+\dots+a_N}{\sqrt{N}} \quad \textit{and} \quad S_N^1 := S_N^* = \frac{a_1^*+\dots+a_N^*}{\sqrt{N}},$$

we have for all $r\in\mathbb{N}$ *and* $k(1),\dots,k(r)\in\{-1,1\}$,

$$\lim_{N\to\infty}\phi\big(S_N^{k(1)}\dots S_N^{k(r)}\big)=\begin{cases}0, & \textit{for } r \textit{ odd}\\ \displaystyle\sum_{V=\{(e_1,z_1),\dots,(e_n,z_n)\}\in P_2(1,\dots,2n)} t(V)\prod_{i=1}^{n}\phi(a^{k(e_i)}a^{k(z_i)}), & \textit{for } r=2n.\end{cases}$$

Proof. Let r be a positive integer and let $k(1),\dots,k(r)\in\{-1,1\}$. In order to show that the assertion holds we have to calculate the expression

$$\begin{aligned}M_N:=\phi\big(S_N^{k(1)}\dots S_N^{k(r)}\big) &= \phi\left[\left(\frac{a_1^{k(1)}+\dots+a_N^{k(1)}}{\sqrt{N}}\right)\dots\left(\frac{a_1^{k(r)}+\dots+a_N^{k(r)}}{\sqrt{N}}\right)\right]\\ &= \frac{1}{N^{r/2}}\sum_{i(1),\dots,i(r)=1}^{N}\phi\big(a_{i(1)}^{k(1)}\dots a_{i(r)}^{k(r)}\big).\end{aligned}$$

For every r-tuple $i=(i(1),\dots,i(r))\in\{1,\dots,N\}^r$ there exists a unique partition $V\in P(1,\dots,r)$ such that $\ker(i)=V$, since, by definition, for $p,q\in\{1,\dots,r\}$ we have $p\sim_{\ker(i)}q$ if and only if $i(p)=i(q)$. Therefore we have,

$$M_N=\sum_{V\in P(1,\dots,r)}\frac{1}{N^{r/2}}\sum_{\substack{i(1),\dots,i(r)=1\\ \ker(i)=V}}^{N}\phi\big(a_{i(1)}^{k(1)}\dots a_{i(r)}^{k(r)}\big). \tag{5.4}$$

Since the set $P(1,\dots,r)$ is finite and independent of N and we focus on the computation of M_N, as $N\to\infty$, we will concentrate to the last sum of the right hand side of (5.4), separetely for all $V\in P(1,\dots,r)$.

Let $V=\{V_1,\dots,V_p\}\in P(1,\dots,r)$ such that $\#V_i=1$ for some $i\in\{1,\dots,p\}$. Then, since $\phi(a_i^k)=0$, for every $i\in\mathbb{N}$ and $k\in\{-1,1\}$, using the relations stated in (5.3) and our assumption that ϕ of naturally ordered products factorizes, we will have $\phi\big(a_{i(1)}^{k(1)}\dots a_{i(r)}^{k(r)}\big)=0$, for all $i(1),\dots,i(r)\in\{1,\dots,N\}$ such that $\ker(i)=V$. Hence, in order to compute M_N it suffices to take the sum over all $V=\{V_1,\dots,V_p\}\in P(1,\dots,r)$ such that $\#V_i\geq 2$ for all $i=1,\dots,p$. For such a partition V we have

$$2p\leq\sum_{i=1}^{p}\#V_i=r.$$

By our assumptions, for $V=\{V_1,\dots,V_p\}\in P(1,\dots,r)$ the expression $\big|\phi\big(a_{i(1)}^{k(1)}\dots a_{i(r)}^{k(r)}\big)\big|$ has the same value m_V, for all $i(1),\dots,i(r)\in\{1,\dots,N\}$ such that $\ker(i)=V$. Also, choosing an r-tuple $i=(i(1),\dots,i(r))$ of elements of $\{1,\dots,N\}$ such that $\ker(i)=V$ is equivalent to choosing p distinct numbers from the set $\{1,\dots,N\}$. This is true, because for such an i, it's values at two points of $\{1,\dots,r\}$ are the same if and only if these points belong to the same block of V. Therefore by the triangle inequality we deduce that,

$$\left|\frac{1}{N^{r/2}}\sum_{\substack{i(1),\dots,i(r)=1\\ \ker(i)=V}}^{N}\phi\big(a_{i(1)}^{k(1)}\dots a_{i(r)}^{k(r)}\big)\right|\leq\frac{1}{N^{r/2}}\sum_{\substack{i(1),\dots,i(r)=1\\ \ker(i)=V}}^{N}m_V=m_V\frac{A_{p;N}}{N^{r/2}},$$

where $A_{p;N} := N(N-1)\dots(N-p+1)$. Since,

$$\lim_{N\to\infty} \frac{A_{p:N}}{N^{r/2}} = 0, \quad \text{for } p < \frac{r}{2},$$

the partitions $V = \{V_1, \dots, V_p\} \in P(1, \dots, r)$, with $p < r/2$, do not contribute to the sum as $N \to \infty$. Hence, only the partitions V of $\{1, \dots, r\}$, with $\#V = r/2$, may contribute to the sum as $N \to \infty$ (if they exist). This means that $M_N \to 0$ as $N \to \infty$, for r odd and

$$\lim_{N\to\infty} M_N = \sum_{\substack{V=\{(e_1,z_1),\dots,(e_n,z_n)\}\\ \in P_2(1,\dots,2n)}} \lim_{N\to\infty} \left(\frac{1}{N^{r/2}} \sum_{\substack{i(1),\dots,i(2n)=1\\ \ker(i)=V}}^{N} \phi\big(a_{i(1)}^{k(1)} \dots a_{i(2n)}^{k(2n)}\big) \right), \tag{5.5}$$

for $r = 2n$.

Let $V = \{(e_1, z_1), \dots, (e_n, z_n)\} \in \mathrm{NC}_2(1, \dots, 2n)$. Then, there exist a $m \in \{1, \dots, n\}$ such that $z_m = e_m + 1$. Hence, for $i(1), \dots, i(2n) \in \{1, \dots, N\}$ with $\ker(i) = V$, since $i(e_m) = i(z_m)$ the relations stated in (5.3) imply that the element $a_{i(e_m)}^{k(e_m)} a_{i(z_m)}^{k(z_m)}$ commutes with everything. Since V is a non-crossing partition, $V \backslash (e_m, z_m)$ can be identified with a non-crossing partition of $\{1, \dots, 2n-2\}$ and as a consequence it will also have an interval block. Therefore, using that ϕ of naturally ordered products factorizes, by induction we have that

$$\phi\big(a_{i(1)}^{k(1)} \dots a_{i(2n)}^{k(2n)}\big) = \phi(a^{k(e_1)} a^{k(z_1)}) \dots \phi(a^{k(e_n)} a^{k(z_n)}),$$

which is independent of N. Taking into account that

$$\lim_{N\to\infty} \frac{A_{N;r/2}}{N^{r/2}} = 1, \quad \text{for } r = 2n,$$

and $t(V) = 1$ for $V \in \mathrm{NC}_2(1,...,2n)$, we see that for non-crossing partitions the corresponding limit stated in the right hand side of (5.5) is equal to the expression that we claimed.

It remains to compute the corresponding limit for crossing partitions $V = \{(e_1, z_1), \dots, (e_n, z_n)\} \in P_2(1, \dots, 2n)$. We consider such a V and $i(1), \dots, i(2n) \in \{1, \dots, N\}$ such that $\ker(i) = V$. Then, using that $a_{i(e_m)}^{k(e_m)} a_{i(z_m)}^{k(z_m)}$ commutes with everything and applying the relations $a_i^k a_j^l = s(i, j) a_j^l a_i^k$ in order to bring the factors $a_{i(e_p)}^{k(e_p)}, a_{i(z_p)}^{k(z_p)}$ $(p \in \{1, \dots, n\})$ in neighbouring positions in case where there exist $q \in \{1, \dots, n\}$ with $(\min\{p, q\}, \max\{p, q\}) \in I(V)$, we have

$$a_{i(1)}^{k(1)} \dots a_{i(2n)}^{k(2n)} = a_{i(e_1)}^{k(e_1)} a_{i(z_1)}^{k(z_1)} \dots a_{i(e_n)}^{k(e_n)} a_{i(z_n)}^{k(z_n)} \prod_{(k,l)\in I(V)} s(i(e_k), i(e_l)).$$

Since ϕ of naturally ordered products factorizes we have

$$\phi\big(a_{i(1)}^{k(1)} \dots a_{i(2n)}^{k(2n)}\big) = \prod_{j=1}^{n} \phi(a^{k(e_j)} a^{k(z_j)}) \prod_{(k,l)\in I(V)} s(i(e_k), i(e_l))$$

which implies

$$\lim_{N\to\infty} \frac{1}{N^n} \sum_{\substack{i(1),\dots,i(2n)=1\\ \ker(i)=V}}^{N} \phi\big(a_{i(1)}^{k(1)} \dots a_{i(2n)}^{k(2n)}\big) = \phi(a^{k(e_1)} a^{k(z_1)}) \dots \phi(a^{k(e_n)} a^{k(z_n)}) t(V).$$

Therefore, the assertion holds. □

In the context of the previous theorem, we make the additional assumption

$$\begin{pmatrix} \phi(a^{-1}a^{-1}) & \phi(a^{-1}a^{1}) \\ \phi(a^{1}a^{-1}) & \phi(a^{1}a^{1}) \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}, \quad \text{for some } C \geq 0.$$

Consider $f \in L^2(\mathbb{R})$ with norm $\|f\|^2 = C$. Then, the relation

$$t(V) = \begin{cases} 1, & \text{if } s(i,j) = 1 \text{ for all } i,j \in \mathbb{N} \\ (-1)^{i(V)}, & \text{if } s(i,j) = -1 \text{ for all } i,j \in \mathbb{N} \end{cases}$$

implies that, for all $r \in \mathbb{N}$ and all $k(1), \ldots, k(r) \in \{-1, 1\}$,

$$\lim_{N\to\infty} \phi\big(S_N^{k(1)} \ldots S_N^{k(r)}\big) = \begin{cases} \rho_1(c^{k(1)}(f) \ldots c^{k(r)}(f)), & \text{if } s(i,j) = 1 \text{ for all } i,j \in \mathbb{N} \\ \\ \rho_{-1}(c^{k(1)}(f) \ldots c^{k(r)}(f)), & \text{if } s(i,j) = -1 \text{ for all } i,j \in \mathbb{N}. \end{cases}$$

We shall now examine the computation of $t(V)$, in order to verify whether under appropriate conditions for the signs $s(i,j)$, the distribution that arises from the non-commutative central limit theorem is equal to the distribution of $c^{-1}(f) \in (\mathcal{C}_\mu, \rho_\mu)$, for some $-1 < \mu < 1$. As we will see, a possibility is to interpolate stochastically, in the sense that, for every i, $j \in \mathbb{N}$, with $i < j$ and $s(i,j) \in \{-1, 1\}$, we choose in a probabilistic way if $s(i,j) = 1$ or $s(i,j) = -1$.

We recall that $\boldsymbol{s} = (s(i,j))_{i,j=1}^{\infty}$ is said an infinite symmetric matrix if $s(i,j) = s(j,i)$ and $s(i,i) = 0$, for all $i, j \in \mathbb{N}$.

Lemma 5.3. *Let $\mathcal{S}$ be the set of infinite symmetric matrices and let $\mathfrak{F}$ be the σ-algebra on $\mathcal{S}$, generated by the functions $\mathcal{S} \ni (s(k,l))_{k,l=1}^{\infty} = \boldsymbol{s} \mapsto s(i,j)$, for $i, j \in \mathbb{N}$. Moreover, let $\mathbb{P}$ be a probability measure on $(\mathcal{S}, \mathfrak{F})$ such that for $i > j$, the random variables $\mathcal{S} \ni \boldsymbol{s} \mapsto s(i,j)$ are independent, with probability distribution*

$$\mathbb{P}(\{\boldsymbol{s} \in \mathcal{S} \mid s(i,j) = 1\}) = p, \qquad \mathbb{P}(\{\boldsymbol{s} \in \mathcal{S} \mid s(i,j) = -1\}) = q := 1 - p.$$

Then, for almost all $\boldsymbol{s}$, we have for all $n \in \mathbb{N}$ and all $V \in P_2(1, \ldots, 2n)$,

$$t(V) = (p - q)^{i(V)}.$$

Proof. Let n be a positive integer and let $V = \{(e_1, z_1), \ldots, (e_n, z_n)\} \in P_2(1, \ldots, 2n)$ with $e_1 < \cdots < e_n$ and $e_k < z_k$ for all $k = 1, \ldots, n$. For $N \in \mathbb{N}$, we define the random variables $X_N : \mathcal{S} \to \mathbb{R}$, by demanding for all $\boldsymbol{s} \in \mathcal{S}$ that,

$$X_N(\boldsymbol{s}) := \frac{1}{N^n} \sum_{\substack{i(e_1), \ldots, i(e_n) = 1 \\ i(e_j) \neq i(e_m) \text{ for } j \neq m}}^{N} \prod_{(k,l) \in I(V)} s(i(e_k), i(e_l)).$$

For $i>j$, the assumptions about the probability distriburion of $\mathcal{S}\ni \boldsymbol{s}\mapsto s(i,j)$ imply that $\int_{\mathcal{S}} s(i,j)\mathbb{P}(ds)=p-q$. Therefore, using our assumption about the independence of the random variables $\mathcal{S}\ni s\mapsto s(i,j)$ $(i,j\in\mathbb{N}, i>j)$, we have,

$$\begin{aligned}
\mathbb{E}[X_N] &= \frac{1}{N^n}\sum_{\substack{i(e_1),\ldots,i(e_n)=1\\ i(e_j)\neq i(e_m)\text{ for } j\neq m}}^{N} \int_{\mathcal{S}}\prod_{(k,l)\in I(V)} s(i(e_k),i(e_l))\mathbb{P}(ds)\\
&= \frac{1}{N^n}\sum_{\substack{i(e_1),\ldots,i(e_n)=1\\ i(e_j)\neq i(e_m)\text{ for } j\neq m}}^{N} \prod_{(k,l)\in I(V)} \int_{\mathcal{S}} s(i(e_k),i(e_l))\mathbb{P}(ds)\\
&= \frac{N(N-1)\ldots(N-n+1)}{N^n}(p-q)^{i(V)}\\
&\to (p-q)^{i(V)},\ \text{as } N\to\infty.
\end{aligned}$$

Hence, it suffices to show that $\lim_{N\to\infty} X_N=\lim_{N\to\infty}\mathbb{E}[X_N]$, for almost all $\boldsymbol{s}$. We will prove the strongest condition

$$\lim_{N\to\infty}\mathbb{P}\Big(\Big\{\sup_{M\geq N}|X_M-\mathbb{E}[X_M]|\geq\alpha\Big\}\Big)=0,\quad \text{for all } \alpha>0.$$

For $\alpha>0$, we have

$$\begin{aligned}
\mathbb{P}\Big(\Big\{\sup_{M\geq N}|X_M-\mathbb{E}[X_M]|\geq\alpha\Big\}\Big) &\leq \mathbb{P}\Big(\bigcup_{M=N}^{\infty}\Big\{|X_M-\mathbb{E}[X_M]|\geq\frac{\alpha}{2}\Big\}\Big)\\
&\leq \sum_{M=N}^{\infty}\mathbb{P}\Big(\Big\{|X_M-\mathbb{E}[X_M]|\geq\frac{a}{2}\Big\}\Big)\\
&\leq \frac{4}{a^2}\sum_{M=N}^{\infty}\mathrm{Var}[X_M],
\end{aligned}$$

where in the last inequality we used Chebyshev's inequality. Now, using the formula $\mathrm{Var}[X_M]=\mathbb{E}[X_M^2]-(\mathbb{E}[X_M])^2$, we get

$$\begin{aligned}
\mathrm{Var}[X_M]=\frac{1}{M^{2n}}&\sum_{\substack{i(e_1),\ldots,i(e_n)=1\\ i(e_j)\neq i(e_m)\text{ for } j\neq m}}^{M}\ \sum_{\substack{j(e_1),\ldots,j(e_n)=1\\ j(e_i)\neq j(e_m)\text{ for } i\neq m}}^{M}\\
&\cdot\Big\{\int_{\mathcal{S}}\prod_{(k,l)\in I(V)} s(i(e_k),i(e_l))s(j(e_k),j(e_l))\mathbb{P}(ds)-(p-q)^{2i(V)}\Big\}.
\end{aligned}$$

Let $(i(e_1),\ldots,i(e_n))$ and $(j(e_1),\ldots,j(e_n))$ be allowed indices. Since the random variables $\mathcal{S}\ni s\mapsto s(i,j)$ $(i,j\in\mathbb{N},i>j)$ are independent and $s(k,l)\in\{-1,1\}$, almost surely for $k\neq l$, we get that the corresponding integral is equal to $(p-q)^r$, for some $r\in\{0,\ldots,2i(V)\}$. Therefore, it has finitely many possible values which does not depend on M. If $r=2i(V)$, then such indices do not contribute to the sum. By the independence condition, $r<2i(V)$ only if there exist $(k,l),(\acute{k},\acute{l})\in I(V)$ (which may be equal) such that $(i(e_k),i(e_l))=(j(e_{\acute{k}}), j(e_{\acute{l}}))$ or $(i(e_k),i(e_l))=(j(e_{\acute{l}}),j(e_{\acute{k}}))$. The number of such $(i(e_1),\ldots,i(e_n)),(j(e_1),\ldots,$

$j(e_n))$ is of order M^{2n-2}. Hence, from the above, we deduce that there exist $C>0$ such that $\operatorname{Var}[X_M]\le C/M^2$, which implies,

$$\lim_{N\to\infty}\mathbb{P}\bigg(\bigg\{\sup_{M\ge N}|X_M-\mathbb{E}[X_M]|\ge\alpha\bigg\}\bigg)\le\lim_{N\to\infty}\frac{4}{\alpha^2}\sum_{M=N}^{\infty}\frac{C}{M^2}=0,$$

because $\sum_{M=1}^{\infty}\frac{1}{M^2}<+\infty$. □

The non-commutative distribution that arises from the non-commutative central limit theorem under the additional assumptions $t(V)=(p-q)^{i(V)}$ and $\phi(a^{-1}a^{-1})=\phi(a^1a^{-1})=\phi(a^1a^1)=0$, $\phi(a^{-1}a^1)=C$ $(C\ge0)$, is exactly the non-commutative distribution of $c^-(f)\in(\mathcal{C}_\mu,\rho_\mu)$, in the case $\mu=p-q$ and $f\in L^2(\mathbb{R})$ with $\|f\|^2=C$. Therefore, Lemma 5.4 shows that our stochastic interpolation gives that the non-commutative distribution of $c^-(f)\in(\mathcal{C}_\mu,\rho_\mu)$ can be derived from a central limit theorem.

A possible choice of a $*$-probability space and of a sequence of non-commutative random variables, such that the assumptions of Theorem 5.2 are satisfied, are the following: Consider the $*$-probability space $(\mathcal{A},\phi)=\left(\bigotimes_{j=1}^{\infty}M_2(\mathbb{C}),\bigotimes_{j=1}^{\infty}\mathrm{tr}_w\right)$, where for $w\in[0,1]$, $\mathrm{tr}_w\colon M_2(\mathbb{C})\to\mathbb{C}$ is the state given by the density matrix $W:=\begin{pmatrix}w&0\\0&1-w\end{pmatrix}$, i.e. $\mathrm{tr}_w(A):=\mathrm{tr}(W\cdot A)$. We also consider the non-commutative random variables $a_i\in\bigotimes_{j=1}^{N}M_2(\mathbb{C})\subset\bigotimes_{j=1}^{\infty}M_2(\mathbb{C})$, where

$$a_i=1_{2\times2}\otimes\cdots\otimes1_{2\times2}\otimes\underset{i\text{-th}}{a}\otimes1_{2\times2}\otimes\cdots\otimes1_{2\times2},\ \text{with } a=\begin{pmatrix}0&1\\0&0\end{pmatrix}\text{ and }1_{2\times2}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Because $a\cdot1_{2\times2}=1_{2\times2}\cdot a$, $a^*\cdot1_{2\times2}=1_{2\times2}\cdot a^*$, we have that a_i^k and a_j^l commute for $i\ne j$, $k,l\in\{1,*\}$ and it is also easy to note that ϕ of naturally ordered products factorizes. Since $\phi(a_i\,a_i)=\phi(a_i^*a_i)=\phi(a_i^*a_i^*)=0$, $\phi(a_i\,a_i^*)=1$, for $w=1$, Theorem 5.2 implies that the sum $N^{-1/2}(a_1+\cdots+a_N)\in\left(\bigotimes_{j=1}^{\infty}M_2(\mathbb{C}),\bigotimes_{j=1}^{\infty}\mathrm{tr}_1\right)$ converges in distribution to $c^-(f)\in(\mathcal{C}_1,\rho_1)$, for $f\in L^2(\mathbb{R})$ with $\|f\|=1$.

For the anti-commuting case, we can consider the non-commutative random variables $b_i\in\left(\bigotimes_{j=1}^{\infty}M_2(\mathbb{C}),\bigotimes_{j=1}^{\infty}\mathrm{tr}_w\right)$, where

$$b_i=\sigma_3\otimes\cdots\otimes\sigma_3\otimes\underset{i\text{-th}}{a}\otimes1_{2\times2}\otimes\cdots\otimes1_{2\times2},\ \text{with } \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Because $a\cdot\sigma_3=-\sigma_3\cdot a$, $a^*\cdot\sigma_3=-\sigma_3\cdot a^*$, we have that b_i^k and b_j^l anticommute for $i\ne j$ and $k,l\in\{1,*\}$. Similarly with the above example, we have that ϕ of naturally ordered products factorizes and $\phi(b_i\,b_i)=\phi(b_i^*b_i)=\phi(b_i^*b_i^*)=0$, $\phi(b_i\,b_i^*)=1$, for $w=1$. Therefore, Theorem 5.3 implies that the sum $N^{-1/2}(b_1+\cdots+b_N)\in\left(\bigotimes_{j=1}^{\infty}M_2(\mathbb{C}),\bigotimes_{j=1}^{\infty}\mathrm{tr}_1\right)$ converges in distribution to $c^-(f)\in(\mathcal{C}_{-1},\rho_{-1})$, for $f\in L^2(\mathbb{R})$ with $\|f\|=1$.

The Lemma 5.3 shows that if we interpolate stochastically, in the sense that we consider $c_i \in \left(\bigotimes_{j=1}^{\infty} M_2(\mathbb{C}), \bigotimes_{j=1}^{\infty} \mathrm{tr}_1 \right)$, where

$$c_i = c \otimes \cdots \otimes c \otimes a \otimes 1_{2\times 2} \otimes \cdots \otimes 1_{2\times 2},$$

and we randomly choose $c = 1_{2\times 2}$ with probability p or $c = \sigma_3$ with probability $q := 1 - p$, then we have that almost surely, the sum $N^{-1/2}(c_1 + \cdots + c_N) \in \left(\bigotimes_{j=1}^{\infty} M_2(\mathbb{C}), \bigotimes_{j=1}^{\infty} \mathrm{tr}_1 \right)$ converges in distribution to $c^-(f) \in (\mathcal{C}_\mu, \rho_\mu)$, for $f \in L^2(\mathbb{R})$, with $\| f \| = 1$ and $\mu = (p-q)^{i(V)}$.

5.3 μ-Gaussian and μ-Poisson distribution

Now, we consider $\mu \in (-1, 1)$ and we want to find the probability measure ν_μ on $\mathbb{R}$ (with compact support) which characterizes the non-commutative distribution of $c^+(f) + c^-(f) \in (\mathcal{C}_\mu, \rho_\mu)$, where $f \in L^2(\mathbb{R})$ with $\| f \| = 1$.

Definition 5.4. *Let $(\mathcal{A}, \phi)$ be a $*$-probability space. A self-adjoint random variable $a \in \mathcal{A}$ is called standard μ-Gaussian variable if it's moments are of the form*

$$\phi(a^k) = \rho_\mu[(c^+(f) + c^-(f))^k] = \begin{cases} 0, & \text{if } k \text{ is odd} \\ \sum\limits_{V \in P_2(1, \ldots, 2n)} \mu^{i(V)}, & \text{if } k = 2n. \end{cases}$$

Our attention concentrates on finding the measure ν_μ which characterizes the non-commutative distribution of standard μ-Gaussian variables. We recall that in classical probability theory, a standard Gaussian measure γ can be characterized by the Hermite polynomials H_k, where

$$H_k(x) := (-1)^k \exp\left(\frac{x^2}{2} \right) \frac{\mathrm{d}^k}{\mathrm{d}x^k} \exp\left(-\frac{x^2}{2} \right), \quad k \geq 0.$$

For the sequence $\{H_k\}_{k=0}^{\infty}$ we have,

$$\int_{\mathbb{R}} H_n(x) H_m(x) \gamma(dx) = n!\, \delta_{n,m}, \quad \text{for all } n, m \geq 0 \tag{5.6}$$

and for $n \geq 1$ they satisfy the recurrence relations $H_{n+1}(x) = x H_n(x) - n H_{n-1}(x)$ [...]. Note that the relation stated in (5.6) allows us to compute explicitly all moments of γ. Similarly, in order to find ν_μ we will rely on μ-analogues of Hermite polynomials.

Before we define μ-Hermite polynomials, we first introduce some notation. For $n \in \mathbb{N} \cup \{0\}$ we put

$$[n]_\mu := \frac{1 - \mu^n}{1 - \mu} = 1 + \mu + \cdots + \mu^{n-1}, \quad \text{for } n > 0$$

and $[0]_\mu := 0$. Then, we define the μ-factorial

$$[n]_\mu := \begin{cases} [1]_\mu \ldots [n]_\mu, & \text{for } n > 0 \\ 0, & \text{for } n = 0. \end{cases}$$

We also define the μ-binomial coefficient

$$\binom{n}{k}_\mu := \frac{[n]_\mu!}{[k]_\mu!\,[n-k]_\mu!} = \prod_{i=1}^{n-k} \frac{1-\mu^{k+i}}{1-\mu^i}.$$

The importance of the above definitions comes from the fact that for indeterminates x, y such that $xy = q\,y\,x$, we have the μ-binomial formula,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_\mu y^k x^{n-k}, \quad \text{for all } n \in \mathbb{N}. \tag{5.7}$$

The equality stated in (5.7) can be easily verified by induction and by the easily checked equality

$$\binom{n}{k}_\mu + \mu^k \binom{n}{k+1}_\mu = \binom{n+1}{k+1}_\mu.$$

Since bosonic relations are connected with the Gaussian distribution, taking into account the μ-binomial formula and the fact that the operators $c^-(f), c^+(g)$ $(f, g \in L^2(\mathbb{R}))$ satisfy the generalized commutation relations, we define μ-Hermite polynomials in a similar way.

Definition 5.5. *For $\mu \in (-1,1)$, we define the μ-Hermite polynomials $\{H_n^{(\mu)}\}_{n=0}^\infty$ to be the one variable polynomials which are determined by*

$$H_0^{(\mu)}(x) = 1, \qquad H_1^{(\mu)}(x) = x,$$

and

$$H_{n+1}^{(\mu)}(x) = x H_n^{(\mu)}(x) - [n]_\mu H_{n-1}^{(\mu)}(x), \qquad \text{for all } n \geq 1. \tag{5.8}$$

As we will see, similarly with the classical case, the μ-Hermite polynomials $\{H_n^{(\mu)}\}_{n=0}^\infty$ are orthogonal elements of $L^2(\nu_\mu)$.

Remark 5.6. For the fermionic case $(\mu = -1)$ where it is valid

$$\nu_{-1}(dx) = \frac{1}{2}(\delta_{-1}(dx) + \delta_1(dx)),$$

we see that the corresponding sequence of (-1)-Hermite polynomials is essentialy finite,

$$H_0^{(-1)}(x) = 1 \quad \text{and} \quad H_1^{(-1)}(x) = x.$$

Lemma 5.7. *Let $f \in L^2(\mathbb{R})$ with $\|f\| = 1$. Then, for all $n \geq 0$ we have,*

$$H_n^{(\mu)}(c^+(f) + c^-(f))\Omega = f^{\otimes n}.$$

Proof. We will show the claim by induction on $n \in \mathbb{N}$. For $n = 0, 1$ we have nothing to prove. Let $n \geq 2$ and we assume that the assertion holds for every $k < n$. Since $\|f\| = 1$, by the definition of $c^-(f)$ we have $c^-(f) f^{\otimes m} = [m]_\mu f^{\otimes(m-1)}$ for every $m \in \mathbb{N}$. Hence, our induction hypothesis and the relations (5.8) imply that

$$\begin{aligned} H_n^{(\mu)}(c^+(f) + c^-(f))\Omega &= (c^+(f) + c^-(f)) f^{\otimes(n-1)} - [n-1]_\mu f^{\otimes(n-2)} \\ &= f^{\otimes n} + c^-(f) f^{\otimes(n-1)} - [n-1]_\mu f^{\otimes(n-2)} \\ &= f^{\otimes n}. \end{aligned}$$

Therefore, the assertion holds. □

Let $f \in L^2(\mathbb{R})$ with $\|f\| = 1$. Then, the previous lemma implies that for the self-adjoint operator $x = c^+(f) + c^-(f)$ we have,

$$\begin{aligned} \rho_\mu\big(H_n^{(\mu)}(x)H_m^{(\mu)}(x)\big) = \big\langle \Omega, H_n^{(\mu)}(x)H_m^{(\mu)}(x)\Omega\big\rangle_\mu &= \big\langle H_n^{(\mu)}(x)\Omega, H_m^{(\mu)}(x)\Omega\big\rangle_\mu \\ &= \langle f^{\otimes n}, f^{\otimes m}\rangle_\mu \\ &= [n]_\mu!\, \delta_{n,m}, \end{aligned}$$

which implies that

$$\int_{\mathbb{R}} H_n^{(\mu)}(t)H_m^{(\mu)}(t)\nu_\mu(dt) = [n]_\mu!\, \delta_{n,m}.$$

Therefore, by the above formula, the moments of ν_μ can be calculated and it arises [...] that ν_μ is the measure on the interval $[-2/\sqrt{1-\mu}, 2/\sqrt{1-\mu}]$ given by

$$\nu_\mu(dt) = \frac{\sqrt{1-\mu}}{\pi} \sin\theta \prod_{n=1}^{\infty} (1-\mu^n)|1-\mu^n e^{2i\theta}|^2 dt,$$

where

$$t = \frac{2}{\sqrt{1-\mu}} \cos\theta \qquad \text{with } \theta \in [0, \pi].$$

In the first section we saw that the sum of the one-sided shift operator on $l^2(\mathbb{N} \cup \{0\})$ with it's adjoint gives a standard semicircular variable (i.e. a standard 0-Gaussian variable). More generally, we can obtain standard μ-Gaussian variables by considering the sum of a weighted shift operator on $l^2(\mathbb{N} \cup \{0\})$ with it's adjoint. If $\{e_n\}_{n \geq 0}$ is the standard orthonormal basis on $l^2(\mathbb{N} \cup \{0\})$, the weighted shift operator S_μ is determined by

$$S_\mu e_n = \sqrt{[n+1]_\mu}\, e_{n+1} \quad (n \geq 0).$$

The adjoint operator S_μ^* of S_μ is determined by

$$S_\mu^* e_n = \begin{cases} \sqrt{[n]_\mu}\, e_{n-1}, & \text{for } n \geq 1 \\ 0, & \text{for } n = 0. \end{cases}$$

The operator S_μ is a bounded operator with norm

$$\|S_\mu\| = \frac{1}{\sqrt{1-\mu}}, \quad \text{for } \mu \in [0, 1) \quad \text{and} \quad \|S_\mu\| = 1, \quad \text{for } \mu \in (-1, 0].$$

Moreover, the operators S_μ, S_μ^* satisfy the generalized commutation relations, i.e. we have $S_\mu^* S_\mu - \mu S_\mu S_\mu^* = 1$.

Lemma 5.8. *The operator $S_\mu + S_\mu^* \in B(l^2(\mathbb{N} \cup \{0\}))$ is a standard μ-Gaussian variable with respect to the vacuum state $\phi_0(\cdot) = \langle e_0, \cdot e_0\rangle_{l^2}$.*

Proof. First, we will show that for all $n \geq 0$ we have,

$$H_n^{(\mu)}(S_\mu + S_\mu^*)e_0 = \sqrt{[n]_\mu!}\, e_n.$$

For $n=0,1$ we have nothing to prove. Let $n\geq 2$ and we assume that the claim holds for all $k<n$. Then, our induction hypothesis and the relations (5.8) imply that,

$$\begin{aligned} H_n^{(\mu)}(S_\mu+S_\mu^*)e_n &= \sqrt{[n-1]_\mu!}\,(S_\mu+S_\mu^*)e_{n-1}-[n-1]_\mu\sqrt{[n-2]_\mu!}\,e_{n-2} \\ &= \sqrt{[n-1]_\mu!}\sqrt{[n]_\mu}\,e_n+\sqrt{[n-1]_\mu}\sqrt{[n-1]_\mu!}\,e_{n-2} \\ &\quad -[n-1]_\mu\sqrt{[n-2]_\mu!}\,e_{n-2} \\ &= \sqrt{[n]_\mu!}\,e_n. \end{aligned}$$

Therefore, for the self-adjoint operator $x=S_\mu+S_\mu^*$ we have,

$$\begin{aligned} \phi_0\big(H_n^{(\mu)}(x)H_m^{(\mu)}(x)\big)=\big\langle e_0,H_n^{(\mu)}(x)H_m^{(\mu)}(x)e_0\big\rangle_{l^2} &= \big\langle H_n^{(\mu)}(x)e_0,H_m^{(\mu)}(x)e_0\big\rangle_{l^2} \\ &= \sqrt{[n]_\mu!}\sqrt{[m]_\mu!}\langle e_n,e_m\rangle_{l^2} \\ &= [n]_\mu!\,\delta_{n,m}. \end{aligned}$$

Hence, the assertion holds. ☐

We saw that the μ-Gaussian distribution ν_μ can be characterized as the orthogonalizing probability measure for the sequence of μ-Hermite polynomials. We recall that the classical Poisson distribution is the orthogonalizing probability measure for the sequence of Charlier-Poisson polynomials [...]. Inspired by that, we can define the μ-Poisson distribution.

For $\mu\in(-1,1)$ and $\lambda>0$ we define the μ-analogues of the Charlier-Poisson polynomials, as the polynomials $C_n^{(\mu)}$ $(n\in\mathbb{N}\cup\{0\})$, determined by

$$C_0^{(\mu)}(x)=1,\qquad C_1^{(\mu)}(x)=x-\lambda$$

and

$$C_{n+1}^{(\mu)}(x)=(x-\lambda-[n]_\mu)C_n^{(\mu)}(x)-\lambda[n]_\mu C_{n-1}^{(\mu)}(x),\quad \text{for all } n\geq 1. \tag{5.9}$$

Definition 5.9. *Let $\mu\in(-1,1)$ and $\lambda>0$. We define the μ-Poisson distribution with parameter λ, as the probability measure $\gamma_{\mu,\lambda}$ on $\mathbb{R}$, determined by the relations*

$$\int_{\mathbb{R}} C_n^{(\mu)}(x)C_m^{(\mu)}(x)\gamma_{\mu,\lambda}(dx)=\lambda^n[n]_\mu!\,\delta_{n,m}.$$

The existence of such a probability measure emerges from Favard's theorem. Similarly with the Gaussian case, a self-adjoint random variable a of a $*$-probability space $(\mathcal{A},\phi)$ will be said a μ-Poisson variable with parameter $\lambda>0$, if it's non-commutative distribution is characterized by the μ-Poisson distribution $\gamma_{\mu,\lambda}$, in the sense that

$$\phi(a^k)=\int_{\mathbb{R}} t^k\gamma_{\mu,\lambda}(dt),\quad \text{for all } k\in\mathbb{N}.$$

In the free case $(\mu=0)$, we saw that for $f\in L^2(\mathbb{R})$ with $\|f\|=1$, the self-adjoint operator $l^+(f)l^-(f)+\sqrt{\lambda}(l^+(f)+l^-(f))+\lambda\cdot 1$ is a free Poisson variable with parameter $\lambda>0$, with respect to the vacuum state. Therefore, that's a similar situation with the bosonic case, where a non-commuatative realization of the classical Poisson distribution with parameter λ can be given by considering the operator $a^+(f)a^-(f)+\sqrt{\lambda}(a^+(f)+a^-(f))+\lambda\cdot 1$ [...]. We recall that $a^+(f)$ and $a^-(f)$ stand for the bosonic creation operator and the bosonic annihilation operator, respectively. In the same way we will give the operator on the μ-Fock space of $L^2(\mathbb{R})$ which is a μ-Poisson variable with parameter λ, with respect to the vacuum state ρ_μ.

Theorem 5.10. *Let $f\in L^2(\mathbb{R})$ with $\|f\|=1$ and let λ be a positive number. Then, the self-sdjoint operator $c^+(f)c^-(f)+\sqrt{\lambda}(c^+(f)+c^-(f))+\lambda\cdot 1\in B(\mathcal{F}_\mu(L^2(\mathbb{R})))$ is a μ-Poisson variable with parameter λ, with respect to ρ_μ.*

Proof. Let $x=c^+(f)c^-(f)+\sqrt{\lambda}(c^+(f)+c^-(f))+\lambda\cdot 1$. First, we will show that

$$C_n^{(\mu)}(x)\Omega=\sqrt{\lambda^n}\,f^{\otimes n},\qquad \text{for all } n\geq 0. \tag{5.10}$$

For $n=0,1$ we have nothing to prove. Let $n\geq 2$ and we assume that the equality stated in (5.10) holds for all $k<n$. By the definition of $c^+(f), c^-(f)$ we have,

$$c^+(f)f^{\otimes m}=f^{\otimes(m+1)},\ \ c^-(f)f^{\otimes m}=[m]_\mu f^{\otimes(m-1)} \text{ and } c^+(f)c^-(f)f^{\otimes m}=[m]_\mu f^{\otimes m},$$

for all $m\in\mathbb{N}$. Then, our induction hypothesis and the relations (5.9) imply that

$$\begin{aligned}
C_n^{(\mu)}(x)\Omega &= (x-(\lambda+[n-1]_\mu)\cdot 1)C_{n-1}^{(\mu)}(x)\Omega-\lambda[n-1]_\mu C_{n-2}^{(\mu)}(x)\Omega\\
&= \sqrt{\lambda^{n-1}}xf^{\otimes(n-1)}-(\lambda+[n-1]_\mu)\sqrt{\lambda^{n-1}}f^{\otimes(n-1)}-\lambda[n-1]_\mu\sqrt{\lambda^{n-2}}f^{\otimes(n-2)}\\
&= \sqrt{\lambda^{n-1}}[n-1]_\mu f^{\otimes(n-1)}+\sqrt{\lambda^n}f^{\otimes n}+\sqrt{\lambda^n}[n-1]_\mu f^{\otimes(n-2)}+\lambda\sqrt{\lambda^{n-1}}f^{\otimes(n-1)}\\
&\quad -\lambda\sqrt{\lambda^{n-1}}f^{\otimes(n-1)}-\sqrt{\lambda^{n-1}}[n-1]_\mu f^{\otimes(n-1)}-\sqrt{\lambda^n}[n-1]_\mu f^{\otimes(n-2)}\\
&= \sqrt{\lambda^n}f^{\otimes n}.
\end{aligned}$$

Hence, we have,

$$\begin{aligned}
\rho_\mu\big(C_n^{(\mu)}(x)C_m^{(\mu)}(x)\big)=\big\langle\Omega, C_n^{(\mu)}(x)C_m^{(\mu)}(x)\Omega\big\rangle_\mu &= \big\langle C_n^{(\mu)}(x)\Omega, C_m^{(\mu)}(x)\Omega\big\rangle_\mu\\
&= \sqrt{\lambda^{n+m}}\langle f^{\otimes n}, f^{\otimes m}\rangle_\mu\\
&= \lambda^n[n]_\mu!\,\delta_{n,m}.
\end{aligned}$$

Therefore, we deduce that the assertion holds. □

Since for $\mu=0$ we have $c^+(f)=l^+(f)$ and $c^-(f)=l^-(f)$, we see that the polynomials $\{C_n^{(0)}\}_{n=0}^\infty$ are the orthogonal polynomials for the free Poisson distribution.

In the same way, using an induction argument, we can show that the self-adjoint operator $S_\mu S_\mu^*+\sqrt{\lambda}(S_\mu+S_\mu^*)+\lambda\cdot 1\in B(l^2(\mathbb{N}\cup\{0\}))$ is a μ-Poisson variable with parameter λ, with respect to the vacuum state ϕ_0.

Remark 5.11. For the fermionic case ($\mu=-1$) we regard the fermionic Fock space as the (-1)-Fock space because, as we saw, for $\mu=-1$, we get a kernel of our scalar product $\langle\cdot,\cdot\rangle_\mu$ consisting of antisymmetric tensors. In this case we can consider the self-adjoint operator $b^+(f)b^-(f)+\sqrt{\lambda}(b^+(f)+b^-(f))+\lambda\cdot 1$ ($\|f\|=1, \lambda>0$) as the fermionic Poisson variable with parameter λ. We recall that $b^+(f)$ and $b^-(f)$ stand for the fermionic creation operator and the fermionic annihilation operator, respectively. Then, for

$$\gamma_\pm=\frac{2\lambda\pm 1+\sqrt{4\lambda+1}}{2}, \quad m_\pm=\frac{1}{2}\pm\frac{1}{2\sqrt{4\lambda+1}},$$

the probability measure $\gamma_{-1,\lambda}(dt):=m_+\delta_{\gamma_-}(dt)+m_-\delta_{\gamma_+}(dt)$ is the fermionic Poisson distribution, i.e. we have,

$$\rho_\wedge[(b^+(f)b^-(f)+\sqrt{\lambda}\,b^+(f)+\sqrt{\lambda}\,b^-(f)+\lambda\cdot 1)^k]=\int_{\mathbb{R}} t^k\,\gamma_{-1,\lambda}(dt), \text{ for all } k\in\mathbb{N}.$$

As a consequence, the corresponding orthogonal polynomials are given by,

$$C_0^{(-1)}(x)=1, \quad C_1^{(-1)}(x)=x-\lambda \quad \text{and} \quad C_2^{(-1)}(x)=x^2-(2\lambda+1)x+\lambda^2.$$

5.4 Invariance principle

In subsection 5.2, using our stochastic interpolation, we saw that the non-commutative distribution of $c^-(f)\in(\mathcal{C}_\mu,\rho_\mu)$ ($f\in L^2(\mathbb{R}), -1<\mu<1$) can be derived from a central limit theorem. Now we are interested in the generalization of Theorem 5.2 to an invariance principle which will lead us to the whole non-commutative process $(\mathcal{C}_\mu,\rho_\mu,(c_I^{-1},c_I^1)_{I\in\mathcal{R}})$. In order to obtain this generalization, we will mimic in the non-commutative framework, the standard procedure from Donsker's theorem, which allows a passage from classical random walk to classical Brownian motion. To be more precise, let $(\mathcal{A},\phi)$ be a $*$-probability space and let $(a_i)_{i\in\mathbb{N}}$ be a sequence of non-commutative random variables which satisfy the assumptions of Theorem 5.2. In this context, for $N\in\mathbb{N}$ and $I=[s,t)\subseteq\mathbb{R}$ we define,

$$S_I^{(N)}:=\frac{1}{\sqrt{N}}\sum_{i=[N\cdot s]+1}^{[N\cdot t]} a_i\,. \tag{5.11}$$

For $x\in\mathbb{R}$, we denote by $[x]$ the largest integer which is less or equal than x. For notation homogeneity, for every $N\in\mathbb{N}$ and $I=[s,t)\subseteq\mathbb{R}$, we define

$$S_I^{(N),-1}:=S_I^{(N)} \quad \text{and} \quad S_I^{(N),1}:=\left(S_I^{(N)}\right)^*.$$

Then for $i=-1,1$ and $N\in\mathbb{N}$, the definition of $S_I^{(N),i}$ extends to $I\in\mathcal{R}$, in such a way that the mapping $\mathcal{R}\ni I\mapsto\left(S_I^{(N),-1},S_I^{(N),1}\right)$ is finitely additive.

Using our stochastic interpolation, we will show that the non-commutative stochastic process $\left(\mathcal{A},\phi,\left(S_I^{(N),-1},S_I^{(N),1}\right)_{I\in\mathcal{R}}\right)$ almost surely converges in distribution to the non-commutative stochastic process $(\mathcal{C}_\mu,\rho_\mu,(c_I^{-1},c_I^1)_{I\in\mathcal{R}})$, as $N\to\infty$.

Theorem 5.12. *Let $\mathcal{S}$ be the set of infinite symmetric matrices $(s(i,j))_{i,j=1}^{\infty}$ and let $\mathfrak{F}$ be the σ-algebra on $\mathcal{S}$ generated by $\mathcal{S} \ni (s(k,l))_{k,l=1}^{\infty} = \boldsymbol{s} \mapsto s(i,j)$, for $i,j \in \mathbb{N}$. Moreover, let $\mathbb{P}$ be a probability measure on $(\mathcal{S}, \mathfrak{F})$ such that for $i > j$, the random variables $\mathcal{S} \ni \boldsymbol{s} \mapsto s(i, j)$ are independent, with probability distribution*

$$\mathbb{P}(\{\boldsymbol{s} \in \mathcal{S} \mid s(i,j) = 1\}) = p, \qquad \mathbb{P}(\{\boldsymbol{s} \in \mathcal{S} \mid s(i,j) = -1\}) = q := 1 - p.$$

Let $(\mathcal{A}, \phi)$ be a $*$-probability space and for every $\boldsymbol{s} = (s(i,j))_{i,j=1}^{\infty} \in \mathcal{S}$ with $s(i,j) \in \{-1, 1\}$, consider a sequence of non-commutative random variables $(a_i)_{i \in \mathbb{N}}$ such that we have for all $i, j \in \mathbb{N}$ with $i \neq j$ and all $k, l \in \{-1, 1\}$,

$$a_i^k a_j^l = s(i,j) a_j^l a_i^k$$

and such that ϕ of naturally ordered products factorizes. Assume further that for all $i \in \mathbb{N}$, we have

$$(\phi(a_i), \phi(a_i^*)) = (0, 0), \qquad \begin{pmatrix} \phi(a_i a_i) & \phi(a_i a_i^*) \\ \phi(a_i^* a_i) & \phi(a_i^* a_i^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then for almost all $(s(i,j))_{i,j=1}^{\infty}$, we have for all $r \in \mathbb{N}$, $I_1, \ldots, I_r \in \mathcal{R}$ and $k(1), \ldots, k(r) \in \{-1, 1\}$,

$$\lim_{N \to \infty} \phi\big(S_{I_1}^{(N),k(1)} \ldots S_{I_r}^{(N),k(r)}\big) = \rho_\mu\big(c_{I_1}^{k(1)} \ldots c_{I_r}^{k(r)}\big),$$

where $\mu := p - q$.

Proof. We will only sketch the proof since in order to prove this theorem we just have to use similar arguments as in the proofs of Theorem 5.2 and of Lemma 5.3. Let $r \in \mathbb{N}$, $I_1, \ldots, I_r \in \mathcal{R}$ and $k(1), \ldots, k(r) \in \{-1, 1\}$. Without loss of generality, we can assume that for $i \neq j$ we have $I_i = I_j$ or $I_i \cap I_j = \emptyset$ and that every interval I_i has the form $I_i = [s_i, t_i)$. This is so because for every $N \in \mathbb{N}$ the maps $I \mapsto \big(S_I^{(N),-1}, S_I^{(N),1}\big)$ and $I \mapsto (c_I^{-1}, c_I^1)$ are finitely additive. We have to calculate the following expression, as $N \to \infty$:

$$\begin{aligned} M_N &:= \phi\big(S_{I_1}^{(N),k(1)} \ldots S_{I_r}^{(N),k(r)}\big) \\ &= \frac{1}{N^{r/2}} \sum_{i(1)=[N\cdot s_1]+1}^{[N\cdot t_1]} \cdots \sum_{i(r)=[N\cdot s_r]+1}^{[N\cdot t_r]} \phi\big(a_{i(1)}^{k(1)} \ldots a_{i(r)}^{k(r)}\big). \\ &= \sum_{V \in P(1,\ldots,r)} \frac{1}{N^{r/2}} \sum_{\substack{(i(1),\ldots,i(r)) \in A_1 \times \cdots \times A_r \\ \ker(i) = V}} \phi\big(a_{i(1)}^{k(1)} \ldots a_{i(r)}^{k(r)}\big), \end{aligned}$$

where $A_i := \{[N \cdot s_i] + 1, [N \cdot s_i] + 2, \ldots, [N \cdot t_i]\}$ for all $i = 1, \ldots, r$. We consider the last sum separateley, for all $V \in P(1, \ldots, r)$ and we use the same arguments as in the proof of Theorem 5.2. Then, for r odd we have that M_N is equal to zero, as $N \to \infty$. Moreover, for $r = 2n$, only the partitions $V = \{(e_1, z_1), \ldots, (e_n, z_n)\} \in P_2(1, \ldots, 2n)$ contribute to the sum, as $N \to \infty$. Since the allowed indices $(i(1), \ldots, i(r))$ belong to a more complicated domain, using the same arguments as in Lemma 5.3, we can see that for $V = \{(e_1, z_1), \ldots, (e_n, z_n)\} \in P_2(1,\ldots,2n)$, the analogue of $t(V)$ is now equal to $(p - q)^{i(V)} \lambda(I_{e_1} \cap I_{z_1}) \ldots \lambda(I_{e_n} \cap I_{z_n})$, almost surely. Therefore, the assertion holds. $\square$

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