The FBSDE approach to stochastic quantisation

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Introduction

For the scope of these lectures we will agree that (scalar bosonic) Euclidean quantum fields are a class of probability measures on $\mathcal{S}'(\mathbb{R}^d)$ endowed with three basic properties: Euclidean invariance, reflection positivity (RP) and exponential integrability of some distributional norm. These ingredients are the base for the reconstruction theorem of Osterwalder–Schrader that allows to recover a quantum field theory from this probabilistic object. Working with a probability measure is much easier than dealing with unbounded operators in Hilbert spaces, the disadvantage is that the construction is quite far removed from the physical meaning of the various objects. The reconstruction performs a complex mathematical procedure to link the two sides.

We know of several examples of EQF in dimensions $d = 1, 2, 3$. Their mathematical understanding was initiated long ago, essentially in the '70 and '80 by Nelson, Glimm, Jaffe, Guerra, Rosen, Simon and many others, which established results about their existence and basic properties. The branch of mathematics which they developed goes with the name of constructive quantum field theory, in which the Euclidean approach played a major role. The book of Glimm and Jaffe [?], that of Rivasseau [?] and the reviews of Guerra [?] and of Summers [?] are useful starting points to capture the extents and the approaches used in the constructive EQFT literature.

We do not have a classification of possible EQFs. All the known examples, are obtained via perturbations of the simplest of them, the (massive) Gaussian free field μ (GFF). This is the Gaussian measure on $\mathcal{S}'(\mathbb{R}^d)$ with covariance given by

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = (m^2 - \Delta)^{-1}(x - y), \qquad x, y \in \mathbb{R}^d,$$

where Δ is the Laplacian in $\mathbb{R}^d$ and $m \geqslant 0$ is a parameter called the mass. When $d > 2$ we can take $m = 0$ but otherwise this expression makes sense as the kernel of a bilinear form on $\mathcal{S}(\mathbb{R}^d)$ only when $m > 0$. When $d > 1$ this kernel is singular on the diagonal $x = y$ which signals the fact that the measure μ is not supported on functions but only on spaces of distributions of negative regularity.

The GFF is reflection positive, Euclidean invariant and has a suitable (exponential) integrability. Its OS reconstruction gives rise to the simplest QFT, the free field of a family of non-interacting particles of spin zero and mass m.

With the hope to obtain interesting theories where the particles are interacting we are led to consider non-Gaussian EQF. Unfortunately Euclidean invariance and reflection positivity, taken together, impose strong restrictions on the class of perturbations which are allowed and in particular it is not know how to perturb the GFF in a way compatible with these two constraint without introducing an approximation procedure in which one of them is at least partially broken.

There are many ways to do this. We consider first to set up the theory on a finite discrete lattice $\mathbb{R}^d_{\varepsilon,L} = \{-L+\varepsilon, -L+2\varepsilon, \dots, L-\varepsilon, L\}^d$ of mesh size ε and size L, with the understanding that there exists $N \in \mathbb{N}$ such that $L = \varepsilon N$. We let $\mu_{\varepsilon,L}$ the centred Gaussian measure on $\mathbb{R}^{\mathbb{R}^d_{\varepsilon,L}}$ with covariance given by $(m^2 - \Delta_{\varepsilon,L})^{-1}(x - y)$, $x, y \in \mathbb{R}^d_{\varepsilon,L}$ where $\Delta_{\varepsilon,L}$ is the discrete periodic Laplacian on $\mathbb{R}^d_{\varepsilon,L}$.

Consider perturbations of the measure $\mu_{\varepsilon,L}$ given by the Gibbsian tilting

$$\nu^V_{\varepsilon,L}(\mathrm{d}\varphi) := \frac{e^{-V(\varphi)}\mu_{\varepsilon,L}(\mathrm{d}\varphi)}{\int e^{-V(\varphi)}\mu_{\varepsilon,L}(\mathrm{d}\varphi)} \tag{1}$$

where $V(\varphi) = \sum_{x \in \mathbb{R}^d_{\varepsilon,L}} v(\varphi(x))$ with $v \colon \mathbb{R} \to \mathbb{R}$ any nice function bounded from below. These measures are still (discretely) RP and Euclidean invariant. The proof goes via a simple computation involving the "locality" of the *interaction potential* V and the RP of the GFF (see e.g. [?] for details).

A natural strategy to obtain non-Gaussian EQF is therefore that of considering measures $\nu^V_{\varepsilon,L}$ and letting $L\to\infty$ and $\varepsilon\to 0$, possibly adjusting along the way the interaction potential V in order to obtain non-trivial limits. Once the limit is shown to exists, one can proceed to establish the basic properties, like RP, Euclidean invariance, regularity but also obtain more detailed information like decay of correlations, uniqueness of the limit, and eventually also an intrinsic characterisation of the limiting measure. This of course does not exhaust the list of interesting questions one could ask.

It turns out that this strategy for the existence of EQF works in many situations and we can make a list of the theories known so far for a scalar field:

- $d=1$: all v bounded below and sufficiently nice functions. This corresponds to (finite-dimensional) quantum mechanics, and a nice account of these measures can be found in the book [?].
- $d=2$: all even polynomials bounded from below ($P(\varphi)_2$ models) and also the following functions $\lambda\cos(\beta\varphi)$ (the sine-Gordon model), $\lambda\exp(\beta\varphi)$ (the exponential or Høegh–Krohn model or Liouville model), $\lambda\cosh(\beta\varphi)$.
- $d=3$: fourth order polynomials bounded from below (the φ^4_3 model).
- $d\geqslant 4$: only Gaussian (triviality of φ^4_d).

The $d=1$ setting is special in that the GFF is supported on a space of Hölder continuous functions, essentially matching the small scale regularity of the Brownian motion. This makes much easier the definition of the interaction potential and the only serious problem is to address the large scale limit $L\to\infty$.

The $P(\varphi)_2$ models has been the first models investigated with Euclidean methods, especially after the pioneering work of Symanzik [?] on Euclidean methods in QFT and the seminal paper of Guerra, Rosen and Simon [?] which showed the power of Euclidean methods over Hilbert space techniques.

As d increases however, the severity of the singularities due to the local behaviour of the Gaussian free field (GFF) μ become more important and this creates a great deal of technical problems, first solved by Glimm and Jaffe "by heroic work with cluster expansions" as Nelson puts it in [?]. For $d\geqslant 4$, the quartic interaction is trivial: the limiting theory with cutoffs removed is non-interacting, as established for $d>4$ by Aizenmann and recently for $d=4$ by Aizenmann and Duminil–Copin [?].

We will focus from now on on the case $d=2,3$ where the samples of the GFF are not supported on a function space. This creates difficulties in controlling the $\varepsilon\to 0$ limit and we need to introduce an ε-dependent interaction potential to obtain non-trivial limits. This particular modification of the limiting procedure is usually called *renormalisation*.

The main technical difficulty to deal with is the fact that the limiting measures are not necessarily absolutely continuous with respect to the GFF measure.

Plan of these lectures. These notes are organised in five parts.

Part I sets up the general framework: we recall the basics of stochastic quantisation and introduce the forward-backward stochastic differential equation (FBSDE) that will be our main tool throughout. The variational/Boué-Dupuis point of view on the FBSDE is explained and related to the Polchinski flow equation of renormalisation group theory.

Part II applies the FBSDE method to the two-dimensional sine-Gordon model. We prove existence and uniqueness of the infinite-volume measure via Γ-convergence of the variational problem and study the Laplace transform of the field using Gaussian multiplicative chaos techniques.

Part III treats φ^4 theories in dimensions two and three. After a detailed discussion of the Gaussian free field and Besov-space regularity, we develop the stochastic control approach in finite and infinite volume, proving tightness and non-Gaussianity of the limiting measures.

Part IV extends the FBSDE framework to Grassmann (fermionic) field theories. We develop the necessary Grassmann stochastic analysis, including non-commutative probability spaces, Grassmann Brownian martingales, and Itô calculus, and use it to construct measures corresponding to interacting fermionic models on the lattice.

Part V presents the Wilson–Itô diffusion (WIDE) approach, which replaces the Euclidean field with a stochastic process indexed by an observation resolution. This leads to a formulation of Euclidean quantum field theory via stochastic differential equations rather than functional integrals, with connections to the Polchinski renormalisation group equation.

Part I

Stochastic quantisation and the FBSDE approach

Chapter 1
Stochastic quantisation

We can summarise the basic idea of stochastic quantisation as:

realise the EQF as a random variable on some Gaussian probability space.

In this respect can be considered an "horizontal" perturbation instead of a "vertical" perturbation of a Gaussian measure, as would be the case in the Gibbsian description provided by eq. (1), that is we are looking at a push-forward which will realise a given random variable from a Gaussian and not for a description of the law of such random variable as a Radon–Nykodym density wrt. a Gaussian.

The study of the mathematics of stochastic quantisation is yet not well developed and there are still many open problems. The goal of these lectures is to provide an introduction to this subject by analysing a particular kind of stochastic quantisation for EQF which we will refer to as the *FBSDE method* where FBSDE stands for forward-backwards stochastic differential equation.

Before discussing the details of this method it can be useful to have a wider perspective on other stochastic quantisation methods for which we have some rigorous results.

- In **parabolic stochastic quantisation** the discretised EQF $\nu^V_{\varepsilon,L}$ is produced as the fixed-"time" marginal $\phi(t,\cdot)$ of an infinite-dimensional Langevin SPDE of the form

$$\partial_t\phi(t,x) = -(m^2-\Delta)\phi(t,x) - v'_\varepsilon(\phi(t,x)) + 2^{1/2}\xi(t,x), \qquad t\geqslant 0, x\in\mathbb{R}^d_{\varepsilon,L},$$

  where ξ is a space-time white noise, v' the derivative of v and where $2^{1/2}$ is put to ensure that, at least formally the measure $\nu^V_{\varepsilon,L}$ is invariant under the stochastic dynamics. Under suitable conditions on the potential, there exists a stationary solution of this equation with $\nu^V_{\varepsilon,L}$ as invariant measure. A mix of probabilistic and analytic techniques allow to deduce properties of the measure from those of the equation and of the white noise. This approach has been introduced in the seminal work of Parisi–Wu [?]. They used an equation of Langevin type to construct the correlations functions of Euclidean gauge theories without the need to fix a gauge. Since then, the idea was extensively explored in the physics literature as an alternative quantisation method for EQFTs. Damgaard and Hüffel reviewed these early developments in a report [?] and their subsequent book [?] presents a collection of influential papers. The first paper to rigorously study parabolic SQ is the work of Jona–Lasinio and Mitter [?] on the Φ^4_2 model in a finite volume. Then Da Prato–Debussche [?] obtained the first results on strong solutions to the Langevin SPDE for the Φ^4_2 in finite volume. The construction of global solutions in time of the dynamical Φ^4_2 model in finite volume was achieved in the work of Mourrat–Weber [?]. Only much later Hairer [?, ?] was able to devise a theory to tackle the more challenging singularities present in three dimensions (the Φ^4_3 model) and finally the first construction of the Φ^4_3 model in the full space via stochastic techniques was achieved via a improved understanding of the coercivity for the singular Langevin dynamics in the work of Gubinelli–Hofmanova [?] which used seminal ideas introduced by Mourrat–Weber [?].

- **Elliptic stochastic quantisation** exploits an elliptic equation of the form

$$-\partial_z^2\phi(z,x) = -(m^2-\Delta)\phi(t,x) - v'(\phi(z,x)) + (4\pi)^{1/2}\xi(z,x), \qquad z\in\mathbb{R}^2, x\in\mathbb{R}^d_{\varepsilon,L}$$

where z has to be a continuous variable and ξ a white noise in $\mathbb{R}^2 \times \mathbb{R}^d_{\varepsilon,L}$. Provided appropriate conditions holds for the potential v it is possible to prove that there exists a stationary solution of the SPDE for which the marginal law of $\phi(z,\cdot)$ for fixed $z \in \mathbb{R}^2$ is given by $\nu^V_{\varepsilon,L}$. This approach has been essentially introduce by Parisi–Sourlas [?] via the study of a supersymmetry hidden in the path integral representation of the solution to the elliptic equation. The rigorous analysis of this problem has been initiated by Klein–Landau–Perez [?] and partially completed by Albeverio–De Vecchi–Gubinelli [?, ?]. Proofs of SQ via elliptic equations have been established for the $\exp(\varphi)_2$ [?] and for the $\sinh(\varphi)_2$ [?] models. A supersymmetric analysis of parabolic stochastic quantisation can be found in [?].

These methods have been used in various recent works to provide a (still partial) alternative approach to the construction of Euclidean quantum fields. It is difficult here to give the full extent of the recent developments in the field of stochastic quantisation, in particular in the very interesting direction of Yang–Mills theories, which is nowadays an active field of research (see e.g. [?, ?]). For a more comprehensive account please check the recent review [?].

Chapter 2
The FBSDE approach

We turn now to the analysis of a further method of stochastic quantisation which will occupy us for the rest of these lectures. We describe it for the regularised measure $\nu^V_{\varepsilon,L}$ and then discuss, for several models, the removal of the cutoffs and other properties.

The method starts by fixing a continuous decomposition of the GFF conveniently provided by a Gaussian martingale $(X_t)_{t\in[0,\infty]}$ such that X_∞ is distributed as $\mu_{\varepsilon,L}$ and

$$\mathrm{d}\langle X(x), X(y)\rangle_t = \dot G_t(x,y)\mathrm{d}t$$

where $\dot G_t = \partial_t G_t$, $G_\infty = (m^2-\Delta)^{-1}$ and $G_0 = 0$.

For example we can take

$$G_t = \int_0^t \dot G_s^{1/2}\mathrm{d}s, \qquad \dot G_s^{1/2} = \frac{1}{s^2}e^{-(m^2-\Delta)/s}, \qquad t,s\geqslant 0. \tag{2.1}$$

Other choices are also possible, depending on the context.

For nice test functions f, we can write

$$\int f(\varphi)\nu_{\varepsilon,L}(\mathrm{d}\varphi) = \frac{\mathbb{E}[f(X_\infty)\,e^{-V(X_\infty)}]}{\mathbb{E}[e^{-V(X_\infty)}]} = \mathbb{E}_{\mathbb{Q}}[f(X_\infty)]$$

where we defined

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{e^{-V(X_\infty)}}{\mathbb{E}[e^{-V(X_\infty)}]}.$$

If we let

$$\mathcal{V}_t(\varphi) := -\log \mathbb{E}e^{-V(\varphi+X_\infty-X_t)},$$

then $\mathbb{E}[e^{-V(X_\infty)}|\mathcal{F}_t] = e^{-\mathcal{V}_t(X_t)}$ and $\mathcal{V}_t$ is C^2 and satisfies the non-linear PDE

$$\partial_t\mathcal{V}_t(\varphi) + \frac{1}{2}\dot G_t\mathrm{D}^2\mathcal{V}_t(\varphi) - \frac{1}{2}\dot G_t\mathrm{D}\mathcal{V}_t(\varphi)\,\mathrm{D}\mathcal{V}_t(\varphi) = 0, \tag{2.2}$$

which do not pose particular problem as we are dealing with an essentially finite-dimensional situation thanks to the presence of the cutoffs.

By Itô's formula

$$f_t(X_t)e^{-\mathcal{V}_t(X_t)} = \left[\partial_t f_t(X_t) - \mathrm{D}f(X_t)\dot G_t\mathrm{D}\mathcal{V}_t(X_t) + \frac{1}{2}\dot G_t\mathrm{D}^2 f(X_t)\right]e^{-\mathcal{V}_t(X_t)}\mathrm{d}t + \text{martingale}$$

so under the measure $\mathbb{Q}$ the process $(X_t)_{t\geqslant 0}$ is a weak solution to the SDE

$$\mathrm{d}X_t = -\dot G_t\mathrm{D}\mathcal{V}_t(X_t) + \mathrm{d}\hat X_t,$$

where $(\hat X_t)_{t\geqslant 0}$ is a Brownian martingale with quadratic variation

$$\mathrm{d}\langle\hat X\rangle_t = \dot G_t\mathrm{d}t.$$

Taking $f_t := \mathrm{D}\mathcal{V}_t$ we deduce also that $\mathrm{D}\mathcal{V}_t(X_t)$ is a martingale and therefore that

$$\mathrm{D}\mathcal{V}_t(X_t) = \mathbb{E}_{\mathbb{Q},t}[\mathrm{D}\mathcal{V}_\infty(X_\infty)] = \mathbb{E}_{\mathbb{Q},t}[\mathrm{D}V(X_\infty)] \tag{2.3}$$

where $\mathbb{E}_{\mathbb{Q},t} = \mathbb{E}_{\mathbb{Q}}[\cdot|\mathcal{F}_t]$. Under the condition that the time-dependent SDE

$$\mathrm{d}\Phi_t = -\dot G_t\mathrm{D}\mathcal{V}_t(\Phi_t)\mathrm{d}t + \mathrm{d}X_t, \tag{2.4}$$

has a unique strong solution, the above argument gives the following representation for the measure $\nu^V_{\varepsilon,L}$:

$$\int f(\varphi)\nu_{\varepsilon,L}(\mathrm{d}\varphi) = \mathbb{E}[f(\Phi_\infty)],$$

which establishes (2.4) as a stochastic quantisation formula.

When $(G_t)_t$ is chosen as to give a scale decomposition of the GFF, as in (2.1), the Markov process generated by (2.4) is called the Polchinski process [?].

Alternatively, the process $(\Phi_t)_t$ is described by the stochastic equation

$$\mathrm{d}\Phi_t = -\dot{G}_t\mathbb{E}_t[\mathrm{D}V(\Phi_\infty)] + \mathrm{d}X_t \tag{2.5}$$

as the conditional expectation $\mathbb{E}_t = \mathbb{E}[* \,|\, \mathcal{F}_t]$ is computed explicitly by the function $\mathrm{D}\mathcal{V}_t$ as $\mathrm{D}\mathcal{V}_t(\Phi_t) = \mathbb{E}_t[\mathrm{D}V(\Phi_\infty)]$ (cfr. eq. (2.3)).

This equation is not a standard SDE as it is written since it depends on its "terminal" value Φ_∞. The computation of the conditional expectation giving the drift can be cast into a backwards differential equation (BSDE), and in this sense the equation can be interpreted as a forward-backward SDE (FBSDE).

The advantage of this formulation is that it avoids to solve the PDE for $(\mathcal{V}_t)_t$.

Remark 2.1. Note that, by Itô formula,

$$\begin{aligned}\mathbb{E}_t[V(\Phi_\infty)] &= \mathcal{V}_t(\Phi_t) + \mathbb{E}_t\bigg[\int_t^\infty \Big(\partial_s\mathcal{V}_s(\Phi_s) - \mathrm{D}\mathcal{V}_s(\Phi_s)\dot{G}_s\mathrm{D}\mathcal{V}_s(\Phi_s) + \frac{1}{2}\dot{G}_s\mathrm{D}^2\mathcal{V}_s(\Phi_s)\Big)\mathrm{d}s\bigg] \\ &= \mathcal{V}_t(\Phi_t) - \frac{1}{2}\mathbb{E}_t\bigg[\int_t^\infty \mathrm{D}\mathcal{V}_s(\Phi_s)\dot{G}_s\mathrm{D}\mathcal{V}_s(\Phi_s)\mathrm{d}s\bigg] \leqslant \mathcal{V}_t(\Phi_t).\end{aligned}$$

Which in particular implies that $\mathbb{E}_t[V(\Phi_\infty)]$ is a supermartingale.

If V is bounded and smooth then $\mathrm{D}\mathcal{V}_t$ is Lipschitz and standard results on SDEs guarantee that the SDE (2.4) has a unique global strong solution.

We will have soon to move away from this easy setting since the removal of the cutoffs ε, L and the consideration of more general problems will drive us to less well-behaved functions.

Chapter 3
Variational structure

First of all we look for the variational structure underlying the FBSDE (2.5). Assume that $\dot{G}_t$ has a positive and symmetric square root $\dot{G}_t^{1/2}$ and let

$$u_t^* := -\dot{G}_t^{1/2}\mathbb{E}_t[\mathrm{D}V(\Phi_\infty)].$$

We can then write

$$\Phi_t = X_t + I_t(u^*), \qquad \text{with } u \mapsto I_t(u) := \int_0^t \dot{G}_s^{1/2} u_s \mathrm{d}s.$$

Let v be any L^2 adapted process and consider the scalar product

$$\begin{aligned}\mathbb{E}\int_0^\infty \langle v_s, u_s^*\rangle \mathrm{d}s &= -\mathbb{E}\int_0^\infty \big\langle v_s, \dot{G}_s^{1/2}\mathbb{E}_s[\mathrm{D}V(\Phi_\infty)]\big\rangle \mathrm{d}s \\ = -\mathbb{E}\int_0^\infty \big\langle \dot{G}_s^{1/2} v_s, \mathbb{E}_s[\mathrm{D}V(\Phi_\infty)]\big\rangle \mathrm{d}s &= -\mathbb{E}\int_0^\infty \big\langle \dot{G}_s^{1/2} v_s, \mathrm{D}V(\Phi_\infty)\big\rangle \mathrm{d}s \\ = -\mathbb{E}\langle I_\infty(v), \mathrm{D}V(\Phi_\infty)\rangle,&\end{aligned}$$

where we used the symmetry of $\dot{G}_s^{1/2}$. This computation implies that u^* is a critical point of the functional

$$\mathcal{J}^V(u) := \mathbb{E}\left[V(\Psi_\infty^u) + \frac{1}{2}\int_0^\infty \langle u_s, u_s\rangle \mathrm{d}s\right], \qquad \Psi_t^u := X_t + I_t(u).$$

For the solution $(\mathcal{V}_t)_t$ of the PDE (2.2), by Itô's formula, we have the interpolation

$$V(\Psi_\infty^u) = \mathcal{V}_0(0) + \int_0^\infty \mathrm{D}\mathcal{V}_t(\Psi_t^u)\mathrm{d}X_t + \int_0^\infty \left[\partial_t \mathcal{V}_t(\Psi_t^u) + \frac{1}{2}\dot{G}_t \mathrm{D}^2\mathcal{V}_t(\Psi_t^u) + u_t \dot{G}_t^{1/2}\mathrm{D}\mathcal{V}_t(\Psi_t^u)\right]\mathrm{d}t.$$

Taking expectations and using the PDE (2.2) it gives

$$\mathbb{E}V(\Psi_\infty^u) = \mathcal{V}_0(0) + \mathbb{E}\int_0^\infty \left(u_t + \frac{1}{2}\dot{G}_t^{1/2}\mathrm{D}\mathcal{V}_t(\Psi_t^u)\right)\dot{G}_t^{1/2}\mathrm{D}\mathcal{V}_t(\Psi_t^u)\mathrm{d}t$$

and therefore

$$\begin{aligned}\mathcal{J}^V(u) &= \mathcal{V}_0(0) + \mathbb{E}\left[\int_0^\infty \left(u_t + \frac{1}{2}\dot{G}_t^{1/2}\mathrm{D}\mathcal{V}_t(\Psi_t^u)\right)\dot{G}_t^{1/2}\mathrm{D}\mathcal{V}_t(\Psi_t^u)\mathrm{d}t + \frac{1}{2}\int_0^\infty \langle u_t, u_t\rangle \mathrm{d}t\right] \\ &= \mathcal{V}_0(0) + \frac{1}{2}\mathbb{E}\left[\int_0^\infty \left\| u_t + \dot{G}_t^{1/2}\mathrm{D}\mathcal{V}_t(\Psi_t^u)\right\|^2 \mathrm{d}t\right]\end{aligned}$$

and the equation

$$u_t = -\dot{G}_t^{1/2}\mathrm{D}\mathcal{V}_t(\Psi_t^u), \tag{3.1}$$

characterises an optimal feedback control for the minimisation of the functional $\mathcal{J}$.

For any control $u \in \mathbb{H}_a$ (where $\mathbb{H}_a$ is a space of L^2 in ω, t, x and predictable processes) satisfying (3.1) we have that

$$\Psi_t^u = X_t - \int_0^t \dot{G}_s^{1/2}\mathrm{D}\mathcal{V}_s(\Psi_s^u)\mathrm{d}s,$$

so by the Lipshitzianity of $\mathrm{D}\mathcal{V}_s$ for all $s \in [0,\infty]$ we can infer that this equation has a unique solution and that the solution u of (3.1) is (a.s.) unique.

Note that at the minimum $\mathcal{J}^V(u) = \mathcal{V}_0(0)$ so by the martingale property of $e^{-\mathcal{V}_t(X_t)}$ we also have

$$-\log \mathbb{E}[e^{-V(X_\infty)}] = -\log \mathbb{E}[e^{-\mathcal{V}_0(0)}] = \mathcal{V}_0(0) = \inf_{u \in \mathbb{H}_a} \mathcal{J}^V(u) \tag{3.2}$$

We collect these findings in the following theorem

Theorem 3.1. *Let V be a bounded and C^2 potential, then the FBSDE (2.5) has a unique strong solution $(\Psi_t)_{t\in[0,\infty]}$. This solution satisfies*

$$\mathrm{Law}(\Psi_\infty) = \nu^V_{\varepsilon,L},$$

and the process

$$u^*_t = -\dot{G}^{1/2}_t \mathbb{E}_t[\mathrm{D}V(\Psi^{u^*_t}_t)], \qquad t \geqslant 0,$$

is the unique optimiser of the stochastic control problem

$$\inf_{u \in \mathbb{H}_a} \mathcal{J}^V(u) = \mathcal{J}^V(u^*).$$

From (3.2) we have the following formula for the Laplace transform of $\nu^V_{\varepsilon,L}$:

$$\mathcal{W}(f) := -\log \int e^{-f(\varphi)} \nu^V_{\varepsilon,L}(\mathrm{d}\varphi) = \inf_{u \in \mathbb{H}_a} \mathcal{J}^{f+V}(u) - \inf_{u \in \mathbb{H}_a} \mathcal{J}^V(u).$$

The above discussion seems an overly complicated way to deal with a relatively simple measure as $\mu_{\varepsilon,L}$. However, we will see that we have uncovered a representation of this measure in which the removal of the cutoffs ε, L can be performed with relative ease.

Having established the general FBSDE framework and its connection to variational problems and the Polchinski equation, we now turn to concrete applications. Our first example is the sine-Gordon model in two dimensions.

Part II

The sine-Gordon model

Chapter 4
The sine-Gordon model

The variational representation, or the associated FBSDE computing the optimally controlled process, have been used to provide stochastic quantisations of various models, see e.g. the work of Barashkov and the author on $\Phi^4_{2,3}$ models both in finite and infinite volume [?, ?, ?], the work of De Vecchi, Fresta and the author on the construction of Fermionic models [?].

More recently Barashkov and Gunaratman [?] used the variational approach in a remarkable proof of the Markov property of Φ^4_3. It remains an open problem to understand fully the FBSDE associated to models with unbounded force, in particular $\Phi^4_{2,3}$ models.

In the following we will focus on a specific model with a bounded interaction potential which is interesting enough to provide a lot of discussion but which avoids the difficulties of models with unbounded interaction potential.

Before continuing lets make a small change of our setting. The lattice regularisation we chose is not ideal for a pedagogical exposition as it forces us to discuss irrelevant technical details. Looking for a simpler setting where to expose some basic ideas, we are going to change the way we regularise the EQF both in small and large scales.

In particular let's focus on measures $\nu_{T,\rho}$ on $\mathcal{S}'(\mathbb{R}^d)$ defined by choosing $0<T<\infty$ and ρ: $\mathbb{R}^d\to\mathbb{R}_+$ of bounded support and letting

$$\int f(\varphi)\nu_{T,\rho}(\mathrm{d}\varphi)=\frac{\mathbb{E}[f(X_T)e^{-V^\rho(X_T)}]}{\mathbb{E}[e^{-V^\rho(X_T)}]},\qquad V^\rho(\varphi)=\int_{\mathbb{R}^d}\rho(x)v(\varphi(x))\mathrm{d}x$$

where $(X_t)_{t\geqslant 0}$ is a Brownian martingale closed by a Gaussian free field X_∞ of mass m on $\mathbb{R}^d$. The infrared cutoff ρ plays the role of L while the finite regularisation time $T<\infty$ that of ε^{-1}.

These measures are not reflection positive nor Euclidean invariant, but they provide a convenient pedagogical setting to study the problem of the removal of the cutoffs and the associated renormalisation.

The infinite volume limit will be taken by taking $\rho=\rho_L$ where $\rho_L\colon\mathbb{R}^d\to[0,1]$ such that $\rho_L=1$ in a ball of radius L centred at the origin and then letting this $L\to\infty$.

With small changes the discussion in the previous section can be extended to this new situation, which is right away infinite dimensional, but which, for any finite T, involves only smooth fields provided the scale decomposition $(G_t)_t$ is chosen appropriately (like in our example). This is left as an exercise to the reader.

We are now ready to consider a specific model which will constitute the test-bed for the FBSDE approach and the principal topic of these notes. We will follow closely the analysis of Meyer and the author [?]. The reader is invited to peruse the paper where all the proofs are given with full details.

Let $d=2$ and take

$$v(\varphi(x))=\lambda\cos(\beta\varphi(x))$$

for some fixed β such that $\beta^2\in(0,8\pi)$ and constant $\lambda=\lambda_T$ to be adjusted as $T\to\infty$ to obtain a nontrivial limit. This prescription defines a cutoff version of the so called *sine-Gordon model*. For a short literature review, please refer to Section-? below.

The parameter β has a special role in this model and via a specific mechanism determines the distance to criticality of this model, where criticality is identified, informally, as the situation where the perturbation to the measure, induced by the interaction, is comparable to the GFF at small scale.

To apply our stochastic quantisation method of choice we have to deal with the FBSDE

$$\mathrm{d}\Phi_t=-\dot{G}_t\mathbb{E}_t[\mathrm{D}V(\Phi_T)]+\mathrm{d}X_t,\qquad t\in[0,T].$$

What happens after time T is not very important for us as the measure $\nu_{T,\rho}$ is given by the law of Φ_T.

As $T\to\infty$ the *force* $F:=\mathrm{D}V$ given by

$$F(\Phi_T)=\mathrm{D}V(\Phi_T)=\rho(x)\lambda\beta\sin(\beta\Phi_T(x)),$$

become very singular. A preliminary computation to understand what is going on is given by substituting X_T for Φ_T as a first approximation (e.g. in a Picard iteration). In average this would give

$$\begin{aligned}\mathbb{E}_t[\mathrm{D}V(X_T)]=&\rho(x)\lambda\beta\mathbb{E}_t[\sin(\beta X_T(x))]\\ =&\rho(x)\lambda\beta\,\hat{\mathbb{E}}[\sin(\beta X_t(x)+\beta(\hat{X}_T-\hat{X}_t)(x))]\\ =&\rho(x)\lambda\beta\sin(\beta X_t(x))\hat{\mathbb{E}}[\cos(\beta(\hat{X}_T-\hat{X}_t)(x))]\\ =&\rho(x)\lambda\beta\sin(\beta X_t(x))\exp\Big[-\frac{\beta^2}{2}(G_T(0)-G_t(0))\Big].\end{aligned}$$

Using the explicit expression (2.1) for $\dot{G}_t$, we have

$$G_T(0)\approx\int_0^T\frac{1}{4\pi s}e^{-m^2/s}\mathrm{d}s\approx\frac{1}{4\pi}\log(T)\to\infty,\qquad T\to\infty.$$

We see that $\mathbb{E}_t[\mathrm{D}V(X_T)]\to 0$ and we can (heuristically) conclude that in this limit we converge to a GFF unless we "tune up" the interaction along the way, that is we counterbalance the factor $\exp\Big(-\frac{\beta^2}{2}G_T(0)\Big)$ by choosing

$$\lambda=\lambda_T=\hat{\lambda}\exp\Big(\frac{\beta^2}{2}G_T(0)\Big)$$

in which case we will, in this first approximation, have the well defined limit

$$\lim_{T\to\infty}\mathbb{E}_t[\mathrm{D}V(X_T)(x)]=\rho(x)\hat{\lambda}\exp\Big(\frac{\beta^2}{2}G_t(0)\Big)\beta\sin(\beta X_t(x))=\rho(x)\lambda_t\,\beta\sin(\beta X_t(x)).$$

The "renormalisation" we put in place allowed us to replace the diverging "bare coupling" λ_T with a scale-dependent ("running") coupling constant

$$\lambda_t:=\hat{\lambda}\exp\Big(\frac{\beta^2}{2}G_t(0)\Big).$$

While this computation hints to a possible approach to a non-trivial limit, we still need to confirm it rigorously for the full FBSDE. We should however keep with us the idea of the running coupling and the possible non-existence of the limiting potential V in the ultraviolet limit.

Instead of using the potential V we prefer for the moment to focus on its derivative $F=\mathrm{D}V$ which we call the (microscopic) force and try to obtain a representation of the FBSDE which will allow us to perform non-perturbatively the passage to the $T\to\infty$ limit, which is usually called the UV limit.

For any smooth functional $(F_t)_{t\in[0,T]}$ for which $F_T=F=-\mathrm{D}V$ we have, by Itô formula:

$$\mathbb{E}_t[F_T(\Phi_T)]=F_t(\Phi_t)+\int_t^T\mathbb{E}_t\Big[\partial_sF_s(\Phi_s)+\mathrm{D}F_s(\Phi_s)\dot{G}_s\mathbb{E}_s[F_T(\Phi_T)]\mathrm{d}s+\frac{1}{2}\dot{G}_s\mathrm{D}^2F_s(\Phi_s)\Big]\mathrm{d}s.$$

If we introduce the new variable

$$R_t:=\mathbb{E}_t[F_T(\Phi_T)]-F_t(\Phi_t),$$

we cast the FBSDE into a system of two equations for the unknown (Φ,R):

$$\begin{cases}\Phi_t=X_t+\int_0^t\dot{G}_sF_s(\Phi_s)\mathrm{d}s+\int_0^t\dot{G}_sR_s\mathrm{d}s,\\ R_t=\int_t^T\mathbb{E}_t[H_s(\Phi_s)]\mathrm{d}s+\int_t^T\mathbb{E}_t[\mathrm{D}F_s(\Phi_s)\dot{G}_sR_s]\mathrm{d}s\end{cases}\tag{4.1}$$

where

$$H_s(\Phi_s) := \partial_s F_s(\Phi_s) + \mathrm{D}F_s(\Phi_s)\dot{G}_s F_s(\Phi_s)\mathrm{d}s + \frac{1}{2}\dot{G}_s \mathrm{D}^2 F_s(\Phi_s).$$

In this system the UV cutoff T appears only as bound of integration, so we can imagine that, provided we obtain sufficient integrability for $H_s(\Phi_s)$ as $s \to \infty$ and a nice behaviour of the function $(F_t)_t$ for finite t, this systems could be under control.

As safety check note that if we take $F_t = -D\mathcal{V}_t$ then indeed $H_t(\Phi_t) = 0$ and $R_t = 0$ for all $t \in [0,T]$ (since the equation for R behaves like a linear ODE with final condition $R_T = 0$), and we re-obtain the fact that Φ_t has a Markovian drift given by $-\dot{G}_t D\mathcal{V}_t$. Therefore the new formulation give the possibility to work with *approximate* solutions of the HJB equation (2.2) or better, with approximate solutions $(F_s)_s$ of the equation solved by the derivative of the solution:

$$\partial_s F_s(\Phi_s) + \frac{1}{2}\mathrm{D}[F_s \dot{G}_s F_s](\Phi_s) + \frac{1}{2}\dot{G}_s \mathrm{D}^2 F_s(\Phi_s) \approx 0, \tag{4.2}$$

how well we need to approximate these solutions will depend on the integrability of $H_s(\Phi_s)$ (or more precisely of $\mathbb{E}_t[H_s(\Phi_s)]$) for $s \to \infty$.

We call (4.2) the flow equation for the *effective force* of the FBSDE and in the next section we sketch how to approximate its solutions well enough to control the UV limit.

Chapter 5
Solving the flow equation

We need *any* (approximate) solution with prescribed final condition

$$F_T(\varphi)(x) = -\mathrm{D}V(\varphi)(x) = \rho(x)\,\beta\lambda_T \sin(\beta\varphi(x)) = \frac{\rho(x)\,\beta\lambda_T}{2i}(e^{i\beta\varphi(x)} - e^{-i\beta\varphi(x)})$$

(and recall that we agreed to take $\lambda = \lambda_T$ even if we do not stress it in the notation for V). Note that $\mathrm{D}e^{i\beta\varphi(x)} = e^{i\beta\varphi(x)}(i\beta\delta(x))$, so we can look for solutions (or approximate solution), in the space of polynomials in $e^{\pm i\beta\varphi(x)}$. To ease the notation we let $\xi = (\sigma, x)$ and

$$\psi(\xi) = \psi^\sigma(x) := e^{\sigma i\beta\varphi(x)},$$

moreover if we have multi-indices $\xi_{1:n} = (\xi_1, \dots, \xi_n)$, then

$$\psi(\xi_{1:n}) = \prod_{i=1}^{n} \psi(\xi_i).$$

With these notation we can introduce the functionals $\big(F_t^{[\ell]}\big)_{\ell=1,\dots}$

$$\begin{aligned} F_t^{[\ell]}(\varphi)(x_1) &= \sum_{\sigma_i\in\{-1,1\}^\ell} \sigma_1 \int_{(\mathbb{R}^2)^{\ell-1}} f_t^{[\ell]}(\xi_{1:\ell})\, \mathrm{e}^{i\beta\sigma_1\varphi(x_1)} \dots \mathrm{e}^{i\beta\sigma_\ell\varphi(x_\ell)} \mathrm{d}x_{2:\ell} \\ &= \sum_{\sigma_1\in\{-1,1\}} \sigma_1 \int f_t^{[\ell]}(\xi_{1:\ell})\psi(\xi_{1:\ell})\mathrm{d}\xi_{2:\ell} \end{aligned}$$

where it will be clear from below that the kernels $f_t^{[\ell]}(\xi_{1:\ell})$ can be chosen to be symmetric wrt. permutations of the ξ variables. We require that they satisfy the equation

$$\partial_s F_s^{[\ell]} + \frac{1}{2}\sum_{\ell_1+\ell_2=\ell} \mathrm{D}\big[F_s^{[\ell_1]}\dot{G}_s F_s^{[\ell_2]}\big] + \frac{1}{2}\dot{G}_s\mathrm{D}^2 F_s^{[\ell]} = 0 \tag{5.1}$$

This is an almost triangular system which can be solved explicitly via a variation of constants:

$$F_t^{[\ell]} = \exp\Big[\frac{1}{2}(G_T - G_t)\mathrm{D}^2\Big]F_T^\ell - \frac{1}{2}\int_t^T \exp\Big[\frac{1}{2}(G_s - G_t)\mathrm{D}^2\Big] \sum_{\ell_1+\ell_2=\ell} \mathrm{D}\big[F_s^{[\ell_1]}\dot{G}_s F_s^{[\ell_2]}\big]\mathrm{d}s.$$

A simple computation shows that the "Laplacian" acts on the kernel as

$$\dot{G}_t\mathrm{D}^2 F_t^{[\ell]}(\varphi)(x_1) = \sum_{\sigma_1\in\{-1,1\}} \sigma_1 \int \mathrm{d}\xi_{2:\ell} \Bigg[-\beta^2 \sum_{k,l=1,\dots\ell} \sigma_k\sigma_l \dot{G}_t(x_k - x_l)\Bigg] f_t^{[\ell]}(\xi_{1:\ell})\psi(\xi_{1:\ell})$$

which together with the mild formulation and introducing the function

$$W_{t,s}(\xi_{1:\ell}) := -\frac{\beta^2}{2}\sum_{i,j} \sigma_i\,\sigma_j (G_s - G_t)(x_i - x_j), \quad t \leqslant s. \tag{5.2}$$

gives us the following equation for the kernels. For $\ell = 1$,

$$f_t^{[1]}(\xi_1) = \mathrm{e}^{W_{t,T}(\xi_1)}\lambda_T \frac{\beta}{2\,i} = \exp\Big(-\frac{\beta^2}{2}(G_T(0) - G_t(0))\Big)\lambda_T\frac{\beta}{2\,i} = \lambda_t\frac{\beta}{2\,i},$$

while for $\ell > 1$,

$$f_t^{[\ell]}(\xi_{1:\ell}) = -\beta^2 \sum_{I_1+I_2=[\ell]} C(|I_1|,|I_2|) \int_t^T \mathrm{e}^{W_{t,s}(\xi_{1:\ell})} f_s(\xi_{I_1}) \Big[\sum_{i\in I_1} \sum_{j\in I_2} \sigma_i\sigma_j \dot{G}_s(x_i - x_j) \Big] f_s(\xi_{I_2}) \mathrm{d}s. \tag{5.3}$$

modulo positive combinatorial coefficients which we hide in the constant $C(|I_1|,|I_2|)$. Note that here we are using the running coupling

$$\lambda_t = \hat{\lambda} \exp\Big(\frac{\beta^2}{2} G_t(0) \Big). \tag{5.4}$$

At this point we can form the partial sum

$$F^{[\leqslant \ell^*]} = \sum_{\ell=1}^{\ell^*} F^{[\ell]},$$

and observe that it satisfies

$$\partial_s F^{[\leqslant \ell^*]} + \frac{1}{2} \mathrm{D}[F^{[\leqslant \ell^*]} \dot{G}_s F^{[\leqslant \ell^*]}] + \frac{1}{2} \dot{G}_s \mathrm{D}^2 F^{[\leqslant \ell^*]} = \frac{1}{2} \sum_{\substack{\ell_1+\ell_2 > \ell^* \\ \ell_1,\ell_2 \leqslant \ell^*}} \mathrm{D}\big[F_s^{[\ell_1]} \dot{G}_s F_s^{[\ell_2]} \big],$$

which will contribute as a source term to the remainder term in the FBSDE (4.1).

What kind of estimates we could hope for in terms of these functionals? Since $F^{[1]}$ is explicit, this is easy:

$$\big\| \mathrm{D} F_t^{[1]}(\varphi) \big\|_{L^\infty} + \big\| F_t^{[1]}(\varphi) \big\|_{L^\infty} \lesssim \lambda_t \lesssim \langle t \rangle^{1-\delta},$$

where

$$\delta := 1 - \frac{\beta^2}{8\pi} > 0.$$

The structure of the equation (5.1) and the fact that we have the bound

$$\| \dot{G}_s \|_{L^1} \lesssim \langle s \rangle^{-2},$$

for the smoothing operator, suggests that

$$\big\| \mathrm{D} F_t^{[\ell]}(\varphi) \big\|_{L^\infty} + \big\| F_t^{[\ell]}(\varphi) \big\|_{L^\infty} \lesssim \lambda_t^\ell \langle t \rangle^{-(\ell-1)} \lesssim \langle t \rangle^{1-\ell\delta} \tag{5.5}$$

could propagate inductively provided

$$\delta \ell > 1. \tag{5.6}$$

Indeed in this case, the non-linear term can be estimated as

$$\sum_{\ell_1+\ell_2=\ell} \int_t^T \big\| \mathrm{D}\big[F_s^{[\ell_1]} \dot{G}_s F_s^{[\ell_2]} \big] \big\|_{L^\infty} \mathrm{d}s \lesssim \int_t^T \lambda_s^\ell \langle s \rangle^{-\ell} \mathrm{d}s \lesssim \int_t^T \langle s \rangle^{-\delta\ell} \mathrm{d}s \lesssim \langle s \rangle^{1-\delta\ell} \approx \lambda_s^\ell \langle s \rangle^{-(\ell-1)},$$

while we will see below that the linear term will not pose particular problems and actually will help.

Due to (5.6), we will not be able to propagate the regularity when ℓ is small. The smaller ℓ which we have to determine is $\ell = 2$ so this constraint give $\delta > 1/2$, that is $\beta^2 < 4\pi$. This is called the first threshold of the sine-Gordon model.

We will show how to handle the first non-trivial case, that is beyond the first threshold an up to the second, where $\beta^2 \in (4\pi, 6\pi)$ and for which $\delta \in (1/4, 1/2)$. To achieve this we need a careful consideration of the equations for $f^{[2]}$ and $f^{[3]}$ to look for gains which allow better estimates.

The relevant, translation invariant norm is:

$$\| f \| := \sup_{\xi_1} \int \mathrm{d}\xi_{2:\ell} \Big| f(\xi_{1:\ell}) \Big|,$$

and, using the positivity of $\dot{G}_s$, the function W can be estimated by

$$W_{t,s}(\xi_{1:\ell}) \leqslant 0, \quad \text{for } t \leqslant s.$$

For this reason the exponential $\mathrm{e}^{W_{t,s}(\xi_{1:\ell})} \leqslant 1$ will not pose problems to us. This bound is sharp, since if the *total charge* of the configuration $\xi_{1:\ell}$ defined as

$$q(\xi_{1:\ell}) := \sum_{k=1}^{\ell} \sigma_k,$$

is zero, then $W_{t,s}(\xi_{1:\ell}) = 0$ when all the x_i are equal. Such a configuration is neutral. A non-neutral configuration is called *charged* and the key point is that charged configuration have a better behaviour for $W_{t,s}(\xi_{1:\ell})$ captured in the following lemma whose proof is given by a specific computation which will not provide.

Lemma 5.1. *Suppose that $\xi_{1:\ell}$ is charged. Then there is a constant $C > 0$ such that for all $s \geqslant t$,*

$$W_{t,s}(\xi_1, \dots, \xi_\ell) \leqslant -\frac{\beta^2}{8\,\pi}(G_s(0) - G_t(0)) + C, \tag{5.7}$$

and in particular

$$\mathrm{e}^{W_{t,s}(\xi_{1:\ell})} \lesssim \lambda_t\, \lambda_s^{-1}.$$

Charged configuration gain a better UV behaviour from the linear part of the equation. This helps us with integrability, as can be easily shown. Recall the definition (5.3), that the kernel $f^{[2]}$ is given by

$$f_t^{[2]}(\xi_1, \xi_2) = C \int_t^T \mathrm{d}s\, \mathrm{e}^{W_{t,s}(\xi_1,\xi_2)}\, f_s^{[1]}(\xi_1)\, \sigma_1\, \sigma_2\, \dot G_s(x_1 - x_2)\, f_s^{[1]}(\xi_2).$$

If the pair is charged then we estimate as follows (with $\ell = 2$)

$$\Big\| f_t^{[2]\mathrm{(charged)}} \Big\| \lesssim \sup_{\xi_1} \lambda_t \int_t^T \mathrm{d}s\, \lambda_s^{-1} \|\!| f_s^{[1]} |\!\|^2 \|\dot G_s\|_{L^1} \lesssim \lambda_t \int_t^T \mathrm{d}s\, \lambda_s \langle s\rangle^{-2} \lesssim \lambda_t^\ell \langle t\rangle^{-(\ell-1)}.$$

It remains to deal with neutral contributions. The first thing to note is that in this case

$$W_{t,s}(\xi_1, \xi_2) + \beta^2 G_s(0) = \beta^2 G_t(0) - \beta^2 G_t(x_1 - x_2) + \beta^2 G_s(x_1 - x_2) \leqslant \beta^2 G_s(x_1 - x_2)$$

so we can absorb the divergence of λ_s^2 coming from the two factors $f^{[1]}$ as

$$\begin{aligned}
\Big\| k_t f_t^{[2](0)} \Big\| &= C \sup_{\xi_1} \left| \int \mathrm{d}\xi_2 k_t(\xi_1, \xi_2) \int_t^T \mathrm{d}s \mathrm{e}^{W_{t,s}(\xi_1,\xi_2)}\, f_s^{[1]}(\xi_1)\, f_s^{[1]}(\xi_2)\, \dot G_s(x_1 - x_2) \right| \\
&\lesssim \sup_{x_1} \left| \int_t^T \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}x_2\, \dot G_s(x_1 - x_2) |x_1 - x_2|^{2\alpha}\, t^\alpha \mathrm{e}^{ct|x_1 - x_2|^2}\, \mathrm{e}^{W_{t,s}(\xi_1,\xi_2) + \beta^2 G_s(0)} \right| \\
&\lesssim \mathrm{e}^{\beta^2 G_t(0)}\, t^\alpha \int_t^T \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}x |x|^{2\alpha}\, \mathrm{e}^{ct|x|^2}\, \dot G_s(x) \mathrm{e}^{\beta^2 G_s(x)}.
\end{aligned}$$

where we introduced in the norm the weight

$$k_t(\xi_1, \xi_2) := \begin{cases} t^\alpha |\delta_{12}\, x|^{2\alpha} \mathrm{e}^{ct|\delta_{12} x|^2}, & q(\xi_1, \xi_2) = 0 \\ 1, & \text{otherwise.} \end{cases}$$

which allows us to propagate the bound for $\alpha > (1 - 2\delta) \vee 0$ since in this case a direct computation shows that

$$\int_{\mathbb{R}^2} \mathrm{d}x |x|^{2\alpha}\, \mathrm{e}^{ct|x|^2}\, \dot G_s(x) \mathrm{e}^{\beta^2 G_s(x)} \lesssim \langle s\rangle^{-2-\alpha}$$

so we get

$$\Big\| k_t f_t^{[2](0)} \Big\| \lesssim \lambda_t^2\, t^\alpha \int_t^T \mathrm{d}s \langle s\rangle^{-2-\alpha} \lesssim \lambda_t^2 \langle t\rangle^{-1} = \lambda_t^\ell \langle t\rangle^{-(\ell-1)}.$$

We skip the simpler analysis of $f^{[3]}$ which will lead to the fact that for $\alpha = 1/2$ and $\delta > 1/4$ ($\beta^2 < 6\pi$), it holds that

$$\Big\| f_t^{[3]} \Big\| \lesssim \lambda_t^3 \langle t\rangle^{-2} = \lambda_t^\ell \langle t\rangle^{-(\ell-1)}$$

so also this kernel is under control. Recall that α is the exponent in the weight for $f^{[2]}$. This estimate is actually true also for all others $\ell > 2$ but we will not need it in the following. If we want to improve on $\beta^2 < 6\pi$ these simple estimate are not more possible, there are other strategies available which however spoil some uniform bound for the coefficients of the FBSDEs.

Remark 5.2. It is currently an open problem to devise a "simple" approach to the regime $\beta^2 \in (6\pi, 8\pi)$ in $d = 2$.

This leads directly to basic estimates on force which reads

Proposition 5.3. *For any $\varphi \in \mathcal{S}'(\mathbb{R}^2)$, the following estimates apply uniformly in $\varphi \in \mathcal{S}'(\mathbb{R}^2), \rho \leqslant 1$ and $T < \infty$ (with $Q_t = \dot{G}_t^{1/2}$)*

$$\begin{aligned} \left\| Q_t F_t^{[\ell]}(\varphi) \right\|_{L^\infty} &\lesssim \lambda_t \langle t \rangle^{-1}, \\ \left\| \mathrm{D} F_t^{[\ell]}(\varphi)\, Q_t \right\|_{L^\infty} &\lesssim \lambda_t \langle t \rangle^{-1}, \\ \| H_t(\varphi) \|_{L^\infty} &\lesssim (\lambda_t \langle t \rangle^{-1})^4, \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} \left\| Q_t F_t^{[\ell]}(\varphi) - Q_t F_t^{[\ell]}(\tilde{\varphi}) \right\|_{L^\infty} &\lesssim \lambda_t \langle t \rangle^{-1} \| \varphi - \tilde{\varphi} \|_{L^\infty}, \\ \left\| \mathrm{D} F_t^{[\ell]}(\varphi)\, Q_t - \mathrm{D} F_t^{[\ell]}(\tilde{\varphi})\, Q_t \right\|_{L^\infty} &\lesssim \lambda_t \langle t \rangle^{-1} \| \varphi - \tilde{\varphi} \|_{L^\infty}, \\ \| H_t(\varphi) - H_t(\tilde{\varphi}) \|_{L^\infty} &\lesssim (\lambda_t \langle t \rangle^{-1})^4 \| \varphi - \tilde{\varphi} \|_{L^\infty}. \end{aligned} \tag{5.9}$$

The only tricky part is how to estimate the neutral pair kernel, and there we use crucially the presence of the operator $Q_t = \dot{G}_t^{1/2}$ to conclude.

The Lipschitz bounds holds also in weighted norms.

For the dependence on ρ we need to observe the following proposition, introducing explicitly this dependence with the notation F^ρ, H^ρ.

Proposition 5.4. *For $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ and $\rho_1, \rho_2 \leqslant 1$, it holds that*

$$\begin{aligned} \| Q_t (F_t^{\rho_1} - F_t^{\rho_2})(\varphi) \|_{L^{2,-n}} &\lesssim \lambda_t \langle t \rangle^{-1} \| \rho_1 - \rho_2 \|_{L^{2,-n}}, \\ \| (\mathrm{D} F_t^{\rho_1} - \mathrm{D} F_t^{\rho_2})(\varphi)\, Q_t \|_{L^{2,-n}} &\lesssim \lambda_t \langle t \rangle^{-1}, \| \rho_1 - \rho_2 \|_{L^{2,-n}}, \\ \| (H_t^{\rho_1} - H_t^{\rho_2})(\varphi) \|_{L^{2,-n}} &\lesssim (\lambda_t \langle t \rangle^{-1})^4 \| \rho_1 - \rho_2 \|_{L^{2,-n}}. \end{aligned} \tag{5.10}$$

where

$$\| f \|_{L^{2,-n}}^2 = \int_{\mathbb{R}^d} |\langle x \rangle^{-n} f(x)|^2 \mathrm{d}x$$

is a weighted norm which focuses on what happens locally around the origin and for which when n is large enough ($n > 2$) one has $\left\| \rho_L - \rho_{L'} \right\|_{L^{2,-n}} \to 0$ when $L > L' \to \infty$.

Chapter 6

Well-posedness for the FBSDE for weak interactions

The bounds for the force give right away existence and uniqueness of solutions to the FBSDEs for small λ uniformly in the regularisation parameter T.

With these informations we go back to the analysis of the FBSDE (4.1) which is now straightforward. In L^∞ we can indeed estimate, with Proposition 5.3, and recalling that $\|\dot G_s\|_{L^1} \lesssim \langle s\rangle^{-2}$, $\|\dot G_s^{1/2}\|_{L^1} \lesssim \langle s\rangle^{-1}$, and letting

$$\Phi_t = X_t + Z_t,$$

we have

$$\begin{aligned}\|Z_t\|_{L^\infty} &\leqslant \int_0^t \|\dot G_s^{1/2}\|_{L^1} \|\dot G_s^{1/2} F_s(\Phi_s)\|_{L^\infty} \mathrm{d}s + \int_0^t \|\dot G_s\|_{L^1} \|R_s\|_{L^\infty} \mathrm{d}s \\ &\leqslant C \int_0^t \lambda_s \langle s\rangle^{-2} \mathrm{d}s + \int_0^t \langle s\rangle^{-2} \|R_s\|_{L^\infty} \mathrm{d}s\end{aligned}$$

$$\begin{aligned}\|R_t\|_{L^\infty} &\leqslant \int_t^T \|H_s(\Phi_s)\|_{L^\infty} \mathrm{d}s + \int_t^T \|\mathrm{D}F_s(\Phi_s)\dot G_s^{1/2}\|_{L^\infty} \|\dot G_s^{1/2}\|_{L^1} \|R_t\|_{L^\infty} \mathrm{d}s \\ &\lesssim \int_t^T (\lambda_s \langle s\rangle^{-1})^4 \mathrm{d}s + \int_t^T \lambda_s \langle s\rangle^{-2} \|R_t\|_{L^\infty} \mathrm{d}s.\end{aligned}$$

The inequality for $\|R_t\|_{L^\infty}$ gives via a (reverse) Gronwall (and $\delta > 1/4$)

$$\|R_t\|_{L^\infty} \lesssim \exp\Big(\int_t^\infty \lambda_s \langle s\rangle^{-2} \mathrm{d}s\Big) \int_t^\infty (\lambda_s \langle s\rangle^{-1})^4 \mathrm{d}s \lesssim \int_t^\infty (\lambda_s \langle s\rangle^{-1})^4 \mathrm{d}s \lesssim \langle t\rangle^{1-4\delta}$$

and then that for $\|Z_t\|_{L^\infty}$

$$\|Z_t\|_{L^\infty} \lesssim C \int_0^\infty \lambda_s \langle s\rangle^{-2} \mathrm{d}s + \int_0^\infty \langle s\rangle^{-2} \|R_s\|_{L^\infty} \mathrm{d}s \lesssim C.$$

We have been careful not to estimate Φ_t in L^∞ because it wouldn't work. Stationary Gaussian fields like X_t grow at infinity, at least with some power of log.

Remark 6.1. These preliminary estimates are particularly pleasant. They show that the interacting multi-scale field $(\Phi_t)_{t\in[0,T]}$ is an a.s. bounded perturbation $(Z_t)_{t\in[0,T]}$ of the multi-scale GFF $(X_t)_{t\in[0,T]}$, a very powerful control which is available essentially on in this model and in some other Grassmann theory, see. [?].

A similar argument involving the study of a fix-point equation for the solutions and the Lipschitz estimates in Proposition 5.3 shows also that for small $\hat\lambda$ there is a unique solution to the FBSDE.

Therefore at this point we have a tight control of the interacting field $(\Phi_t^{\rho,T})_t$ where we introduced explicitly the dependence on the UV cutoff T coming from the initial condition of the flow equation for the effective force $(F_t^{\rho,T})_t$ and on the IR cutoff ρ which appears in the kernels of the effective force. Proving that $(\Phi_t^{\rho,T})_t$ has a limit when $T\to\infty$ and $\rho\to\infty$ involves a comparison of two solutions Φ^{ρ_1,T_1} and Φ^{ρ_2,T_2} via the equations they satisfy (together with the associated remainders) and the use of the informations gathered on the stability of the flow equation.

We note that the estimates we performed in L^∞ could have been done also in any weighted space $L^{p,-n}$ with $pn > d$ and $p \geqslant 1$. In particular the removal of the IR cutoff requires to use weighted spaces as it can hold only in a local topology.

For small $\hat{\lambda}$ we can then prove that, for any $n > 0$,

$$\lim_{\substack{\rho\to 1\\ T\to\infty}} \sup_t \{\|Z_t^{\rho,T} - Z_t\|_{L^{\infty,-n}} + \|R_t^{\rho,T} - R_t\|_{L^{\infty,-n}}\} = 0,$$

where (Z, R) is the unique solution of the FBSDE with the effective force $F^{1,\infty}$ obtained as the limit of $F^{\rho,T}$ as $T \to \infty$ and $\rho \to 1$.

Since

$$\mathrm{Law}(Z_T^{\rho,T} + X_T) = \nu^{\rho,T},$$

this implies the weak convergence of $(\nu^{\rho,T})_{\rho,T}$ to a limiting measure

$$\nu_{\mathrm{SG}} = \mathrm{Law}(Z_\infty + X_\infty)$$

in the space of probability measures on $H^{-\varepsilon,-n}$ (for example, with $\varepsilon > 0$ and $n > d$) since it is in this space that $X_T \to X_\infty$. The measure ν_{SG} is the sine-Gordon EQF.

The choice of $H^{-\varepsilon,-n}$ is a bit arbitrary, dictated by the properties of the GFF more than by the convergence of $Z_T^{\rho,T} \to Z_\infty$ which holds almost surely in the stronger space $L^{\infty,-n}$, as stated above.

Note that $Z_\infty \in L^\infty$, so this measure has at least the (negative) regularity of the GFF. It is interesting to obtain a more precise regularity for Z_∞, from the FBSDE we have for $\alpha > 0$ and $n > d$, and provided $\alpha < 2\delta$

$$\begin{aligned}
\|Z_\infty\|_{H^{\alpha,-n}} &\leqslant \left\| \int_0^\infty \mathrm{d}s\, \dot{G}_s(F_s(X_s) + R_s) \right\|^2_{H^{\alpha,-n}} \\
&\leqslant \int_0^\infty \mathrm{d}s\, \langle s\rangle^{-1+\alpha} \left\| G_s^{1/2}(F_s(X_s) + R_s) \right\|^2_{L^{2,-n}} \\
&\lesssim \int_0^\infty \mathrm{d}s\, \langle s\rangle^{-1+\alpha} \left\| G_s^{1/2}(F_s(X_s) + R_s) \right\|^2_{L^\infty} \\
&\lesssim \int_0^\infty \mathrm{d}s\, \langle s\rangle^{-1+\alpha} (\langle s\rangle^{-2\delta} + \langle s\rangle^{-1}) < \infty.
\end{aligned}$$

Therefore $Z_\infty \in H^{\alpha,-n}$. (And a similar estimate would also show that $Z_\infty \in B^\alpha_{\infty,\infty}$).

As a byproduct of this kind of construction we always obtain a *coupling* $\pi = \mathrm{Law}(X_\infty, X_\infty + Z_\infty)$ of the GFF μ with the EQF ν_{SG}, i.e. a measure on a product space for which the two marginals are the two given measures.

This coupling is quite nice and actually much better than the trivial coupling obtained by taking the product of the two probability measures, indeed under $\pi(\mathrm{d}\varphi, \mathrm{d}\psi)$ the difference $\chi := \varphi - \psi$ lives almost surely in L^∞ as can be readily checked, which is not the case for the product coupling since in that case χ would have, in general, the regularity of the worst of the two factors.

The existence of the coupling implies for example that there exists a small $\gamma > 0$ for which

$$\int_{\mathcal{S}'(\mathbb{R}^2)} \mathrm{e}^{\gamma\|\varphi\|^2_{H^{-\varepsilon,-n}}} \nu_{\mathrm{SG}}(\mathrm{d}\varphi) \lesssim \mathbb{E}[\exp(2\gamma\|X_\infty\|^2_{H^{-\varepsilon,-n}} + 2\gamma\|Z_\infty\|^2_{L^\infty})] < \infty$$

where the integrability of X_∞ follows from Fernique's theorem (or from Boué–Dupuis formula). This integrability condition is well enough to satisfy the regularity axiom needed for OS reconstruction.

We note also that, if $\hat{\lambda}$ is not small, our estimates of the effective force and of the approximate solutions of the FBDSEs give good bounds on $(Z_T^{\rho,T})_{\rho,T}$ allowing to prove the tightness of the family $(\nu^{\rho,T})_{\rho,T}$ in $H^{-\varepsilon,-n}$. As a consequence the UV limit exists for the sine-Gordon model but uniqueness then it is not clear (at least in this approach). See Section ? for more on the available literature on the subject.

In general, any limit will be described by a solution of the unregularised FBSDE where $(F_t)_t$ is the limiting approximate effective force. Note that there is no ambiguity in defining this effective force for any $\hat{\lambda}$ since it is specified by a triangular system which has always a unique solution.

The possibility to have more than limiting measure or more than one solution of the FBSDE is linked also to the phenomenon of phase transitions. However in finite volume uniqueness of the Sine–Gordon measure can be established via a stochastic approach, as the next section will show.

Chapter 7
Uniqueness of the measure in finite volume

In the presence of a finite volume ($\rho \prec 1$), we can prove the existence of a unique limit in law as $T \to \infty$ for the SG measure for any value $\hat{\lambda}$ of the coupling constant.

Keeping a fixed compactly supported cutoff function $\rho \in C_c^\infty(\mathbb{R}^2)$ means we allow interaction only in a bounded domain.

Let $\mathbb{Q}$ be a probability space where we have a copy $W^{\mathbb{Q}}$ of our regularised GFF $(W_t)_{t \geqslant 0}$. Consider the (approximate) effective force $(F_s)_{s \geqslant 0}$ with cutoff ρ (that we omit to indicate, for simplicity, but keep it in place and fixed in the whole of this section).

Let now $(\Phi_t)_{t \geqslant 0}$ be the solution to the "standard" SDE

$$\Phi_t = W_t^{\mathbb{Q}} - \int_0^t \dot{G}_s F_s(\Phi_s) \mathrm{d}s, \tag{7.1}$$

which exists and its unique since our estimates imply that the drift term is globally Lipschitz and bounded. Construct now a new measure $\mathbb{P}^T$ obtained from $\mathbb{Q}$ via the density

$$\mathcal{E}_T := \frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{Q}} = \exp\Big(\int_0^T F_s(\Phi_s) \mathrm{d}W_s^{\mathbb{Q}} - \frac{1}{2} \int_0^T F_s(\Phi_s) \dot{G}_s F_s(\Phi_s) \mathrm{d}s \Big).$$

By Girsanov's theorem $(\Phi_t)_{t \in [0,T]}$ is a Brownian martingale under $\mathbb{P}^T$ such that

$$\mathrm{Law}_{\mathbb{P}^T}(\Phi) = \mathrm{Law}_{\mathbb{Q}}(W^{\mathbb{Q}}),$$

i.e. $\mathrm{Cov}(\Phi_t) = G_t$. Using that $F_s = \mathrm{D}V_s$, where $(V_s)_s$ is an approximate solution of Polchinski's flow equation [?], we have

$$V_T(\Phi_T) - V_0(\Phi_0) - \int_0^T \mathcal{H}_s(\Phi_s) \mathrm{d}s = \int_0^T F_s(\Phi_s) \mathrm{d}W_s^{\mathbb{Q}} - \frac{1}{2} \int_0^T F_s(\Phi_s) \dot{G}_s F_s(\Phi_s) \mathrm{d}s$$

with

$$\mathcal{H}_s := \partial_s V_s + \frac{1}{2} \dot{G}_s \mathrm{D}^2 V_s + \frac{1}{2} \mathrm{D}V_s \dot{G}_s \mathrm{D}V_s.$$

By argument similar to those used in the construction of the effective force we can prove that the approximate effective potential $(V_s)_s$ satisfies

$$|\mathcal{H}_s(\Phi_s)| \lesssim \langle s \rangle^{1-4\delta}.$$

This implies that

$$\left| \int_0^\infty \mathcal{H}_s(\Phi_s) \mathrm{d}s \right| \lesssim 1$$

almost surely, from which we deduce that

$$\mathbb{E}_{\mathbb{Q}}\big[\mathcal{O}(W_T^{\mathbb{Q}}) e^{-V_T(W_T^{\mathbb{Q}})} \big] = \mathbb{E}_{\mathbb{P}^T}[\mathcal{O}(\Phi_T) e^{-V_T(\Phi_T)}]$$

$$= \mathbb{E}_{\mathbb{Q}}[\mathcal{O}(\Phi_T) e^{-V_T(\Phi_T)} \mathcal{E}_T] = \mathbb{E}_{\mathbb{Q}}\Big[\mathcal{O}(\Phi_T) e^{-V_0(\Phi_0) - \int_0^T \mathcal{H}_s(\Phi_s) \mathrm{d}s} \Big].$$

We can then introduce the measure $\tilde{\mathbb{P}}^T$ by the density

$$\frac{\mathrm{d}\tilde{\mathbb{P}}^T}{\mathrm{d}\mathbb{Q}} = \frac{e^{-V_0(\Phi_0) - \int_0^T \mathcal{H}_s(\Phi_s) \mathrm{d}s}}{\mathbb{E}_{\mathbb{Q}}\big[e^{-V_0(\Phi_0) - \int_0^T \mathcal{H}_s(\Phi_s) \mathrm{d}s} \big]} = \frac{e^{-\int_0^T \mathcal{H}_s(\Phi_s) \mathrm{d}s}}{\mathbb{E}_{\mathbb{Q}}\big[e^{-\int_0^T \mathcal{H}_s(\Phi_s) \mathrm{d}s} \big]}$$

and obtain the representation

$$\int \mathcal{O}(\varphi)\nu_{\mathrm{SG}}^{T}(\mathrm{d}\varphi) = \frac{\int \mathcal{O}(\varphi)e^{-V_T(\varphi)}\mu^T(\mathrm{d}\varphi)}{\int e^{-V_T(\varphi)}\mu^T(\mathrm{d}\varphi)}$$

$$= \frac{\mathbb{E}_{\mathbb{Q}}\big[\mathcal{O}(W_T^{\mathbb{Q}})e^{-V_T(W_T^{\mathbb{Q}})}\big]}{\mathbb{E}_{\mathbb{Q}}\big[e^{-V_T(W_T^{\mathbb{Q}})}\big]} = \mathbb{E}_{\tilde{\mathbb{P}}^T}[\mathcal{O}(\Phi_T)],$$

for the approximate Sine–Gordon measure ν^T. Moreover we can directly take the limit as $T\to\infty$ to obtain that $\tilde{\mathbb{P}}^T \to \tilde{\mathbb{P}}$ weakly, where

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{Q}} = \frac{e^{-\int_0^\infty \mathcal{H}_s(\Phi_s)\mathrm{d}s}}{\mathbb{E}_{\mathbb{Q}}\big[e^{-\int_0^\infty \mathcal{H}_s(\Phi_s)\mathrm{d}s}\big]},$$

and therefore that $\nu_{\mathrm{SG}}^{T} \to \nu_{\mathrm{SG}}$ weakly.

We conclude that ν_{SG} has the representation

$$\int \mathcal{O}(\varphi)\nu_{\mathrm{SG}}(\mathrm{d}\varphi) = \frac{\mathbb{E}_{\mathbb{Q}}\big[\mathcal{O}(\Phi_\infty)e^{-\int_0^\infty \mathcal{H}_s(\Phi_s)\mathrm{d}s}\big]}{\mathbb{E}_{\mathbb{Q}}\big[e^{-\int_0^\infty \mathcal{H}_s(\Phi_s)\mathrm{d}s}\big]} = \mathbb{E}_{\tilde{\mathbb{P}}}[\mathcal{O}(\Phi_\infty)] \tag{7.2}$$

that is it is obtained from the law of the endpoint Φ_∞ of the SDE (7.1) via a scale-dependent tilt

$$e^{-\int_0^\infty \mathcal{H}_s(\Phi_s)\mathrm{d}s},$$

which is bounded above and away from zero.

Chapter 8

Absolute continuity and singularity in finite volume

The discussion in Section 7 gave a representation of the Sine–Gordon measure as an absolutely continuous perturbation of the measure $\mathbb{Q}$ under which the (canonical) process $(\Phi_t)_{t\geqslant 0}$ satisfies an SDE driven by a Brownian martingale. In particular this shows that there is a bounded coupling between Φ_∞ and W_∞ which under $\mathbb{Q}$ is distributed as a GFF.

Note that the quantity

$$\int_0^T F_s(\Phi_s)\dot{G}_s F_s(\Phi_s)\mathrm{d}s$$

can be estimated by

$$\int_0^T F_s(\Phi_s)\dot{G}_s F_s(\Phi_s)\mathrm{d}s \lesssim \int_0^T \langle s\rangle^{-2\delta}\mathrm{d}s$$

and it is uniformly bounded as $T\to\infty$ only if $\delta<1/2$, i.e. $\beta^2<4\pi$. In this case, by Novikov's criterion, the $\mathbb{Q}$-martingale $(\mathcal{E}_T)_T$ is uniformly integrable with limit $\mathcal{E}_\infty$ which is the Radon–Nykodym derivative of the measure $\mathbb{P}=\lim_{T\to\infty}\mathbb{P}^T$ wrt. $\mathbb{Q}$. Under $\mathbb{P}$ the process $(\Phi_t)_{t\geqslant 0}$ is a Brownian martingale with covariance $(G_t)_{t\geqslant 0}$ and

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}=\mathcal{E}_\infty^{-1}=\exp\Big[-\int_0^\infty F_s(\Phi_s)\mathrm{d}\Phi_s-\frac{1}{2}\int_0^\infty F_s(\Phi_s)\dot{G}_s F_s(\Phi_s)\mathrm{d}s\Big].$$

In particular this implies that ν_{SG} is absolutely continuous wrt. the GFF due to eq. (7.2) and the fact that

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}=\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{Q}}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}=\frac{e^{-\int_0^\infty \mathcal{H}_s(\Phi_s)\mathrm{d}s}}{\mathbb{E}_{\mathbb{Q}}\big[e^{-\int_0^\infty \mathcal{H}_s(\Phi_s)\mathrm{d}s}\big]}\mathcal{E}_\infty^{-1}.$$

This result is optimal, indeed we are going to prove now that when $\beta^2\in[4\pi,6\pi)$ the (finite-volume) Sine–Gordon measure is singular with respect to the GFF. Singularity does not seems to have any particular physical significance, but it is an interesting probabilistic property.

The proof proceeds by finding an event which has full measure with respect to SG but zero measure for the GFF. From the point of view of the coupling $\Phi_\infty=W_\infty+Z_\infty$ obtained via the FBSDE, the singularity is due to a low regularity of the shift Z_∞, which does not belong to the Cameron–Martin space of the GFF. A witness of this behaviour can be obtained by looking at specific regularised observables and quantify their degree of divergence as a regularisation is removed.

For any $\varepsilon\in(0,1]$ and $t\geqslant 0$ consider the functional

$$U_t^\varepsilon(\varphi):=r(\varepsilon)\int\rho(x)\Big[\lambda_t^\varepsilon\cos(\beta(\chi^\varepsilon*\varphi)(x))-1\Big]\mathrm{d}x$$

where φ is a scalar field, $\chi^\varepsilon(x)=\chi(x/\varepsilon)\varepsilon^{-d}$ with $\chi\colon\mathbb{R}^2\to\mathbb{R}$ a nice test function with compact support in Fourier space, $(r(\varepsilon))_\varepsilon$ some function of ε to be chosen below, and finally where we let

$$\lambda_t^\varepsilon:=\exp\Big[\frac{\beta^2}{2}G_t^\varepsilon(0)\Big],$$

with $G_t^\varepsilon(x):=\mathrm{Cov}\big(W_t^\varepsilon(x),W_t^\varepsilon(0)\big)$, and $W_t^\varepsilon:=\chi^\varepsilon*W_t$. This is regularised version of the covariance of the GFF, and it is not difficult to show that

$$G_t^\varepsilon(0)=\frac{1}{4\pi}\log((\varepsilon^{-2}\wedge t)\vee 1)+O(1),$$

see e.g. Lemma 6.3 in [?]. As a consequence we have also the asymptotic

$$\lambda_t^\varepsilon \approx (\varepsilon^{-2} \wedge t)^{1-\delta}$$

as $t \to \infty$.

With such a definition, the factor λ_t^ε ensures that $(U_t^\varepsilon(W_t))_{t\geqslant 0}$ is a martingale with zero mean:

$$U_t^\varepsilon(W_t) = \int_0^t \mathrm{D}U_s^\varepsilon(W_s)\mathrm{d}W_s =: M_t^\varepsilon$$

where

$$\mathrm{D}U_t^\varepsilon(\varphi)\cdot h = r(\varepsilon)\int \rho(x)\big[\lambda_t^\varepsilon \sin(\beta(\chi^\varepsilon * \varphi)(x))\beta(\chi^\varepsilon * h)(x)\big]\mathrm{d}x,$$

so

$$\begin{aligned}\mathrm{d}_t\langle M^\varepsilon\rangle_t \;&=\; \beta^2 r(\varepsilon)^2(\lambda_t^\varepsilon)^2\int\rho(x)\int\rho(x')\sin(\beta W_t^\varepsilon(x))\sin(\beta W_t^\varepsilon(x'))\times\\ &\qquad\times \mathrm{d}_t\langle W^\varepsilon(x), W^\varepsilon(x')\rangle_t \mathrm{d}x\mathrm{d}x',\end{aligned}$$

We can then estimate

$$\begin{aligned}\mathbb{E}\langle M^\varepsilon\rangle_t \;&\lesssim\; \int_0^t \mathrm{d}s(\varepsilon)^2(\lambda_s^\varepsilon)^2\int\rho(x)\mathrm{d}x\int\rho(x')\mathrm{d}x'\times\\ &\qquad\times\int \mathrm{d}y\mathrm{d}y'|\chi^\varepsilon(x-y)\chi^\varepsilon(x'-y')\dot G_s^\varepsilon(y-y')|\end{aligned}$$

and by Young's inequality, using that $\big\|\chi^\varepsilon\big\|_{L^1} = \big\|\chi\big\|_{L^1} \lesssim 1$,

$$\begin{aligned}\mathbb{E}\langle M^\varepsilon\rangle_\infty \;&\lesssim\; r(\varepsilon)^2\int_0^\infty \mathrm{d}s \underbrace{(\lambda_s^\varepsilon)^2}_{\lesssim(\varepsilon^{-2}\wedge s)^{2(1-\delta)}} \big\|\chi^\varepsilon\big\|_{L^1}^2\big\|\dot G_t^\varepsilon\big\|_{L^1}\\ &\lesssim\; r(\varepsilon)^2\left[\int_0^{\varepsilon^{-2}} e^{-m^2/s}s^{2(1-\delta)-2}\mathrm{d}s + \int_{\varepsilon^{-2}}^\infty e^{-m^2/s}\varepsilon^{-4(1-\delta)}s^{-2}\mathrm{d}s\right]\\ &\lesssim\; r(\varepsilon)^2\varepsilon^{-2(1-2\delta)}\end{aligned} \tag{8.1}$$

(this when $\delta < 1/2$, if $\delta = 1/2$ we have a logarithmic factor in place of $\varepsilon^{-2(1-2\delta)}$).

As a consequence as long as we choose $r(\varepsilon)$ such that $r(\varepsilon)\varepsilon^{-(1-2\delta)} \to 0$ as $\varepsilon \to 0$ we can extract a subsequence $(\varepsilon_n)_n$ and obtain that, $M_\infty^{\varepsilon_n} \to 0$ almost surely as $n \to \infty$.

On the other hand, again by Itô formula, we have that $U_t^\varepsilon(\Phi_t)$ behaves differently as $t \to \infty$, in particular

$$U_t^\varepsilon(\Phi_t) = \underbrace{\int_0^t \mathrm{D}U_s^\varepsilon(\Phi_s)\mathrm{d}Z_s}_{\mathbb{I}_t^\varepsilon} + \underbrace{\int_0^t \mathrm{D}U_s^\varepsilon(\Phi_s)\mathrm{d}W_s}_{\tilde M_t^\varepsilon},$$

the martingale term

$$\tilde M_t^\varepsilon := \int_0^\infty \mathrm{D}U_s^\varepsilon(\Phi_s)\mathrm{d}W_s,$$

can be controlled essentially as M_t^ε and therefore we can make it go to zero in the same way, i.e.

$$\mathbb{E}\langle M^\varepsilon\rangle_\infty \lesssim r(\varepsilon)^2\varepsilon^{-2(1-2\delta)}. \tag{8.2}$$

The analysis of the contribution $\mathbb{I}_t^\varepsilon$ coming from the drift term is a bit more involved and crucial to the argument. First observe that

$$\mathrm{d}Z_s = \dot G_s F_s^{[1]}(\Phi_s)\mathrm{d}s + \dot G_s\big(F_s^{[>1]}(\Phi_s) + R_s\big)\mathrm{d}s$$

where

$$F_s^{[1]}(\varphi)(x) = \lambda_s\sum_{\sigma=\pm 1} e^{i\sigma\varphi(x)}$$

and where from the estimates on the remainder R_s and on the contributions $F_s^{[\ell]}$ for $\ell \geqslant 2$ we have

$$\big|\mathrm{d}Z_s - \dot G_s F_s^{[1]}(\Phi_s)\mathrm{d}s\big| \lesssim \langle s\rangle^{-2+1-2\delta},$$

so we can split

$$\mathbb{I}_t^\varepsilon = \int_0^t \mathrm{D}U_s^\varepsilon(\Phi_s)\dot{G}_s F_s^{[1]}(\Phi_s)\mathrm{d}s + \int_0^t \mathrm{D}U_s^\varepsilon(\Phi_s)\big(\mathrm{d}Z_s - \dot{G}_s F_s^{[1]}(\Phi_s)\mathrm{d}s\big) =: \mathbb{A}_t^\varepsilon + \mathbb{B}_t^\varepsilon. \tag{8.3}$$

Let us first consider the term $\mathbb{A}_t^\varepsilon$. Expanding both $\mathrm{D}U_s^\varepsilon(\Phi_s)$ and $F_s^{[1]}(\Phi_s)$ in terms of complex exponentials we get

$$\begin{aligned}\mathbb{A}_t^\varepsilon &= \int_0^t \mathrm{D}U_s^\varepsilon(\Phi_s)\dot{G}_s F_s^{[1]}(\Phi_s)\mathrm{d}s \\ &= -\beta^2 r(\varepsilon)\sum_{\sigma_1,\sigma_2=\pm 1}\sigma_1\sigma_2\int_0^t \lambda_s^\varepsilon\lambda_s\int\rho(x)\mathrm{d}x\int\mathrm{d}x'\, e^{i\beta\sigma_1\Phi_s(x)+i\beta\sigma_2\Phi_s^\varepsilon(x')}(\dot{G}_s * \chi^\varepsilon)(x-x')\mathrm{d}s.\end{aligned}$$

The main contribution to the sums over σ_1, σ_2 comes from the term where the oscillations of the complex exponential are minimal near the diagonal $x \approx x'$, i.e. from the terms where $\sigma_1 = -\sigma_2$, in this case we can, in first approximation let

$$e^{i\beta\sigma_1\Phi_s(x)+i\beta\sigma_2\Phi_s^\varepsilon(x')} \approx 1, \tag{8.4}$$

and observe that the contributions D_t^ε to $\mathbb{A}_t^\varepsilon$ coming from these terms behave as

$$\begin{aligned} D_t^\varepsilon &\approx r(\varepsilon)\int_0^\infty \lambda_s^\varepsilon\lambda_s\int\rho(x)\mathrm{d}x\int\mathrm{d}x'(\dot{G}_s * \chi^\varepsilon)(x-x')\mathrm{d}s \\ &\approx r(\varepsilon)\int_0^\infty \lambda_s^\varepsilon\lambda_s\langle s\rangle^{-2}\mathrm{d}s \\ &\approx r(\varepsilon)\int_0^{\varepsilon^{-2}}\langle s\rangle^{-2\delta}\mathrm{d}s + r(\varepsilon)\varepsilon^{-2(1-\delta)}\int_{\varepsilon^{-2}}^\infty\langle s\rangle^{-1-\delta}\mathrm{d}s \\ &\approx r(\varepsilon)\varepsilon^{-2(1-2\delta)}. \end{aligned} \tag{8.5}$$

A careful analysis of the other contributions and the errors in the approximation (8.4) shows that

$$\big|\mathbb{A}_\infty^\varepsilon - D_\infty^\varepsilon\big| \approx r(\varepsilon)\varepsilon^{-2(1-2\delta)}o_\varepsilon(1).$$

Remark 8.1. The proof requires some attention in order to extract additional powers of ε. For example when $\sigma_1 = \sigma_2$, the exponentials behave, for $x \approx x'$ and $\varepsilon \to 0$ as

$$e^{i\beta\sigma_1\Phi_s(x)+i\beta\sigma_2\Phi_s^\varepsilon(x')} \approx e^{2i\beta\sigma_1\Phi_s(x)}$$

modulo lower order terms. From here we see that the term in the integral behaves as

$$r(\varepsilon)\int_0^t \lambda_s^\varepsilon\lambda_s\lambda_s^{-4}\int\rho(x)\mathrm{d}x\Big[\lambda_s^4 e^{2i\beta\sigma_1 W_s(x)}\Big]e^{2i\beta\sigma_1 Z_s(x)}\int\mathrm{d}x'(\dot{G}_s * \chi^\varepsilon)(x-x')\mathrm{d}s \tag{8.6}$$

where we split the interacting field and introduced enough powers of λ_s as to construct the Wick-renormalised exponential $\lambda_s^4 e^{2i\beta\sigma_1 W_s(x)}$. Now $\lambda_s^4 e^{2i\beta\sigma_1 W_s}$ converges as $s \to \infty$ to a well defined random distribution $[\![e^{2i\beta\sigma_1 W_\infty}]\!]$, while $e^{2i\beta\sigma_1 Z_s}$ converges to a relatively smooth function $e^{2i\beta\sigma_1 Z_\infty}$. Ignoring these spatial regularity issues, and using that

$$\left|\int\mathrm{d}x'(\dot{G}_s * \chi^\varepsilon)(x-x')\right| \lesssim \big\|\dot{G}_s\big\|_{L^1} \lesssim \langle s\rangle^{-2},$$

we expect, roughly, that the quantity (8.6) can be bounded by

$$r(\varepsilon)\int_0^t \lambda_s^\varepsilon\lambda_s\lambda_s^{-4}\langle s\rangle^{-2}\mathrm{d}s \lesssim r(\varepsilon)\int_0^t \lambda_s^{-2}\langle s\rangle^{-2}\mathrm{d}s \lesssim r(\varepsilon)\int_0^t \langle s\rangle^{-2+2\delta}\langle s\rangle^{-2}\mathrm{d}s$$

which shows indeed a gain in ε. The proper analysis of the interaction between the asymptotic irregularity of $\lambda_s^4 e^{2i\beta\sigma_1 W_s(x)}$ with the spatial behaviour of $e^{2i\beta\sigma_1 Z_s}$ and $(\dot{G}_s * \chi^\varepsilon)$ produces additional positive powers of ε. Finally however the net gain is in our favour. A similar analysis is needed for the corrections coming from the terms where $\sigma_1 = -\sigma_2$, to the main contributions.

Let us now go back to (8.3) to consider the term $\mathbb{B}^\varepsilon_t$. Observe that, making rigorous the estimates for the effective force in (5.5), we have

$$\begin{aligned}\sup_t \left|\mathbb{B}^\varepsilon_t\right| &\lesssim r(\varepsilon)\int_0^\infty \lambda^\varepsilon_s \langle s\rangle^{-1-2\delta}\mathrm{d}s \lesssim r(\varepsilon)\int_0^\infty (s\wedge\varepsilon^{-2})^{1-\delta}\langle s\rangle^{-1-2\delta}\mathrm{d}s \\ &\lesssim r(\varepsilon)\Big[\int_0^{\varepsilon^{-2}}\langle s\rangle^{-3\delta}\mathrm{d}s + \int_{\varepsilon^{-2}}^\infty(\varepsilon)^{-2(1-\delta)}\langle s\rangle^{-1-2\delta}\mathrm{d}s\Big] \\ &\lesssim r(\varepsilon)\Big[1+\varepsilon^{-2(1-3\delta)}\Big].\end{aligned} \tag{8.7}$$

Overall we see now that can choose $r(\varepsilon)\to 0$ such that (cfr. eq. (8.5))

$$r(\varepsilon)\varepsilon^{-2(1-2\delta)}\to\infty,$$

while (cfr. eqns. (8.1), (8.2) and (8.7)),

$$r(\varepsilon)\varepsilon^{-(1-2\delta)}\to 0, \qquad r(\varepsilon)\Big[1+\varepsilon^{-2(1-3\delta)}\Big]\to 0.$$

This guarantees that as $\varepsilon\to 0$ (e.g. almost surely along a fixed subsequence $(\varepsilon_n)_n$)

$$\mathbb{III}^{\varepsilon_n}_\infty - D^{\varepsilon_n}_\infty\to 0, \qquad \tilde{M}^{\varepsilon_n}_\infty\to 0, \qquad M^{\varepsilon_n}_\infty\to 0,$$

and therefore that, almost surely

$$U^{\varepsilon_n}_\infty(\Phi_\infty)\to\infty, \qquad \text{while} \qquad U^{\varepsilon_n}_\infty(W_\infty)\to 0,$$

proving the singularity of SG wrt. the GFF.

Chapter 9
Renormalised variational structure

We give a look now at what happens to the variational structure of Theorem 3.1 as $T \to \infty$ and $\rho \to 1$. Recall that the variational problem is given by the minimisation of the functional

$$\mathcal{J}_T(u) = \mathbb{E}\left[V(\Phi_T^u) + \frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s \right],$$

with

$$\Phi_t^u = X_t + \int_0^t \dot{G}_s^{1/2} u_s \mathrm{d}s.$$

If $\beta^2 < 4\pi$, the variational description in the finite volume is essentially a direct consequence of the convergence of the Wick-ordered cosine using results from [?]. More in detail, one can establish, using a continuity argument that

$$\lim_{T\to\infty} \inf_{u\in\mathbb{H}_a} \mathcal{J}_T^\rho(u) = \inf_{u\in\mathbb{H}_a} \mathcal{J}_\infty^\rho(u) \tag{9.1}$$

where

$$\mathcal{J}_\infty(u) = \mathbb{E}\left[\hat{\lambda} \int \rho [\![\cos(\beta \Phi_\infty^u)]\!] \mathrm{d}x + \frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s \right],$$

where $\Phi_\infty^u = X_\infty + Z_\infty^u$, $Z_\infty^u = \int_0^\infty \dot{G}_s u_s \mathrm{d}s$, and the quantity $[\![\cos(\beta\Phi_\infty^u)]\!]$ is defined via the formula

$$[\![\cos(\beta\Phi_\infty^u)]\!] = [\![\cos(\beta X_\infty)]\!]\cos(\beta Z_\infty^u) - [\![\sin(\beta X_\infty)]\!]\sin(\beta Z_\infty^u)$$

where $[\![\cos(\beta X_\infty)]\!], [\![\sin(\beta X_\infty)]\!]$ are the Wick renormalised cosine and sine of the GFF X_∞, while $Z_\infty^u \in H^1$, since we have the estimate

$$\|(m^2 - \Delta)^{1/2} Z_\infty^u\|^2 \lesssim \int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s, \tag{9.2}$$

which can be checked via Parseval's theorem and the definition of $\dot{G}_t$. This regularity allows to prove that the quantity $[\![\cos(\beta\Phi_\infty^u)]\!]$ is well defined for any control u such that $\int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s < \infty$ and that we have, for some $\delta > 0$,

$$\mathcal{J}_\infty(u) \geqslant -C + \delta \int_0^T \|u_s\|^2 \mathrm{d}s,$$

which implies coercivity of $\mathcal{J}_\infty(u)$ and a good control of the variational problem (9.1).

Beyond the first threshold ($\beta^2 \geqslant 4\pi$) this representation does not hold anymore (in correspondence of the singularity of the Sine–Gordon measure, see Section 8) and we need to obtain a renormalised description.

To discuss the removal of the cutoff we employ Itô formula with an approximate effective potential $(V_s)_s$ which for the moment is an arbitrary scale-dependent functional such that $V_T = V$. We obtain

$$\begin{aligned}\mathcal{J}_T(u) &= V_0(\Phi_0^u) + \mathbb{E}\left\{ \int_0^T \left[\mathcal{H}_s(\Phi_s^u) + \mathrm{D}V_s(\Phi_s^u)\, \dot{G}_s^{1/2} u_s + \frac{1}{2}(\mathrm{D}V_s \dot{G}_s \mathrm{D}V_s)(\Phi_s^u) \right] \mathrm{d}s + \frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s \right\} \\ &= V_0(\Phi_0^u) + \mathbb{E}\left[\int_0^T \mathcal{H}_s(\Phi_s^u)\mathrm{d}s + \frac{1}{2}\int_0^T \left\| u_s + \dot{G}_s^{1/2} \mathrm{D}V_s(\Phi_s^u) \right\|^2 \mathrm{d}s \right],\end{aligned}$$

where

$$\mathcal{H}_t := \partial_t V_t + \frac{1}{2}\operatorname{Tr} \dot{G}_t \mathrm{D}^2 V_t - \frac{1}{2}\mathrm{D}V_t \dot{G}_t \mathrm{D}V_t.$$

In this new formulation we see that the running cost does not depends anymore on the L^2 norm of u but on that of the *remainder* $r=(r_s)_s$ defined as

$$r_s := u_s + G_s^{1/2}\mathrm{D}V_s(\Phi_s^u).$$

Now we can invert this dependence and let u be a function of r, since the equation

$$\hat{Z}_t^r = -\int_0^t \dot{G}_s \mathrm{D}V_s(X_s+\hat{Z}_s^r)\mathrm{d}s + \int_0^t \dot{G}_s^{1/2} r_s \mathrm{d}s$$

has Lipschitz coefficients and unique global solutions. Additionally, the control

$$\hat{u}_s = r_s - \dot{G}_s^{1/2}\mathrm{D}V_s(X_s+\hat{Z}_s^r),$$

is admissible for the original control problem and $\Phi_t^{\hat{u}} = X_t + \hat{Z}_t^r =: \hat{\Phi}_t^r$. So we can reformulate the control problem as

$$\hat{\mathcal{J}}_T(r) = V_0(\hat{\Phi}_0^r) + \mathbb{E}\bigg[\int_0^T \mathcal{H}_s(\hat{\Phi}_s^r)\mathrm{d}s + \frac{1}{2}\int_0^T \big\|r_s\big\|_{L^2}^2 \mathrm{d}s\bigg],$$

and

$$\inf_{u\in\mathbb{H}_a} \mathcal{J}_T(u) = \inf_{r\in\mathbb{H}_a} \hat{\mathcal{J}}_T(r).$$

Moreover we have the bound

$$\hat{\mathcal{J}}_T(r) \geqslant -C + \frac{1}{2}\int_0^T \|r_s\|_{L^2}^2 \mathrm{d}s,$$

uniformly in T. If we recall that the optimiser for $\mathcal{J}(u)$ is

$$u_t = -\dot{G}_t^{1/2}\mathrm{D}V_t(\Phi_t^u) - \dot{G}_t^{1/2} R_t$$

we can deduce that the optimiser for $\hat{\mathcal{J}}(r)$ is

$$r_t = -\dot{G}_t^{1/2}\mathrm{D}V_t(\Phi_t^u) - \dot{G}_t^{1/2} R_t + \dot{G}_t^{1/2}\mathrm{D}V_t(\Phi_t^u) = -\dot{G}_t^{1/2} R_t.$$

From these observations we deduce that

$$\lim_{T\to\infty}\inf_{u\in\mathbb{H}_a} \mathcal{J}_T^\rho(u) = \inf_{r\in\mathbb{H}_a} \hat{\mathcal{J}}_\infty^\rho(r),$$

this allows to extend the variational description to the unregularised theory (i.e. $T\to\infty$).

Note that by our estimates we only have

$$\big\|\dot{G}_s^{1/2}\mathrm{D}V_s(\Phi_s^u)\big\|_{L^2} \lesssim \langle s\rangle^{-\delta},$$

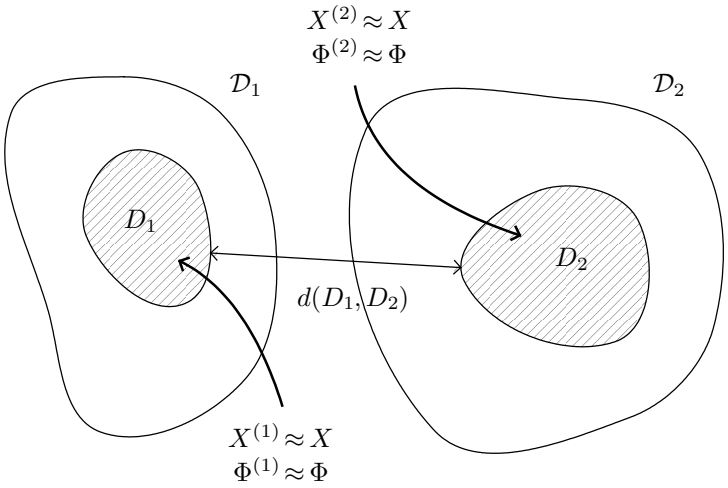
which is in L_t^2 only when $2\delta>1$ that is $\beta^2<4\pi$. However the renormalised control r still retains L^2 regularity beyond this range and it is expected to do so for all $\beta^2<8\pi$ provided suitable approximate solutions of the flow equation can be found.

Chapter 10
Decay of correlations

We present another use of the FBSDE to analyse in a stochastic approach the spatial decay of correlations for the EQF. We learned about this approach in [?] in the context of regular SPDEs and similar ideas have been explored in the setting of singular SPDEs in [?] and [?]. The present proof is taken, as much of these notes, from [?].

Decay of correlations means that sufficiently regular functions of the the random field Φ_∞ which are localised in far apart regions behave almost independently, in particular their covariance decay with the distance.

Let us assume that we have two disjoint (nice) regions $D_1, D_2 \subset \mathbb{R}^2$ at some positive distance from each other, for convenience we also define $D_0 := \mathbb{R}^2$. For $i = 1, 2$, let $\mathcal{D}_i \supset D_i$ with $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ be sets to be fixed later:



We want to prove the following:

Theorem 10.1. *For any $\hat{\lambda}$ small enough, the (unique) Sine–Gordon field Φ_∞ has exponential decay of correlation, namely, there exists $\gamma > 0$ such that for globally Lipshitz observables $\mathcal{O}^{(i)}$ of Φ_∞, $i = 1, 2$ respectively localised in D_i (i.e. such that they depends only on the restriction of the distribution Φ_∞ on some open subset of D_i) we have*

$$|\mathrm{Cov}(\mathcal{O}^{(1)}(\Phi_\infty), \mathcal{O}^{(2)}(\Phi_\infty))| \lesssim e^{-\gamma d(D_1, D_2)}$$

where the implicit constant does not depend on the distance among the observables $\mathcal{O}^{(i)}$ but only on their Lipshitz norm (e.g. as functionals on $\mathcal{S}'(\mathbb{R}^2)$).

In order sketch a stochastic proof of this theorem we consider the solution $(\Phi_t)_{t \geqslant 0}$ of the FBSDE for the Sine–Gordon measure. Its uniqueness in law is guaranteed by the fact that $\hat{\lambda}$ can be chosen small enough so that the FBSDE has path-wise uniqueness.

We introduce also other two solutions to the same FBDSE, called $\Phi^{(i)}$, $i=1,2$, but driven by different Gaussian source terms $X^{(i)}$. We couple the three driving Gaussian processes $X^{(0)}, X^{(1)}, X^{(2)}$ in such a way that they form a Gaussian vector with covariance

$$\mathbb{E}\big[X_t^{(i)}(x)\,X_t^{(j)}(y)\big]=\int_0^t \mathrm{d}s\int_{\mathbb{R}^2} G_s^{1/2}(z-x)\,\mathbb{1}_{\mathcal{D}_i\cap\mathcal{D}_j}(z)\,G_s^{1/2}(z-y)\mathrm{d}z.$$

For $X=X^{(0)}$, we just recover the same driving noise as before, while $X^{(1)}, X^{(2)}$ are independent multi-scale GFFs such that $X^{(i)}\approx X$ on D_i. More precisely one can prove that $X_\infty - X_\infty^{(i)}$ is a smooth field well inside $\mathcal{D}_i$ and that for $\gamma>0$ sufficiently small, and provided $d(x,\mathcal{D}_i^c)\geqslant a>0$, we have

$$\mathbb{E}\big|X_s(x)-X_s^{(i)}(x)\big|^2\lesssim \log(a)\exp(-\gamma d(x,\mathcal{D}_i^c)), \tag{10.1}$$

for $i=1,2$ and all $s\geqslant 0$.

Fix for the moment $i=1,2$ and write a FBSDE for the differences

$$\begin{cases}\Phi_t-\Phi_t^{(i)}= & -\displaystyle\int_0^t \mathrm{d}s\,\dot G_s\big(F_s(\Phi_s)-F_s\big(\Phi_s^{(i)}\big)+R_s-R_s^{(i)}\big)+X_t-X_t^{(i)},\\ R_s-R_s^{(i)}= & \mathbb{E}_t\displaystyle\int_t^\infty \mathrm{d}s\,\big[H_s(\Phi_s)-H_s\big(\Phi_s^{(i)}\big)\big],\\ & +\mathbb{E}_t\displaystyle\int_t^\infty \mathrm{d}s\,\big[\mathrm{D}F_s(\Phi_s)\,\dot G_s\,R_s-\mathrm{D}F_s\big(\Phi_s^{(i)}\big)\,\dot G_s\,R_s^{(i)}\big].\end{cases} \tag{10.2}$$

Using a Lipschitz estimates on the effective force similar to Proposition 5.3 and eq. (10.1) to control the source term in this system, we would like to argue that we have $\Phi_t\approx\Phi_t^{(i)}$ in the region D_i. This is not difficult to believe as the source term is relatively large only outsider the region $\mathcal{D}_i$ which is far from D_i and both the kernel $\dot G_s$ and the kernels appearing in the effective force F_s are very well localised and decay exponentially at macroscopic scales (i.e. of size of $O(1)$).

The concrete way to extract this information is to introduce a weight of the form

$$q^{(i)}(x)=\exp(-\gamma d(x,D_i))$$

for some $\gamma>0$ to be chosen small enough and extract from (10.2), a bound for $q^{(i)}\big(Z_t-Z_t^{(i)}\big)$ and $q^{(i)}\big(R_t-R_t^{(i)}\big)$ where we let, as usual, $Z=\Phi-X$ and $Z^{(i)}=\Phi^{(i)}-X^{(i)}$.

This allows us to use simple L^∞ bounds to get

$$\begin{aligned}\big\|q^{(i)}\big(Z_t-Z_t^{(i)}\big)\big\|_{L^\infty}\leqslant & \int_0^t\langle s\rangle^{-2}\lambda_s\big(\big\|q^{(i)}\big(\Phi_s-\Phi_s^{(i)}\big)\big\|_{L^\infty}+\big\|q^{(i)}\big(R_s-R_s^{(i)}\big)\big\|_{L^\infty}\big)\mathrm{d}s,\\ \big\|q^{(i)}\big(R_t-R_t^{(i)}\big)\big\|_{L^\infty}\leqslant & \mathbb{E}_t\int_t^\infty\langle s\rangle^{-4}\lambda_s^4\big(\big\|q^{(i)}\big(\Phi_s-\Phi_s^{(i)}\big)\big\|_{L^\infty}\big)\mathrm{d}s\\ & +\mathbb{E}_t\int_t^\infty\langle s\rangle^{-2}\lambda_s\big(\big\|q^{(i)}\big(\Phi_s-\Phi_s^{(i)}\big)\big\|_{L^\infty}+\big\|q^{(i)}\big(R_s-R_s^{(i)}\big)\big\|_{L^\infty}\big)\mathrm{d}s.\end{aligned} \tag{10.3}$$

The commutation of the weight $q^{(i)}$ with the non-local kernels is obtained by imposing that γ is small enough.

Observe that

$$\big\|q^{(i)}\big(\Phi_t-\Phi_t^{(i)}\big)\big\|_{L^\infty}\leqslant\big\|q^{(i)}\big(X_t-X_t^{(i)}\big)\big\|_{L^\infty}+\big\|q^{(i)}\big(Z_t-Z_t^{(i)}\big)\big\|_{L^\infty},$$

and that an extension of the estimate (10.1) gives also

$$\big\|q^{(i)}\big(X_t-X_t^{(i)}\big)\big\|_{L^\infty}\lesssim_\varepsilon\langle t\rangle^\varepsilon\exp(-c\gamma d(D_i,\mathcal{D}_i^c)),$$

for some small constant c and any small $\varepsilon>0$, almost surely. This tells us that the source terms are small enough when $d(D_i,\mathcal{D}_i^c)$ is large.

The system (10.3) cannot be handled via Gronwall, as it mixes forward and backwards inequalities. If we impose that $\hat\lambda$ is small we can however still deduce that both l.h.s. are controlled by the source term $q^{(i)}\big(X_t-X_t^{(i)}\big)$, i.e. a bound of the form

$$\sup_t\big\|q^{(i)}\big(Z_t-Z_t^{(i)}\big)\big\|_{L^\infty}+\big\|q^{(i)}\big(R_t-R_t^{(i)}\big)\big\|_{L^\infty}\lesssim\exp(-c\gamma d(D_i,\mathcal{D}_i^c))$$

which in particular implies that on the set D_i, $Z_t \approx Z_t^{(i)}$ as in this case $q^{(i)} \approx 1$, and as a consequence that $\Phi_t \approx \Phi_t^{(i)}$ on the same set, since we will also have $X_t \approx X_t^{(i)}$.

Therefore for small $\hat{\lambda}$ we can replace Φ by $\Phi^{(i)}$ on $D^{(i)}$, for $i = 1, 2$ by paying an error which in (spatial) L^∞ is of size $O(\exp(-c\gamma d(D_i, \mathcal{D}_i^c)))$. By retracing the steps above one can extract a probabilistic statement in which this error terms holds in $L^p(\mathbb{P})$. This means that for globally Lipshitz observables $\mathcal{O}^{(i)}$, $i = 1, 2$ respectively localised in D_i, we have that

$$\left\| \mathcal{O}^{(i)}(\Phi_\infty) - \mathcal{O}^{(i)}\big(\Phi_\infty^{(i)}\big) \right\|_{L^p(\mathbb{P})} \lesssim \exp(-c\gamma d(D_i, \mathcal{D}_i^c)), \qquad i = 1, 2.$$

By simply tracking these errors we establish that

$$\left| \mathrm{Cov}(\mathcal{O}^{(1)}(\Phi_\infty), \mathcal{O}^{(2)}(\Phi_\infty)) - \mathrm{Cov}\big(\mathcal{O}^{(1)}\big(\Phi_\infty^{(1)}\big), \mathcal{O}^{(2)}\big(\Phi_\infty^{(2)}\big)\big) \right| \lesssim \sup_{i=1,2} \exp(-c\gamma d(D_i, \mathcal{D}_i^c))$$

and since $\Phi^{(1)}$ is independent of $\Phi^{(2)}$ we also have

$$\mathrm{Cov}\big(\mathcal{O}^{(1)}\big(\Phi_\infty^{(1)}\big), \mathcal{O}^{(2)}\big(\Phi_\infty^{(2)}\big)\big) = 0$$

which allows to conclude the exponential decay of correlations

$$|\mathrm{Cov}(\mathcal{O}^{(1)}(\Phi_\infty), \mathcal{O}^{(2)}(\Phi_\infty))| \lesssim e^{-\frac{c\gamma}{2} d(D_1, D_2)}$$

by taking the regions $\mathcal{D}_i$ such that $d(D_i, \mathcal{D}_i^c) = d(D_1, D_2)/2$.

Chapter 11
Semi-classical limit in finite volume

The notion of semiclassical limit refers to the limit where the size of quantum effects in a quantum field theory, as measured by Planck's constant $\hbar$, becomes negligible. In the Euclidean picture, the value of $\hbar > 0$ is associated to the size of the fluctuations. Formally the path-integral for a scalar theory takes the form

$$\nu^{\hbar}(\mathrm{d}\varphi) = \frac{e^{-\frac{S(\varphi)}{\hbar}}}{Z}\mathrm{d}\varphi, \tag{11.1}$$

where

$$S(\varphi) = \int \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi + v(\varphi)\mathrm{d}x, \tag{11.2}$$

and where we reintroduced $\hbar$ which was taken to be $\hbar = 1$ (or really any fixed constant), in the previous sections of these notes.

The limit $\hbar \to 0$ corresponds therefore to a saddle-point limit of the path-integral (11.1) and we expect that the main contributions should come from the minima of the *classical Euclidean action* (11.2).

The sine–Gordon model $\nu^{\hbar}_{\infty}$ is obtained as (any) accumulation point of the family of measures $\nu^{\hbar}_{T,\rho}$ defined by

$$\nu^{\hbar}_{T,\rho}(\mathrm{d}\varphi) = \frac{e^{-\frac{1}{\hbar}V^{\rho}_{T}(\varphi)}}{Z^{\hbar}_{T,\rho}}\mu^{\hbar}_{T}(\mathrm{d}\varphi),$$

where $\mu^{\hbar}_{T}$ is a regularised (massive) GFF with covariance $\hbar G_T$.

In order to simplify our discussion we will restrict our argument to the finite volume measures and keep ρ fixed and with compact support. Then the finite volume Sine–Gordon measure is unique (see Section 7) and given by the weak limit

$$\nu^{\hbar}_{\rho} := \lim_{T\to\infty} \nu^{\hbar}_{\rho,T}.$$

The study of the semiclassical limit reduces, therefore, to establish *the Laplace principle*

$$\lim_{\hbar\to 0} -\hbar \log \int e^{-\frac{1}{\hbar}g(\varphi)}\nu^{\hbar}_{\rho}(\mathrm{d}\varphi) = \inf_{\varphi}\left[g(\varphi) + S_{\rho}(\varphi)\right] \tag{11.3}$$

where

$$S_{\rho}(\varphi) = \int \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2\varphi(x) + \rho(x)\hat{\lambda}\cos(\beta\varphi(x))\mathrm{d}x, \tag{11.4}$$

is the classical action of the Sine–Gordon model with a finite volume cutoff ρ.

The stochastic approach to establish (11.3) starts with noting that

$$-\hbar \log \int e^{-\frac{1}{\hbar}g(\varphi)}\nu^{\hbar}_{\rho}(\mathrm{d}\varphi) = -\hbar \log \mathbb{E}\Big[e^{-\frac{1}{\hbar}g(\Phi^{\hbar}_{\infty})}\Big],$$

where $\Phi^{\hbar}_{\infty}$ is the solution to the FBSDE,

$$\begin{cases} \Phi^{\hbar}_{\infty} = \hbar^{1/2}X_{\infty} + \hbar\int_0^{\infty}\dot{G}_s\big[F^{\hbar}_s(\Phi^{\hbar}_s) + R^{\hbar}_s\big]\mathrm{d}s, \\ R^{\hbar}_t = \hbar\int_0^{\infty}H^{\hbar}_s(\Phi^{\hbar}_s)\mathrm{d}s + \hbar\int_0^{\infty}\mathrm{D}F^{\hbar}_s(\Phi^{\hbar}_s)\dot{G}_s R^{\hbar}_s\mathrm{d}s, \end{cases}$$

and where $(F^{\hbar}_s)_s$ is the approximate effective force obtained by solving the flow equation with "propagator" $\hbar\dot{G}_t$ instead of $\dot{G}_t$ and such that

$$F^{\hbar,[1]}_s(\varphi) = \frac{1}{\hbar}\lambda_s\cos(\beta\varphi).$$

An inspection of this representation shows that the model behaves as if the parameter β has been replaced by $\beta_{\text{eff}} = \beta\hbar^{1/2}$, in particular, as $\hbar \to 0$ the model is well into the first threshold region where $\beta_{\text{eff}}^2 < 4\pi$.

As a consequence, for $\hbar$ small enough $[\![\sin(\beta\Phi_\infty^\hbar)]\!]$ is a well defined distribution of regularity $H_{\text{loc}}^{-1+\varepsilon}$ for some $\varepsilon > 0$. Since

$$\Phi_\infty^\hbar = \hbar^{1/2} X_\infty + Z_\infty^\hbar$$

we have

$$[\![\sin(\beta\Phi_\infty^\hbar)]\!] = [\![\sin(\beta\hbar^{1/2}X_\infty^\hbar)]\!]\cos(\beta Z_\infty^\hbar) + [\![\cos(\beta\hbar^{1/2}X_\infty^\hbar)]\!]\sin(\beta Z_\infty^\hbar),$$

and by the regularity of $Z_\infty^\hbar$ the products in this formula are well defined. In this regime we can therefore "undo" the FBSDE representation and write

$$F_s^\hbar(\Phi_s^\hbar) + R_s^\hbar = \mathbb{E}_s[F_\infty^\hbar(\Phi_\infty^\hbar)] = \frac{\hat\lambda\beta}{\hbar}\,\mathbb{E}_s\Big[\,\rho[\![\sin(\beta\Phi_\infty^\hbar)]\!]\Big]$$

as a closed martingale. Next, if we let $\Psi_\infty^\hbar = \hbar^{-1/2}\Phi_\infty^\hbar$ we have

$$\Psi_\infty^\hbar = X_\infty + \hat\lambda\beta\hbar^{1/2}\hbar^{-1}\int_0^\infty \dot G_s \mathbb{E}_s\Big[\,\rho[\![\sin(\beta\hbar^{1/2}\Psi_\infty^\hbar)]\!]\Big]\mathrm{d}s,$$

which also shows that $\Psi_\infty^\hbar$ has the law of the Sine–Gordon model for with parameters $\beta, \hat\lambda$ replaced by $\beta\hbar^{1/2}$ and $\hat\lambda/\hbar$ (as the reader could have also guessed from a direct change of variables of the approximate path-integral (11.2)).

We can now look back at the Laplace asymptotic and observe that for sufficiently small $\hbar$ we can use the variational representation (9.1) to write

$$-\hbar\log\int e^{-\frac{1}{\hbar}g(\varphi)}\nu_\rho^\hbar(\mathrm{d}\varphi) = -\hbar\log\mathbb{E}\Big[\,e^{-\frac{1}{\hbar}g(\hbar^{1/2}\Psi_\infty^\hbar)}\Big] =$$

$$= \inf_u\ \mathbb{E}\bigg[\,g(\hbar^{1/2}\Psi_\infty^\hbar(u)) + \int_{\mathbb{R}^2}\hat\lambda\rho[\![\cos(\beta\hbar^{1/2}\Psi_\infty^\hbar)]\!]\mathrm{d}x + \frac{1}{2}\hbar\int_0^\infty\|u_s\|^2\mathrm{d}s\bigg]$$

where $\Psi_\infty^\hbar(u)$ solves

$$\Psi_\infty^\hbar(u) = X_\infty + \int_0^\infty \dot G_s^{1/2}u_s\mathrm{d}s.$$

Next, making the change of variables $u \to \hbar^{-1/2}u$ we have

$$-\hbar\log\int e^{-\frac{1}{\hbar}g(\varphi)}\nu_\rho^\hbar(\mathrm{d}\varphi) =$$

$$= \inf_u\ \mathbb{E}\bigg[\,g(\hbar^{1/2}X_\infty + Z_\infty^u) + \int_{\mathbb{R}^2}\hat\lambda\rho[\![\cos(\beta\hbar^{1/2}X_\infty + \beta Z_\infty^u)]\!]\mathrm{d}x + \frac{1}{2}\int_0^\infty\|u_s\|^2\mathrm{d}s\bigg].$$

From this formula it is now easy to take the limit $\hbar \to 0$ observing that

a) we have $\hbar^{1/2}X_\infty \to 0$ in $H_{\text{loc}}^{-\varepsilon}$;

b) we have

$$\rho[\![\cos(\beta\hbar^{1/2}X_\infty)]\!] \to 1, \qquad \rho[\![\sin(\beta\hbar^{1/2}X_\infty)]\!] \to 0$$

in $H^{-1+\varepsilon}$ as $\hbar \to 0$ (in $L^p(\mathbb{P})$ for some $p \geqslant 2$);

c) we have also enough regularity for Z_∞^u (which is in H^1 due to an estimate similar to (9.2)).

As a consequence we have

$$\lim_{\hbar\to 0} -\hbar\log\int e^{-\frac{1}{\hbar}g(\varphi)}\nu_\rho^\hbar(\mathrm{d}\varphi) = \inf_u\ \mathbb{E}\bigg[\,g(Z_\infty^u) + \int_{\mathbb{R}^2}\hat\lambda\rho\cos(\beta Z_\infty^u)\mathrm{d}x + \frac{1}{2}\int_0^\infty\|u_s\|^2\mathrm{d}s\bigg]$$

and since there is no source of randomness in the variational functional it is an exercise to show that the infimum can be computed in two steps:

$$\inf_u\Big[\cdots\Big] = \inf_{\varphi\in H^1}\bigg[\,g(\varphi) + \int_{\mathbb{R}^2}\hat\lambda\rho\cos(\beta\varphi)\mathrm{d}x + \inf_{u:Z_\infty^u=\varphi}\frac{1}{2}\int_0^\infty\|u_s\|^2\mathrm{d}s\bigg]$$

and that it is an exercise to show that

$$\inf_{u:Z_\infty^u=\varphi}\frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s = \frac{1}{2}\|(m^2-\Delta)\varphi\|_{L^2}^2,$$

(hint: consider $u_s = \dot{G}_s^{1/2}(m^2-\Delta)\varphi$). It follows that the Laplace asymptotic (11.3) holds.

Chapter 12
Literature on the Sine–Gordon model

The sine–Gordon model has been subject to many studies in the constructive literature, covering finite or infinite volume interactions and allowing various ranges for $\beta^2 \in [0, 8\,\pi)$ and the coupling constant $\lambda \in \mathbb{R}$. However, the full mathematical understanding of this model is still lacking and none of these works cover the theory on the full space $\mathbb{R}^2$ for all $\beta^2 \in (0, 8\,\pi)$ and all $\lambda \in \mathbb{R}$. We single out the pioneering work of Benfatto et al. [?] and Nicoló et al. [?] who establish existence of the model for a finite volume interaction and small coupling constants in the full subcritical range $\beta^2 < 8\,\pi$ via a probabilistic method initiated by the Roman school of Gallavotti and co-authors. A more modern account is the martingale method of [?] which covers the full subcritical regime in the case $d = 1$, in a bounded domain but without restrictions on the coupling constant $\lambda \in \mathbb{R}^2$. A comprehensive review of the vast literature on the model can be found in the paper [?] where in the reader will also find a description of the correspondence with certain fermionic Euclidean models.

Due to the analytic treatability of the sine-Gordon interaction, there have been several accounts based on renormalisation group ideas and a direct analysis of the Polchinski flow equation. In this regard, we want to mention the analysis of Brydges and Kennedy [?], where they lay the foundations for this approach relying on a majorant method to establish convergence of the Mayer expansion up to $\beta^2 < \frac{2}{3}\,8\,\pi$. More recently, Bauerschmidt and Bodineau [?] showed convergence for the Mayer expansion up to $6\,\pi$ which allows them to establish a uniform log-Sobolev inequality for a lattice approximations of the model. In a related work, Bauerschmidt and Hofstetter [?] use the solution obtained from the Mayer expansion to construct a multi-scale coupling between the Gaussian free field and the sine-Gordon model and analyse the maximum of the sine-Gordon measure.

The sine-Gordon analysis has illustrated the power of the FBSDE approach in two dimensions. We now tackle the more challenging φ^4 theories, where ultraviolet divergences require a more sophisticated renormalisation procedure.

Part III

φ^4 theories in finite and infinite volume

Chapter 13
Gaussian Free Field

We have already encountered the Gaussian free field (GFF) in the Introduction. Here we develop its theory in more detail, starting with its one dimensional version, which is the Gaussian Markov process known as the Ornstein–Uhlenbeck process, i.e. the stationary solution to the SDE:

$$\mathrm{d}X_t = -X_t \mathrm{d}t + 2^{1/2}\mathrm{d}B_t, \qquad t \in \mathbb{R},$$

which can be expressed as the Wiener integral

$$X_t = 2^{1/2}\int_{-\infty}^{t} e^{-(t-s)}\mathrm{d}B_s, \qquad t \in \mathbb{R},$$

or as the Gaussian process with covariance

$$C(t-s) = \mathbb{E}[X_t X_s] = e^{-|t-s|}, \qquad t, s \in \mathbb{R}.$$

This process is Markov and its covariance has the Fourier transform given by

$$C(t) = \int_{\mathbb{R}} \frac{e^{i\omega t}}{1+\omega^2}\frac{\mathrm{d}\omega}{\pi}$$

as can be checked via an application of the residue formula. This means that the covariance C is a fundamental solution for the operator $(1-\partial_t^2)$.

By analogy, the multidimensional version $(\varphi(x))_{x\in\mathbb{R}^d}$ of this random field can be defined as the Gaussian field on $\mathbb{R}^d$ whose covariance $C(x)$ has the Fourier transform given by

$$C(x) = \int_{\mathbb{R}^d} \frac{e^{ik\cdot x}}{1+|k|^2}\frac{\mathrm{d}x}{\pi}$$

which gives a fundamental solution of the operator $(1-\Delta)$ where Δ is the Laplacian on $\mathbb{R}^d$. However, when $d \geqslant 2$ we have that $C(0) = +\infty$ and more generally that

$$C(x) \approx \begin{cases} -\log|x| & (d=2) \\ |x|^{2-d} & (d>2) \end{cases}, \qquad |x| \ll 1,$$

which means that the random field φ does not have continuous sample paths and it must be realised as a generalised function taking values a.s. in $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions.

The rigorous definition of GFF is therefore that of a Gaussian process indexed by $\mathcal{S}(\mathbb{R}^d)$ and such that

$$\mathbb{E}[\varphi(f)^2] = \langle f, (1-\Delta)^{-1} f\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^d)$$

where we denoted by $(1-\Delta)^{-1}$ the operator whose L^2-kernel is given by C. We denote by $\mu = \mu^{\mathrm{GFF}}$ the law of the GFF as a probability measure on $\mathcal{S}'(\mathbb{R}^d)$. Later on we will consider also the GFF on $\mathbb{T}^d = [-1/2, 1/2]^d$ (with periodic b.c.) which has a similar definition.

If we denote by C_Λ the covariance of the GFF on $\Lambda = \mathbb{T}^d$ or $\mathbb{R}^d$ then they are related by the method of images:

$$C_{\mathbb{T}^d}(x) = \sum_{k\in\mathbb{Z}^d} C_{\mathbb{R}^d}(x+k)$$

where the sum converges as $C_{\mathbb{R}^d}$ has exponential decay for large distances.

13.1 Besov spaces

Besov spaces provide a convenient setting to measure regularity of random fields. For simplicity we discuss the case of $\mathbb{R}^d$ and discuss later how to adapt it to the case of $\mathbb{T}^d$. The reader can find full details and proofs in [?, ?] and for weighted spaces in [?, ?].

Let $\langle\alpha\rangle := (1+|\alpha|^2)^{1/2}$. Let $\mathcal{S}(\mathbb{R}^d)$ the Schwarz space of test functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ the corresponding space of tempered distributions. For any $\alpha\in\mathbb{R}$ we define fractional derivatives $\langle \mathrm{D}\rangle^\alpha = (1+\mathrm{D}^2)^{\alpha/2}$ on $\mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \mathrm{D}\rangle^\alpha f = \mathcal{F}^{-1}(\xi\mapsto\langle\xi\rangle^{\alpha/2}(\mathcal{F}f)(\xi)), \qquad f\in\mathcal{S}'(\mathbb{R}^2)$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^d$. More generally if $h\colon\mathbb{R}^d\to\mathbb{R}$ is a nice radial function we let

$$h(\mathrm{D})f = \mathcal{F}^{-1}(\xi\mapsto h(\xi)(\mathcal{F}f)(\xi)), \qquad f\in\mathcal{S}'(\mathbb{R}^2).$$

To localise functions in frequency space, we need a partition of unity whose existence can be proved but we shall take this for granted here. Let χ,ϱ be smooth radial functions $\mathbb{R}^d\to\mathbb{R}$ such that

- $\mathrm{supp}\,\chi\subseteq B(0,R)$, $\mathrm{supp}\,\varrho\subseteq B(0,2R)\setminus B(0,R)$;
- $0\leqslant\chi,\varrho\leqslant 1$, $\chi(\xi)+\sum_{j\geqslant 0}\varrho(2^{-j}\xi)=1$ for any $\xi\in\mathbb{R}^d$;
- $\mathrm{supp}\,\varrho(2^{-j}\cdot)\cap\mathrm{supp}\,\varrho(2^{-i}\cdot)=\varnothing$ if $|i-j|>1$.

Introduce the notations $\varrho_{-1}=\chi$, $\varrho_j=\varrho(2^{-j}\cdot)$ for $j\geqslant 0$.

Given such a partition of unity, we define the associated ***Littlewood–Paley–projectors*** as

$$\Delta_j = \varrho_j(2^{-j}D), \qquad j\geqslant -1,$$

and write

$$K_j = \mathcal{F}^{-1}(\varrho_j(\mathrm{D})), \qquad j\geqslant -1,$$

for the resulting convolution kernels in position space.

For any distribution we can then write for any $u\in S'(\mathbb{R}^d)$,

$$u = \sum_{j\geqslant -1}\Delta_j u = \sum_{j\geqslant -1} K_j * u. \tag{13.1}$$

where convergence is in the sense of $S'(\mathbb{R}^d)$.

Note that $\Delta_j u$ is now an holomorphic function since its Fourier transform is supported on a ball (for $j=-1$) or on an annulus (for $j\geqslant 0$), indeed we have for any $u\in S'(\mathbb{R}^d)$

$$(\Delta_j u)(x) := \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\hat{u}(\cdot)).$$

From (13.1) we finally define the Besov spaces.

Definition 13.1. *Let $s\in\mathbb{R}, p,q\in[1,\infty]$. For a Schwarz distribution $f\in\mathscr{S}'(\Lambda)$ define the norm*

$$\|f\|_{B^s_{p,q}} = \|(2^{js}\|\Delta_j f\|_{L^p})_{j\geqslant -1}\|_{\ell^q}$$

where $\|\|_{L^p}$ denotes the normalised $L^p(\Lambda)$ norm. The space $B^s_{p,q}$ is the set of functions $f\in\mathscr{S}'(\Lambda)$ such that $\|f\|_{B^s_{p,q}}<\infty$ moreover $H^s=B^s_{2,2}$ are the usual Sobolev spaces, and we denote by $\mathscr{C}^s$ the closure of smooth functions in the $B^s_{\infty,\infty}$ norm.

Remark 13.2.

- The specific norm $\|\cdot\|_{B^s_{p.q}}$ depends on the partition of unity, but all such norms are equivalent and the space $B^s_{p,q}$ does not depend on the specific choice.
- The space $(B^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{B^s_{p.q}})$ is a Banach space.

Let us quickly note that the parameter $s \in \mathbb{R}$ indeed measures regularity in the usual sense.

Theorem 13.3. *For any $s, \alpha \in \mathbb{R}$ and $u \in S'(\mathbb{R}^d)$*

$$\|u\|_{B^s_{p,q}} \approx \|(1-\Delta)^{\alpha/2} u\|_{B^{s-\alpha}_{p,q}}.$$

Remark 13.4. There is a relation between Besov spaces and Sobolev and Holder spaces: in particular one can prove that,

$$B^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d), \qquad s \in \mathbb{R}_+ \setminus \mathbb{Z} \qquad B^\alpha_{\infty,\infty}(\mathbb{R}^d) = C^\alpha(\mathbb{R}^d), \qquad \alpha \in \mathbb{R} \setminus \mathbb{Z}.$$

Easy embeddings among Besov spaces are as follows:

- Since $\ell^{q_1}(\mathbb{Z}) \hookrightarrow \ell^{q_2}(\mathbb{Z})$ for $q_1 \leqslant q_2$, we directly see $B^s_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^d)$.
- Moreover, for $s_2 \leqslant s_1$, $2^{qs_2 j} \leqslant 2^{qs_1 j}$ so that $B^{s_1}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p,q}(\mathbb{R}^d)$.

More generally, functions with localised Fourier transform have equivalent Lebesgue norms (this is called Bernstein's theorem) and this generates embeddings similar to the well-known Sobolev embeddings:

Lemma 13.5. *Let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$. If $s_2 \leqslant s_1 - d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)$, then*

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d).$$

Besov spaces on $\mathbb{T}^d$ have similar definition using the LP decomposition on the torus.

All along these notes we need to consider Banach subspaces of $\mathcal{S}'(\mathbb{R}^d)$ associated with a positive spatial weight ρ.

Definition 13.6. *A polynomial weight ρ is a function $\rho : \mathbb{R}^d \to \mathbb{R}_+$ of the form $\rho(x) = c\langle \ell x \rangle^{-\sigma}$ for $\sigma, c, \ell \geqslant 0$. For a polynomial weight ρ let*

$$\|f\|_{L^p(\rho)} = \bigg(\int_{\mathbb{R}^d} |f(x)\rho(x)|^p \, \mathrm{d}x \bigg)^{1/p}$$

and by $L^p(\rho)$ the space of functions for which this norm is finite. For function defined on a torus in $\mathbb{R}^d$ we consider their periodic extensions on $\mathbb{R}^d$. Similarly we define the weighted Besov spaces $B^s_{p,q}(\rho)$ as the set of elements of $\mathscr{S}'(\mathbb{R}^d)$ for which the norm

$$\|f\|_{B^s_{p,q}(\rho)} = \|(2^{js}\|\Delta_j f\|_{L^p(\rho)})_{j \geqslant -1}\|_{\ell^q}$$

is finite and by $\mathscr{C}^s(\rho)$ those such that the norm

$$\|f\|_{\mathscr{C}^s(\rho)} = \|(2^{js}\|\rho \Delta_j f\|_{L^\infty})_{j \geqslant -1}\|_{\ell^\infty}$$

is finite.

We will fix a weight ρ such that for some sufficiently small $\varepsilon > 0$ we have

$$|\Delta \, \rho^{1/2}| + |\nabla \rho^{1/2}| \leqslant \varepsilon \, \rho^{1/2}, \qquad \rho \lesssim \langle \cdot \rangle^{-32}. \tag{13.2}$$

The number ε can depend on various parameters but not on the volume cutoffs we introduce to regularise the Euclidean QFTs.

We note that the weighted norm $B^s_{p,q}(\rho^\alpha)$ is equivalent to

$$\|f\|_{B^s_{p,q}(\rho^\alpha)} = \sum_{i\geqslant -1} 2^{isq}\|\Delta_i f\|_{L^p(\rho^\alpha)} = \sum_{i\geqslant -1} 2^{isq}\|\rho^\alpha \Delta_i f\|_{L^p(\mathbb{R}^d)} \approx = \sum_{i\geqslant -1} 2^{isq}\|\Delta_i(\rho^\alpha f)\|_{L^p(\mathbb{R}^d)},$$

that is, the weight ρ^α can be freely commuted with the LP projectors as far as Besov norms are concerned.

We collect here some basic estimates which will be useful below.

Proposition 13.7. *Let $\delta > 0$. We have for any $q_1, q_2 \in [1,\infty]$, $q_1 < q_2$*

$$\|f\|_{B^s_{p,q_2}} \leq \|f\|_{B^s_{p,q_1}} \leq \|f\|_{B^{s+\delta}_{p,\infty}}.$$

Furthermore, if we denote by $W^{s,p}$ the normalised fractional Sobolev spaces then for any $q \in [1,\infty]$

$$\|f\|_{B^s_{p,q}} \leq \|f\|_{W^{s+\delta,p}} \leq \|f\|_{B^{s+2\delta}_{p,\infty}}.$$

Proposition 13.8. *For any $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$, any $p, q \in [1,\infty]$ the Besov space $B^{s_1}_{p,q}$ is compactly embedded into $B^{s_2}_{p,q}$.*

Definition 13.9. *A smooth function η is said to be an S^m multiplier if for every multi-index α there exists a constant C_α such that*

$$\left|\frac{\partial^\alpha}{\partial \xi^\alpha}\eta(\xi)\right| \lesssim_\alpha (1+|\xi|)^{m-|\alpha|}, \qquad \xi \in \mathbb{R}^d. \tag{13.3}$$

We say that a family η_t is a uniformly S^m multiplier if (13.3) is satisfied for every t with C_α independent of t.

Proposition 13.10. *Let η be an S^m multiplier, $s \in \mathbb{R}$, $p, q \in [1,\infty]$, and $f \in B^s_{p,q}$, then*

$$\|\eta(\mathrm{D})f\|_{B^{s-m}_{p,q}} \lesssim \|f\|_{B^s_{p,q}}.$$

Furthermore the constant depends only on s, p, q, d and the constants C_α in eq. (13.3).

For a proof see [?] Lemma 2.78.

Proposition 13.11. *Let θ p, p_1, p_2 and s, s_1, s_2 be such that $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $s = \theta s_1 + (1-\theta)s_2$ and assume that $f \in W^{s_1,p_1} \cap W^{s_2,p_2}$. Then*

$$\|f\|_{W^{s,p}} \leqslant \|f\|^\theta_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}.$$

For a proof see [?].

13.2 Regularity of the GFF

The basic argument to obtain a Besov regularity for the GFF is to compute the following average:

$$\mathbb{E}\big[\|\varphi\|^{2p}_{B^\alpha_{2p,2p}(\mathbb{T}^d)}\big].$$

We have

$$\mathbb{E}\big[\|\varphi\|^{2p}_{B^\alpha_{2p,2p}}\big] = \mathbb{E}\left[\sum_{j\geqslant -1} 2^{2p\alpha j}\|\Delta_j\varphi\|^{2p}_{L^{2p}}\right] = \sum_{j\geqslant -1} 2^{2p\alpha j}\mathbb{E}[\|\Delta_j\varphi\|^{2p}_{L^{2p}}].$$

Now we observe that, by Fubini theorem and **hypercontractivity** of Gaussian vectors (see e.g. Janson's book on Gaussian measures [?])

$$\mathbb{E}[\|\Delta_j\varphi\|^{2p}_{L^{2p}}] \lesssim (\mathbb{E}[\|\Delta_j\varphi\|^2_{L^2}])^p,$$

so we reduced ourselves to computing covariances. Now we observe that, by standard Fourier analysis

$$\mathbb{E}[\|\Delta_j\varphi\|_{L^2}^2] = \frac{1}{2\pi}\sum_{k,k'} \varrho_j(k)\,\varrho_j(k')\mathbb{E}[\hat{\varphi}(k)\overline{\hat{\varphi}(k')}] = \frac{1}{2\pi}\sum_{k} \frac{\varrho_j{}^2(k)}{1+|k|^2}$$

But now the sum is over 2^{jd} points thanks to the support of $\varphi_j(k)$ and in addition this forces $|k|$ to be of order 2^j, these two remarks give

$$\mathbb{E}[\|\Delta_j\varphi\|_{L^2}^2] \lesssim 2^{jd}2^{-2j} = 2^{j(d-2)}.$$

This means that

$$\mathbb{E}\big[\,\|\varphi\|_{B^\alpha_{2p,2p}}^{2p}\big] \lesssim \sum_{j\geqslant -1} 2^{2p\alpha j}\mathbb{E}[\|\Delta_j\varphi\|_{L^{2p}}^{2p}] \lesssim \sum_{j\geqslant -1} 2^{2p\alpha j}2^{jp(d-2)} = \sum_{j\geqslant -1} 2^{2pj(\alpha+d/2-1)}.$$

So we see that in order to have a convergent series we need

$$\alpha + d/2 - 1 < 0 \qquad \Rightarrow \qquad \alpha < \frac{2-d}{2}.$$

To conclude, we shall use Besov embedding recalling that if $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, $s_2 \leqslant s_1 - d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)$, then

$$B^{s_1}_{p_1,q_1}(\mathbb{T}^d) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{T}^d).$$

We are interested in the case

$$p_2 = q_2 = \infty, \qquad p_1 = q_1 = 2p, \qquad s_1 = \alpha$$

then we see that

$$B^\alpha_{2p,2p}(\mathbb{T}^d) \hookrightarrow B^s_{\infty,\infty}(\mathbb{T}^d) = C^{s_2}(\mathbb{T}^d),$$

for any $s \leqslant \alpha - \frac{d}{2p}$. But since p can be chosen to be arbitrarily large, we conclude that the above embedding is true for any

$$s < \alpha < -\frac{d-2}{2}.$$

Remark 13.12. We see that:

- for $d = 1$ the GFF can be realised as a continuous function of time taking values in a space of continuous function of space;
- for $d = 2$, the GFF can be realised as a continuous function of time taking values in a space of distribution which are not much singular, namely C^α for any $\alpha < 0$, so arbitrarily close to 0. From the viewpoint we discussed here, this is why the φ^4 model is less difficult in 2 dimension wrt to the higher dimensional case;
- for $d \geqslant 2$ we have a singular behaviour getting progressively worse with d.

13.3 Regularity of the Wick powers

Since the GFF is a random distribution, it is not easy to construct local non-linear functionals. The non-trivial fact is that they indeed exists and can be constructed by approximations.

Consider a mollifier operator R_N (see below for precise definitions), e.g. one can take

$$R_N = \Delta_{\leqslant N} = \sum_{i\leqslant N} \Delta_i$$

via partial sums of LP blocks. Then, if φ is a GFF,

$$\varphi_N := R_N\varphi$$

is a smooth Gaussian process with a translation invariant covariance given by $C_N = R_N(1-\Delta)^{-1}R_N^*$. A direct computation, e.g. in Fourier space, shows that

$$C_N(x) \propto \begin{cases} -\log(|x|+N^{-1}) & d=2 \\ (|x|+N^{-1})^{2-d} & d\geqslant 3 \end{cases}$$

Recall also that the **Hermite polynomials** $(H_n(x))_{n\geqslant 0}$ are the orthogonal polynomials of the standard Gaussian measure over $\mathbb{R}$ and that the first of them (in probabilist's normalisation) are:

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= x, \\ H_2(x) &= x^2-1, \\ H_3(x) &= x^3-3x, \\ H_4(x) &= x^4-6x^2+3, \\ H_5(x) &= x^5-10x^3+15x, \\ &\vdots \end{aligned}$$

generated by the series

$$\exp\left(tx-\frac{t^2}{2}\right) = \sum_{n\geqslant 0} t^n H_n(x).$$

We define now the (regularised) Wick powers of the GFF as

$$[\![(\varphi_N)^n(x)]\!] := C_N^{n/2}(0) H_n\Big(\varphi_N(x)/C_N^{1/2}(0)\Big).$$

For example:

$$[\![(\varphi_N)^2(x)]\!] = \varphi_N^2(x) - C_N(0)$$

$$[\![(\varphi_N)^3(x)]\!] = \varphi_N^3(x) - 3\varphi_N(x)C_N(0)$$

$$[\![(\varphi_N)^4(x)]\!] = \varphi_N^4(x) - 6\varphi_N^2(x)C_N(0) + 3C_N^2(0).$$

In particular the covariance of these local functions of φ_N is given by

$$\mathbb{E}\Big[[\![(\varphi_N)^n(x)]\!][\![(\varphi_N)^k(y)]\!]\Big] = \delta_{n,k}(n!)C_N^n(x-y)$$

and by hypercontractivity we also have, for any nice test function $f:\mathbb{R}^d\to\mathbb{R}$

$$\begin{aligned} \mathbb{E}\left[\left|\int_{\mathbb{R}^d} [\![(\varphi_N)^n(x)]\!] f(x)\mathrm{d}x\right|^p\right] &\lesssim_{p,n} \left\{\mathbb{E}\left[\left|\int_{\mathbb{R}^d} [\![(\varphi_N)^n(x)]\!] f(x)\mathrm{d}x\right|^2\right]\right\}^{p/2} \\ &\lesssim \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} C_N(x-y)f(x)f(y)\mathrm{d}x\mathrm{d}y\right|^p. \end{aligned}$$

We will need to estimate the average size of the random fields $[\![\varphi_N^n]\!]$ uniformly in N. As with the GFF, we do not expect them to be bounded in some function space and therefore we will use negative regularity Besov–Hölder spaces to estimate them. First we note that by Besov embedding, for all $\alpha\in\mathbb{R}$ and $N\geqslant 1$,

$$\big\|[\![\varphi_N^n]\!]\big\|_{C^\alpha(\rho^\varepsilon)}^p \lesssim \big\|[\![\varphi_N^n]\!]\big\|_{B^{\alpha+d/N}_{p,p}(\rho^\varepsilon)}^p = \sum_{i\geqslant -1} 2^{i\left(\alpha+\frac{d}{p}\right)p}\int \rho^{\varepsilon p}(x)|\Delta_i[\![\varphi_N^n]\!](x)|^p\mathrm{d}x$$

Taking expectations and using Fubini's theorem we have now

$$\mathbb{E}\big\|[\![\varphi_N^n]\!]\big\|_{C^\alpha(\rho^\varepsilon)}^p \lesssim \sum_{i\geqslant -1} 2^{i\left(\alpha+\frac{d}{p}\right)p}\int \rho^{\varepsilon p}(x)\mathbb{E}[|\Delta_i[\![\varphi_N^n]\!](x)|^p]\mathrm{d}x$$

Now Gaussian hypercontractivity applied to the Gaussian polynomial $\Delta_i[\![W_T^n]\!](x)$, gives

$$\mathbb{E}[|\Delta_i[\![\varphi_N^n]\!](x)|^p] \lesssim \Big\{\mathbb{E}[|\Delta_i[\![\varphi_N^n]\!](x)|^2]\Big\}^{p/2}$$

which reduces the computation to an L^2 bound. Now, when $d=2$,

$$\begin{aligned}\mathbb{E}[|\Delta_i[\![\varphi_N^n]\!](x)|^2] &= \iint K_i(x-y)K_i(x-y')\mathbb{E}[[\![\varphi_N^n]\!](y)[\![\varphi_N^n]\!](y')]\mathrm{d}y\mathrm{d}y' \\ &\propto \iint K_i(x-y)K_i(x-y')\big\{\mathbb{E}[\varphi_N(y)\varphi_N(y')]\big\}^n\mathrm{d}y\mathrm{d}y' \\ &\lesssim \iint K_i(x-y)K_i(x-y')\big|\log^n|y-y'|\big|\mathrm{d}y\mathrm{d}y' \\ &\lesssim \iint K_i(y)K_i(y')\big|\log^n|y-y'|\big|\mathrm{d}y\mathrm{d}y' \\ &\lesssim \Big[\iint (K_i(y)K_i(y'))^{p/(p-1)}\mathrm{d}y\mathrm{d}y'\Big]^{(q-1)/q} \lesssim 2^{4i/q}\end{aligned}$$

where we used that in two dimensions

$$\mathbb{E}[\varphi_N(y)\varphi_N(y')] \lesssim \big|\log[|y-y'|+N^{-1}]\big| \lesssim \big|\log[|y-y'|]\big|$$

and that this function is in L^q for the product measure $\mathrm{d}y\times\mathrm{d}y'$ for any $q\geqslant 1$.

Putting all together we arrive at

$$\mathbb{E}\big\|[\![\varphi_N^n]\!]\big\|^p_{C^\alpha(\rho^\varepsilon)} \lesssim \sum_{i\geqslant -1} 2^{i\left(\alpha+\frac{2}{p}+\frac{4}{q}\right)p}\int_{\mathbb{R}^2}\rho^{\varepsilon p}(x)\mathrm{d}x < +\infty$$

for any $\alpha<0$ as soon as p and q are large enough, since then we have also $\rho^{\varepsilon p}\in L^1(\mathbb{R}^2)$.

This sketches the proof that in $d=2$, for all $\kappa>0,\varepsilon>0$ we have, for any $N\geqslant 1$ and any $n\geqslant 1$,

$$\sup_N \mathbb{E}\big\|[\![\varphi_N^n]\!]\big\|^p_{C^{-\kappa}(\rho^\varepsilon)} < +\infty. \tag{13.4}$$

A variant of these computations shows that

$$\sup_{M\geqslant N}\mathbb{E}\big\|[\![\varphi_N^n]\!]-[\![\varphi_M^n]\!]\big\|^p_{C^{-\kappa-1}(\rho^\varepsilon)} \lesssim N^{-p},$$

that is, the family $([\![\varphi_N^n]\!])_N$ is Cauchy and converges in L^p to a (unique) limit denoted $[\![\varphi^n]\!]\in C^{-\kappa}(\rho^\varepsilon)$ and called the n-th Wick power of the GFF.

Similar computations in $d=3$ shows that for all $\kappa,\varepsilon>0$ and $p\geqslant 1$,

$$\sup_N \mathbb{E}\big\|[\![\varphi_N^2]\!]\big\|^p_{C^{-1-\kappa}(\rho^\varepsilon)} < +\infty,$$

and similarly can be used to establish the existence of the random field $[\![\varphi^2]\!]\in C^{-1-\kappa}$. However for any $\alpha>0,\varepsilon>0$ we have for $n=3$:

$$\mathbb{E}\big\|[\![\varphi_N^3]\!]\big\|^p_{C^{-\alpha}(\rho^\varepsilon)} \approx (\log N)^{p/2}$$

which hints to the fact that in $d=3$ the family $([\![\varphi_N^3]\!])_N$ does not converge to a well-defined random distribution. A similar phenomenon affects all the higher Wick powers. In $d=4$ things get worser and not even $[\![\varphi^2]\!]$ exists.

13.4 Nelson's approach to φ_2^4

To illustrate the basic ideas of the construction of Euclidean QFTs let's us review the proof of Nelson that the φ_2^4 model exists in finite volume. Consider the functional

$$V_T(\varphi) = \int_\Lambda [\varphi^4(x) - r_T\varphi^2(x)]\mathrm{d}x$$

and let's try to take the limit $T \to \infty$ for the family of measures

$$\nu^T(\mathrm{d}\varphi) = \frac{\exp(-\lambda V_T(R_T\varphi))}{Z_T}\mu(\mathrm{d}\varphi),$$

for some $\lambda > 0$. We let

$$r_T = 3\mathbb{E}[(R_T\varphi(x))^2]$$

where we denote by $\mathbb{E}$ the expectation wrt the measure $\mu = \mu^{\mathrm{GFF}}$ with canonical process φ.

In this case, we denote

$$V_T(R_T\varphi) = \int_\Lambda [\![(R_T\varphi(x))^4]\!]\mathrm{d}x$$

since this corresponds to the 4-th Hermite polynomial of the random variable $R_T\varphi(x)$.

Recall that Hermite polynomials are defined to be the orthogonal polynomials of a Gaussian measure, see e.g. [?].

A basic result is to show that the density

$$\exp(-\lambda V_T(R_T\varphi))$$

belongs to $L^p(\mu^{\mathrm{GFF}})$ for all $p \geqslant 1$ uniformly in $T \to \infty$. The strategy of Nelson [?] is based on two observations:

1. The first is that the polynomial $\xi^4 - r_T\xi^2$ is bounded below by a quantity proportional to $-r_T^2$, which in our case diverges to $-\infty$ as $-\log^2 T$.
2. The second is hypercontractivity, which we take to mean the following inequality for polynomials P of order d of a Gaussian vector X:

$$\mathbb{E}[|P|^{2p}] \leqslant (2p-1)^{dp}\,\mathbb{E}[|P|^2]^p \tag{13.5}$$

   see e.g. [?] or the lecture notes of Hairer [?].

Take $K < T$ and write

$$V_T(R_T\varphi) = (V_T(R_T\varphi) - V_K(R_K\varphi)) + V_K(R_K\varphi)$$

with $V_K(R_K\varphi) \gtrsim -\log^2 K$. Therefore for all $l > 0$

$$V_T(R_T\varphi) < -l \Rightarrow$$

$$(V_T(R_T\varphi) - V_K(R_K\varphi)) < -l - V_K(R_K\varphi) < -l + C\log^2 K$$

and using Markov+hypercontractivity (13.5),

$$\mathbb{P}\Big(V_T(R_T\varphi) < -l\Big) \leqslant \mathbb{P}\Big(|(V_T(R_T\varphi) - V_K(R_K\varphi))| > l - C\log^2 K\Big)$$

$$\leqslant \frac{\mathbb{E}[|V_T(R_T\varphi) - V_K(R_K\varphi)|^p]}{(l - C\log^2 K)^p}$$

$$\lesssim \frac{\{\mathbb{E}[|V_T(R_T\varphi) - V_K(R_K\varphi)|^2]\}^{p/2}}{(l - C\log^2 K)^p} \lesssim p^p K^{-p/2}(l - C\log^2 K)^{-p}.$$

Take $K = e^{((l-1)/C)^{1/2}}$ so $l - C\log^2 K = 1$ and

$$\mathbb{P}\Big(V_T(R_T\varphi) < -l\Big) \lesssim p^p e^{-(p/2)((l-1)/C)^{1/2}} \lesssim \big(pe^{-cl^{1/2}}\big)^p,$$

and taking $p = e^{cl^{1/2}-1}$ we have

$$\Big(V_T(R_T\varphi) < -l\Big) \lesssim e^{-e^{cl^{1/2}-1}}$$

from which we conclude that

$$\sup_T \|\exp(-\lambda V_T(R_T\varphi))\|_{L^q(\mu)} < \infty$$

for all $q \geqslant 1$.

Chapter 14
The variational approach

We have seen in Part I a first derivation of the variational structure underlying stochastic quantisation via the Boué-Dupuis formula. In this chapter we develop the variational approach more systematically in the setting needed for the construction of Euclidean field theories.

Gibbs measures are solution to a variational problem: If μ is probability measure on a measure space $(\Omega, \mathcal{F})$ and V a bounded measurable function, bounded from below, the variational functional

$$F_V(\nu) := \int V \mathrm{d}\nu + H(\nu|\mu),$$

is defined for all probability measures ν on $(\Omega, \mathcal{F})$. Here H is the relative entropy of ν wrt. μ defined as

$$H(\nu|\mu) := \sup_f \left[\int f \mathrm{d}\nu - \log \int \exp(f) \mathrm{d}\mu \right],$$

where the supremum runs over bounded measurable functions $f: \Omega \to \mathbb{R}$. The relative entropy is finite if $\nu \ll \mu$ and in this case it holds

$$H(\nu|\mu) = \int \log\left[\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right] \mathrm{d}\nu.$$

From the definition one also deduce that H is non-negative and convex and that $H(\nu|\mu) = 0$ iff $\nu = \mu$. It is also continuous for the weak topology as the supremum in the definition can be taken over continuous functions (see e.g. [?]).

The functional F_V is convex and it admits a unique minimum ν^V given by

$$\mathrm{d}\nu^V = \frac{e^{-V}}{Z^V} \mathrm{d}\mu \tag{14.1}$$

and

$$F_V(\nu^V) = -\log Z^V = -\log \int e^{-V} \mathrm{d}\mu.$$

As a consequence Gibbs measure (14.1) can be characterised as the minimum of the functional F^V, i.e. as the result of an optimisation problem.

When the measure μ is a Wiener measure, the variational formula acquires a more specific form which is amenable to interesting stochastic analysis.

By a theorem of Föllmer if $(\Omega, \mathcal{F}, \mu)$ is a Brownian probability space with Wiener measure μ we have that every ν with finite entropy $H(\nu|\mu) < +\infty$ can be represented as a Girsanov transform, namely there exists a previsible function $u \in L^2(\Omega \times [0,T]; \mathbb{R}^n)$ such that

$$H(\nu|\mu) = \frac{1}{2} \mathbb{E} \int_0^\infty |u_s|^2 \mathrm{d}s < +\infty,$$

and $\nu = \mu^u$ with μ^u the Girsanov shift of μ in the direction u:

$$\frac{\mathrm{d}\mu^u}{\mathrm{d}\mu} = \exp\left(\int_0^\infty u_s \mathrm{d}B_s - \frac{1}{2} \int_0^\infty |u_s|^2 \mathrm{d}s \right),$$

where we denote B the canonical process on $(\Omega, \mathcal{F})$. By this result the variational problem takes the form

$$\inf_{u\in\mathbb{H}_a} \int_\Omega V \mathrm{d}\mu^u + \frac{1}{2}\mathbb{E}\int_0^\infty |u_s|^2 \mathrm{d}s. \tag{14.2}$$

Under μ^u the process $W^u := B - I(u)$ with $I(u)_t := \int_0^t u_s \mathrm{d}s$ is a Brownian motion and $\mathrm{Law}_{\mu^u}(W^u) = \mathrm{Law}_\mu(B)$.

A simila formula, but based on a fixed probability space and a fixed Brownian motion, was found by Boué–Dupuis [?]. We need the notion of *tame* functional:

Definition 14.1. *Let $(X, \mathcal{F}, \gamma)$ be a probability space. A measurable function $F: X \to [-\infty, \infty]$ is called tame with respect to γ if there exists $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and*

$$\mathbb{E}_\gamma[\exp(-qF)] + \mathbb{E}_\gamma[|F|^p] < \infty.$$

Tameness has been introduced by Üstunel [?] as a convenient condition for the validity of the Boué–Dupuis variational formula (**BD formula**) for Brownian functionals. It has been improved by Hariya and Watanabe [?] to allow for $p = q = 1$.

Theorem 14.2. *[Boué–Dupuis] Let V be a tame functional for μ, i.e. such that*

$$\int |V|^p + e^{-qV} \mathrm{d}\mu < +\infty,$$

for some $p, q \geqslant 1$ with $1/p + 1/q = 1$, then

$$-\log \int e^{-V} \mathrm{d}\mu = \inf_{u\in\mathbb{H}_a} \int \left[V(B + I(u)) + \frac{1}{2}\int_0^\infty \|u_s\|^2 \mathrm{d}s \right] \mathrm{d}\mu.$$

Proof. This is essentially Theorem 7 from Üstunel [?]. Note that the functional $V(B + I(u))$ makes sense for any $u \in \mathbb{H}_a$ since by Theorem 2 from Üstunel [?] we have that the push-forward measure $\nu = (B_\cdot + I(u)_\cdot)_\# \mu$ has finite relative entropy with respect to μ and therefore is absolutely continuous, i.e. $\nu \ll \mu$. □

Remark 14.3. Some observations on these variational formulas.

a) They originates directly from the variational formula for the free energy of a statistical mechanical systems: V playing the role of the internal energy and the quadratic term playing the role of the entropy.

b) The infimum might not be attained in Theorem ? (see e.g. Theorem 8 in [?]) while it is attained in (14.2).

c) The drift generated by absolutely continuous perturbations of the Wiener measure has been introduced and studied by Föllmer [?].

d) They are a non–Markovian and infinite dimensional extension of the well known stochastic control problem representation of the Hamilton–Jacobi–Bellman equation in finite dimensions [?].

e) The BD formula is easier to use than the formula (14.2) since the probability do not depend on the drift u. Going from one formulation to the other requires proving that certain SDEs with functional drift admits strong solutions and that one is able to approximate unbounded functionals V by bounded ones.

f) See Üstünel [?] and Lehec [?] for a streamlined proof of the BD formula and for applications to functional inequalities on Gaussian measures.

14.1 An example

As an example application of the BD formula, consider the problem of proving the exponential integrability of a functional $F(B)$ of a Brownian motion $(B_t)_{t\geqslant 0}$. We will assume that this functional allows the following bound

$$F(B+I(u))\leqslant H(B)+L\|u\|_{L^2(\mathbb{R})}, \tag{14.3}$$

for some integrable $H(B)$ and a constant $L>0$. Essentially, the functional should be of linear growth along the directions of the Cameron-Martin space of the Brownian motion. In this case by BD we have, for all $\alpha\geqslant 0$,

$$\log \mathbb{E}e^{\alpha F(B)}=\sup_{u\in\mathbb{H}_a}\mathbb{E}\bigg[\alpha F(B+I(u))-\frac{1}{2}\int_0^\infty \|u_s\|^2\mathrm{d}s\bigg]$$

where we denoted by $\mathbb{E}$ the expectation wrt. the Wiener measure μ. Using the bound (14.3) for F we note that, *path-wise* (i.e. for almost every $\omega\in\Omega$)

$$\begin{aligned}\alpha F(B+I(u))+\frac{1}{2}\int_0^\infty \|u_s\|^2\mathrm{d}s&\leqslant \alpha H(B)+\alpha L\|u\|_{L^2(\mathbb{R})}-\frac{1}{2}\|u\|^2_{L^2(\mathbb{R})}\\ &\leqslant \alpha H(B)+\sup_{\eta\geqslant 0}\bigg[\alpha L\eta-\frac{1}{2}\eta^2\bigg]\leqslant \alpha H(B)+\frac{L^2\alpha^2}{2}\end{aligned}$$

which gives a bound uniform in the control process u. As a consequence we readily derive the bound

$$\mathbb{E}e^{\alpha F(B)}\leqslant \exp\bigg(\alpha\mathbb{E}[H(B)]+\frac{L^2\alpha^2}{2}\bigg),\qquad \alpha>0,$$

which in particular implies that F has a Gaussian upper tail. What is remarkable is that the BD variational formula allows an analytic line of reasoning (find a bound the variational functional for all the controls) to derive a probabilistic consequence. Moreover the integrability question has been rephrased in a control of the Cameron–Martin shifts of the functional.

14.2 Euclidean fields as minimisers

Consider the Gibbs measure

$$\theta^V(\mathrm{d}\phi):=\frac{e^{-V(\phi)}\,\theta^0(\mathrm{d}\phi)}{\int e^{-V(\phi)}\,\theta^0(\mathrm{d}\phi)},\qquad \phi\in\mathcal{S}'(\mathbb{R}^2), \tag{14.4}$$

where θ^0 is the law of the Gaussian field ϕ (GFF) on $\mathbb{R}^2$ with covariance $(1-\Delta)^{-1}$ and where Δ denotes the Laplacian on $\mathbb{R}^2$. The function $V\colon\mathcal{S}'(\mathbb{R}^2)\to\mathbb{R}$ represent an interaction potential and for now we will only assume that it is measurable and *tame* with respect to θ^0.

To introduce a useful variational description of this class of measures we need to enlarge our probability space and realise the GFF θ^0 as the terminal value of a Brownian martingale $(W_t)_{t\in[0,1]}$ with values in $\mathcal{S}'(\mathbb{R}^2)$ and covariance given by

$$\mathbb{E}[W_t(x)W_s(y)]=(t\wedge s)(1-\Delta)^{-1}(x-y),\qquad x,y\in\mathbb{R}^2,\quad t,s\in[0,1].$$

We construct this Brownian martingale on the measure $\mathbb{P}$ given by the law of a cylindrical Brownian motion $(X_t)_{t\in[0,1]}$ on $L^2(\mathbb{R}^2)$. It can be realised on $C([0,1],C^{-1-\varepsilon}(\rho^\varepsilon))$ where $\varepsilon>0$ can be taken arbitrarily small. Let

$$W_t:=(1-\Delta)^{-1/2}\,X_t,\qquad t\in[0,1],$$

and note that indeed $\theta^0=\mathrm{Law}_{\mathbb{P}}(W_1)$. We will consider W as a random variable in $\mathfrak{S}_0:=C([0,1],C^{-\varepsilon}(\rho^\varepsilon))$.

The measure θ can be then interpreted as the law of W_1 under the Gibbs measure

$$\mathbb{Q}^V(\mathrm{d}W) = \frac{1}{Z} e^{-V(W_1)} \mathbb{P}(\mathrm{d}W).$$

It is well known that the measure $\mathbb{Q}^V$ is the minimiser of a Gibbs variational problem and that

$$-\log Z^V = -\log \mathbb{E}[e^{-V(W_1)}] = \inf_{\mathbb{R}} \big\{ \mathbb{E}_{\mathbb{R}}[V(W_1)] + H(\mathbb{R}|\mathbb{P}) \big\},$$

where $\mathbb{E}_{\mathbb{R}}$ denotes the expectation with respect to a probability measure $\mathbb{R}$ on the path space $C([0,1], \mathcal{S}'(\mathbb{R}^2))$ and $H(\mathbb{R}|\mathbb{P})$ the relative entropy of $\mathbb{R}$ with respect to $\mathbb{P}$.

Let $\mathfrak{H}(\rho^\alpha) = H^1_t H^1_x(\rho^\alpha)$, $\mathfrak{H} = \mathfrak{H}(\rho^0)$. Denote with $\mathfrak{H}^a(\rho^\alpha)$ the space of all processes $u : \mathfrak{S}_0 \times [0,1] \to \mathbb{R}$ progressively measurable with respect to the filtration generated by $(W_t)_{t\in[0,1]}$, starting from 0 and equipped with the norm of $L^2(\mathbb{P}, \mathfrak{H}(\rho^\alpha))$. Moreover define the bilinear form $\mathcal{E} : \mathfrak{H}^a \to L^0(\mathbb{P})$ giving the "energy" of the *drift* Z as

$$\mathcal{E}(Z) := \frac{1}{2} \int_0^1 \|(1-\Delta)^{1/2} \dot{Z}_s\|^2_{L^2} \mathrm{d}s = \frac{1}{2} \int_0^1 (\|\nabla \dot{Z}_s\|^2_{L^2} + \|\dot{Z}_s\|^2_{L^2}) \mathrm{d}s$$

where $\dot{Z}$ denotes the time-derivative of Z.

With these definitions, the Boué–Dupuis formula [?, ?] reads as follows.

Lemma 14.4. ***(BD formula)*** *For any tame functional $G : \mathfrak{S}_0 \to \mathbb{R}$ we have*

$$-\log \mathbb{E}[e^{-G(W_\cdot)}] = \inf_{Z\in\mathfrak{H}^a} \mathbb{E}[G(W_\cdot + Z_\cdot) + \mathcal{E}(Z)].$$

Proof. Theorem 14.2 specialised in our context gives

$$-\log \mathbb{E}[e^{-G(W_\cdot)}] = \inf_{u\in\mathbb{H}^a} \mathbb{E}\bigg[G(W_\cdot + Z_\cdot) + \frac{1}{2} \int_0^1 \|u_s\|^2_{L^2} \mathrm{d}s \bigg],$$

where $\mathbb{H}^a$ is the space of all progressively measurable processes starting from 0 equipped with the norm of $L^2(\mathbb{P}, L^2_t L^2_x)$ and

$$Z_t := (1-\Delta)^{-1/2} \int_0^t u_s \mathrm{d}s.$$

Observing that $u_t = (1-\Delta)^{1/2} \dot{Z}_t$ we obtain the claim. □

We call a random variable with values in $\mathfrak{H}^a$ a *control* (or *drift*) and we will understand that $W = W_1$ and $Z = Z_1$ whenever this does not lead to confusion.

To simplify certain technical considerations later is convenient to augment the process W with some additional informations. Consider a Polish space $\mathfrak{S}$ endowed with a continuous projection $\pi_W : \mathfrak{S} \to \mathfrak{S}_0$ and let $\mathbb{W} : \mathfrak{S}_0 \to \mathfrak{S}$ a measurable map such that $\pi_W \mathbb{W} = W$, i.e. $\mathfrak{S} \xrightarrow{W} \mathfrak{S}_0$ is a fibered topological space and $\mathbb{W}$ a measurable section. We will understand $\mathbb{W}$ as an ***enhanced GFF***. The kind of enhancements needed will be different in different models. We will always consider the space $\mathfrak{S}$ endowed with the measure given by the law of $\mathbb{W}$ under $\mathbb{P}$. A functional on $\mathfrak{S}$ is tame if it is tame for this measure.

For any measurable $f : \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$ bounded below and in $L^2(\theta^0)$ (not necessarily linear) define $\mathcal{W}^V(f)$ as

$$e^{-\mathcal{W}^V(f)} := \int e^{-f(\phi)} \theta^V(\mathrm{d}\phi).$$

As a direct consequence of Lemma 14.4, the functional $\mathcal{W}^V$ has the variational representation

$$\mathcal{W}^V(f) = \inf_{Z\in\mathfrak{H}^a} F^{f+V}(Z) - \inf_{Z\in\mathfrak{H}^a} F^V(Z).$$

for any f such that $f+V$ is tame and where for a tame functional G on $\mathfrak{S}_0$ we let

$$F^G(Z) := \mathbb{E}[G(W+Z)+\mathcal{E}(Z)], \qquad Z \in \mathfrak{H}^a.$$

In the formulation of Gibbs variational problem, the optimiser corresponds to the Gibbs measure. Correspondingly we would like to show that the optimiser in the BD formula, i.e. an optimal drift Z describes the law of the "interacting field" via the equality

$$\mathrm{Law}_{\mathbb{Q}^V}(W) = \mathrm{Law}_{\mathbb{P}}(W+Z) \tag{14.5}$$

valid for any minimiser Z of the problem in Lemma 14.4 and in particular

$$\theta^V = \mathrm{Law}_{\mathbb{P}}(W_1+Z_1).$$

The existence of such minimisers will be guaranteed, in the rigorous discussion below, by suitably relaxing the variational problem into a problem on the space of probability measures.

Before proceeding with the rigorous development, let us pause a moment to give an overview of what follows assuming that minimisers exists, then:

1. any minimiser Z satisfy the Euler–Lagrange equation (first variation)

$$\mathbb{E}\big[\nabla V(W_1+Z_1)\cdot K_1 + 2\mathcal{E}(Z,K)\big] = 0 \tag{14.6}$$

for all "test controls" K, where $\nabla V(W_1+Z_1)\cdot K_1$ denotes the derivative of $\varphi \mapsto V(\varphi)$ at W_1+Z_1 and in the direction of K_1. In the following we will refer to the equation (14.6) as a *weak forward–backwards stochastic (partial) differential equation* (wFBSDE). The reason for this name derives from observing that (14.6) can be formally rewritten as

$$\int_0^1 \int_{\mathbb{R}^2} \mathbb{E}\big[(\nabla V(W_1+Z_1) + (1-\Delta)\dot{Z}_s)\dot{K}_s\big]\mathrm{d}s = 0$$

and since this holds for any nice adapted control K it implies that

$$(1-\Delta)\dot{Z}_s = -\mathbb{E}\big[\nabla V(W_1+Z_1)\,|\,\mathcal{F}_s\big] \tag{14.7}$$

which is a kind of elliptic equation for the optimal control $(Z_s)_{s\in[0,1]}$. It can be cast into a system of a forward SDE for Z and a backward SDE which computes the conditional expectation. In these notes we will not exploit further this specific interpretation of (14.6) as a FBSDE so it is not necessary to make it more precise. Let us stress that our wFBSDE is a **stochastic quantisation** for Euclidean QFTs of a new kind.

2. additionally, we will be able to introduce a variational description of the measure $\mathbb{Q}^V$ with drift Z via a renormalised BD formula of the form

$$-\log \mathbb{E}[e^{-f(X_1)}] = \inf_K \mathbb{E}\big[f(X_1+K_1)+V(X_1+K_1)-V(X_1)+\mathcal{E}(Z+K)-\mathcal{E}(Z)\big] \tag{14.8}$$

with $X_s = W_s + Z_s$ being the "interacting field" with potential V and $f(\varphi)$ a test function which depends only locally on the field φ and where the infimum is taken among all drifts K which satisfy suitable decay at infinity. Given this decay one expects that the quantities $V(X_1+K_1)-V(X_1)$ and $\mathcal{E}(Z+K)-\mathcal{E}(Z)$ are finite even if each summand is infinite.

This reformulation will show its full power as soon as we try to describe Euclidean quantum fields in infinite volume. In that case the original variational problem loose its meaning as the interaction potential is almost surely infinite, due to an integration over the unbounded set $\mathbb{R}^2$.

On the other hand the first-order condition (14.6) remains meaningful also in this setting, as the interaction is local and can be used, very much like the weak formulation of an elliptic problem, to get estimates on the minimiser Z, e.g. by testing with $K=\rho Z$ where $\rho\colon \mathbb{R}^2 \to \mathbb{R}_+$ is some weight function decaying at spatial infinity. In particular we will obtain estimates of the form

$$\mathbb{E}\|\rho Z\|^2_{H^1_t H^1_x} < \infty,$$

uniformly in the volume. This estimate together with the identification (14.5) is at the core of the proof of tightness of the family of finite volume measures $(\mathbb{Q}^\Lambda)_\Lambda$ with $\Lambda \nearrow \mathbb{R}^2$. Any accumulation point $\mathbb{Q}$ represents an infinite volume Euclidean field, moreover the corresponding drift Z gives, via the formula (14.5) an explicit coupling of $\mathbb{Q}$ and the GFF given by $\mathbb{P}$.

The point of view which we advocate in these lectures is that the wFBSDE equation (14.6) provides a complete control over the drift Z describing the infinite volume Euclidean field, for which a naive variational description is meaningless. This will be exemplified by the analysis of the quartic interaction and of the exponential model in infinite volume, respectively in Section ? and Section ?.

Moreover the renormalised cost functional $\check{H}^V(\upsilon)$ can be made finite in the infinite volume by choosing localised perturbations K. This remark will be clearer by looking at how the determine the infinite volume limit of the exponential model in Section ?.

The rest of this section is dedicated to provide a rigorous proof of these claims.

The standard approach to guarantee the existence of minimisers is to relax our problem, i.e. minimise over the *law* of the pair (W, Z) with W fixed and Z varying over $\mathfrak{H}^a$. In order to ensure compactness we need to complete the space of trial laws for a suitable topology for which our target functional is coercive.

Let $\mathfrak{H}_w(\rho^\alpha)$ to be $\mathfrak{H}(\rho^\alpha)$ equipped with the weak topology and let $\hat{\mathfrak{H}} := \mathfrak{H}(\rho^{1/2})$, $\hat{\mathfrak{H}}_w := \mathfrak{H}_w(\rho^{1/2})$ and $\hat{\mathfrak{H}}^a := \mathfrak{H}^a(\rho^{1/2})$. With a slight abuse of notation we denote $(\mathbb{W}, Z)$ the canonical coordinates on $\mathfrak{S} \times \hat{\mathfrak{H}}_w$.

Definition 14.5. *Let*

$$\mathcal{X} := \Big\{ \mu = \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, Z) \in \mathcal{P}(\mathfrak{S} \times \hat{\mathfrak{H}}_w) : Z \in \hat{\mathfrak{H}}^a, \mathbb{E}_\mu[\|Z\|^2_{\hat{\mathfrak{H}}}] < \infty \Big\},$$

and

$$\bar{\mathcal{X}} := \Big\{ \mu \in \mathcal{P}(\mathfrak{S} \times \hat{\mathfrak{H}}_w) : \exists \mu_n \in \mathcal{X},\, \mu_n \to \mu \text{ weakly}, \sup_{n\in\mathbb{N}} \mathbb{E}_{\mu_n}[\|Z\|^2_{\hat{\mathfrak{H}}}] < \infty \Big\}.$$

We equip $\bar{\mathcal{X}}$ with the following notion of convergence: a sequence $(\mu_n)_n$ in $\bar{\mathcal{X}}$ converges to μ if

a) $\mu_n \to \mu$ *weakly on* $\mathfrak{S} \times \hat{\mathfrak{H}}_w$,

b) $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mu_n}[\|Z\|^2_{\hat{\mathfrak{H}}}] < \infty$.

Note that the probability space in the definition of $\mathcal{X}$ is fixed, so all elements in $\mathcal{X}$ are laws of random variables living on a fixed probability space. Note that all elements of $\mathcal{X}$ have first marginal given by $\mathrm{Law}(\mathbb{W})$*, and this also carries over to $\bar{\mathcal{X}}$ by weak convergence.*

We will need to realise couplings of two elements of $\bar{\mathcal{X}}$. These are constructed by the next lemma which needs the following preliminary definition of the space hosting the coupling.

We denote the canonical coordinates on the product space $\mathfrak{S} \times \hat{\mathfrak{H}}_w \times \hat{\mathfrak{H}}_w$ by $(\mathbb{W}, Z, K)$.

Definition 14.6. *For $\mu \in \bar{\mathcal{X}}$ let $\bar{\mathcal{Y}}(\mu) \subseteq \mathcal{P}(\mathfrak{S} \times \hat{\mathfrak{H}}_w \times \hat{\mathfrak{H}}_w)$ the set of all υ such that* $\mathrm{Law}_\upsilon(\mathbb{W}, Z) = \mu$ *and for which there exists sequences $(Z^n)_n, (K^n)_n \subseteq \hat{\mathfrak{H}}^a$ with*

$$\mathrm{Law}_{\mathbb{P}}(\mathbb{W}, Z^n, K^n) \to \upsilon \text{ and } \sup_{n\in\mathbb{N}} \mathbb{E}[\|Z^n\|^2_{\hat{\mathfrak{H}}} + \|K^n\|^2_{\hat{\mathfrak{H}}}] < \infty.$$

Lemma 14.7. *For any $\nu, \mu \in \bar{\mathcal{X}}$ there exists $\upsilon \in \bar{\mathcal{Y}}(\mu)$ such that*

$$\mathrm{Law}_\upsilon(\mathbb{W}, Z + K) = \mathrm{Law}_\nu(\mathbb{W}, Z).$$

Proof. Since μ, ν are in $\bar{\mathcal{X}}$ we can find sequences $Z^n, \bar{Z}^n \in \hat{\mathfrak{H}}^a$ such that $\mu_n := \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, Z^n) \to \mu$ and $\nu_n := \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, \bar{Z}^n) \to \nu$ weakly in $\hat{\mathfrak{H}}_w$ and furthermore $\sup_n \mathbb{E}[\|Z^n\|^2_{\hat{\mathfrak{H}}} + \|\bar{Z}^n\|^2_{\hat{\mathfrak{H}}}] < \infty$. Now define

$$K^n = \bar{Z}^n - Z^n,$$

and set

$$\upsilon^n = \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, Z^n, K^n).$$

We have that $\sup_n \mathbb{E}[\|K^n\|^2_{\hat{\mathfrak{H}}}] \leqslant \sup_n \mathbb{E}[\|Z^n\|^2_{\hat{\mathfrak{H}}} + \|\bar{Z}^n\|^2_{\hat{\mathfrak{H}}}] < \infty$ so υ^n is tight on $\mathfrak{S} \times \hat{\mathfrak{H}}_w \times \hat{\mathfrak{H}}_w$. Taking a convergent subsequence (not relabelled) we can assume that $\upsilon^n \to \upsilon$. We then have

$$\begin{aligned} \mathrm{Law}_{\upsilon}(\mathbb{W}, Z + K) &= \lim_{n\to\infty} \mathrm{Law}_{\upsilon^n}(\mathbb{W}, Z + K) \\ &= \lim_{n\to\infty} \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, Z^n + K^n) \\ &= \lim_{n\to\infty} \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, \bar{Z}^n) = \nu \end{aligned}$$

which proves the claim. □

On functionals on $\bar{\mathcal{X}}$ we can define first and second order derivatives as follows. Let $\mu \in \bar{\mathcal{X}}$ and let $K \colon \mathfrak{S} \times \hat{\mathfrak{H}}_w \to \hat{\mathfrak{H}}_w$ a measurable process adapted to the filtration generated by $(\mathbb{W}_t, Z_t)_{t\in[0,1]}$, continuous in the second component and such that $K \in L^2(\mu, \hat{\mathfrak{H}})$. We call such K an admissible control for μ. For all $\sigma \in \mathbb{R}$ let $\mu^{\sigma K}$ the law of $(\mathbb{W}, Z + \sigma K)$ under μ. We have $\mu^{\sigma K} \in \bar{\mathcal{X}}$. A functional $\check{F} \colon \bar{\mathcal{X}} \to \mathbb{R}$ is differentiable in $\bar{\mathcal{X}}$ in the direction of the admissible control K if the map $\sigma \mapsto \check{F}(\mu^{\sigma K})$ is differentiable in $\sigma = 0$ and in this case we define

$$\nabla \check{F}(\mu)(K) = \frac{\mathrm{d}}{\mathrm{d}\sigma}\bigg|_{\sigma=0} \check{F}(\mu^{\sigma K}).$$

Second order derivatives can be similarly defined by taking two admissible controls K, K' and letting

$$\nabla^2 \check{F}(\mu)(K, K') = \frac{\mathrm{d}}{\mathrm{d}\sigma'}\bigg|_{\sigma=0} \frac{\mathrm{d}}{\mathrm{d}\sigma}\bigg|_{\sigma=0} \check{F}(\mu^{\sigma K + \sigma' K'}),$$

whenever $(\sigma, \sigma') \mapsto \check{F}(\mu^{\sigma K + \sigma' K})$ is C^2 in $(\sigma, \sigma') = 0$.

We are ready to relax the variational problem and prove an alternative representation of $\mathcal{W}_\Lambda(f)$. For a tame $G \colon \mathfrak{S} \to \mathbb{R}$ define the functional $\check{F}^G \colon \bar{\mathcal{X}} \to \mathbb{R}$ by

$$\check{F}^G(\mu) := \mathbb{E}_\mu[G(W + Z) + \mathcal{E}(Z)], \qquad \mu \in \bar{\mathcal{X}}.$$

Definition 14.8. *We say that a functional $\check{F} \colon \bar{\mathcal{X}} \to \mathbb{R}$ is admissible if it is lower semicontinuous and such that*

$$\check{F}(\mu) \gtrsim 1 + \mathbb{E}_\mu[\mathcal{E}(Z)], \tag{14.9}$$

this ensures in particular that the functional $\check{F} \colon \bar{\mathcal{X}} \to \mathbb{R}$ is coercive, that is for all $L < \infty$ the level sets $\{\check{F} \leqslant L\}$ are compact in $\bar{\mathcal{X}}$.

Definition 14.9. *For a tame $G \colon \mathfrak{S} \to \mathbb{R}$ define the functional $\check{F}^G \colon \bar{\mathcal{X}} \to \mathbb{R}$ by*

$$\check{F}^G(\mu) := \mathbb{E}_\mu[G(W + Z) + \mathcal{E}(Z)], \qquad \mu \in \bar{\mathcal{X}}.$$

We say that G is admissible if G is tame, $\check{F}^G$ is admissible and $(W, Z) \to G(W + Z)$ is continuous on $\mathfrak{S} \times \hat{\mathfrak{H}}_w$.

The following lemma is an obvious consequence of this relaxed variational setting.

Lemma 14.10. *Assume that $G \colon \mathfrak{S} \to \mathbb{R}$ is admissible, then*

$$\inf_{Z\in\hat{\mathfrak{H}}^a} F^G(Z) = \inf_{Z\in\hat{\mathfrak{H}}^a} F^G(Z) = \inf_{\mu\in\mathcal{X}} \check{F}^G(\mu) = \min_{\mu\in\bar{\mathcal{X}}} \check{F}^G(\mu). \tag{14.10}$$

Let $\mu^G \in \bar{\mathcal{X}}$ be any minimiser of $\check{F}^G$. If $\check{F}^G$ is differentiable at μ^G it satisfies the equation $\nabla \check{F}^G(\mu^G)(K) = 0$ for all admissible controls K. Moreover if it is twice differentiable at μ^G we also have the positivity condition $\nabla^2 \check{F}^G(\mu^G)(K, K') \geqslant 0$ for all admissible controls K, K'.

Proof. The equality $\inf_{Z \in \mathfrak{H}^a} F^G(Z) = \inf_{Z \in \hat{\mathfrak{H}}^a} F^G(Z)$ follows from the property (14.9) of the admissible functional $\check{F}^G$ since $\mathfrak{H}^a$ can be viewed as a subspace of $\bar{\mathcal{X}}$ via the map $u \mapsto \mathrm{Law}(\mathbb{W}, u)$. The equality $\inf_{Z \in \hat{\mathfrak{H}}^a} F^G(Z) = \inf_{\mu \in \mathcal{X}} \check{F}^G(\mu)$ follows directly from the definition of $\mathcal{X}$. We have that $\inf_{\mu \in \bar{\mathcal{X}}} \check{F}^G(\mu) = \min_{\mu \in \bar{\mathcal{X}}} \check{F}^G(\mu)$ again from the admissibility of G which ensures that sequences $(\mu_n)_n \subseteq \mathcal{X}$ of approximate minimisers of $\check{F}^G$ converge in $\bar{\mathcal{X}}$ and that any limit μ^G is a minimiser. The first and second order conditions are then consequences of the differentiability of $\check{F}^G$. It remains to show

$$\inf_{\mu \in \mathcal{X}} \check{F}^G(\mu) = \inf_{\mu \in \bar{\mathcal{X}}} \check{F}^G(\mu).$$

We fist show this statement for bounded G. We want to show that for any μ in $\bar{\mathcal{X}}$ any $\delta > 0$ there exists $\mu_n \in \mathcal{X}$

$$\mathbb{E}_{\mu_n}[G(W + Z) + \mathcal{E}(Z)] \leqslant \mathbb{E}_{\mu}[G(W + Z) + \mathcal{E}(Z)] + \delta.$$

So given $\mu \in \bar{\mathcal{X}}$ we take a sequence $\mu_n \in \mathcal{X}$ such that $\mu_n \to \mu$. We define μ_n^N to be

$$\mathrm{Law}(\mathbb{W}, Z^N) \quad \text{where } Z_t^N = Z_{t \wedge \tau^N} \quad \text{where } \tau_N = \inf_{t \in [0,1]} \int_0^t \|\nabla \dot{Z}_s\|_{L^2}^2 \mathrm{d}s \geqslant N.$$

Then $\mu_n^N = \mu_n$ on the event that $\mathcal{E}(Z) \leqslant N$. Select a subsequence such that $\mu_n^N \to \upsilon^N$ in $\mathcal{X}$. Define $\theta_*^\delta \nu = \mathrm{Law}_\nu(W, \theta^\delta * Z)$ where θ^δ is a sequence of standard mollifiers. Now by weak convergence

$$\lim_{n \to \infty} \mathbb{E}_{\theta_*^\delta \mu_n^N}[G(W + Z) + \mathcal{E}(Z)] = \mathbb{E}_{\theta_*^\delta \upsilon^N}[G(W + Z) + \mathcal{E}(Z)],$$

since the functional inside the expectation is continuous and bounded on the joint support of the measures $\theta_*^\delta \mu_n^N$. We have to show that

$$\mathbb{E}_{\theta_*^\delta \upsilon^N}[G(W + Z) + \mathcal{E}(Z)] \leqslant \mathbb{E}_{\mu}[G(W + Z) + \mathcal{E}(Z)] + \bar{\delta}.$$

First we establish that

$$\lim_{\delta \to 0} \mathbb{E}_{\theta_*^\delta \upsilon^N}[G(W + Z) + \mathcal{E}(Z)] = \mathbb{E}_{\upsilon^N}[G(W + Z) + \mathcal{E}(Z)]$$

Since $\mathcal{E}(\theta^\delta * Z) \leqslant \mathcal{E}(Z)$ and G is continuous on $\mathfrak{S} \times \hat{\mathfrak{H}}_w$ we have

$$\lim_{\delta \to 0} \mathbb{E}_{\theta_*^\delta \upsilon^N}[G(W + Z)] = \mathbb{E}_{\upsilon^N}[G(W + Z)] \qquad \lim_{\delta \to 0} \mathbb{E}_{\upsilon^N}[\mathcal{E}(\theta^\delta * Z)] = \mathbb{E}[\mathcal{E}(Z)].$$

Now we have to show that

$$\mathbb{E}_{\upsilon^N}[G(W + Z) + \mathcal{E}(Z)] \leqslant \mathbb{E}_{\mu}[G(W + Z) + \mathcal{E}(Z)] + \delta.$$

Since υ^N coincides with μ on any open set contained in $\mathcal{E}(Z) \leqslant N$ we have

$$|\mathbb{E}_{\upsilon^N}[G(W + Z)] - \mathbb{E}_{\mu}[G(W + Z)]| \leqslant \|G\|_{L^\infty} \mathbb{E}_{\mu}[\mathbb{1}_{\mathcal{E}(Z) > N/2}] \lesssim N^{-1}.$$

For the energy

$$\begin{aligned} \mathbb{E}_{\upsilon^N}[\mathcal{E}(Z)] &\leqslant \liminf_{n \to \infty} \mathbb{E}_{\mu_n^N}[\mathcal{E}(Z)] \leqslant \liminf_{n \to \infty} \mathbb{E}_{\mu_n}\left[\int_0^{\tau_N} \|\nabla \dot{Z}_s^\delta\|_{L^2}^2 \mathrm{d}s\right] \\ &\leqslant \liminf_{n \to \infty} \mathbb{E}_{\mu_n}[\mathcal{E}(Z) \wedge N] \leqslant \mathbb{E}_{\mu}[\mathcal{E}(Z) \wedge N]. \end{aligned}$$

Now

$$\mathbb{E}_{\mu}[\mathcal{E}(Z) \wedge N] \leqslant \mathbb{E}_{\mu}[\mathcal{E}(Z)].$$

We now remove the restriction that G is bounded. To this end let $G^N = (G \wedge N) \vee (-N)$. Then we have established that already that

$$-\log \int \exp(-G^N) \mathrm{d}\theta^0 = \inf_{\mu \in \mathcal{X}} \check{F}^{G^N}(\mu) = \min_{\mu \in \bar{\mathcal{X}}} \check{F}^{G^N}(\mu).$$

Now let ν be a minimiser of $\check{F}^G$ in $\bar{\mathcal{X}}$ (existence of this was established above). Then

$$\begin{aligned}\inf_{\mu\in\bar{\mathcal{X}}}\mathbb{E}_\mu[G(W+Z)+\mathcal{E}(Z)] &= \mathbb{E}_\nu[G(W+Z)+\mathcal{E}(Z)]\\ &= \lim_{N\to\infty}\mathbb{E}_\nu[G^N(W+Z)+\mathcal{E}(Z)]\\ &\geqslant \lim_{N\to\infty}\inf_{\mu\in\bar{\mathcal{X}}}\mathbb{E}_\mu[G^N(W+Z)+\mathcal{E}(Z)]\\ &= \lim_{N\to\infty}-\log\int\exp(-G^N)\mathrm{d}\theta_0\\ &= -\log\int\exp(-G)\mathrm{d}\theta_0\\ &= \inf_{\mu\in\mathcal{X}}\mathbb{E}_\mu[G(W+Z)+\mathcal{E}(Z)]\end{aligned}$$

Where we have used dominated convergence. □

Remark 14.11. Note that

$$\begin{aligned}\check{F}^G(\mu^{\sigma K})-\check{F}^G(\mu)&=\mathbb{E}_\mu[G(W+Z+\sigma K)-G(W+Z)+\mathcal{E}(Z+\sigma K)-\mathcal{E}(Z)]\\ &=\mathbb{E}_\mu[G(W+Z+\sigma K)-G(W+Z)+2\sigma\mathcal{E}(Z,K)+\sigma^2\mathcal{E}(K)].\end{aligned}$$

If G is differentiable in the direction of H^1 we have

$$\nabla\check{F}^G(\mu)(K)=\mathbb{E}_\mu[\nabla G(W+Z)\cdot K+2\mathcal{E}(Z,K)].$$

If G is twice differentiable we have also

$$\nabla^2\check{F}^G(\mu)(K,K')=\mathbb{E}_\mu[\nabla^2 G(W+Z)\cdot(K,K')+2\mathcal{E}(K,K')].$$

The consequences of the existence of this minimiser are clarified by the following lemmas.

Lemma 14.12. *If $f+V$ is admissible and μ^{f+V} a minimiser of $\check{F}^{f+V}$ then we have*

$$\mathbb{E}_{\mu^{f+V}}[f(W+Z)]\leqslant\mathcal{W}^V(f)\leqslant\mathbb{E}_{\mu^V}[f(W+Z)]. \tag{14.11}$$

Proof. If $f+V$ is admissible we have proven that

$$\mathcal{W}^V(f)=\inf_{\mu\in\bar{\mathcal{X}}}\check{F}^{f+V}(\mu)-\inf_{\mu\in\bar{\mathcal{X}}}\check{F}^V(\mu)=\check{F}^{f+V}(\mu^{f+V})-\check{F}^V(\mu^V). \tag{14.12}$$

Eq. (14.12) implies in turn

$$\check{F}^{f+V}(\mu^{f+V})-\check{F}^V(\mu^{f+V})\leqslant\mathcal{W}^V(f)\leqslant\check{F}^{f+V}(\mu^V)-\check{F}^V(\mu^V)$$

and since

$$\begin{aligned}\check{F}^{f+V}(\mu^{g+V})-\check{F}^V(\mu^{g+V})&=\mathbb{E}_{\mu^{g+V}}[f(W+Z)+V(W+Z)+\mathcal{E}(Z)]\\ &-\mathbb{E}_{\mu^{g+V}}[V(W+Z)+\mathcal{E}(Z)]=\mathbb{E}_{\mu^{g+V}}[f(W+Z)]\end{aligned}$$

for any g (in particular admissible g), we obtain eq. (14.11). □

Lemma 14.13. *Let $V:\mathfrak{S}\to\mathbb{R}$ be admissible. Then for all $f:\mathfrak{S}\to\mathbb{R}$ continuous and bounded $f+V$ is admissible. Let μ^{f+V} be a minimiser for $\check{F}^{f+V}$, then for all $\alpha\in\mathbb{R}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^V(\alpha f)=\mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)]. \tag{14.13}$$

Proof. The admissibility of $f+V$ is easy to obtain given that f is bounded and continuous and V is admissible. It is clear that $\alpha\mapsto\mathcal{W}^V(\alpha f)$ is differentiable for bounded f, we need only to establish the representation of this derivative via the variational minimiser $\mu^{\alpha f}$. For any $\gamma\geqslant 0$ we have, by direct computation

$$\mathcal{W}^V((\alpha+\gamma)f)-\mathcal{W}^V(\alpha f)\leqslant\check{F}^{(\alpha+\gamma)f+V}(\mu^{\alpha f+V})-\check{F}^{\alpha f+V}(\mu^{\alpha f+V})=\gamma\mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)],$$

and

$$\mathcal{W}^V(\alpha f)-\mathcal{W}^V((\alpha-\gamma)f)\geqslant \check{F}^{\alpha f+V}(\mu^{\alpha f+V})-\check{F}^{(\alpha-\gamma)f+V}(\mu^{\alpha f+V})=\gamma\mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)].$$

By differentiability of $\mathcal{W}^V(\alpha f)$ we have

$$\mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)]\leqslant \lim_{\gamma\to 0+}\frac{\mathcal{W}^V((\alpha+\gamma)f)-\mathcal{W}^V(\alpha f)}{\gamma}=\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^V(\alpha f)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^V(\alpha f)=\lim_{\gamma\to 0+}\frac{\mathcal{W}^V(\alpha f)-\mathcal{W}^V((\alpha-\gamma)f)}{\gamma}\leqslant \mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)]$$

from which we can conclude. □

There are several interesting consequences of this formula. First of all, integrating (14.13) in α, we have

$$\mathcal{W}^V(f)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^V(\alpha f)\mathrm{d}\alpha=\int_0^1\mathbb{E}_{\mu^{\alpha f+V}}[f(W+Z)]\mathrm{d}\alpha, \tag{14.14}$$

which is a representation of $\mathcal{W}^V(f)$ alternative to the variational formula. On the other hand, taking $\alpha=0$ in (14.13) we obtain

$$\int f(\phi)\theta^V(\mathrm{d}\phi)=\frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0}\mathcal{W}^V(\alpha f)=\mathbb{E}_{\mu^V}[f(W_1+Z_1)].$$

Theorem 14.14. *The law of W_1+Z_1 under μ^V is indeed θ^V. In particular for all $f\colon\mathfrak{S}\to\mathbb{R}$ continuous and bounded*

$$\mathcal{W}^V(f)=-\log\int e^{-f(\phi)}\theta^V(\mathrm{d}\phi)=-\log\mathbb{E}_{\mu^V}[e^{-f(W_1+Z_1)}].$$

This last formula naturally calls for a variational description of the average $\mathbb{E}_{\mu^V}[e^{-f(W_1+Z_1)}]$ with respect to the base measure μ^V. This extension of the BD formula to the interacting field $W+Z$ is the subject of the next lemma.

Let $\mathcal{E}(Z,K)$ the symmetric bilinear form associated to $\mathcal{E}(Z)$, in such a way that

$$\mathcal{E}(Z+K)=\mathcal{E}(Z)+\mathcal{E}(K)+2\mathcal{E}(Z,K).$$

Lemma 14.15. *Let V be admissible and denote by $\mu^V\in\bar{\mathcal{X}}$ a minimiser of $\check{F}^V$. Then we have the following direct variational description of μ^V:*

$$\mathcal{W}^V(f)=-\log\mathbb{E}_{\mu^V}[e^{-f(W+Z)}]=\inf_{\upsilon\in\bar{\mathcal{Y}}(\mu^V)}\check{G}^{f,V}(\upsilon)$$

with

$$\check{G}^{f,V}(\upsilon):=\mathbb{E}_\upsilon[f(W+Z+K)]+\check{H}^V(\upsilon),$$

and

$$\check{H}^V(\upsilon):=\mathbb{E}_\upsilon[V(W+Z+K)-V(W+Z)+2\mathcal{E}(Z,K)+\mathcal{E}(K)].$$

Moreover ν is a minimiser of $\check{F}^V$ iff $\check{H}^V(\upsilon)\geqslant 0$ for all $\upsilon\in\bar{\mathcal{Y}}(\nu)$ and if V is differentiable in the directions of H^1 we also have the weak forward–backward SDE (wFBSDE):

$$\mathbb{E}_\upsilon[\nabla V(W+Z)\cdot K+2\mathcal{E}(Z,K)]=0,\qquad \upsilon\in\bar{\mathcal{Y}}(\nu), \tag{14.15}$$

and $\check{H}^V(\upsilon)$ takes the alternative form

$$\check{H}^V(\upsilon)=\mathbb{E}_\upsilon[V(W+Z+K)-V(W+Z)-\nabla V(W+Z)\cdot K+\mathcal{E}(K)].$$

Proof. Recall that

$$\begin{aligned}\mathcal{W}^V(f) &= \inf_{\mu\in\bar{\mathcal{X}}} \check{F}^{f+V} - \inf_{\mu\in\bar{\mathcal{X}}} \check{F}^V \\ &= \inf_{\mu\in\bar{\mathcal{X}}} \mathbb{E}_\mu[f(W+Z)+V(W+Z)+\mathcal{E}(Z)] - \mathbb{E}_{\mu^V}[V(W+Z)+\mathcal{E}(Z)] \\ &= \inf_{\upsilon\in\bar{\mathcal{Y}}(\mu^V)} \mathbb{E}_\upsilon[f(W+Z+K)+V(W+Z+K)+\mathcal{E}(Z+K)] \\ &\quad -\mathbb{E}_{\mu^V}[V(W+Z)+\mathcal{E}(Z)]\end{aligned}$$

where in the last line we used Lemma 14.7 to justify the change of variables in the minimisation problem. Note that for any $\upsilon\in\bar{\mathcal{Y}}(\mu^V)$ we have that

$$\mathbb{E}_{\mu^V}[V(W+Z)+\mathcal{E}(Z)] = \mathbb{E}_\upsilon[V(W+Z)+\mathcal{E}(Z)]$$

so

$$\mathcal{W}^V(f) = \inf_{\upsilon\in\mathcal{Y}(\mu^V)} \Big\{\mathbb{E}_\upsilon[f(W+Z+K)] + \check{H}^V(\upsilon)\Big\}.$$

The fact that $\check{H}^V(\upsilon)\geqslant 0$ iff $\mathrm{Law}_\upsilon(W,Z)=\nu$ is a minimiser of $\check{F}^V$ follows from its definition, indeed note that

$$\check{H}^V(\upsilon) = \check{F}^V(\mathrm{Law}_\upsilon(W,Z+K)) - \check{F}^V(\mathrm{Law}_\upsilon(W,Z)).$$

If V is differentiable in the direction of H^1 we have that for any minimiser ν of $\check{F}^V$ and any $\upsilon\in\bar{\mathcal{Y}}(\nu)$:

$$0 = \nabla\check{F}^V(\upsilon) = \mathbb{E}_\upsilon[\nabla V(W+Z)\cdot K + 2\mathcal{E}(Z,K)]$$

which allow also to simplify the expression of $\check{H}^V$. □

Chapter 15
The quartic interaction

In this section we will consider the general setting of Section 14.2 in the case where the interaction potential V is given by

$$V_\Lambda(\phi) := \lambda \int_\Lambda [\![\phi^4]\!],$$

where $\Lambda \subseteq \mathbb{R}^2$ is a measurable bounded set (of sufficient regularity, see below) and $[\![\phi^n]\!]$ denotes the n-th Wick power of the Gaussian free field ϕ [?] and $\lambda > 0$ an arbitrary fixed constant. Let $\mathcal{W}_\Lambda := \mathcal{W}^{V_\Lambda}$ and $\theta_\Lambda := \theta^{V_\Lambda}$.

The measure θ_Λ on $\mathcal{S}'(\mathbb{R}^2)$ is called the φ^4_2 Euclidean quantum field with a finite volume cutoff and we will discuss in detail the problem of the removal of the finite volume cutoff, i.e. the limit $\Lambda \nearrow \mathbb{R}^2$. In particular:

1. using the wFBSDE (14.6) we will derive a-priori bounds for minimisers strong enough to give tightness of the family $(\theta_\Lambda)_\Lambda$ as $\Lambda \nearrow \mathbb{R}^2$;
2. we will describe any accumulation point via a formula for its Laplace transform involving the wFBSDE obtained in the previous section.

Unfortunately, establishing rigorously the form (14.8) of the limiting variational problem requires certain locality properties of the solutions to the wFBSDE (14.6) in infinite volume. This is usually possible when V is convex. Ultraviolet renormalisation in φ^4_2 however spoils any "easy" convexity of the potential and it is not clear how to gain sufficient control of (14.6) in this case. Similar issues appear in other methods of stochastic quantisation as soon as the infinite volume limit is considered for $P(\varphi)_2$ Euclidean quantum fields [?, ?, ?]. Despite these difficulties, we show that the wFBSDE is still a useful tool which can be used effectively to prove the large deviations of the φ^4_2 measure in the semiclassical limits. Let us remark that Lacoin Rhodes and Vargas obtained large deviations results in the case of the Liouville model in [?, ?].

Remark 15.1. Let us note that all the discussion in this section can be extended to general $P(\varphi)_2$ models with a positive coefficient in the highest order polynomial. Moreover also $O(N)$ invariant vector models could be handled in similar ways. Convexity of the classical potential (i.e. before Wick ordering) is only needed in Section ? below to prove large deviations.

We will need the following enhancement $\mathbb{W}$ of the extended GFF W. Let

$$\mathbb{W}^2_t := [\![W^2_t]\!], \quad \text{and} \quad \mathbb{W}^3_t := [\![W^3_t]\!],$$

where here $[\![\cdot]\!]$ denotes the Wick powers with respect to the Gaussian field W. Then

$$\mathbb{W} := (W, \mathbb{W}^2, \mathbb{W}^3) \in L^p(\mathbb{P}, \mathfrak{S})$$

for all $p \geqslant 1$, where $\mathfrak{S} := C([0,1], (C^{-\delta}(\rho^{-1/4}))^3)$ for a fixed $\delta > 0$ which can be chosen as small as we like. By abuse of notation we will denote also $\mathbb{W} = (W, \mathbb{W}^2, \mathbb{W}^3)$ the canonical process on $\mathfrak{S}$.

In the following we will always consider a family of $(\Lambda)_{|\Lambda| \in \mathbb{R}}$ given by $\Lambda = |\Lambda|^{1/2} S$ for some fixed set S of volume 1 with C^1-boundary. This allow to prove that $\mathbb{1}_\Lambda$ is sufficiently regular to make sense of quantities like $\int_\Lambda [\![\phi^4]\!]$, $\int_\Lambda [\![W^4_1]\!]$ and in general of products with Wick monomials of the GFF.

Lemma 15.2. *For any $p \geqslant 1$, $s < \frac{1}{p}$ and $\iota > \frac{1}{16}$ we have that $\mathbb{1}_\Lambda \in B^s_{p,p}(\rho^\iota)$, furthermore*

$$\sup_{|\Lambda|} \|\mathbb{1}_\Lambda\|_{B^s_{p,p}(\rho^\iota)} < \infty.$$

Proof. We have that $\nabla \mathbb{1}_\Lambda = \nu \mathrm{d}\sigma$ where $\mathrm{d}\sigma$ is the surface measure on Λ and ν is the outer unit normal. It is known that $\|f\|_{B^s_{p,p}(\rho)} \lesssim \|f\|_{B^s_{p,p}(\rho)} + \|\nabla f\|_{B^{s-1}_{p,p}(\rho)}$ (see e.g. [?], Lemma A.5). This implies that for any $s < 1$, $\sup_{|\Lambda|} \|\mathbb{1}_\Lambda\|_{B^s_{1,1}(\rho^\iota)} < \infty$. Clearly $\sup_{|\Lambda|} \|\mathbb{1}_\Lambda\|_{L^\infty(\rho^\iota)} < \infty$ and by Besov space interpolation (see e.g. [?] Lemma A.3)

$$\|\mathbb{1}_\Lambda\|_{B^{\theta s}_{p,p}(\rho^\iota)} \lesssim \|\mathbb{1}_\Lambda\|^\theta_{B^s_{1,1}(\rho^\iota)} \|\mathbb{1}_\Lambda\|^{1-\theta}_{L^\infty(\rho^\iota)}$$

$\|\mathbb{1}_\Lambda\|_{L^\infty(\rho^\iota)}$where

$$\frac{1}{p} = \theta. \qquad \square$$

Lemma 15.3. *The functional $\check{F}^\Lambda := \check{F}^{V_\Lambda}$ is admissible and for all $\mu \in \bar{\mathcal{X}}$*

$$\check{F}^\Lambda(\mu) = \check{F}^{\mathcal{V}_\Lambda}(\mu) = \mathbb{E}_\mu\left[\mathcal{V}_\Lambda(\mathbb{W}, Z) + \mathcal{E}(Z)\right],$$

with

$$\mathcal{V}_\Lambda(\mathbb{W}, Z) := 4\lambda \int_\Lambda \mathbb{W}_1^3 Z_1 + 6\lambda \int_\Lambda \mathbb{W}_1^2 (Z_1)^2 + 4\lambda \int_\Lambda W_1 (Z_1)^3 + \lambda \int_\Lambda (Z_1)^4. \tag{15.1}$$

Proof. It is known that

$$\mathbb{E}\big[e^{-q\lambda \int_\Lambda [\![W^4]\!]}\big] < \infty,$$

for any $q > 0$ (see e.g [?] or [?]) and

$$\mathbb{E}[|V_\Lambda(W)|^p] \leqslant \mathbb{E}\big[\big|\|[\![W^4]\!]\|_{B^{-\delta}_{\infty,\infty}(\rho^\delta)}\big|^p\big] < \infty,$$

for some $\delta > 0$, which shows that F is a tame functional. Now it remains to prove the equality in (15.1). For this it is enough to show that for Z such that $\|Z\|_{H^1(\rho^{1/2})} < \infty$ μ-almost surely, we have

$$\int_\Lambda [\![(W_1 + Z_1)^4]\!] \;=\; \int_\Lambda [\![W_1^4]\!] + 4\int_\Lambda [\![W_1^3]\!] Z_1 + 6\int_\Lambda [\![W_1^2]\!] (Z_1)^2 + 4\int_\Lambda W_1 (Z_1)^3 + \int_\Lambda (Z_1)^4.$$

To see this, recall that if $C^\infty_c \ni \psi_\varepsilon \to \delta_0$, is a standard mollifier, $W^\varepsilon = \psi_\varepsilon * W$ and $a_\varepsilon = 6\mathbb{E}[(W_1^\varepsilon(0))^2]$, $b_\varepsilon = 3\mathbb{E}[(W_1^\varepsilon(0))^2]^2$

$$\begin{aligned}
\int_\Lambda [\![(W_1 + Z_1)^4]\!] \;&=\; \lim_{\varepsilon \to 0} \int_\Lambda (\psi_\varepsilon * (W_1 + Z_1))^4 - a_\varepsilon \int_\Lambda (\psi_\varepsilon * (W_1 + Z_1))^2 - b_\varepsilon \\
&=\; \lim_{\varepsilon \to 0} \int_\Lambda (W_1^\varepsilon)^4 - a_\varepsilon (W_1^\varepsilon)^2 - b_\varepsilon + 4\int_\Lambda \Big((W_1^\varepsilon)^3 - \frac{a_\varepsilon}{2} W_1^\varepsilon \Big) Z_1^\varepsilon \\
&\quad + 6\int_\Lambda \Big((W_1^\varepsilon)^2 - \frac{a_\varepsilon}{6} \Big) (Z_1^\varepsilon)^2 + 4\int_\Lambda W_1^\varepsilon (Z_1^\varepsilon)^3 + 4\int_\Lambda (Z_1^\varepsilon)^4
\end{aligned}$$

where we have denoted $Z_1^\varepsilon = \psi_\varepsilon * Z_1$. Now it is well known that almost surely in $C^{-\delta}(\rho^\delta)$ we have $W_1^\varepsilon \to W_1$ and

$$(W_1^\varepsilon)^4 - a_\varepsilon (W_1^\varepsilon)^2 - b_\varepsilon \to [\![W_1^4]\!], \quad \Big((W_1^\varepsilon)^3 - \frac{a_\varepsilon}{2} W_1^\varepsilon \Big) \to [\![W_1^3]\!], \quad \Big((W_1^\varepsilon)^2 - \frac{a_\varepsilon}{6} W_1^\varepsilon \Big) \to [\![W_1^2]\!].$$

Furthermore $Z^\varepsilon \to Z$ in H^1 which implies $(Z_1^\varepsilon)^2 \to Z_1^2, (Z_1^\varepsilon)^3 \to (Z_1^\varepsilon)^3$ in $W^{2\delta,1}$ for some $\delta > 0$, which gives the statement. From this calculation it also follows that $(W, Z) \to G(W + Z)$ is continuous on $\mathfrak{S} \times \mathfrak{H}_w$.

To check the lower semicontinuity of $\check{F}^{\mathcal{V}_\Lambda}$ recall from the Section 3 of [?] that

$$\mathcal{V}_\Lambda(\mathbb{W}, Z) + \mathcal{E}(Z) \geqslant -Q(\mathbb{W})$$

with $Q(\mathbb{W}) = C(\|\mathbb{W}\|^p_{\mathfrak{S}} + 1)$ for some $p \geqslant 1$. Furthermore observe that the function

$$(\mathbb{W}, Z) \mapsto \mathcal{V}_\Lambda(\mathbb{W}, Z) + \mathcal{E}(Z)$$

is lower semicontinuous on $\mathfrak{S}\times\mathfrak{H}_w$. Since the first marginal in $\bar{\mathcal{X}}$ is fixed, the Portmanteau theorem gives, for any $\mu_n\to\mu$ in $\bar{\mathcal{X}}$:

$$\begin{aligned}\liminf_{n\to\infty}\check{F}^{\mathcal{V}_\Lambda}(\mu_n) &= \liminf_{n\to\infty}\mathbb{E}_{\mu_n}[\mathcal{V}_\Lambda(\mathbb{W},Z)+\mathcal{E}(Z)+Q(\mathbb{W})]-\mathbb{E}_{\mu_n}[Q(\mathbb{W})]\\ &\geqslant \mathbb{E}_{\mu}[\mathcal{V}_\Lambda(\mathbb{W},Z)+\mathcal{E}(Z)+Q(\mathbb{W})]-\mathbb{E}_{\mu}[Q(\mathbb{W})]\\ &= \check{F}^{\mathcal{V}_\Lambda}(\mu).\end{aligned}$$

Finally, by a slight adaptation of the proof of Lemma ? below we see that there exists $\delta>0$ and $C_{\delta,|\Lambda|}<\infty$ such that

$$\check{F}^{\mathcal{V}_\Lambda}(\mu)\geqslant\delta\,\mathbb{E}_\mu[\mathcal{E}(Z)+\lambda\|\mathbb{1}_\Lambda Z_1\|_{L^4}^4]-C_{\delta,|\Lambda|},$$

for all $\mu\in\bar{\mathcal{X}}$, from which the strong coercivity condition (14.9) follows. □

The admissibility of $\check{F}^\Lambda$ implies the variational representation for $\mathcal{W}_\Lambda$ and the existence of minimisers $\mu^{f,\Lambda}$ of $\check{F}^{f,\Lambda}:=\check{F}^{f+\mathcal{V}_\Lambda}$ with the corresponding wFBSDE equation

Lemma 15.4. *For any bounded and continuous $f:\mathfrak{S}_0\to\mathbb{R}$ there exists a minimiser $\mu^{f,\Lambda}\in\bar{\mathcal{X}}$ of $\check{F}^{f,\Lambda}:=\check{F}^{f+\mathcal{V}_\Lambda}$. Moreover if f is differentiable in the directions of H^1 any minimiser μ satisfies the wFBSDE equation*

$$\begin{aligned}&\mathbb{E}_\mu\Big[4\,\lambda\int_\Lambda Z^3\,K+2\mathcal{E}(Z,K)\Big]\\ &\quad=-\mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot K+\lambda\int_\Lambda(4\,\mathbb{W}^3+12\,\mathbb{W}^2\,Z+12\,WZ^2)\,K\Big]\end{aligned}\tag{15.2}$$

for any admissible control K.

Proof. As we already remarked the existence of $\mu^{f,\Lambda}$ follows from the admissibility of $\check{F}^{f,\Lambda}$. The functional $Z\mapsto\mathcal{V}_\Lambda(\mathbb{W},Z)$ is differentiable in H^1 and

$$\nabla\mathcal{V}_\Lambda(\mathbb{W},Z)\cdot K=4\lambda\int_\Lambda\mathbb{W}_1^3K_1+12\lambda\int_\Lambda\mathbb{W}_1^2Z_1K_1+12\lambda\int_\Lambda W_1(Z_1)^2K_1+4\lambda\int_\Lambda(Z_1)^3K_1.$$

Provided $Z\mapsto f(W+Z)$ is also differentiable, we have

$$0=\nabla\check{F}^{f,\Lambda}(\mu)(K)=\mathbb{E}_\mu[\nabla f(W+Z)\cdot K+\nabla\mathcal{V}_\Lambda(\mathbb{W},Z)\cdot K+2\mathcal{E}(Z,K)]$$

which is (15.2). □

Remark 15.5. Note that $Z\mapsto\mathcal{V}_\Lambda(\mathbb{W},Z)$ is twice differentiable, with

$$\nabla^2\mathcal{V}_\Lambda(\mathbb{W},Z)(K,K')=12\,\lambda\int_\Lambda\mathbb{W}_1^2K_1'K_1+24\,\lambda\int_\Lambda W_1Z_1K_1'K_1+4\,\lambda\int_\Lambda(Z_1)^2K_1K_2.$$

Therefore if f is twice differentiable then we have also the second order condition:

$$0\leqslant\nabla^2\check{F}^{f,\Lambda}(\mu)(K,K')=\mathbb{E}_\mu[\nabla^2f(W+Z)(K,K')+\nabla^2\mathcal{V}_\Lambda(\mathbb{W},Z)(K,K')+2\mathcal{E}(K',K)]$$

for any minimiser μ and any pair K,K' of admissible controls.

We record also the normalised variational representation for $\mathcal{W}_\Lambda(f)$.

Lemma 15.6. *Denote by $\mu^\Lambda\in\bar{\mathcal{X}}$ a minimiser of $\check{F}^{0,\Lambda}$. Then we have*

$$\mathcal{W}_\Lambda(f)=\inf_{\upsilon\in\bar{\mathcal{Y}}(\mu^\Lambda)}\Big\{\mathbb{E}_\upsilon\Big[\int_\Lambda f(W+Z+K)\Big]+\check{H}^\Lambda(\rho)\Big\}$$

with

$$\check{H}^\Lambda(\upsilon):=\mathbb{E}_\upsilon\Big[\int_0^1\int_{\mathbb{R}^2}\dot{Z}_s(1-\Delta)\,\dot{K}_s\mathrm{d}s+\frac{1}{2}\int_0^1\int_{\mathbb{R}^2}\dot{K}_s(1-\Delta)\,\dot{K}_s\mathrm{d}s\Big]$$

$$+\mathbb{E}_\upsilon\Big[4\lambda\int_\Lambda[\![(W+Z)^3]\!]\,K+6\lambda\int_\Lambda[\![(W+Z)^2]\!]\,K^2+4\lambda\int_\Lambda[\![(W+Z)]\!]\,K^3+\lambda\int_\Lambda K^4\Big].$$

Proof. This follows from Lemma 14.15 by direct computation. □

Next we now show that the wFBSDE (15.2) gives us an a-priori bound uniform in $|\Lambda|$ on the minimiser in terms of weighted norms of $\mathbb{W}$. For this we first need some analytic bounds.

Lemma 15.7. *Assume that, for $\delta > 0$ small enough, either*

a) $\operatorname{supp}(\nabla f(W+Z)) \subset \Lambda$ *and*

$$\|\nabla f(W+Z)\|_{L^{4/3}(\rho^{3/4})} \leqslant \delta \|\mathbb{1}_\Lambda (W+Z)\|^3_{W^{-\varepsilon,4}(\rho^{1/4})} + C,$$

b)

$$\|\nabla f(W+Z)\|_{H^{-1}(\rho^{1/2})} \leqslant \delta \|W+Z\|_{H^{-\varepsilon}(\rho^{1/2})}.$$

Then there exists $p \in (1,\infty)$ such that

$$\begin{aligned} &\mathbb{E}\Big|\nabla f(W+Z)\cdot(Z\rho) + \lambda \int_\Lambda (\mathbb{W}^3 \rho Z + 12\mathbb{W}^2 \rho Z^2 + 12 W \rho Z^3)\Big| \\ \leqslant\ & \delta\, \mathbb{E}\Big(\lambda \int_\Lambda \rho Z^4 + \mathcal{E}(\rho^{1/2} Z)\Big) + C. \end{aligned}$$

Proof. First observe that we have that $\int_\Lambda \rho Z^4 = \|\mathbb{1}_\Lambda \rho^{1/4} Z^4\|^4_{L^4}$, and $\mathcal{E}(\rho^{1/2}Z) \geqslant C\|\rho^{1/2} Z\|^2_{H^1}$, for instance by fundamental theorem of calculus. By our assumptions on f we have, in the first case

$$\begin{aligned} |\nabla f(W+Z)\cdot(Z\rho)| &\leqslant \|\nabla f(W+Z)\|_{L^{4/3}(\rho^{3/4})} \|\mathbb{1}_\Lambda Z\|_{L^4(\rho^{1/4})} \\ &\leqslant \delta \|\mathbb{1}_\Lambda (W+Z)\|^3_{W^{-\varepsilon,4}(\rho^{1/4})} \|\mathbb{1}_\Lambda Z\|_{L^4(\rho^{1/4})} + C \\ &\leqslant C\delta\big(\|\mathbb{1}_\Lambda Z\|^4_{L^4(\rho^{1/4})} + \|\mathbb{1}_\Lambda W\|^3_{W^{-\varepsilon,4}(\rho^{1/4})} \|\mathbb{1}_\Lambda Z\|_{L^4(\rho^{1/4})}\big) + C \\ &\leqslant 2\delta \|\mathbb{1}_\Lambda Z\|^4_{L^4(\rho^{1/4})} + C\|\mathbb{1}_\Lambda W\|^4_{W^{-\varepsilon,4}(\rho^{1/4})} + C. \end{aligned}$$

While in the second case

$$\begin{aligned} |\nabla f(W+Z)\cdot(Z\rho)| &\leqslant \|\nabla f(W+Z)\|_{H^{-1}(\rho^{1/2})} \|Z\|_{H^1(\rho^{1/2})} \\ &\leqslant \delta \|W+Z\|_{H^{-\varepsilon}(\rho^{1/2})} \|Z\|_{H^1(\rho^{1/2})} \\ &\leqslant \delta \|W\|_{H^{-\varepsilon}(\rho^{1/2})} \|Z\|_{H^1(\rho^{1/2})} + \delta \|Z\|^2_{H^1(\rho^{1/2})} \\ &\leqslant \|W\|^2_{H^{-\varepsilon}(\rho^{1/2})} + 2\delta \|Z\|^2_{H^1(\rho^{1/2})}. \end{aligned}$$

For the second term, by Cauchy–Schwarz we obtain

$$\Big|\int_\Lambda \rho\, \mathbb{W}^3 Z\Big| \leqslant C_\delta \|\mathbb{1}_\Lambda \mathbb{W}^3\|^2_{H^{-1}(\rho^{1/2})} + \delta \|Z\|^2_{H^1(\rho^{1/2})}.$$

For the third term, we use Lemma A.7 from [?] to get

$$\begin{aligned} \Big|\int_\Lambda \mathbb{W}^2 \rho Z^2\Big| &\leqslant \|\rho^{1/8} \mathbb{W}^2\|_{C^{-\varepsilon}} \|\rho^{7/8} \mathbb{1}_\Lambda Z^2\|_{B^\varepsilon_{1,1}} \\ &\leqslant \|\rho^{1/8} \mathbb{W}^2\|_{C^{-\varepsilon}} \|\rho^{3/8} \mathbb{1}_\Lambda Z\|_{L^2} \|\rho^{1/2} Z\|_{H^{2\varepsilon}} \\ &\leqslant C\|\rho^{1/8} \mathbb{W}^2\|_{C^{-\varepsilon}} \|\rho^{1/4} \mathbb{1}_\Lambda Z\|_{L^4} \|\rho^{1/2} Z\|_{H^{2\varepsilon}} \\ &\leqslant C_\delta \|\rho^{1/8} \mathbb{W}^2\|^4_{C^{-\varepsilon}} + \delta(\|\rho^{1/4} \mathbb{1}_\Lambda Z\|^4_{L^4} + \|\rho^{1/2} Z\|^2_{H^{2\varepsilon}}). \end{aligned}$$

For the final term we proceed similarly. By Lemma 15.2 we have $\sup_{|\Lambda|} \|\mathbb{1}_\Lambda\|_{B^s_{p,p}(\rho^\iota)} < \infty$ for any $s < 1/p$ and any $\iota > 1/16$. Then with $\varepsilon < 1/8$ and for any $\iota > 1/16$ we have by fractional Leibniz rule

$$\begin{aligned} \Big|\int_\Lambda \rho W Z^3\Big| &\leqslant \|\rho^{1/8-\iota} W\|_{\mathcal{C}^{-\varepsilon}} \|\rho^{3/8+\iota} \mathbb{1}_\Lambda Z\|_{H^\varepsilon} \|\rho^{1/4} \mathbb{1}_\Lambda Z\|^2_{L^4} \\ &\lesssim \|\rho^{1/8-\iota} W\|_{\mathcal{C}^{-\varepsilon}} \|\rho^{3/8} Z\|_{B^{1/2}_{8/3,8/3}} \|\rho^{1/4} \mathbb{1}_\Lambda Z\|^2_{L^4} \\ &\leqslant C\|\rho^{1/8-\iota} W\|_{\mathcal{C}^{-\varepsilon}} \|\rho^{1/2} Z\|^{1/2}_{H^1} \|\rho^{1/4} \mathbb{1}_\Lambda Z\|^{5/2}_{L^4} \\ &\leqslant C_\delta \|\rho^{1/8-\iota} W\|^8_{\mathcal{C}^{-\varepsilon}} + \delta(\|\rho^{1/2} Z\|^2_{H^1} + \|\rho^{1/4} \mathbb{1}_\Lambda Z\|^4_{L^4}). \end{aligned}$$

and we can choose $\iota = 1/8 - \varepsilon > 1/16$ to get the claim. □

Theorem 15.8. *Assume that f satisfies the conditions of Lemma 15.7. There exists a constant C independent of $|\Lambda|$ such that, for any minimiser $\mu^{f,\Lambda} \in \bar{\mathcal{X}}$ of $\check{F}^{f,\Lambda}(\mu)$ we have*

$$\mathbb{E}_{\mu^{f,\Lambda}}\Big[2\,\lambda \int_\Lambda \rho\, Z^4 + \mathcal{E}(\rho^{1/2} Z)\Big] \leqslant C. \tag{15.3}$$

Proof. Note that the map $Z \mapsto \rho\, Z$ is continuous from $\mathfrak{H}$ to $\mathfrak{H}$. Take $K = \rho\, Z$ in (15.2) to get that any minimiser $\mu^{f,\Lambda} \in \bar{\mathcal{X}}$ satisfies

$$\begin{aligned} & \mathbb{E}_{\mu^{f,\Lambda}}\Big[4\,\lambda \int_\Lambda \rho\, Z^4 + \int_0^1 \int_{\mathbb{R}^2} ((1-\Delta)^{1/2}\, \rho^{1/2}\, \dot{Z}_s)^2 \mathrm{d}s\Big] \\ = \; & \mathbb{E}_{\mu^{f,\Lambda}}\Big[\int_0^1 \int_{\mathbb{R}^2} \dot{Z}_s [(1-\Delta), \rho^{1/2}]\, \rho^{1/2}\, \dot{Z}_s \mathrm{d}s\Big] \\ & + \mathbb{E}_{\mu^{f,\Lambda}}\Big[\nabla f(W+Z)\,(Z\,\rho) + \lambda \int_\Lambda \rho(4\mathbb{W}^3\, Z + 12 \mathbb{W}^2\, Z^2 + 12\, W Z^3)\Big]. \end{aligned} \tag{15.4}$$

From Lemma ? and Lemma 15.7 below we deduce that the r.h.s. is controlled by a small fraction of the l.h.s. + a constant, from which we obtain the claim. □

Lemma 15.9.

$$\Big|\int_0^1 \int_{\mathbb{R}^2} \dot{Z}_s [(1-\Delta), \rho^{1/2}]\, \rho^{1/2}\, \dot{Z}_s \mathrm{d}s\Big| \leqslant \varepsilon \Big[\int_0^1 \|\dot{Z}_s\|^2_{H^1(\rho^{1/2})} \mathrm{d}s\Big]$$

Proof. We observe that 1 obviously commutes with $\rho^{1/2}$, and

$$[\Delta, \rho^{1/2}] = (\Delta\, \rho^{1/2}) + \nabla \rho^{1/2} \cdot \nabla.$$

Using the assumptions (13.2) on the weight ρ we deduce

$$\|[\Delta, \rho^{1/2}]\, f\|_{L^2} \leqslant \varepsilon(\|\rho^{1/2}\, f\|_{L^2} + \|\rho^{1/2} \nabla f\|_{L^2}).$$ □

Let us now show that these estimates give tightness of $(\theta_\Lambda)_\Lambda$ as $|\Lambda| \to \infty$ on a suitable weighted Sobolev space.

Lemma 15.10. *Consider $(\theta_\Lambda)_\Lambda$ as a family of measures on $H^{-\varepsilon}(\rho)$. We have for some $\delta > 0$,*

$$\sup_\Lambda \Big[\int \exp(\delta \|\mathbb{1}_\Lambda \phi\|^4_{W^{-\varepsilon,4}(\rho)}) \mathrm{d}\theta_\Lambda + \int \exp(\delta \|\phi\|^2_{H^{-\varepsilon}(\rho)}) \mathrm{d}\theta_\Lambda\Big] < \infty.$$

In particular $(\theta_\Lambda)_\Lambda$ is exponentially tight on $H^{-2\varepsilon}(\rho)$ and any accumulation point θ satisfies

$$\int \exp(\delta \|\phi\|^4_{W^{-\varepsilon,4}(\rho)}) \mathrm{d}\theta < \infty.$$

Proof. Let $g_N \in C^\infty(\mathbb{R}, R)$ be such that $g_N(x) = x$ for $|x| < N$ and g. Define $f^N(\phi) = g_N\big(\delta \|\phi\|^2_{H^\varepsilon(\rho^{1/2})}\big)$ we have $\nabla f^N(\phi) = (g_N)'\big(\|\phi\|^2_{H^\varepsilon(\rho^{1/2})}\big)\delta \langle D\rangle^{-2\varepsilon}\phi$ so

$$\|\nabla f^N(W+Z)\|_{H^{-\varepsilon}} \leqslant \delta \|W+Z\|_{H^{-\varepsilon}},$$

and this bound is uniform in N. This implies by Lemma 15.7 that

$$\sup_\Lambda \mathbb{E}_{\mu^{f^N,\Lambda}}[\|Z\|^2_{H^{-\varepsilon}}] < \infty,$$

and

$$\log \int \exp(f^N) \mathrm{d}\theta_\Lambda \leqslant \mathbb{E}_{\mu^{0,\Lambda}}[f^N(W+Z)] + \mathbb{E}_{\mu^{f^N,\Lambda}}[f^N(W+Z)] \leqslant C < +\infty,$$

And taking sup over N we get

$$\log \int \exp\big(\delta \|\phi\|^2_{H^\varepsilon(\rho^{1/2})}\big) \mathrm{d}\theta_\Lambda \leqslant C$$

a fact that implies exponential tightness of $(\theta_\Lambda)_\Lambda$. For the second statement we proceed similarly. Indeed if θ is an accumulation point of $(\theta_\Lambda)_\Lambda$ then

$$\begin{aligned}\int \exp(\delta\|\phi\|^4_{W^{-\varepsilon,4}(\rho)})\mathrm{d}\theta &\leqslant \sup_{\tilde\Lambda}\int \exp(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho)})\mathrm{d}\theta\\ &\leqslant \sup_{\tilde\Lambda}\limsup_{\Lambda\to\infty}\int \exp(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho)})\mathrm{d}\theta_\Lambda\\ &\leqslant \sup_{\tilde\Lambda}\sup_{\Lambda\supseteq\tilde\Lambda}\int \exp(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho)})\mathrm{d}\theta_\Lambda.\end{aligned}$$

This gives the claim provided that for $f^N(\phi)=g^N\big(\delta\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho^{1/4})}\big)$ we have

$$\sup_N \log\int \exp(f^N)\mathrm{d}\theta_\Lambda \leqslant C.$$

Indeed we compute $\nabla f^N(\phi)=(g^N)'\big(\|\mathbb{1}_{\tilde\Lambda}\phi\|^4_{W^{-\varepsilon,4}(\rho^{1/4})}\big)\delta\rho\mathbb{1}_\Lambda\langle\mathrm{D}\rangle^{-\varepsilon}(\langle\mathrm{D}\rangle^{-\varepsilon}\phi)^3$, from which we can check that

$$\|\nabla f^N(\phi)\|_{L^{4/3}(\rho^{3/4})}\leqslant C\delta\|\rho\mathbb{1}_\Lambda(\langle\mathrm{D}\rangle^{-\varepsilon}\phi)^3\|_{L^{4/3}}\leqslant C\delta\|\rho^{1/4}\langle\mathrm{D}\rangle^{-\varepsilon}\phi\|^3_{L^4}\leqslant C\delta\|\phi\|^3_{W^{-\varepsilon/2,4}},$$

and $\mathrm{supp}(\nabla f^N(\phi))\subseteq\Lambda$. Applying Lemma 15.7 we get that the minimiser $\mu^{f^N,\Lambda}$ satisfies

$$\sup_{\tilde\Lambda}\sup_{\Lambda\supseteq\tilde\Lambda}\mathbb{E}_{\mu^{f^N,\Lambda}}\big[\|\mathbb{1}_{\tilde\Lambda}Z\|^4_{L^4(\rho^{1/4})}\big]\leqslant\sup_\Lambda\mathbb{E}_{\mu^{f^N,\Lambda}}\big[\|\mathbb{1}_\Lambda Z\|^4_{L^4(\rho^{1/4})}\big]=:C<\infty.$$

As a consequence, eq. (14.11) gives

$$\log\int\exp(f^N)\mathrm{d}\theta_\Lambda\leqslant\mathbb{E}_{\mu^{0,\Lambda}}[f^N(W+Z)]+\mathbb{E}_{\mu^{f^N,\Lambda}}[f^N(W+Z)]\leqslant 2C<+\infty,$$

which implies the claim. □

We have actually a more refined result: let $\mu^{f,\Lambda}$ be a minimiser for $\check F^{f,\Lambda}$. By Theorem 15.8 the family $(\mu^{f,\Lambda})_\Lambda$ is tight in $\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w)$.

Lemma 15.11. *Denote by $\mu\in\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w)$ any accumulation point of $(\mu^{f,\Lambda})_\Lambda$ as $|\Lambda|\to\infty$. Then the following equation holds:*

$$\mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot K+4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,K+2\,\mathcal{E}(Z,K)\Big]=0. \tag{15.5}$$

for any $K\colon\mathfrak{S}\times\hat{\mathfrak{H}}_w\to\hat{\mathfrak{H}}_w$ continuous, adapted to the filtration generated by $(\mathbb{W}_t,Z_t)_{t\in[0,1]}$ and compactly supported.

Proof. Let the support of K be contained in $B_N(0)$ for some $N\geqslant 0$. Let $\chi\in C^\infty_c(\mathbb{R}^2)$ such that $\chi=1$ on $B_N(0)$ so that $K=\chi K$. Then Lemma 15.7 implies that

$$\mathbb{E}_{\mu^{f,\Lambda}}[\|\chi\,Z\|^2_{H^1}+\|\chi\,Z\|^4_{L^4}]\leqslant C,$$

where the constant depends on N but not on Λ and we have

$$\begin{aligned}&\mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot K+4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,K+2\,\mathcal{E}(Z,K)\Big]\\ =\;&\mathbb{E}_\mu\Big[\nabla f(W+Z)\cdot(\chi K)+4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,\chi K+2\,\mathcal{E}(Z,\chi K)\Big]\end{aligned}$$

Now observe that

$$4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\,\chi K=H(\mathbb{W},Z,K)+4\lambda\int Z^3\,\chi K$$

where

$$H(\mathbb{W},Z,K)=4\lambda\int_{\mathbb{R}^2}\mathbb{W}^3\,\chi K+12\lambda\int\mathbb{W}^2Z\,\chi K+12\lambda\int\mathbb{W}Z^2\,\chi K.$$

Let $\tilde\chi \in C_c^\infty(\mathbb{R}^2)$ and $\tilde\chi = 1$ on the support of χ. It is not hard to see from the definition that

$$|H(\mathbb{W}, Z, K)| \lesssim \|\tilde\chi \mathbb{W}\|_{\mathfrak{S}}\big(1 + \|\chi^{1/3} Z\|_{L^4}^{4/3}\|\chi^{1/3} Z\|_{H^1}^{2/3}\big)\|\chi^{1/3} K\|_{B^{1/3}_{3,3}}$$

Indeed,

$$\left|\int_{\mathbb{R}^2} \mathbb{W}^3 \chi K\right| = \left|\int_{\mathbb{R}^2} \tilde\chi \mathbb{W}^3 \chi K\right| \lesssim \|\tilde\chi \mathbb{W}\|_{\mathfrak{S}}\|\chi K\|_{B^{1/3}_{3,3}},$$

while, by using interpolation from the first to the second line, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \mathbb{W}^2 Z \chi K\right| &\lesssim \|\chi^{1/3}\mathbb{W}\|_{\mathfrak{S}}\|\chi^{1/3} Z\|_{B^{1/3}_{4/3,4/3}}\|\chi^{1/3} K\|_{B^{1/3}_{3,3}} \\ &\lesssim \|\chi^{1/3}\mathbb{W}\|_{\mathfrak{S}}\|\chi^{1/3} Z\|_{L^4}^{2/3}\|\chi^{1/3} Z\|_{H^1}^{1/3}\|\chi^{1/3} K\|_{B^{1/3}_{3,3}} \\ &\lesssim \|\tilde\chi \mathbb{W}\|_{\mathfrak{S}}\big(1 + \|\chi^{1/3} Z\|_{L^4}^{4/3}\|Z\|_{H^1}^{2/3}\big)\|\chi^{1/3} K\|_{B^{1/3}_{3,3}}\end{aligned}$$

and similarly

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} W Z^2 \chi K\right| \\ \lesssim\ & \|\tilde\chi \mathbb{W}\|_{\mathfrak{S}}\|\chi^{1/3} Z\|_{B^{1/3}_{4/3,4/3}}^2\|\chi^{1/3} K\|_{B^{1/3}_{3,3}} \\ \lesssim\ & \|\tilde\chi \mathbb{W}\|_{\mathfrak{S}}\|\chi^{1/3} Z\|_{L^4}^{4/3}\|\chi^{1/3} Z\|_{H^1}^{2/3}\|\chi^{1/3} K\|_{B^{1/3}_{3,3}}.\end{aligned}$$

From this we get

$$\begin{aligned}&\left|\nabla f(W+Z)\cdot(\chi K) + 4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\chi K + 2\mathcal{E}(Z, \chi K)\right| \\ \lesssim\ & \|\tilde\chi(W+Z)\|_{\mathcal{C}^{-\varepsilon}}\|\chi K\|_{H^1} + \left(\int_0^1 \|\chi^{1/2}\dot Z_t\|_{H^1}^2 \mathrm{d}t\right)^{1/2}\left(\int_0^1 \|\chi^{1/2}\dot K_t\|_{H^1}^2 \mathrm{d}t\right)^{1/2} \\ &+ \|\tilde\chi \mathbb{W}\|_{\mathfrak{S}}\big(1 + \|\chi^{1/3} Z\|_{L^4}^{4/3}\|\chi^{1/3} Z\|_{H^1}^{2/3}\big)\|\chi^{1/3} K\|_{B^{1/3}_{3,3}} \\ &+ \|\chi^{1/4} Z\|_{L^4}^3\|\chi^{1/4} K\|_{L^4}.\end{aligned}$$

The r.h.s. of this last equation is integrable by assumptions on K and bounds on Z. From Lemmas 12 and 14 of [?] we know that there exist a family $(K^M)_{M\geqslant 1}$ adapted to the filtration generated by $(W_t)_t$ with

$$\begin{aligned}\mathbb{E}_\mu[\|\chi^{1/4}K\|_{L^4}^4 - \|\chi^{1/4}K^M\|_{L^4}^4] &\to 0, \\ \mathbb{E}_\mu\left[\left(\int_0^1 \|\chi^{1/2}(\dot K_t)\|_{H^1}^2 \mathrm{d}t\right) - \int_0^1 \|\chi^{1/2}(\dot K_t^M)\|_{H^1}^2 \mathrm{d}t\right] &\to 0,\end{aligned} \tag{15.6}$$

as $M \to \infty$. After possibly passing to a subsequence, we can assume that $\chi^{1/4}K^M \to \chi^{1/4}K$ in $L^4_w(\mu, L^4_w(\mathbb{R}^2))$ where L^4_w denotes L^4 with the weak topology, and in $L^2_w(\mu, L^2(\mathbb{R}_+ \times \mathbb{R}^2))$. This together with the bound (15.6) implies that

$$\begin{aligned}&\lim_{M\to\infty} \mathbb{E}_\mu\left[\nabla f(W+Z)\cdot(\chi K^M) + 4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\chi K^M + 2\mathcal{E}(Z, \chi K^M)\right] \\ =\ & \mathbb{E}_\mu\left[\nabla f(W+Z)\cdot(\chi K) + 4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\chi K + 2\mathcal{E}(Z, \chi K)\right].\end{aligned}$$

At this point we have reduced the problem to showing that

$$\begin{aligned}&\lim_{\Lambda\to\infty} \mathbb{E}_{\mu^{f,\Lambda}}\left[\nabla f(W+Z)\cdot(\chi K^M) + 4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\chi K^M + 2\mathcal{E}(Z, \chi K^M)\right] \\ =\ & \mathbb{E}_\mu\left[\nabla f(W+Z)\cdot(\chi K^M) + 4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\chi K^M + 2\mathcal{E}(Z, \chi K^M)\right]\end{aligned}$$

since by assumption

$$\mathbb{E}_{\mu^{f,\Lambda}}\left[\nabla f(W+Z)\cdot(\chi K^M) + 4\lambda\int_{\mathbb{R}^2}[\![(W+Z)^3]\!]\chi K^M + 2\mathcal{E}(Z, \chi K^M)\right] = 0.$$

This follows from the weak convergence $\mu^{f,\Lambda} \to \mu$ and the bound

$$\mathbb{E}_{\mu^{f,\Lambda}}[\|\chi Z\|_{H^1}^2 + \|\chi Z\|_{L^4}^4] \leqslant C,$$

since we can approximate Z by bounded admissible controls $(Z^L)_L$ defined as $Z_t^L = Z_{t\wedge T_L}$ with the stopping time

$$T_L := 1 \wedge \inf\{t \in [0,1] \colon \|\chi Z_t\|_{H^1}^2 + \|\chi Z_t\|_{L^4}^4 \geqslant L\},$$

in such a way that

$$\sup_{t\in[0,1]} \|\chi Z_t^L\|_{H^1}^2 + \|\chi Z_t^L\|_{L^4}^4 \leqslant L,$$

almost surely. Note that $T_L \to 1$ as $L \to \infty$ and therefore indeed $Z_t^L \to Z_t$ as $L \to \infty$. □

To summarise the result of this section: we obtained tightness of the finite volume φ_2^4 measures and described *any* possible infinite volume limit via a novel kind of stochastic equation, the wFBSDE (15.5). Note that it cannot be interpreted anymore as the first order condition of a functional.

Unfortunately, due to the strongly non-convex character of the renormalised potential we have currently limited information on its solutions. One expects that in the "single phase region" [?] (e.g. where λ is small) there should be uniqueness, and alternatively that multiple solutions should appear in "two phase region". Despite these limitations eq. (15.5) is useful and in the next section we show that it allows to prove large deviations for φ_2^4 in the semiclassical limit.

15.1 Large deviations

In this section we want to investigate the large deviations of φ_2^4 in the infinite volume in the semiclassical limit $\hbar \to 0$ where $\hbar > 0$ is a parameter which is inserted in the model as follows. For any $f \colon \mathcal{S}'(\mathbb{R}^2) \to \mathbb{R}$, let $\mathcal{W}_\Lambda^\hbar(f)$ be defined by

$$e^{-\frac{1}{\hbar}\mathcal{W}_\Lambda^\hbar(f)} := \int e^{-\frac{1}{\hbar}f(\phi)}\, \theta_\Lambda^\hbar(\mathrm{d}\phi),$$

where

$$\theta_\Lambda^\hbar(\mathrm{d}\phi) := \frac{\exp\big(-\frac{1}{\hbar}V_\Lambda(\phi)\big)\theta^\hbar(\mathrm{d}\phi)}{\int \exp\big(-\frac{1}{\hbar}V_\Lambda(\phi)\big)\theta^\hbar(\mathrm{d}\phi)}.$$

Here $\theta^\hbar$ is the Gaussian measure on $\mathcal{S}'(\mathbb{R}^2)$ with covariance $\hbar(1-\Delta)^{-1}$ and the Wick ordering is now taken with respect to $\theta^\hbar$. With some minor modifications to the derivation in the previous section we have the identity

$$\mathcal{W}_\Lambda^\hbar(f) = \inf_{\mu\in\bar{\mathcal{X}}} \check{F}^{f,\Lambda,\hbar}(\mu) - \inf_{\mu\in\bar{\mathcal{X}}} \check{F}^{0,\Lambda,\hbar}(\mu),$$

where

$$\check{F}^{f,\Lambda,\hbar}(\mu) := \mathbb{E}_{\mu\in\bar{\mathcal{X}}}[f(\hbar^{1/2}W + Z) + V_\Lambda(\hbar^{1/2}W + Z) + \mathcal{E}(Z)], \qquad \mu \in \bar{\mathcal{X}}.$$

Let $\mu^{f,\Lambda,\hbar}$ a minimiser for this admissible $\check{F}^{f,\Lambda,\hbar}$. Reasoning as in Lemma 15.10 above we have that the family $(\mu^{f,\Lambda,\hbar})_{\Lambda,\hbar}$ is exponentially tight for any bounded and continuous f. For each $\hbar > 0$ we pick a weakly convergent subsequence of $(\mu^{f,\Lambda,\hbar})_\Lambda$ with limit $\mu^{f,\hbar}$. We have then also $\theta_\Lambda^\hbar \to \theta^\hbar = \mathrm{Law}_{\mu^{f,\hbar}}(W_1 + Z_1)$ along the same subsequence and clearly, for all bounded continuous f we have

$$\mathcal{W}_\Lambda^\hbar(f) \to \mathcal{W}^\hbar(f),$$

where $\mathcal{W}^\hbar(f)$ is now defined as

$$e^{-\frac{1}{\hbar}\mathcal{W}^\hbar(f)} := \int e^{-\frac{1}{\hbar}f(\phi)}\, \theta^\hbar(\mathrm{d}\phi).$$

In this setting where we do not have sufficient information to establish uniqueness of the limiting measures $(\mu^{f,\hbar})_\hbar$, nonetheless we are able to obtain a Laplace principle as $\hbar \to 0$ for the corresponding family $(\theta^\hbar)_\hbar$. Note that the family $(\theta^\hbar)_\hbar$ is exponentially tight by Lemma 15.10.

Theorem 15.12. *Any family of accumulation points $(\theta^\hbar)_\hbar$ of $(\theta^\hbar_\Lambda)_{\hbar,\Lambda}$ satisfies a Laplace principle on $\mathcal{S}'(\mathbb{R}^2)$ with rate function*

$$J(\phi)=\begin{cases}\lambda\int_{\mathbb{R}^2}\phi^4+\frac{1}{2}\int_{\mathbb{R}^2}\phi(1-\Delta)\phi, & \text{for } \phi\in H^1(\mathbb{R}^2);\\ +\infty, & \text{otherwise.}\end{cases}$$

That is for $f:\mathcal{S}'(\mathbb{R}^2)\to\mathbb{R}$ continuous and bounded

$$\lim_{\hbar\to 0}\mathcal{W}^\hbar(f)=\inf_{\psi\in\mathcal{S}'(\mathbb{R}^2)}\{f(\psi)+J(\psi)\}.$$

Proof. Assume that f is of the form

$$f(\psi)=C\int\rho((1-\Delta)^{-1/2}(\psi-\varphi))^2 \tag{15.7}$$

for $C\geqslant 0$ and $\varphi\in C_c^\infty(\mathbb{R}^2)$.

It is enough to prove the theorem for this particular family of functions since we already know that $(\theta^\hbar)_\hbar$ are exponentially tight, and the family given by (15.7) is rate function determining in the sense of Definition 3.15 of [?], as it isolates points and is bounded below.

From (14.13) we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^\hbar_\Lambda(\alpha f)=\mathbb{E}_{\mu^{\alpha f,\Lambda,\hbar}}[f(\hbar^{1/2}W+Z)]=\frac{\int f(\phi)e^{-\frac{1}{\hbar}\alpha f(\phi)}\,\theta^\hbar_\Lambda(\mathrm{d}\phi)}{\int e^{-\frac{1}{\hbar}\alpha f(\phi)}\,\theta^\hbar_\Lambda(\mathrm{d}\phi)}.$$

By taking a further subsequence we can also have that $\mu^{\alpha f,\Lambda,\hbar}=\mu^{\alpha f,\Lambda,\hbar}\to\mu^{\alpha f,\hbar}$. We can then pass to the limit and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}^\hbar(\alpha f)=\mathbb{E}_{\mu^{\alpha f,\hbar}}[f(\hbar^{1/2}W+Z)],$$

for any f,α and arbitrarily chosen accumulation point $\mu^{\alpha f,\hbar}$. Note that we do not have to do a diagonal argument over the uncountably many choices of α, f: for any given α, f we need just to take a subsequence of the sequence chosen to have $(\theta^\hbar_\Lambda)_\Lambda$ converge in order to establish the equality.

Now recall that the minimiser $\mu^{f,\hbar}$ satisfies an infinite volume wFBSDE of the form

$$\begin{aligned}0 \;=\; & \mathbb{E}_{\mu^{f,\hbar}}\Big[\nabla f(\hbar W+Z)\cdot H+\lambda\hbar^{3/2}\int\mathbb{W}^3H+\lambda\hbar\int\mathbb{W}^2ZH+\,.\\ & +\lambda\hbar^{1/2}\int W\,(Z)^2H+\lambda\int Z^3H+2\,\mathcal{E}(Z,H)\Big]\end{aligned} \tag{15.8}$$

for any H with compact support. Furthermore we can carry over the proof of Lemma 15.7, and obtain that

$$\sup_{\hbar<1}\mathbb{E}_{\mu^{f,\hbar}}\big[\,\|Z\|^4_{L^4(\rho^{1/4})}+\|Z\|^2_{H^1(\rho^{1/2})}\big]<\infty.$$

This implies that the family $(\mu^{f,\hbar})_\hbar$ is also tight and therefore we can consider any accumulation point μ^f as $\hbar\to 0$ which will satisfy the same estimate. By passing to the limit in (15.8) via by now standard estimates similar to those in Lemma 15.11 we obtain

$$0 \;=\; \mathbb{E}_{\mu^f}\Big[\nabla f(Z)\cdot H+\lambda\int Z^3H+2\,\mathcal{E}(Z,H)\Big], \tag{15.9}$$

valid for any adapted and compactly supported H. Note that this equation does not contain anymore the GFF. This equation has a unique solution. Indeed if ν is another solution we can couple the two solutions in $\rho\in\bar{\mathcal{Y}}(\mu^f)$ such that $\nu=\mathrm{Law}_v(W,Z+K)$ and have

$$0=\mathbb{E}_v\Big[[\nabla f(Z+K)-\nabla f(Z)]\cdot H+\lambda\int[(Z+K)^3-Z^3]H+2\,\mathcal{E}(K,H)\Big].$$

Now taking $H = \chi K$ for $\chi \in C_c^\infty(\mathbb{R}^2)$ and $\chi \geqslant 0$, we obtain

$$0 = \mathbb{E}_v\Big[[\nabla f(Z+K) - \nabla f(Z)] \cdot (\chi K) + \lambda \int \chi[(Z+K)^3 - Z^3]K + 2\,\mathcal{E}(K, \chi K)\Big].$$

By convexity of f and $(\cdot)^4$ we have that

$$\mathbb{E}_v\Big[[\nabla f(Z+K) - \nabla f(Z)] \cdot (\chi K) + \lambda \int \chi[(Z+K)^3 - Z^3]K\Big] \geqslant 0.$$

Sending $\chi \to 1$ we obtain that

$$\mathbb{E}_v[\mathcal{E}(K,K)] \leqslant 0 \Rightarrow K = 0 \Rightarrow \nu = \mu^f.$$

Therefore μ^f is the unique minimiser of the admissible (convex) functional

$$\check{F}(\nu) := \mathbb{E}_\nu\Big[f(Z) + \lambda \int Z^4 + \mathcal{E}(Z,Z)\Big].$$

Let $\nu^\phi = \mathrm{Law}_{\mathbb{P}}(W, s \mapsto s\phi)$ for $\phi \in H^1$ then

$$\check{F}(\nu^\phi) = f(\phi) + \lambda \int \phi^4 + \frac{1}{2}\int \phi(1-\Delta)\phi = f(\phi) + J(\phi).$$

Using the fact that the variational problem in the time variable for the functional

$$\mathcal{E}(Z,Z) = \frac{1}{2}\int_0^1 \int \dot{Z}_s(1-\Delta)\dot{Z}_s \mathrm{d}s$$

is minimised for $\dot{Z}_s$ constant, we have for any ν

$$\check{F}(\nu) = \mathbb{E}_\nu\Big[f(Z) + \lambda \int Z^4 + \mathcal{E}(Z,Z)\Big]$$

$$\geqslant \mathbb{E}_\nu\Big[f(Z) + \lambda \int Z^4 + \frac{1}{2}\int Z_1(1-\Delta)Z_1\Big] = \mathbb{E}_\nu[f(Z) + J(Z)]$$

$$\geqslant \inf_{\phi \in H^1} [f(\phi) + J(\phi)].$$

As a consequence we obtain that

$$\inf_{\nu \in \bar{\mathcal{X}}} \check{F}(\nu) = \inf_{\phi \in H^1} [f(\phi) + J(\phi)],$$

and that $\mu^f = \nu^{\phi^f}$ for the unique ϕ^f which attain the convex deterministic minimisation problem on the r.h.s..

To sum up, we have established that

$$\lim_{\hbar \to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \mathcal{W}^\hbar(\alpha f) = \lim_{\hbar \to 0} \mathbb{E}_{\mu^{\alpha f, \hbar}}[f(\hbar^{1/2}W + Z)] = f(\phi^{\alpha f}). \tag{15.10}$$

Integrating eq. (15.10) and using dominated convergence

$$\lim_{\hbar \to 0} \mathcal{W}^\hbar(\alpha f) = \int_0^\alpha \lim_{\hbar \to 0} \frac{\mathrm{d}}{\mathrm{d}\beta} \mathcal{W}^\hbar(\beta f)\mathrm{d}\beta = \int_0^\alpha \lim_{\hbar \to 0} \mathbb{E}_{\mu^{\beta f, \hbar}}[f(\hbar^{1/2}W + Z)]\mathrm{d}\beta$$

$$= \int_0^\alpha f(\phi^{\beta f})\mathrm{d}\beta =: \mathcal{W}^0(\alpha f).$$

This also implies that the derivative in $\alpha \in [0,1]$ of $\alpha \mapsto \mathcal{W}^0(\alpha f)$ exists and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \mathcal{W}^0(\alpha f) = f(\phi^{\alpha f}).$$

To conclude it is enough to show that this coincides with

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \tilde{\mathcal{W}}^0(\alpha f),$$

where

$$\tilde{\mathcal{W}}^0(\alpha f) = \inf_{\psi \in H^1(\mathbb{R}^2)} \{\alpha f(\psi) + J(\psi)\}.$$

Recall that $\phi^{\alpha f}$ is the minimiser of $\phi \mapsto \alpha f(\phi) + J(\phi)$ so the Euler–Lagrange equation gives

$$\alpha \nabla f(\phi^{\alpha f}) \cdot h + \nabla J(\phi^{\alpha f}) \cdot h = 0.$$

At this point we note that

$$\begin{aligned} 0 = \alpha[\nabla f(\phi^{(\alpha+\gamma)f}) - \nabla f(\phi^{\alpha f})] \cdot (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) \\ + [\nabla J(\phi^{(\alpha+\gamma)f}) - \nabla J(\phi^{\alpha f})] \cdot (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) \\ + \gamma \nabla f(\phi^{(\alpha+\gamma)f}) \cdot (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}). \end{aligned}$$

By convexity of f and strong convexity of J we have

$$\begin{aligned} & (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f})^2 \\ \lesssim\ & \alpha[\nabla f(\phi^{(\alpha+\gamma)f}) - \nabla f(\phi^{\alpha f}) + \nabla J(\phi^{(\alpha+\gamma)f}) - \nabla J(\phi^{\alpha f})] \cdot (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}). \end{aligned}$$

Now

$$\gamma \int \nabla f(\phi^{(\alpha+\gamma)f}) \cdot (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) \mathrm{d}x \lesssim \gamma \|\nabla f(\phi^{(\alpha+\gamma)f})\|_{H^1} \|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1}$$

so

$$\|(\phi^{(\alpha+\gamma)f} - \phi^{\alpha f})\|_{H^1}^2 \lesssim \gamma \|\nabla f(\phi^{(\alpha+\gamma)f})\|_{H^1} \|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1}$$

so dividing by $\|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1}$ we have

$$\|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1} \leqslant \gamma \|\nabla f(\phi^{(\alpha+\gamma)f})\|_{H^1} \leqslant \gamma(\|\phi^{(\alpha+\gamma)f}\|_{H^1} + C) \lesssim \gamma.$$

We can then calculate

$$\begin{aligned} \tilde{\mathcal{W}}^0((\alpha+\gamma)f) - \tilde{\mathcal{W}}^0(\alpha f) &= \gamma f(\phi^{(\alpha+\gamma)f}) + \alpha(f(\phi^{(\alpha+\gamma)f}) - f(\phi^{\alpha f})) \\ &\quad + J(\phi^{(\alpha+\gamma)f}) - J(\phi^{\alpha f}). \end{aligned}$$

Given that $\|\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}\|_{H^1} \lesssim \gamma$ we have, since by Sobolev embedding $\|\cdot\|_{L^4}^4$ is locally Lipschitz in the H^1 norm

$$\begin{aligned} & (\alpha(f(\phi^{(\alpha+\gamma)f}) - f(\phi^{\alpha f})) + J(\phi^{(\alpha+\gamma)f}) - J(\phi^{\alpha f})) \\ =\ & \nabla f(\phi^{\alpha f})(\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) + \int (1-\Delta)(\phi^{\alpha f}) (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) \\ & + \lambda \int (\phi^{\alpha f})^3 (\phi^{(\alpha+\gamma)f} - \phi^{\alpha f}) + O(\gamma^2) \\ =\ & O(\gamma^2), \end{aligned}$$

where in the last line we used the Euler–Lagrange equation for $\phi^{\alpha f}$. This implies in turn

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \tilde{\mathcal{W}}^0(\alpha f) = \lim_{\gamma \to 0} \frac{1}{\gamma} (\tilde{\mathcal{W}}^0((\alpha+\gamma)f) - \tilde{\mathcal{W}}^0(\alpha f)) = \lim_{\gamma \to 0} \frac{1}{\gamma} \gamma f(\phi^{(\alpha+\gamma)f}) = f(\phi^{\alpha f})$$

which concludes the proof. □

Chapter 16
Exponential interaction

In this section we will discuss a variational description of the exponential interaction, that is the measure

$$\nu^{\xi,\exp}(\mathrm{d}\phi) = \frac{e^{-V^\xi(\phi)}\,\theta^0(\mathrm{d}\phi)}{\int e^{-V^\xi(\phi)}\,\theta^0(\mathrm{d}\phi)},$$

where now V^ξ is given by

$$V^\xi(\phi) = \lambda \int_{\mathbb{R}^2} \xi [\![\exp(\beta\phi)]\!],$$

with some $\beta^2 \in (0, 8\pi)$ and where $M^\beta := [\![\exp(\beta\phi)]\!]$ is the Gaussian Multiplicative Chaos (GMC) and $\xi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}_+)$ is a (smooth and compactly supported) spatial cutoff. We start with some preliminaries about the rigorous construction of the GMC.

This model is also known as the $\exp(\beta\varphi)_2$ model (or Høegh–Krohn model), and has been introduced in [?]. See e.g. [?] for a discussion of its stochastic quantisation in the elliptic setting. For this model we repeat the construction of the variational description of the infinite volume measure. At variance with what happens in $P(\varphi)_2$ theories, renormalisation does not spoil the convexity properties of the potential and therefore the study of the related wFBSDE can go further and in particular we will be able to prove the necessary locality of the perturbed equation. This in turn will allow us to prove the Γ-convergence of the variational problem along with the infinite volume limit and establish eq. (14.8) in its rigorous form.

As a result of this analysis, we are able to obtain a concise and well-behaved description of an infinite volume Gibbs measure related to an Euclidean quantum field.

We start with a definition and some properties of the GMC, i.e. the renormalised exponential of the GFF.

Proposition 16.1. *Let W be a GFF and $W_\varepsilon = \psi_\varepsilon * W$ where $\psi_\varepsilon \to \delta_0$ and $\psi_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$ and let $\beta^2 \in (0, 8\pi)$. The Borel measures $(M^{\beta,\varepsilon})_\varepsilon$ on $\mathbb{R}^2$ given by*

$$M^{\beta,\varepsilon}(A) = \int_A \exp(-\beta W_\varepsilon(x) - \beta^2 \mathbb{E}[W_\varepsilon^2(x)])\mathrm{d}x, \qquad A \in \mathcal{B}(\mathbb{R}^2),$$

converges almost surely set-wise to a measure

$$M^\beta(A) = \lim_{\varepsilon\to 0} M^{\beta,\varepsilon}(A), \qquad A \in \mathcal{B}(\mathbb{R}^2). \tag{16.1}$$

We call M^β the Gaussian multiplicative chaos (GMC) with parameter β associated to the GFF W. It enjoys the following properties

a) For $1 < p < 8\pi/\beta^2$ and for any compact set $A \subseteq \mathbb{R}^2$

$$\sup_\varepsilon \mathbb{E}[|M^{\beta,\varepsilon}(A)|^p] < \infty.$$

Consequently

$$\mathbb{E}[|M^\beta(A)|^p] < \infty.$$

b) For $1 < p < 8\pi/\beta^2$

$$\mathbb{E}[|M^{\beta,\varepsilon}(B(0,\lambda))|^p]^{1/p} \leqslant C\,\lambda^{2-(p-1)\beta^2/4\pi}\,\mathbb{E}[|M^{\beta,\varepsilon}(B(0,1))|^p]^{1/p},$$

where $B(x,r)\subseteq\mathbb{R}^2$ denotes the open ball centred in x of radius r.

Proof. For a proof of (16.1) see [?]. The property a) is due to Kahane [?], while the estimate in b) is due to Hoshino, Kawabi and Kusuoka [?]. □

From these properties we can determine easily the Besov regularity of M^β following [?](Section 2).

Proposition 16.2. *Assume that $s>(\beta^2/(4\pi))(p-1)$, $1<p<8\pi/\beta^2$, then*

$$\mathbb{E}\Big[\|M^\beta\|^p_{B^{-s}_{p,p}(\rho^{1/2})}\Big]<\infty.$$

Proof. We have

$$\begin{aligned}\mathbb{E}\Big[\|M^{\beta,\varepsilon}\|^p_{B^{-s}_{p,p}(\rho^{1/2})}\Big] &= \sum_{i\geqslant 0}2^{-isp}\mathbb{E}\|\Delta_iM^{\beta,\varepsilon}\|^p_{L^p(\rho^{1/2})}\\ &\leqslant C\sum_{i\geqslant 0}2^{-isp}\mathbb{E}|\Delta_iM^{\beta,\varepsilon}(0)|^p.\end{aligned}$$

Recall that the Littlewood–Paley operators $(\Delta_i)_{i\geqslant -1}$ are defined by $\Delta_i=\psi(\mathrm{D}/2^i)$ for $i\geqslant 0$ and some Schwarz function $\psi\colon\mathbb{R}^2\to\mathbb{R}_{\geqslant 0}$. Now using translation invariance of the law of $M^{\beta,\varepsilon}$ and the fact that $\mathcal{F}^{-1}\psi$ is a Schwarz function we estimate

$$\begin{aligned}\mathbb{E}|\Delta_iM^{\beta,\varepsilon}(0)|^p &= 2^{2pi}\mathbb{E}\left|\int_{\mathbb{R}^2}(\mathcal{F}^{-1}\psi)(2^ix)M^{\beta,\varepsilon}(x)\mathrm{d}x\right|^p\\ &\leqslant \sum_{n\in\mathbb{Z}^2}(1+|n|^2)^{-2}\mathbb{E}\left|\int_{B(n,1)}M^{\beta,\varepsilon}(2^{-i}x)\mathrm{d}x\right|^p\\ &= \sum_{n\in\mathbb{Z}^2}(1+|n|^2)^{-2}2^{2pi}\mathbb{E}\left|\int_{B(n,2^{-i})}M^{\beta,\varepsilon}(x)\mathrm{d}x\right|^p\\ &\lesssim 2^{p(p-1)\beta^2i/(4\pi)}.\end{aligned}$$

This gives the statement, since by weak convergence of $M^{\beta,\varepsilon}\to M^\beta$ we have

$$\mathbb{E}\|\Delta_iM^\beta\|^p_{L^p(\rho^{1/2})}\leqslant\limsup_{\varepsilon\to 0}\mathbb{E}\|\Delta_iM^{\beta,\varepsilon}\|^p_{L^p(\rho^{1/2})}.$$

□

In this model we let $\mathfrak{S}:=\mathfrak{S}_0\times\mathcal{P}(\mathbb{R}^2)$ and $\mathbb{W}=(W,M^\beta)\colon\mathfrak{S}_0\to\mathfrak{S}$ will be the appropriate enhanced GFF. By the usual abuse of notation we will denote also $\mathbb{W}=(W,M^\beta)$ the canonical process on $\mathfrak{S}$.

We also need a slightly improved definition of the relaxed trial space $\bar{\mathcal{X}}^{\exp}$.

Definition 16.3. *Set*

$$\mathcal{X}^{\exp}:=\{\mu=\mathrm{Law}(W,M^\beta,Z)\in\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w)\colon Z\in\hat{\mathfrak{H}}^a \text{ with } \mathbb{E}_\mu[\|Z\|^2_{\hat{\mathfrak{H}}}]<\infty\},$$

and

$$\bar{\mathcal{X}}^{\exp}:=\left\{\mu\in\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w)\colon\exists(\mu_n)_n\subseteq\mathcal{X}\colon\mu_n\to\mu \text{ weakly}, \sup_{n\in\mathbb{N}}\mathbb{E}_{\mu_n}[\|Z\|^2_{\hat{\mathfrak{H}}}]<\infty\right\}$$

We equip $\bar{\mathcal{X}}^{\exp}$ with the following topology: a sequence $(\mu_n)_n\subseteq\bar{\mathcal{X}}^{\exp}$ converges to μ if

a) $\mu_n\to\mu$ *weakly in* $\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w)$,

b) $\sup_{n\in\mathbb{N}}\mathbb{E}_{\mu_n}[\|Z\|^2_{\hat{\mathfrak{H}}}]<\infty$.

For technical reasons we first establish the variational formulation for the approximate potential $V^{\xi,\varepsilon}$ where

$$V^{\xi,\varepsilon}(\phi)=\lambda\int_{\mathbb{R}^2}\xi[\![\exp(\beta\psi_\varepsilon*\phi)]\!]$$

and the Wick ordering is now taken with respect to the covariance of $W_\varepsilon = \psi_\varepsilon * W$.

Lemma 16.4. *The potential $V^{\xi,\varepsilon}$ is admissible and*

$$\check{F}^{f,\xi,\varepsilon}(\mu) := \check{F}^{V^{\xi,\varepsilon}}(\mu) = \mathbb{E}_\mu\Big[\int_{\mathbb{R}^2} \xi \exp(\beta(\psi_\varepsilon * Z))\mathrm{d}M^{\beta,\varepsilon} + \mathcal{E}(Z)\Big]$$

for all $\mu \in \bar{\mathcal{X}}^{\exp}$.

Proof. Note that $M^{\beta,\varepsilon}$ is positive and that Proposition 16.2 implies that for p close enough to 1 and $\frac{1}{p}+\frac{1}{q}=1$

$$\mathbb{E}[\exp(-qV^{\xi,\varepsilon})] + \mathbb{E}[|V^{\xi,\varepsilon}|^p] < \infty,$$

so V^ξ is a tame functional. From the properties of the exponential it is clear that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \xi[\![\exp(\beta(\psi_\varepsilon * Z) + \beta(\psi_\varepsilon * W_\varepsilon))]\!]\mathrm{d}x \\
=\ &\int_{\mathbb{R}^2} \xi \exp(\beta(\psi_\varepsilon * Z))[\![\exp(\beta(\psi_\varepsilon * W_\varepsilon))]\!]\mathrm{d}x \\
=\ &\int_{\mathbb{R}^2} \xi \exp(\beta(\psi_\varepsilon * Z))\mathrm{d}M^{\beta,\varepsilon}.
\end{aligned}$$

We have for all $\mu \in \bar{\mathcal{X}}^{\exp}$:

$$\begin{aligned}
\check{F}^{f,\xi,\varepsilon}(\mu) &:= \check{F}^{V^{\xi,f,\varepsilon}}(\mu) \\
&= \mathbb{E}_\mu\Big[f(W+Z) + \int_{\mathbb{R}^2} \xi \exp(\beta(\psi_\varepsilon * Z))\mathrm{d}M^{\beta,\varepsilon} + \mathcal{E}(Z)\Big] \\
&\geqslant \mathbb{E}_\mu[\mathcal{E}(Z)] - C \geqslant \frac{1}{2}\mathbb{E}_\mu[\|Z\|^2_{\mathfrak{H}}] - C
\end{aligned}$$

from which also the coercivity of $\check{F}^{V^{\xi,f,\varepsilon}}$ is clear. Furthermore it is not hard to see that

$$(W,Z) \to \int_{\mathbb{R}^2} \xi[\![\exp(\beta(\psi_\varepsilon * Z) + \beta(\psi_\varepsilon * W))]\!]\mathrm{d}x$$

is continuous as a map $(\mathfrak{S}_0, H^1) \to \mathbb{R}$ (note hat here the renormalisation if finite) and convergence of W, Z in $(\mathfrak{S}_0, H^1)$ implies convergence in $L^\infty \times L^\infty$ of $\psi_\varepsilon * Z, \psi_\varepsilon * W$. To conclude admissibility it remains to show that $\check{F}^{V^{\xi,f,\varepsilon}}$ is lower semicontinuous on $\bar{\mathcal{X}}^{\exp}$. Since we have already shown the interaction to be continuous it remains to do this for $\mathcal{E}$. Indeed we have for any $\mu \in \bar{\mathcal{X}}^{\exp}$ a sequence $(\mu_n)_n$ such that $\mu_n \to \mu$ in $\bar{\mathcal{X}}^{\exp}$

$$\liminf_n \mathbb{E}_{\mu^n}[\mathcal{E}(Z)] \geqslant \mathbb{E}_\mu[\mathcal{E}(Z)],$$

this implies lower semicontinuity. □

We now show that we can remove the ultraviolet cutoff in the variational formulation. We introduce the unregularized functional

$$\check{F}^{f,\xi}(\mu) = \check{F}^{V^{\xi,f}}(\mu) = \mathbb{E}_\mu\Big[f(W+Z) + \int_{\mathbb{R}^2} \xi \exp(\beta Z)\mathrm{d}M^\beta + \mathcal{E}(Z)\Big].$$

We will show that as $\varepsilon \to 0$ $\check{F}^{f,\xi,\varepsilon}$ **Γ-converges** to $\check{F}^{f,\xi}$.

Γ-convergence is a notion of convergence introduced by De Giorgi well suited for the study of variational problems. The book [?] is a nice introduction to Γ-convergence in the context of the calculus of variations. For the convenience of the reader we recall here the basic definitions and results.

Definition 16.5. *Let $\mathcal{T}$ be a topological space and let $F, F_n : \mathcal{T} \to (-\infty, \infty]$. We say that the sequence of functionals $(F_n)_n$ Γ-converges to F iff*

i. For every sequence $x_n \to x$ in $\mathcal{T}$

$$F(x) \leqslant \liminf_{n\to\infty} F_n(x_n);$$

ii. For every point x there exists a sequence $x_n \to x$ (called a recovery sequence) such that

$$F(x) \geqslant \limsup_{n\to\infty} F_n(x_n).$$

Definition 16.6. *A sequence of functionals $F_n : \mathcal{T} \to (-\infty, \infty]$ is called equicoercive if there exists a compact set $\mathcal{K} \subseteq \mathcal{T}$ such that for all $n \in \mathbb{N}$*

$$\inf_{x\in\mathcal{K}} F_n(x) = \inf_{x\in\mathcal{T}} F_n(x).$$

A fundamental consequence of Γ-convergence is the convergence of minima.

Theorem 16.7. *If $(F_n)_n$ Γ-converges to F and $(F_n)_n$ is equicoercive, then F admits a minimum and*

$$\min_{\mathcal{T}} F = \lim_{n\to\infty} \inf_{\mathcal{T}} F_n.$$

For a proof see [?].

Lemma 16.8. *$\check{F}^{f,\xi,\varepsilon}$ Γ-converges to $\check{F}^{f,\xi}$ on $\bar{\mathcal{X}}$ as $\varepsilon \to 0$.*

Proof. We first show the liminf inequality. We have for sequence $(\mu_\varepsilon)_\varepsilon$ such that $\mu_\varepsilon \to \mu$ in $\bar{\mathcal{X}}^{\exp}$

$$\liminf_{\varepsilon\to 0} \mathbb{E}_{\mu^\varepsilon}[\mathcal{E}(Z)] \geqslant \mathbb{E}_\mu[\mathcal{E}(Z)], \qquad \lim_{\varepsilon\to 0} \mathbb{E}_{\mu^\varepsilon}[f(W+Z)] = \mathbb{E}_\mu[f(W+Z)].$$

Now let $g^N : \mathbb{R} \to \mathbb{R}$ be defined by $g^N(x) = x$ if $x < N$, $g^N(x) = N$ if $x > N$. Then $g^N \leqslant N$, g^N is Lipschitz with constant 1 and $g^N(x) \to x$. Set $Z_t^N = g^N(Z_t)$ and $\mu_{N,\varepsilon} = \mathrm{Law}_{\mu^\varepsilon}(W, Z^N)$. Then by monotone convergence

$$\begin{aligned} &\lim_{N\to\infty} \mathbb{E}_\mu \int \xi \exp(\beta Z^N) \mathrm{d}M^\beta \\ = \;& \mathbb{E}_\mu \int \xi \exp(\beta Z) \mathrm{d}M^\beta \end{aligned}$$

It remains to prove that

$$\lim_{\varepsilon\to 0} \mathbb{E}_{\mu^\varepsilon} \int \xi \exp(\beta Z^{N,\varepsilon}) \mathrm{d}M^{\beta,\varepsilon} \;=\; \mathbb{E}_\mu \int \xi \exp(\beta Z^N) \mathrm{d}M^\beta,$$

then the statement will follow from

$$\begin{aligned} \mathbb{E}_\mu \int \xi \exp(\beta Z) \mathrm{d}M^\beta \;&=\; \lim_{N\to\infty} \mathbb{E}_\mu \int \xi \exp(\beta Z^N) \mathrm{d}M^\beta \\ &=\; \lim_{N\to\infty} \lim_{\varepsilon\to\infty} \mathbb{E}_{\mu^\varepsilon} \int \xi \exp(\beta Z^{N,\varepsilon}) \mathrm{d}M^{\beta,\varepsilon} \\ &\leqslant\; \lim_{\varepsilon\to 0} \mathbb{E}_{\mu^\varepsilon} \int \xi \exp(\beta Z^\varepsilon) \mathrm{d}M^{\beta,\varepsilon}. \end{aligned}$$

We have that $\|\exp(\beta Z^N)\|_{L^\infty} \leqslant \exp(\beta N)$ and by chain rule

$$\|\rho \nabla \exp(\beta Z^N)\|_{L^2} = \|\rho \beta \nabla Z^N \exp(\beta Z^N)\|_{L^2} \lesssim \exp(\beta N) \|\rho \nabla Z^N\|_{L^2}.$$

By interpolation

$$\|\exp(\beta Z^{N,\varepsilon})\|_{B^\delta_{p',p'}(\rho)} \lesssim \|\exp(\beta Z^{N,\varepsilon})\|_{L^\infty}^{(1-2\delta)} \|\exp(\beta Z^{N,\varepsilon})\|_{H^1(\rho)}^{2\delta},$$

provided $2/p' > \delta$. Now let $p = p'/(p'-1)$. Note that we can choose δ such that $\frac{8\pi}{\beta^2} > p$ and $\frac{\beta^2}{4\pi}(p-1) < \delta$ and $2/p' < \delta$ by taking

$$\frac{2}{p}(p-1) > \delta > \frac{\beta^2}{4\pi}(p-1).$$

Since $M^{\beta,\varepsilon} \to M^\beta$ in $B^s_{p,p}$ by Proposition 16.2 this establishes that $\int \xi \exp(\beta Z^{N,\varepsilon}) \mathrm{d}M^{\beta,\varepsilon} \to \int \xi \exp(\beta Z^N) \mathrm{d}M^\beta$ almost surely. Since

$$\int \xi \exp(\beta Z^{N,\varepsilon}) \mathrm{d}M^{\beta,\varepsilon} \lesssim e^{\beta N} \int \xi \mathrm{d}M^{\beta,\varepsilon},$$

and the r.h.s is uniformly integrable in the probability variable, we have by dominated convergence

$$\lim_{\varepsilon\to 0}\mathbb{E}_{\mu^\varepsilon}\int \xi\exp(\beta Z^{N,\varepsilon})\mathrm{d}M^{\beta,\varepsilon}=\mathbb{E}_{\mu}\int \xi\exp(\beta Z^{N})\mathrm{d}M^{\beta},$$

and we can conclude. To show the limsup inequality, for given μ we need to construct μ^ε such that

$$\operatorname*{limsup}_{\varepsilon\to 0} F^{V^{\xi,f,\varepsilon}}(\mu^\varepsilon)\leqslant F^{V^{\xi,f}}(\mu).$$

To do this we proceed similarly as in the first step. Define $Z^{N,\delta}=g^N(\psi_\delta * Z)$, $\mu^{N,\delta}=\mathrm{Law}(W,M^\beta, Z^{N,\delta})$, where ψ_δ is a sequence of standard mollifiers. Then

$$M^{\beta,\varepsilon}\to M^\beta$$

weakly so

$$\mathbb{E}\int \exp(\beta Z^{N,\delta})\mathrm{d}M^{\beta,\varepsilon}\to\mathbb{E}\int\exp(\beta Z^{N,\delta})\mathrm{d}M^\beta$$

since $Z^{N,\delta}$ is bounded and continuous. Furthermore $\mathbb{E}_{\mu^\varepsilon}[\mathcal{E}(Z^{N,\delta})]\to\mathbb{E}_\mu[\mathcal{E}(Z^{N,\delta})]$. This implies that

$$\lim_{\varepsilon\to 0}F^{V^{\xi,\varepsilon}}(\mu^{N,\delta})=F^{V^\xi}(\mu^{N,\delta})$$

Now by dominated/monotone convergence we have

$$\lim_{N\to\infty}\lim_{\delta\to 0}\mathbb{E}_\mu\int\exp(\beta Z^{N,\delta})\mathrm{d}M^\beta=\mathbb{E}_\mu\int\exp(\beta Z)\mathrm{d}M^\beta,$$

and

$$\lim_{N\to\infty}\lim_{\delta\to 0}\mathbb{E}_\mu[\mathcal{E}(Z^{N,\delta})]=\mathbb{E}_\mu[\mathcal{E}(Z)]\qquad\lim_{N\to\infty}\lim_{\delta\to 0}\mathbb{E}_\mu[f(W+Z^{N,\delta})]=\mathbb{E}_\mu[f(W+Z)],$$

so taking a suitable diagonal sequence implies the statement. □

Corollary 16.9.
$$\lim_{\varepsilon\to 0}\inf_{\mu\in\bar{\mathcal{X}}}\check{F}^{V^{\xi,\varepsilon}}=\inf_{\mu\in\bar{\mathcal{X}}}\check{F}^{V^\xi}(\mu)$$

and the infimum on the r.h.s is attained.

Proof. The statement will follow from the fundamental theorem of Γ-convergence (see e.g. [?]), once we have established that $\check{F}^{V^{\xi,\varepsilon}}$ is equicoercive on $\bar{\mathcal{X}}^{\exp}$, but this follows from

$$\check{F}^{V^{\xi,f,\varepsilon}}(\mu)\geqslant\mathbb{E}_\mu[\mathcal{E}(Z)]-C.$$ □

We can now relate this back to our general framework:

Lemma 16.10. *The functional $\check{F}^{f,\xi}$ has a minimiser $\mu^{f,\xi}\in\bar{\mathcal{X}}^{\exp}$ and*

$$\mathcal{W}^{\xi,\exp}(f):=-\log\int\exp(-f(\phi))\mathrm{d}\nu^\xi=\min_{\mu\in\bar{\mathcal{X}}^{\exp}}\check{F}^{f,\xi}(\mu)=\check{F}^{f,\xi}(\mu^{f,\xi}).$$

Furthermore for any minimiser $\mu\in\bar{\mathcal{X}}^{\exp}$ the following wFBSDE

$$\mathbb{E}_\mu\left[\lambda\beta\int\xi\exp(\beta Z)\,K\mathrm{d}M^\beta+2\mathcal{E}(Z,K)\right]=\mathbb{E}_\mu\left[\int\nabla f(W+Z)\,K\right]$$

holds for any admissible control K.

Proof. This follows *mutatis mutandis* as in Lemma 15.4. □

Lemma 16.11. *The functional $\check{F}^{0,\xi}$ has a unique minimiser $\mu^{0,\xi}$ in $\bar{\mathcal{X}}$. Furthermore*

$$Z\leqslant 0,$$

$\mu^{0,\xi}$ almost surely, and

$$\sup_\xi\mathbb{E}_{\mu^{0,\varepsilon}}[\|Z\|_{\mathfrak{H}}^2]<\infty.$$

Proof. Observe that $F^{0,\xi}(Z)$ is convex in Z. Now let μ,ν be two minimisers for $\check{F}^{0,\xi}$. Recall from Lemma 14.7 that there exists $\upsilon\in\bar{\mathcal{Y}}(\mu)$ such that $\mathrm{Law}_\upsilon(W,Z+K)=\nu$. Now

$$\mathbb{E}_\upsilon\Big[\lambda\int\xi\exp(\beta(Z+K))\mathrm{d}M^\beta+\mathcal{E}(Z+K)\Big]-\mathbb{E}_\upsilon\Big[\lambda\int\xi\exp(\beta Z)\mathrm{d}M^\beta+\mathcal{E}(Z)\Big]$$
$$\geqslant\mathbb{E}_\upsilon\Big[\lambda\beta\int\xi\exp(\beta Z)K\mathrm{d}M^\beta+2\mathcal{E}(Z,K)+\mathcal{E}(K)\Big].$$

By Lemma 16.10 we have that

$$\mathbb{E}_\upsilon\Big[\lambda\beta\int\xi\exp(\beta Z)K\mathrm{d}M^\beta+2\mathcal{E}(Z,K)\Big]=0,$$

so

$$\mathbb{E}[\mathcal{E}(K)]\leqslant\check{F}^{f,\xi}(\mu)-\check{F}^{f,\xi}(\nu)=0,$$

which implies $K=0$ and therefore $\nu=\mu$. For the second statement observe that the mapping A: $Z\to-|Z|$ leaves the energy $\mathcal{E}(Z)$ invariant by chain rule, while

$$\int\xi\exp(\beta A(Z))\mathrm{d}M^\beta\leqslant\int\xi\exp(\beta Z)\mathrm{d}M^\beta,$$

so if $\mu^{0,\xi}$ is a minimiser, so is $\mathrm{Law}_{\mu^{0,\xi}}(W,A(Z))$. By uniqueness we can conclude. For the third claim we test the wFBSDE with $K=\rho Z$ and obtain

$$\mathbb{E}_\mu\Big[4\,\lambda\beta\int\xi\exp(\beta Z)\rho\,Z\mathrm{d}M^\beta+2\mathcal{E}(Z,\rho\,Z)\Big]=\mathbb{E}_\mu\Big[\int\nabla f(W+Z)\rho\,Z\Big].$$

We can again estimate as in the proof of Lemma 15.7

$$2\mathcal{E}(Z,\rho\,Z)\geqslant\frac{1}{2}\|Z\|^2_{\hat{\mathfrak{H}}},$$

and

$$\int\nabla f(W+Z)\rho\,Z\lesssim\|Z\|_{\hat{\mathfrak{H}}}.$$

Furthermore observe that

$$\int_{\mathbb{R}^2}\xi\exp(\beta Z)\rho\,Z\mathrm{d}M^\beta\geqslant-C\int_{\mathbb{R}^2}\xi\rho\,\mathrm{d}M^\beta,$$

since $\exp(\beta Z)Z$ is bounded below. Summing up, we have the estimate

$$\frac{1}{2}\mathbb{E}_{\mu^{0,\xi}}[\|Z\|^2_{\hat{\mathfrak{H}}}]\lesssim\mathbb{E}[\|Z\|^2_{\hat{\mathfrak{H}}}]^{1/2}+\mathbb{E}_\mu\Big[\int_{\mathbb{R}^2}\xi\rho\,\mathrm{d}M^\beta\Big]\lesssim\mathbb{E}[\|Z\|^2_{\hat{\mathfrak{H}}}]^{1/2}+1$$

where in the last bound we used Lemma 16.2. □

Lemma 16.12. *Let $\mu^\xi\in\bar{\mathcal{X}}^{\exp}$. For $\upsilon\in\bar{\mathcal{Y}}(\mu^\xi)$ we have*

$$\check{H}^{\xi,\exp}(\upsilon):=\check{H}^{V^\xi}(\upsilon)=\mathbb{E}_\upsilon[2\mathcal{E}(K,Z)+\mathcal{E}(K)]$$
$$+\lambda\,\mathbb{E}_\upsilon\Big[\int\xi\exp(\beta(Z+K))\mathrm{d}M^\beta-\int\xi\exp(\beta Z)\mathrm{d}M^\beta\Big].$$

Then $\check{H}^{\xi,\exp}(\upsilon)\geqslant0$ for all $\upsilon\in\bar{\mathcal{Y}}(\mu^\xi)$ iff μ^ξ is a minimiser of $\check{F}^{0,\xi}$. Moreover in this case, thanks to the wFBSDE for Z we have

$$\check{H}^{\xi,\exp}(\upsilon)=\mathbb{E}_\upsilon\Big[\mathcal{E}(K)+\lambda\int\xi\exp(\beta Z)(\exp(\beta K)-1-\beta K)\mathrm{d}M^\beta\Big],$$

and

$$\mathcal{W}^{\exp}_\xi(f)=\inf_{\upsilon\in\bar{\mathcal{Y}}(\mu^\xi)}\check{G}^{f,\xi,\exp}(\upsilon),\quad\check{G}^{f,\xi,\exp}(\upsilon):=\mathbb{E}_\upsilon[f(W+Z+K)]+\check{H}^{\xi,\exp}(\upsilon).$$

Any minimiser $\upsilon\in\bar{\mathcal{Y}}(\mu^\xi)$ of $\check{G}^{f,\xi,\exp}$ satisfies the perturbed wFBSDE

$$\begin{aligned}&\mathbb{E}_\upsilon\Big[\mathcal{E}(K)+\lambda\beta\int\xi\exp(\beta Z)(\exp(\beta K)-1)H\mathrm{d}M^\beta\Big]\\&=\mathbb{E}_\upsilon[\nabla f(W+Z+K)\cdot H].\end{aligned}\tag{16.2}$$

The proof of Lemma 16.12 is completely analogous to Lemma 15.6 and Lemma 15.4, thus we omit the details. Thanks to the convexity of the interaction we have a good control of the locality properties of the solutions to the wFBSDE (16.2).

Lemma 16.13. *Assume that $N>0$ and*

$$\sup_{\varphi\in L^2(\mathbb{R}^2)}\|\nabla f(\varphi)\|_{H^{-1}(\rho^{-N})}<\infty.$$

Then for any minimiser υ^f of $\check{G}^{f,\xi,\exp}(\rho)$ we have

$$\mathbb{E}_{\upsilon^f}[\|K\|^2_{H^1(\rho^{-N})}]<\infty.$$

Proof. Testing equation (16.2) with $H=\rho^{-2N}K$ we have

$$\begin{aligned}&\mathbb{E}_\upsilon\Big[\frac{1}{2}\int_0^1\int_{\mathbb{R}^2}\rho^{-2N}\dot{K}_s(1-\Delta)\dot{K}_s\mathrm{d}s\Big]\\&+\lambda\beta\mathbb{E}_\upsilon\Big[\int\xi\exp(\beta Z)(\exp(\beta K)-1)\rho^{-2N}K\mathrm{d}M^\beta\Big]\\=\;&\mathbb{E}_\upsilon\Big[\int\rho^{-2N}\nabla f(W+Z+K)K\Big]\leqslant C\mathbb{E}_\upsilon[\|K\|^2_{H^1(\rho^{-N})}]^{1/2}\end{aligned}$$

Now

$$\begin{aligned}&\mathbb{E}_\upsilon\Big[\frac{1}{2}\int_0^1\int_{\mathbb{R}^2}\rho^{-2N}\dot{K}_s(1-\Delta)\dot{K}_s\mathrm{d}s\Big]\\=\;&\mathbb{E}_\upsilon\Big[\frac{1}{2}\int_0^1\int_{\mathbb{R}^2}\rho^{-2N}((1-\Delta)^{-1/2}\dot{K}_s)^2\mathrm{d}s\Big]\\&+\mathbb{E}_\upsilon\Big[\frac{1}{2}\int_0^1\int_{\mathbb{R}^2}[(1-\Delta)^{1/2},\rho^{-2N}]\dot{K}_s(1-\Delta)^{1/2}\dot{K}_s\mathrm{d}s\Big]\end{aligned}$$

and we can compute, as in the proof of Lemma 15.7,

$$[(1-\Delta)^{1/2},\rho^{-2N}]\dot{K}_s\leqslant\delta\rho^{-2N}\dot{K}_s.$$

Furthermore observe that

$$\mathbb{E}_\upsilon\Big[\int\xi\exp(\beta Z)(\exp(\beta K)-1)\rho^{-2N}K\mathrm{d}M^\beta\Big]\geqslant 0,$$

Putting everything together we have

$$\begin{aligned}(1/2-\delta)\mathbb{E}_\upsilon\|K\|^2_{H^1(\rho^{-N})}&=(1/2-\delta)\mathbb{E}_\upsilon\Big[\frac{1}{2}\int_0^1\int_{\mathbb{R}^2}\rho^{-2N}((1-\Delta)^{-1/2}\dot{K}_s)^2\mathrm{d}s\Big]\\&\leqslant C\mathbb{E}_\upsilon\|K\|_{H^1(\rho^{-N})}\end{aligned}$$

which implies the statement. □

Definition 16.14. *For $N>0$ large (to be chosen suitably below) denote by $\mathcal{Y}^M(\mu)$ the set of measures $\upsilon\in\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w\times\mathfrak{H}_w(\rho^{-N}))$ such that*

a) $\mathrm{Law}_\upsilon(W,M^\beta,Z)=\mu$

b) $\exists(Z^n)_n,(K^n)_n\subseteq\hat{\mathfrak{H}}^a$ *for which*

$$\mathrm{Law}(W,M^{\beta,n},Z^n,K^n)\to\upsilon\;\text{ and }\;\sup_{n\in\mathbb{N}}\mathbb{E}[\|Z^n\|^2_{\hat{\mathfrak{H}}}+\|K^n\|^2_{\mathfrak{H}(\rho^{-N})}]\leqslant 2M.$$

Let $\mathcal{Y}^M := \bigcup_{\mu\in\bar{\mathcal{X}}^{\exp}} \mathcal{Y}^M(\mu)$. *We say that* $(\upsilon^n)_n \subseteq \mathcal{Y}^M$ *converges to* $\upsilon \in \mathcal{Y}^M$ *if* $\upsilon^n \to \upsilon$ *weakly in* $\mathcal{P}(\mathfrak{S}\times\hat{\mathfrak{H}}_w \times \mathfrak{H}_w(\rho^{-N}))$ *and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\upsilon^n}[\|Z\|^2_{\hat{\mathfrak{H}}_w} + \|K\|^2_{\mathfrak{H}(\rho^{-N})}] \leqslant 2M.$$

Remark 16.15. Note that any sequence in $\mathcal{Y}^M$ has a convergent subsequence.

Corollary 16.16. *There exists a sufficiently large* $M = M(f)$ *(not depending on* ξ*) such that*

$$\inf_{\upsilon\in\mathcal{Y}(\mu^{0,\xi})} \check{G}^{f,\xi,\exp}(\upsilon) = \inf_{\upsilon\in\mathcal{Y}^M(\mu^{0,\xi})} \check{G}^{f,\xi,\exp}(\upsilon).$$

Proof. This is a direct consequence of the decay of the minimiser established in Lemma 16.13. □

Definition 16.17. *Define*

$$\bar{G}^{f,\xi,\exp}(\upsilon) = \check{G}^{f,\xi,\exp}(\upsilon) \text{ if } \upsilon \in \mathcal{Y}^{M(f)}(\mu^{0,\xi}) \text{ and } +\infty \text{ otherwise.}$$

The next lemma discusses Γ-convergence of the variational description as the infrared cutoff is removed.

Lemma 16.18. *Let* $(\xi_n)_n$ *be a sequence of spatial cutoffs such that* $\xi_n(x) = 1$ *for* $|x| \leqslant n$, *and*

$$\sup_{n\in\mathbb{N}} \|\xi_n\|_{W^{k,\infty}} < +\infty, \qquad \text{for any } k \in \mathbb{N}.$$

Pick a subsequence ξ_n *(not relabelled) such that* $\mu^n \to \mu$ *weakly on* $\mathfrak{S}\times\hat{\mathfrak{H}}_w$ *where* μ^n *is the minimiser of* $\check{F}^{0,\xi_n}$ *in* $\bar{\mathcal{X}}$. *Then* $\bar{G}^{f,\xi_n,\exp}$ Γ*-converges to* $\bar{G}^{f,\infty,\exp}$ *on* $\mathcal{Y}^M$ *where*

$$\bar{G}^{f,\infty,\exp}(\upsilon) = \mathbb{E}_\upsilon\Big[f(W+Z+K) + \int \exp(\beta Z)(\exp(\beta K)-1)\mathrm{d}M^\beta + \mathcal{E}(K)\Big]$$

if $\upsilon \in \mathcal{Y}^M(\mu)$ *and* $+\infty$ *otherwise.*

Proof. We first prove the liminf inequality. Assume that $\upsilon^n \to \upsilon$ in $\mathcal{Y}^M$. By definition of ξ_n we can restrict ourselves to $\upsilon \in \mathcal{Y}^M(\mu)$. From the definition of convergence in $\mathcal{Y}^M$ it is not hard to see that

$$\mathbb{E}_{\upsilon^n}[f(W+Z^n+K^n)] \to \mathbb{E}_\upsilon[f(W+Z+K)],$$

and

$$\mathbb{E}_\upsilon[\mathcal{E}(K)] \leqslant \liminf_{n\to\infty} \mathbb{E}_{\upsilon^n}[\mathcal{E}(K^n)].$$

by the Portmanteau theorem. Now we can invoke the Skhorohod's embedding theorem from [?] and pass to a probability space $\tilde{\mathbb{P}}$ and on it a sequence $(W^n, M^{\beta,n}, Z^n, K)$ convergent almost surely on $\mathfrak{S}\times\hat{\mathfrak{H}}_w\times\mathfrak{H}_w(\rho^{-N})$ to (W, M^β, Z, K) such that $\mathrm{Law}(W^n, M^{\beta,n}, Z^n, K) = \upsilon^n$. Using the fact that

$$\sup_n \mathbb{E}_{\tilde{\mathbb{P}}}\big[\|Z^n - Z\|^2_{\mathfrak{H}_w(\rho^{1/2})} + \|K^n - K\|^2_{\mathfrak{H}_w(\rho^{-N})}\big] < \infty$$

and almost-sure convergence we can assume that

$$\mathbb{E}_{\tilde{\mathbb{P}}}\|M^{\beta,n} - M^\beta\|^p_{B^\delta_{p,p}} \to 0, \qquad \mathbb{E}_{\tilde{\mathbb{P}}}\big[\|Z^n - Z\|^q_{\mathfrak{H}_w(\rho^{1/2})} + \|K^n - K\|^q_{\mathfrak{H}_w(\rho^{-N})}\big] \to 0,$$

for any δ, p satisfying the assumptions of Proposition 16.2 and $q < 2$. Let $\varsigma_N : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}_+$ be continuous such that

$$\varsigma_N(z,x) = z \text{ if } z, |x| \leqslant N,$$

$\varsigma_N(z,x) = 0$ if either $z \geqslant 2N$ or $|x| \geqslant 2N$ and moreover $\varsigma_N(z,y) \leqslant \varsigma_{N+1}(z,y)$. We will write in the following $\varsigma_N(f,x)$ for the function $x \in \mathbb{R}^2 \mapsto \varsigma_N(f(x),x)$. We claim that

$$\begin{aligned} &\lim_{n\to\infty} \int \xi_n \exp(\beta Z^n)\varsigma_N((\exp(\beta K^n)-1)_+, x)\mathrm{d}M^{\beta,n} \\ =\ &\int \exp(\beta Z)\varsigma_N((\exp(\beta K)-1)_+, x)\mathrm{d}M^\beta. \end{aligned}$$

Since

$$\sup_{n\in\mathbb{N}}\|\exp(\beta Z^n)\|_{L^\infty}\leqslant 1,\qquad \sup_{n\in\mathbb{N}}\|\varsigma_N((\exp(\beta K^n)-1)_+,x)\|_{L^\infty}\leqslant N,$$

we can apply Lemma ? below provided that, for any , $q<2$,

$$\mathbb{E}\big[\|\xi_n\exp(\beta Z^n)\varsigma_N((\exp(\beta K^n)-1)_+,x)-\exp(\beta Z)\varsigma_N((\exp(\beta K)-1)_+,x)\|^q_{W^{1,q}(\rho^{-1})}\big]\to 0,$$

which in turn is ensured by Lemma ? and Lemma ?. Then, recalling that $Z\leqslant 0$ μ^n almost surely and monotone convergence we have,

$$\begin{aligned}
&\int\exp(\beta Z)(\exp(\beta K)-1)_+\mathrm{d}M^\beta\\
=\;&\sup_N\int\exp(\beta Z)\varsigma_N((\exp(\beta K)-1)_+,x)\mathrm{d}M^\beta\\
=\;&\sup_N\lim_{n\to\infty}\int\xi_n\exp(\beta Z^n)\varsigma_N((\exp(\beta K^n)-1)_+,x)\mathrm{d}M^{\beta,n}\\
\leqslant\;&\sup_N\operatorname*{liminf}_{n\to\infty}\int\xi_n\exp(\beta Z^n)(\exp(\beta K^n)-1)_+\mathrm{d}M^{\beta,n}\\
=\;&\operatorname*{liminf}_{n\to\infty}\int\xi_n\exp(\beta Z^n)(\exp(\beta K^n)-1)_+\mathrm{d}M^{\beta,n}.
\end{aligned}$$

Let us now show that

$$\lim_{n\to\infty}\int\xi_n\exp(\beta Z^n)(\exp(\beta K^n)-1)_-\mathrm{d}M^{\beta,n}=\int\exp(\beta Z)(\exp(\beta K)-1)_-\mathrm{d}M^\beta.$$

To start, note that

$$|\exp(\beta Z^n)|\leqslant 1,\qquad |(\exp(\beta K^n)-1)_-|\leqslant 1,$$

so again we have that from Lemmas ?, ? below, for any $p<2$,

$$\mathbb{E}\|\exp(\beta Z^n)(\exp(\beta K^n)-1)_--\exp(\beta Z)(\exp(\beta K)-1)_-\|^p_{W^{1,p}(\rho^{-1})}\to 0.$$

From this and using that $\xi_n=1$ for $|x|\leqslant n$, it is easy to see that

$$\mathbb{E}\|\xi_n\exp(\beta Z^n)(\exp(\beta K^n)-1)_--\exp(\beta Z)(\exp(\beta K)-1)_-\|^p_{W^{1,p}(\rho^{-1/2})}\to 0.$$

we have that,

$$\int\xi_n\exp(\beta Z^n)(\exp(\beta K^n)-1)_-\mathrm{d}M^{\beta,n}\to\int\exp(\beta Z)(\exp(\beta K)-1)_-\mathrm{d}M^\beta.$$

To construct the recovery sequence consider υ such that $\bar{G}^{f,\infty,\exp}(\upsilon)<\infty$. Then

$$\mathbb{E}_\upsilon\int\exp(\beta Z)(\exp(\beta K)-1)\mathrm{d}M^\beta<\infty,\qquad \mathbb{E}_\upsilon\mathcal{E}(K)\leqslant M$$

where the second inequality holds by definition of $\mathcal{Y}^M$. Again, after an application of Lemma ? and interpolation between L^∞ and $H^1(\rho^{-1})$ and provided $N>1/\delta$, we have that

$$\begin{aligned}
&\mathbb{E}_\upsilon\int\exp(\beta Z)(\exp(\beta K)-1)_-\mathrm{d}M^\beta\\
\leqslant\;&\mathbb{E}\Big[\|(\exp(\beta K)-1)_-\|^{p'\delta}_{B^\delta_{p',p'}(\rho^{-1})}\Big]\mathbb{E}\Big[\|M^\beta\|^p_{B^{-\delta}_{p,p}(\rho)}\Big]\\
\leqslant\;&\mathbb{E}[\|(\exp(\beta K)-1)_-\|_{H^1(\rho^{-N})}]\mathbb{E}\Big[\|M^\beta\|^p_{B^{-\delta}_{p,p}(\rho)}\Big]\\
<\;&\infty.
\end{aligned}$$

So

$$\begin{aligned}
&\mathbb{E}_\upsilon\int\exp(\beta Z)(\exp(\beta K)-1)_+\mathrm{d}M^\beta\\
\leqslant\;&\left|\mathbb{E}_\upsilon\int\exp(\beta Z)(\exp(\beta K)-1)\mathrm{d}M^\beta\right|+\left|\mathbb{E}_\upsilon\int\exp(\beta Z)(\exp(\beta K)-1)_-\mathrm{d}M^\beta\right|\\
<\;&\infty.
\end{aligned}$$

Now set $K^N(x) = \varsigma_N(K,x)$, by monotone convergence

$$\lim_{N\to\infty} \mathbb{E}_\upsilon \int \exp(\beta Z)(\exp(\beta K^N)-1)_+ \mathrm{d}M^\beta \to \mathbb{E}_\upsilon \int \exp(\beta Z)(\exp(\beta K)-1)_+ \mathrm{d}M^\beta.$$

Moreover, since $|(\exp(\beta K^N)-1)_-| \leqslant |(\exp(\beta K)-1)_-|$, by dominated convergence also holds

$$\lim_{N\to\infty} \mathbb{E}_\upsilon \int \exp(\beta Z)(\exp(\beta K^N)-1)_- \mathrm{d}M^\beta \to \mathbb{E}_\upsilon \int \exp(\beta Z)(\exp(\beta K)-1)_- \mathrm{d}M^\beta.$$

Invoking [?] there exist a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables (W^n, M^β, Z^n) on it such that $(W^n, M^\beta, Z^n) \to (W, M^\beta, Z)$ almost surely in $\mathfrak{S}\times\mathfrak{H}_w(\rho)$ and $\mathrm{Law}_{\tilde{\mathbb{P}}}(W^n, M^\beta, Z^n) = \mu^n$.

Let K be a random variable $\tilde\Omega \to \mathfrak{H}_w(\rho^{-1})$ such that $\upsilon = \mathrm{Law}_{\tilde{\mathbb{P}}}(W, Z, K)$. Set again $K^N = \varsigma^N(K)$ and $\upsilon^{n,N} = \mathrm{Law}_{\tilde{\mathbb{P}}}(W^n, Z^n, K^N)$. We claim that

$$\bar G^{f,\xi_n,\exp}(\upsilon^{n,N}) \to \bar G^{f,\infty,\exp}(\upsilon^N).$$

From Lemmas ?, ?, ? we get

$$\mathbb{E}_{\tilde{\mathbb{P}}} \int \xi_n \exp(\beta Z^n)(\exp(\beta K^N)-1)_- \mathrm{d}M^{\beta,n} \to \mathbb{E}_{\tilde{\mathbb{P}}} \int \exp(\beta Z)(\exp(\beta K^N)-1)_- \mathrm{d}M^\beta.$$

Furthermore

$$(\exp(\beta K^N)-1)_+ \leqslant \exp(N)\mathbb{1}_{B(0,2N)}, \qquad \exp(\beta Z^n) \leqslant 1,$$

and $(K^N, Z^n) \to (K^N, Z)$ in $H^1(B(0,N))$. So again from Lemmas ?, ?, ?

$$\mathbb{E}_{\tilde{\mathbb{P}}} \int \exp(\beta Z^n)(\exp(\beta K^N)-1)_+ \mathrm{d}M^{\beta,n} \to \mathbb{E}_{\tilde{\mathbb{P}}} \int \exp(\beta Z)(\exp(\beta K^N)-1)_+ \mathrm{d}M^\beta.$$

Finally

$$\lim_{n\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}} f(W^n + Z^n + K^N) \to \mathbb{E}_{\tilde{\mathbb{P}}} f(W + Z + K^N)$$

follows by boundedness and continuity of f. Taking a diagonal sequence we obtain the statement. □

Corollary 16.19. *If $(\xi_n)_n$ is as in Lemma 16.18 we have*

$$\lim_{n\to\infty} -\log \int \exp(-f(\varphi))\mathrm{d}\nu^{\xi_n,\exp} = \inf_{\upsilon\in\mathcal{Y}^M(\mu)} \bar G^{f,\infty,\exp}(\upsilon)$$

Proof. This follows from Γ-convergence, the fact that $\mathcal{Y}^M$ is sequentially compact and the fundamental theorem of Γ-convergence (see e.g. [?]). □

We conclude with some technical results needed in the proofs above.

Lemma 16.20. *Assume that $g\colon \mathbb{R}\to\mathbb{R}$ is continuously differentiable, g, g' are bounded. Define $f = g_-$ (the negative part of g). We have that the map*

$$\varphi \mapsto f(\varphi),$$

continuously sends $W^{1,2}_w(\rho^\alpha)$ in itself with

$$\|f(\varphi)\|_{W^{1,2}(\rho^\alpha)} \lesssim 1 + \|\varphi\|_{W^{1,2}(\rho^\alpha)}.$$

If $g(0) = 0$ we have in particular

$$\|f(\varphi)\|_{W^{1,2}(\rho^\alpha)} \lesssim \|\varphi\|_{W^{1,2}(\rho^\alpha)}.$$

Proof. Assume that $g(0) = 0$. By definition of f we have

$$\int |f(\varphi(x))|^2 \rho^\alpha(x)\mathrm{d}x \lesssim \int |\varphi(x)|^2 \rho^\alpha(x)\mathrm{d}x.$$

Furthermore by [?] p. 291–292, we have

$$\int |\nabla_x f(\varphi(x))|^2 \rho^\alpha(x)\mathrm{d}x = \int \mathbb{1}_{g(\varphi(x))\leqslant 0} |g'(\varphi(x))\nabla_x\varphi(x)|^2\rho^\alpha(x)\mathrm{d}x \lesssim \int |\nabla_x\varphi(x)|^2\rho^\alpha(x)\mathrm{d}x.$$

Now for weak continuity, let $\varphi_n \to \varphi$ weakly in $W^{1,2}(\rho^\alpha)$. We know that

$$\sup_n \|\varphi\|_{W^{1,2}(\rho^\alpha)} < \infty \text{ which implies } \sup_n \|f(\varphi)\|_{W^{1,2}(\rho^\alpha)} < \infty.$$

Now let φ_{n_k} be a subsequence such that $f(\varphi_{n_k}) \to \xi$. We can extract a further subsequence (not relabelled) such that

$$\varphi_{n_k} \to \varphi$$

almost surely. Then $f(\varphi_{n_k}) \to f(\varphi)$ almost surely which in turn implies $\xi = f(\varphi)$. From this we can conclude. The second statement follows from the first by writing $g = \tilde{g} + g(0)$ with $\tilde{g}(0) = 0$. □

Lemma 16.21. *Let $M^{\beta,n}$ be a sequence of random measures in $\mathcal{P}(\mathbb{R}^2)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $M^{\beta,n} \to M^\beta$ $\mathbb{P}$-almost surely in $B^{-s}_{p,p}(\rho)$ for some $s > \frac{\beta^2}{4\pi}(p-1)$ and $p < 8\pi/\beta^2$ close enough to $\frac{\beta^2}{4\pi}(p-1), 8\pi/\beta^2$ respectively. There exists $q < 2$ such that we have*

$$\mathbb{E}\int f_n \mathrm{d}M^{\beta,n} \to \mathbb{E}\int f \mathrm{d}M^\beta$$

whenever

$$\mathbb{E}\big[\|f_n - f\|^q_{W^{1,q}(\rho^{-N})}\big] \to 0$$

and

$$\sup_{n\in\mathbb{N}} \|f_n\|_{L^\infty} + \|f\|_{L^\infty} = C < \infty.$$

Proof. By interpolation we have that for $\delta < q/p'$ and $N > \delta^{-1}$ we have with

$$\|f_n - f\|_{B^\delta_{p',p'}(\rho^{-1})} \leqslant C\|f_n - f\|^\delta_{W^{1,q}(\rho^{-N})}.$$

Now recall that for any $\frac{8\pi}{\beta^2} > p > 1$ and $\delta > \frac{\beta^2}{4\pi}(p-1)$ we have

$$\sup_n \mathbb{E}\Big[\|M^{\beta,n}\|^p_{B^{-\delta}_{p,p}(\rho)}\Big] < \infty.$$

so provided $1/p + 1/p' = 1$

$$\begin{aligned}
\mathbb{E}\Big[\int f_n \mathrm{d}M^{\beta,n} - \int f\mathrm{d}M^\beta\Big] &= \mathbb{E}\Big[\int (f_n - f)\mathrm{d}M^{\beta,n}\Big] + \mathbb{E}\Big[\int f\mathrm{d}M^{\beta,n} - \int f\mathrm{d}M^\beta\Big] \\
&\leqslant \mathbb{E}\big[\|f_n - f\|_{B^\delta_{p',p'}(\rho^{-1})}\|M^{\beta,n}\|_{B^{-\delta}_{p,p}(\rho)}\big] \\
&\quad + \mathbb{E}\big[\|f\|_{B^\delta_{p',p'}(\rho^{-1})}\|M^{\beta,n} - M^\beta\|_{B^{-\delta}_{p,p}(\rho)}\big] \\
&\leqslant \mathbb{E}\big[\|f_n - f\|^{p'\delta}_{W^{1,q}(\rho^{-1})}\big]^{\frac{1}{p'}}\mathbb{E}\Big[\|M^{\beta,n}\|^p_{B^{-\delta}_{p,p}(\rho)}\Big]^{\frac{1}{p}} \\
&\quad + \mathbb{E}\big[\|f\|^{p'\delta}_{W^{1,q}(\rho^{-1})}\big]^{\frac{1}{p'}}\mathbb{E}\Big[\|M^{\beta,n} - M^\beta\|^p_{B^{-\delta}_{p,p}(\rho)}\Big]^{\frac{1}{p}}. \\
&\to 0
\end{aligned}$$

since $p'\delta < q$. Note that for any $\beta^2 < 8\pi$ we can choose $q < 2$ such that

$$q/p' = \frac{q}{p}(p-1) > \frac{\beta^2}{4\pi}(p-1)$$

so if we choose δ such that

$$\frac{q}{p}(p-1) > \delta > \frac{\beta^2}{4\pi}(p-1)$$

we obtain as required

$$\frac{8\pi}{\beta^2} > p > 1 \qquad \delta > \frac{\beta^2}{4\pi}(p-1) \qquad \delta < q/p' = \frac{q}{p}(p-1).$$ □

Lemma 16.22. *For any $q<2$ there exists an $N<\infty$ such that if*

$$\|f\|_{L^\infty}, \|g\|_{L^\infty} \leqslant C,$$

then

$$\|fg\|_{W^{1,p}(\rho^{-1})} \lesssim \|f\|_{H^1(\rho)} \|g\|_{H^1(\rho^{-N})}.$$

Note that the L^∞ norms are unweighted. If furthermore if $f_n \to f$ in $H^1(\rho)$ and $g_n \to g$ in $H^1(\rho^{-N})$ and $\sup_n \|f_n\|_{L^\infty} + \|g_n\|_{L^\infty} \leqslant C$ then

$$f_n g_n \to fg \qquad \text{in } W^{1,q}(\rho^{-1}).$$

Proof. It is enough to prove the statement for smooth functions by density. By Hölder's inequality we have

$$\|fg\|_{L^p(\rho^{-1})} \leqslant \|f\|_{L^\infty} \|g\|_{L^2(\rho^{-N})}.$$

By the Leibniz rule

$$\|\nabla(fg)\|_{L^p(\rho^{-1})} \leqslant \|f\|_{L^\infty} \|\nabla g\|_{L^p(\rho^{-1})} + \|\nabla f\|_{L^2(\rho)} \|g\|_{L^p(\rho^{-\alpha N})},$$

provided that $\alpha N > 2$. By interpolation and provided that $\alpha/p = 1/2$, we also have

$$\|g\|_{L^p(\rho^{-\alpha N})} \leqslant \|g\|_{L^\infty}^{(1-\alpha)} \|g\|_{L^2(\rho^{-N})}^{\alpha} \leqslant C \|g\|_{L^2(\rho^{-N})}^{\alpha}.$$

Choosing N large enough gives the statement. For the second statement we rewrite

$$fg - f_n g_n = g(f - f_n) + f_n(g - g_n)$$

and proceed analogously. □

Chapter 17
A scale-by-scale interpolation

In this intermezzo we prepare for the analysis of the UV limit of the quartic interaction in three dimensions, i.e. for the φ^4_3 model in finite volume $\Lambda = \mathbb{T}^3$.

Recall that the φ^4_d Gibbs measure on the d-dimensional torus $\Lambda = \Lambda_L = \mathbb{T}^d_L = (\mathbb{R}/(2\pi L\mathbb{Z}))^d$ is the probability measure ν obtained as the weak limit for $T \to \infty$ of the family $(\nu_T)_{T>0}$ given by

$$\nu_T(\mathrm{d}\phi) = \frac{\exp[-V_T(\phi_T)]}{\mathscr{Z}_T}\vartheta(\mathrm{d}\phi), \tag{17.1}$$

where

$$V_T(\varphi) := \lambda\int_\Lambda (|\varphi(\xi)|^4 - a_T|\varphi(\xi)|^2 + b_T)\mathrm{d}\xi, \qquad \mathscr{Z}_T := \int e^{-V_T(\phi_T)}\vartheta(\mathrm{d}\phi).$$

Here $\lambda \geqslant 0$ is a fixed constant, Δ is the Laplacian on Λ, ϑ is the centred Gaussian measure with covariance $(1-\Delta)^{-1}$, $\mathscr{Z}_T$ is a normalisation factor, a_T, b_T given constants and $\phi_T = \rho_T * \phi$ with ρ_T some appropriate smooth and compactly supported cutoff function such that $\rho_T \to \delta$ as $T \to \infty$. The measures ϑ and ν_T are realised as probability measures on $\mathscr{S}'(\Lambda)$, the space of tempered distributions on Λ. They are supported on the Hölder–Besov space $\mathscr{C}^{(2-d)/2-\kappa}(\Lambda)$ for all small $\kappa > 0$. The existence of the limit ν is conditioned on the choice of a suitable sequence of *renormalisation constants* $(a_T, b_T)_{T>0}$. The constant b_T is not necessary, but is useful to decouple the behaviour of the numerator from that of the denominator in eq. (17.1).

The aim of the rest of these lecture notes is to give a proof of convergence using a variational formula for the partition function $\mathscr{Z}_T$ and for the generating function of the measure ν_T. As a byproduct we obtain also a variational description for the generating function of the limiting measure ν via Γ-convergence of the variational problem.

Define the normalised free energy $\mathcal{W}_T$ for the cutoff φ^4_d measure, as the functional

$$\mathcal{W}_T(f) := -\frac{1}{|\Lambda|}\log\int_{\mathscr{S}'(\Lambda)} \exp[-|\Lambda| f(\phi) - V_T(\phi_T)]\vartheta(\mathrm{d}\phi), \tag{17.2}$$

for all $f \in C(\mathscr{S}'(\Lambda); \mathbb{R})$.

In order to use the stochastic control problem we need to embed our problem into a setting where we have a Brownian motion. How to do this is quite arbitrary and mostly shaped by technical convenience. In two dimensions, where we have very mild singularities, we have adopted a simplistic approach, introducing a Brownian martingale with GFF marginals. To tackle the three dimensional case, where singularities are more severe, it is convenient to choose an embedding where the Brownian motion encodes a **scale decomposition** of the underlying GFF.

We begin by constructing a probability space $(\Omega, \mathscr{B}, \mathbb{P})$ endowed with a process $(W_t)_{t\in[0,\infty]}$ belonging to $C([0,\infty], \mathscr{C}^{(2-d)/2-\kappa}(\Lambda))$ and such that $\mathrm{Law}_\vartheta(\phi_T) = \mathrm{Law}_{\mathbb{P}}(W_T)$ for all $T \geqslant 0$ and $\mathrm{Law}_{\mathbb{P}}(W_\infty) = \vartheta$, the Gaussian free field with covariance $(1-\Delta)^{-1}$.

Fix $\alpha < -d/2$ and let $\Omega := C(\mathbb{R}_+; H^{-\alpha})$, $(X_t)_{t\geqslant 0}$ the canonical process on Ω and $\mathscr{B}$ the Borel σ–algebra of Ω. On $(\Omega, \mathscr{B})$ consider the probability measure $\mathbb{P}$ which makes the canonical process X a cylindrical Brownian motion in $L^2(\Lambda)$. In the following $\mathbb{E}$ without any qualifiers will denote expectations wrt. $\mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}$ will denote expectations wrt. some other measure $\mathbb{Q}$. On the probability space $(\Omega, \mathscr{B}, \mathbb{P})$ there exists a collection $(B^n_t)_{n\in(L^{-1}\mathbb{Z})^d}$ of complex (2-dimensional) Brownian motions, such that $\overline{B^n_t} = B^{-n}_t$, B^n_t, B^m_t independent for $m \neq \pm n$ and $X_t = |\Lambda|^{-1/2}\sum_{n\in(L^{-1}\mathbb{Z})^d} e^{i\langle n,\cdot\rangle}B^n_t$.

Note that X has a.s. trajectories in $C(\mathbb{R}_+, \mathscr{C}^{-d/2-\varepsilon}(\Lambda))$ for any $\varepsilon > 0$ by standard arguments.

Fix some $\rho \in C_c^\infty(\mathbb{R}_+, \mathbb{R}_+)$, decreasing such that $\rho(s) = 1$ for any $s \leqslant 1/2$ and $\rho(s) = 0$ for any $s \geqslant 1$. For $x \in \mathbb{R}^d$, set, $\rho_t(x) := \rho(\langle x \rangle / t)$ and

$$\sigma_t(x) := \left(\frac{\mathrm{d}}{\mathrm{d}t}(\rho_t^2(x)) \right)^{1/2} = (2\dot{\rho}_t(x)\rho_t(x))^{1/2} = (-2(\langle x \rangle / t)\rho(\langle x \rangle / t)\rho'(\langle x \rangle / t))^{1/2} / t^{1/2},$$

where $\dot{\rho}_t$ is the partial derivative of ρ_t with respect to t. Consider the process $(W_t)_{t \geqslant 0}$ defined by

$$W_t := \frac{1}{|\Lambda|^{1/2}} \sum_{n \in (L^{-1}\mathbb{Z})^d} \int_0^t \frac{\sigma_s(n)}{\langle n \rangle} e^{i\langle n, \cdot \rangle} \mathrm{d}B_s^n, \qquad t \geqslant 0. \tag{17.3}$$

It is a centred Gaussian process with covariance

$$\begin{aligned} \mathbb{E}[\langle W_t, \varphi \rangle \langle W_s, \psi \rangle] &= \frac{1}{|\Lambda|} \sum_{n,m \in (L^{-1}\mathbb{Z})^d} \mathbb{E}\left[\int_0^t \frac{\sigma_u(n)}{\langle n \rangle} \mathrm{d}B_u^n \, \hat{\varphi}(n) \overline{\int_0^s \frac{\sigma_u(m)}{\langle m \rangle} \mathrm{d}B_s^m \, \hat{\psi}(m)} \right] \\ &= \frac{1}{|\Lambda|} \sum_{n \in (L^{-1}\mathbb{Z})^d} \frac{\rho^2_{\min(s,t)}(n)}{\langle n \rangle^2} \hat{\varphi}(n) \overline{\hat{\psi}(n)}, \end{aligned}$$

for any $\varphi, \psi \in \mathscr{S}(\Lambda)$ and $t, s \geqslant 0$, by Fubini theorem and Itô isometry. By dominated convergence

$$\lim_{t \to \infty} \mathbb{E}[\langle W_t, \varphi \rangle \langle W_t, \psi \rangle] = |\Lambda|^{-1} \sum_{n \in (L^{-1}\mathbb{Z})^d} \langle n \rangle^{-2} \hat{\varphi}(n) \overline{\hat{\psi}(n)}$$

for any $\varphi, \psi \in L^2(\Lambda)$.

Note that up to any finite time T the r.v. W_T has a bounded spectral support and the stopped process $W_t^T = W_{t \wedge T}$ for any fixed $T > 0$, is in $C(\mathbb{R}_+, W^{k,2}(\Lambda))$ for any $k \in \mathbb{N}$. Furthermore $(W_t^T)_t$ only depends on a finite subset of the Brownian motions $(B^n)_n$. Denote

$$W_t = \int_0^t J_s \mathrm{d}X_s, \qquad t \geqslant 0, \tag{17.4}$$

with $J_s := \langle \mathrm{D} \rangle^{-1} \sigma_s(\mathrm{D})$.

Observe that W_t has a distribution given by the push-forward $(\rho_t(\mathrm{D}))_* \vartheta$ of ϑ through $\rho_t(\mathrm{D})$. We write the measure ν_T in (17.1) in terms of expectations over $\mathbb{P}$ as

$$\int g(\phi) \nu_T(\mathrm{d}\phi) = \frac{\mathbb{E}[g(W_T)\, e^{-V_T(W_T)}]}{\mathscr{Z}_T}, \tag{17.5}$$

for any bounded measurable $g \colon \mathscr{S}'(\Lambda) \to \mathbb{R}$.

For fixed T the polynomial appearing in the expression for $V_T(W_T)$ is bounded below (since $\lambda > 0$) and $\mathscr{Z}_T$ is well defined and also bounded away from zero (this follows easily from Jensen's inequality). However as $T \to \infty$ we tend to loose both these properties due to the fact that we will be obliged to take $a_T \to +\infty$ to renormalise the non–linear terms. To obtain uniform upper and lower bounds we need a more detailed analysis and we proceed as follows.

Denote by $\mathbb{H}_a$ the space of progressively measurable processes which are $\mathbb{P}$–almost surely in $\mathcal{H} := L^2(\mathbb{R}_+ \times \Lambda)$. We say that an element v of $\mathbb{H}_a$ is a *drift*. Below we will need also (generalised) drifts belonging to $\mathcal{H}^\alpha := L^2(\mathbb{R}_+; H^\alpha(\Lambda))$ for some $\alpha \in \mathbb{R}$, we denote the corresponding space with $\mathbb{H}_a^\alpha$. Consider the measure $\mathbb{Q}_T$ on $(\Omega, \mathscr{B})$ whose Radon–Nykodim derivative wrt. $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \frac{e^{-V_T(W_T)}}{\mathscr{Z}_T}.$$

The Boué–Dupuis formula takes the form:

Theorem 17.1. *For any $f \in C(\mathscr{C}^{-1/2-\kappa};\mathbb{R})$ with linear growth the Boué–Dupuis (BD) variational formula for the free energy holds:*

$$\mathcal{W}_T(f) = -\frac{1}{|\Lambda|}\log \mathbb{E}\left[e^{-V_T^f(W_T)}\right] = \inf_{v\in \mathbb{H}_a} \mathbb{E}\left[\frac{1}{|\Lambda|}V_T^f(W_T + I_T(v)) + \frac{1}{2}\int_0^\infty \|v_s\|_{L^2}^2 \mathrm{d}s\right].$$

where the expectation is taken wrt to the measure $\mathbb{P}$ on Ω.

The next lemma provides a deterministic regularity result for $I(v)$: it says that the drift v generates shifts of the Gaussian free field in directions which belong to H^1 uniformly in the scale parameter up to ∞. The space H^1 is the Cameron–Martin space of the free field [?].

Lemma 17.2. *Let $\alpha \in \mathbb{R}$. For any $v \in L^2([0,\infty), H^\alpha)$ we have*

$$\sup_{0\leqslant t\leqslant T} \|I_t(v)\|_{H^{\alpha+1}}^2 + \sup_{0\leqslant s<t\leqslant T} \frac{\|I_t(v) - I_s(v)\|_{H^{\alpha+1}}^2}{1\wedge(t-s)} \lesssim \int_0^T \|v_r\|_{H^\alpha}^2 \mathrm{d}r.$$

Proof. Using the fact that $\sigma_s(\mathrm{D})$ is diagonal in Fourier space, and denoting with $(e_k)_{k\in(L^{-1}\mathbb{Z})^d}$ the basis of trigonometric polynomials, we have

$$\begin{aligned}
\left\|\int_r^t \sigma_s(\mathrm{D})v_s \mathrm{d}s\right\|_{H^\alpha}^2 &= \frac{1}{|\Lambda|}\sum_{k\in(L^{-1}\mathbb{Z})^d} \langle k\rangle^{2\alpha}\left|\int_r^t \langle \sigma_s(\mathrm{D})e_k, v_s\rangle \mathrm{d}s\right|^2 \\
&\leqslant \frac{1}{|\Lambda|}\sum_{k\in(L^{-1}\mathbb{Z})^d} \langle k\rangle^{2\alpha}\left(\int_r^t |\langle \sigma_s(\mathrm{D})e_k, e_k\rangle|^2 \mathrm{d}s\right)\left(\int_r^t |\langle e_k, v_s\rangle|^2 \mathrm{d}s\right) \\
&\leqslant \int_r^t \|v_s\|_{H^\alpha}^2 \mathrm{d}s \sup_k \int_r^t \langle e_k, \sigma_s(\mathrm{D})^2 e_k\rangle \mathrm{d}s \\
&\leqslant \int_r^t \|v_s\|_{H^\alpha}^2 \mathrm{d}s \sup_k \langle e_k, \rho_t^2(\mathrm{D})e_k\rangle \leqslant \int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.
\end{aligned}$$

Which is the second statement. On the other hand $\sigma_s(\mathrm{D})$ is a smooth Fourier multiplier and using Proposition 13.10 we have the estimate $\|\sigma_s(\mathrm{D})f\|_{H^\alpha} \lesssim \|f\|_{H^\alpha}/\langle s\rangle^{1/2}$ uniformly in $s \geqslant 0$, therefore, for all $0 \leqslant r \leqslant t \leqslant T$, we have

$$\begin{aligned}
\left\|\int_r^t \sigma_s(\mathrm{D})v_s \mathrm{d}s\right\|_{H^\alpha}^2 &\leqslant \left(\int_r^t \|\sigma_s(\mathrm{D})v_s\|_{H^\alpha}\mathrm{d}s\right)^2 \leqslant (t-r)\int_r^t \|\sigma_s(\mathrm{D})v_s\|_{H^\alpha}^2 \mathrm{d}s \\
&\lesssim (t-r)\int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.
\end{aligned}$$

We conclude that

$$\|\, I_t(v) - I_r(v)\|_{H^{\alpha+1}}^2 \lesssim \left\|\int_r^t \sigma_s(\mathrm{D})v_s \mathrm{d}s\right\|_{H^\alpha}^2 \leqslant [1\wedge(t-r)]\int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.$$

□

17.1 Two dimensions

We consider first the case $d=2$ assuming that the test functional f satisfy the following estimate: for sufficiently small $\alpha > 0$

$$|f(\varphi)| \leqslant C + \alpha\left\|\varphi\right\|_{C^{-\kappa}}^4. \tag{17.6}$$

From Theorem 17.1 we see that the relevant quantity to bound is of the form

$$F_T(u) := \mathbb{E}\left[\frac{1}{|\Lambda|}f(W_T + I_T(u)) + \frac{1}{|\Lambda|}V_T(W_T + I_T(u)) + \frac{1}{2}\|u\|_{\mathcal{H}}^2\right], \tag{17.7}$$

for $u \in \mathbb{H}_a$. From now on we leave implicit the integration variable over the spatial domain Λ and let $Z_t = I_t(u)$ for brevity. Choosing

$$a_T = 6\mathbb{E}[W_T(0)^2], \qquad b_T = 3\mathbb{E}[W_T(0)^2]^2, \tag{17.8}$$

we have

$$\frac{1}{|\Lambda|} V_T(W_T + Z_T) = \lambda \fint [\![W_T^4]\!] + 4\lambda \fint [\![W_T^3]\!] Z_T + 6\lambda \fint [\![W_T^2]\!] Z_T^2 + 4\lambda \fint W_T Z_T^3 + \lambda \fint Z_T^4,$$

where

$$\begin{aligned} [\![W_T^4]\!] &:= W_T^4 - 6\mathbb{E}[W_T^2]\, W_T^2 + 3\mathbb{E}[W_T^2]^2, \\ [\![W_T^3]\!] &:= W_T^3 - 3\mathbb{E}[W_T^2] W_T, \\ [\![W_T^2]\!] &:= W_T^2 - \mathbb{E}[W_T^2], \end{aligned}$$

denote the Wick powers of the Gaussian r.v. W_T [?].

These polynomials, when seen as stochastic processes in T, are $\mathbb{P}$–martingales wrt. the filtration of $(W_t)_t$:

$$[\![W_t^n]\!] = n \int_0^t [\![W_s^{n-1}]\!] \mathrm{d}W_s, \qquad n \geqslant 1.$$

In particular they have an expression as iterated stochastic integrals wrt. the Brownian motions $(B_t^n)_{t,n}$ introduced in eq. (17.3).

We will need to estimate the average size of the random fields $[\![W_T^n]\!]$ uniformly in T. Our computations in Section 13.3 show that for all $\kappa > 0$ we have, for any $N \geqslant 1$ and any $n \geqslant 1$,

$$\sup_T \mathbb{E} \big\| [\![W_T^n]\!] \big\|_{C^{-\kappa}}^N < +\infty. \tag{17.9}$$

Using Theorem 17.1 with $u = 0$ we readily have an upper bound for the free energy:

$$-\frac{1}{|\Lambda|} \log \mathscr{Z}_T \leqslant \lambda \mathbb{E}\left[\fint [\![W_T^4]\!] \right] = 0.$$

For a lower bound we need to estimate from below the average under $\mathbb{P}$ of the variational expression

$$\lambda \fint [\![W_T^4]\!] + 4\lambda \fint [\![W_T^3]\!] Z_T + 6\lambda \fint [\![W_T^2]\!] Z_T^2 + 4\lambda \fint W_T Z_T^3 + \lambda \fint Z_T^4 + \frac{1}{2} \|u\|_{\mathcal{H}}^2.$$

The strategy we adopt is to bound *path-wise*, and for a generic drift u, the contributions

$$\Phi_T(Z) := \underbrace{4\lambda \fint [\![W_T^3]\!] Z_T}_{\mathrm{I}} + \underbrace{6\lambda \fint [\![W_T^2]\!] Z_T^2}_{\mathrm{II}} + \underbrace{4\lambda \fint W_T Z_T^3}_{\mathrm{III}},$$

in term of quantities involving only the Wick powers of W which we can control in expectation and the last two *positive terms* (our "good terms")

$$\frac{1}{2} \|u\|_{\mathcal{H}}^2 + \lambda \fint Z_T^4.$$

Any residual positive contribution depending on u can be dropped in the lower bound making the dependence on the drift disappear.

To control term I we see that by duality and Young's inequality, for any $\delta > 0$,

$$\left| \fint [\![W_T^3]\!] Z_T \right| \leqslant \left\| [\![W_T^3]\!] \right\|_{C^{-\kappa}} \|Z_T\|_{B_{1,1}^\kappa} \leqslant C_\delta \left\| [\![W_T^3]\!] \right\|_{C^{-\kappa}}^2 + \delta \|Z_T\|_{H^1}^2. \tag{17.10}$$

where we used that $\|Z_T\|_{B_{1,1}^\kappa} \lesssim \|Z_T\|_{H^1}$ and Young's inequality to have a small factor $0 < \delta \ll 1$ in front of $\|Z_T\|_{H^1}^2$.

The other terms can be estimates with similar arguments, for example it will be useful to use interpolation and the following fractional Leibniz rule:

Proposition 17.3. *Let $1 < p < \infty$ and $p_1, p_2, p_1', p_2' > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p}$. Then for every $s, \alpha \geqslant 0$ there exists a constant C such that*

$$\|\langle \mathrm{D}\rangle^s (fg)\|_{L^p} \leqslant C\,\|\langle \mathrm{D}\rangle^{s+\alpha} f\|_{L^{p_2}}\,\|\langle \mathrm{D}\rangle^{-\alpha} g\|_{L^{p_1}} + C\,\|\langle \mathrm{D}\rangle^{s+\alpha} g\|_{L^{p_1'}}\,\|\langle \mathrm{D}\rangle^{-\alpha} f\|_{L^{p_2'}}.$$

The end product of these analytic bounds are estimates

$$|\Phi_T(Z)| \leqslant Q_T + \delta\Big[\frac{1}{2}\|u\|_{\mathcal{H}}^2 + \lambda \int Z_T^4\Big], \tag{17.11}$$

where

$$Q_T = O(\lambda^2)[1 + \|[\![W_T^3]\!]\|_{C^{-\varepsilon}} + \|[\![W_T^2]\!]\|_{C^{-\varepsilon}} + \|W_T\|_{C^{-\varepsilon}}]^K$$

for some integer $K \geqslant 1$. Therefore

$$F_T(u) \geqslant -\mathbb{E}[Q_T] + (1-\delta)\Big[\frac{1}{2}\|u\|_{\mathcal{H}}^2 + \lambda \int Z_T^4\Big] \geqslant -\mathbb{E}[Q_T].$$

This last average does not depend anymore on the drift and using (17.9) we end up with

$$\sup_T \mathbb{E}[Q_T] < \infty.$$

We have established that

$$\sup_T \mathbb{E}\,[e^{f(W_T) - V_T(W_T)}] \leqslant \sup_T \exp(\mathbb{E}[Q_T]) < +\infty \tag{17.12}$$

which in particular is enough to establish tightness for the family of measures $(\nu^T)_T$, as

$$\sup_T \int e^{f(\varphi)} \nu^T(\mathrm{d}\varphi) = \sup_T \frac{\mathbb{E}\,[e^{f(W_T) - V_T(W_T)}]}{\mathbb{E}\,[e^{-V_T(W_T)}]} \leqslant \sup_T \frac{\mathbb{E}\,[e^{f(W_T) - V_T(W_T)}]}{e^{-\mathbb{E}V_T(W_T)}} < +\infty.$$

We can also observe that the argument leading to (17.12) remains valid upon replacing V_T by pV_T for any $p \geqslant 1$. This implies that $e^{-V_T(W_T)}$ is in all the L^p spaces wrt. the measure $\mathbb{P}$ uniformly in T and for any $p \geqslant 1$. Together with the a.s. convergence of the martingale $(V_T(W_T))_T$ (which is bounded in all L^p spaces), this allows to conclude also the existence of the limit measure

$$\nu^\infty = \lim_T \nu^T,$$

such that

$$\frac{\mathrm{d}\nu^\infty}{\mathrm{d}\mu} = e^{-V_\infty(W_\infty)}$$

where $V_\infty(W_\infty) = \lim_T V_T(W_T)$.

Chapter 18
The quartic interaction in three dimensions

For the quartic interaction in three dimensions the strategy we used in two dimensions fails: the Wick powers are less regular: $[\![W_T^2]\!] \in \mathscr{C}^{-1-\kappa}$ uniformly in T for any small $\kappa > 0$ and $[\![W_T^3]\!]$ does not allow uniform bounds as a well-defined random distribution.

By this we mean that, for any $\alpha > 0$, the only bound we can find looks like

$$\mathbb{E}\,\|[\![W_T^3]\!]\|^N_{B^{-\alpha}_{\infty,\infty}} \lesssim \log T.$$

As a consequence there is no straightforward approach to control the terms

$$\int [\![W_T^3]\!] Z_T, \qquad \text{and} \qquad \int [\![W_T^2]\!] Z_T^2, \tag{18.1}$$

like we did in two dimensions.

The only a-priori estimate on the regularity of $Z_T = I_T(u)$ is in H^1, coming from Lemma 17.2 and the quadratic term in the variational functional $F_T(u)$.

An additional substantial difference is that we expect further divergences beyond the Wick ordering which have to be subtracted in order for the limiting measure to be non-trivial. For these reasons in the energy V_T we introduce further scale dependent renormalisation constants γ_T, δ_T to have

$$\frac{1}{|\Lambda|} V_T^f(Y_T) = f(Y_T) + \int (\lambda [\![Y_T^4]\!] - \lambda^2 \gamma_T [\![Y_T^2]\!] - \delta_T). \tag{18.2}$$

where we Wick products $[\![Y_T^4]\!], [\![Y_T^2]\!]$ are taken with respect to the variance of W_T (as opposed to Y_T).

Repeating the computation from Section 17.1 we arrive at

$$\begin{aligned} F_T(u) \;=\; & \mathbb{E}\Big[f(W_T + Z_T) + \lambda \int \mathbb{W}_T^3 Z_T + \frac{\lambda}{2} \int \mathbb{W}_T^2 Z_T^2 + 4\lambda \int W_T Z_T^3 \Big] \\ & - \mathbb{E}\Big[2\lambda^2 \gamma_T \int W_T Z_T + \lambda^2 \gamma_T \int Z_T^2 + \delta_T \Big] + \mathbb{E}\Big[\lambda \int Z_T^4 + \frac{1}{2} \|u\|^2_{\mathcal{H}} \Big]. \end{aligned} \tag{18.3}$$

where we introduced the convenient notations

$$\mathbb{W}_t^3 := 4[\![W_t^3]\!], \qquad \mathbb{W}_t^2 := 12[\![W_t^2]\!], \qquad t \geqslant 0,$$

and we recall that f is a fixed function belonging to $C(\mathscr{C}^{-1/2-\kappa}; \mathbb{R})$ with linear growth.

We will perform a change of variables in the variational functional in order to obtain some explicit cancellations which will leave only quantities well behaved as $T \to \infty$. The main drawback is that the functional will have a less compact and canonical form.

We are going to absorb the mixed terms (18.1) via the quadratic cost function $\frac{1}{2}\|u\|^2_{\mathcal{H}}$. For the first we have, via Ito formula and then completing the square

$$\lambda \fint \mathbb{W}_T^3 Z_T + \frac{1}{2}\int_0^\infty \|u_s\|^2_{L^2}\mathrm{d}s = \lambda \int_0^T \fint \mathbb{W}_t^3 \dot{Z}_t \mathrm{d}t + \frac{1}{2}\int_0^\infty \|u_s\|^2_{L^2}\mathrm{d}s + \text{martingale}$$

$$= -\frac{\lambda^2}{2}\int_0^T \fint (\mathbb{W}_t^{\langle 3\rangle})^2 \mathrm{d}t + \frac{1}{2}\int_0^\infty \|w_s\|^2_{L^2}\mathrm{d}s + \text{martingale},$$

where we used that J_t is self-adjoint and let

$$w_t := u_t + \lambda \mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle}, \qquad t \geqslant 0. \tag{18.4}$$

Taking into account (here and below) that the martingale term will average to zero, we have replaced the divergent term $\fint \mathbb{W}_T^3 Z_T$ with a divergent but purely stochastic term $\int_0^T \fint (\mathbb{W}_t^{\langle 3\rangle})^2 \mathrm{d}t$ which does not affect anymore the variational problem and can be explicitly removed by adding its average to δ_T.

The price to pay for this change of variables is that now the cost functional is relative to w and not u anymore. This is an important feature of the variational approach in three dimensions as we do not expect anymore that the limiting measure is absolutely continuous wrt. the GFF.

Defining

$$K_t := I_t(w), \qquad \mathbb{W}_T^{[3]} := \int_0^T J_s \mathbb{W}_s^{\langle 3\rangle}\mathrm{d}s$$

from (18.4) we deduce the relation

$$Z_T = -\lambda \mathbb{W}_T^{[3]} + K_T,$$

which can be used to expand the second mixed divergent term in (18.1) as

$$\frac{\lambda}{2}\fint \mathbb{W}_T^2 Z_T^2 = \frac{\lambda^3}{2}\fint \mathbb{W}_T^2 (\mathbb{W}_T^{[3]})^2 - \lambda^2 \fint \mathbb{W}_T^2 \mathbb{W}_T^{[3]} K_T + \frac{\lambda}{2}\fint \mathbb{W}_T^2 K_T^2. \tag{18.5}$$

The first term on the r.h.s. a purely stochastic object and will give a contribution independent of the drift u and absorbed in δ_T.

We are still not done since this operation has left two new divergent terms on the r.h.s. of eq. (18.5): the H^1 regularity of K_T (guaranteed by the cost functional $\frac{1}{2}\|w\|^2_{\mathcal{H}}$) is not enough to control the products with $\mathbb{W}^2$ which has regularity $\mathscr{C}^{-1-\kappa}$, a bit below -1.

In order to proceed further we will isolate the divergent parts of these products via a *paraproduct* decomposition.

Let us pause for a moment in order to introduce **paraproducts**. The basic idea is that a product of Besov functions f, g naturally decomposes in various contributions involving products of LP blocks:

$$fg = \sum_{i,j} \Delta_i f \, \Delta_j g = \sum_{i\lesssim j} \Delta_i f \, \Delta_j g + \sum_{i\gtrsim j} \Delta_i f \, \Delta_j g + \sum_{i\approx j} \Delta_i f \, \Delta_j g$$

where we split contributions above and below a thick diagonal of the double sum. The key observation is that if $i \ll j$ then the product $\Delta_i f \Delta_j g$ has Fourier transform supported in an annulus at scale 2^j, while if $i \approx j$ then the same product is a-priori only spectrally supported in a ball of radius $\approx 2^j$. These different spectral support imply a different behaviour of the bilinear maps:

$$f \prec g = \sum_{i\lesssim j} \Delta_i f \, \Delta_j g$$

and

$$f \circ g = \sum_{i \approx j} \Delta_i f \Delta_j g.$$

Definition 18.1. *Let $f, g \in \mathscr{S}(\Lambda)$. We define the paraproducts and resonant product*

$$f \succ g = g \prec f := \sum_{j<i-1} \Delta_i f \Delta_j g, \qquad \text{and} \qquad f \circ g := \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g.$$

Then

$$fg = f \prec g + f \circ g + f \succ g.$$

The basic analytic bounds for the paraproduct and resonant terms are

Proposition 18.2. *For any polynomial weight ρ, $\beta \leqslant 0, \alpha \in \mathbb{R}$ and $p_1, p_2 \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ we have the estimate*

$$\| f \succ g \|_{B^{\alpha+\beta}_{p,q}(\rho)} \lesssim \| f \|_{B^{\alpha}_{p_1,\infty}(\rho)} \| g \|_{B^{\beta}_{p_2,q}(\rho)}.$$

For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$ the estimate

$$\| f \circ g \|_{B^{\alpha+\beta}_{p,q}(\rho)} \lesssim \| f \|_{B^{\alpha}_{p_1,\infty}(\rho)} \| g \|_{B^{\beta}_{p_2,q}(\rho)}.$$

For a proof see Theorem 3.17 and Remark 3.18 in [?].

Proposition 18.3. *For any polynomial weights ν, ρ and $\beta \leqslant 0, \alpha \in \mathbb{R}$ we have*

$$\| f \succ g \|_{B^{\alpha+\beta}_{p,q}(\rho^p \nu)} \lesssim \| f \|_{\mathscr{C}^{\alpha}(\rho)} \| g \|_{B^{\beta}_{p,q}(\nu)}.$$

The proof is an easy modification of the proof of Theorem 3.17 in [?].

Proposition 18.4. *Assume $m \leqslant 0$, $\alpha \in (0,1), \beta \in \mathbb{R}$. Let η be an S^m multiplier and $q, p_1, p_2 \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $f \in B^{\beta}_{p_1,\infty}$, $g \in B^{\alpha}_{p_1,\infty}$. Then for any $\delta > 0$.*

$$\| \eta(\mathrm{D})(f \succ g) - (\eta(\mathrm{D}) f \succ g) \|_{B^{\alpha+\beta-m-\delta}_{p,q}} \lesssim \| f \|_{B^{\beta}_{p_1,\infty}} \| g \|_{B^{\alpha}_{p_1,\infty}}.$$

The constant depends only on α, β, δ and the constants in (13.3).

For a proof see [?] Lemma 2.99.

To continue our algebraic analysis we expand

$$\begin{aligned} -\lambda^2 \int \mathbb{W}_T^2 \mathbb{W}_T^{[3]} K_T + \frac{\lambda}{2} \int \mathbb{W}_T^2 K_T^2 \;=\; & \lambda \int (\mathbb{W}_T^2 \succ Z_T) K_T - \lambda^2 \int (\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}) K_T \\ & - \lambda^2 \int (\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]}) K_T + \frac{\lambda}{2} \int (\mathbb{W}_T^2 \prec K_T) K_T \\ & + \frac{\lambda}{2} \Big(\int (\mathbb{W}_T^2 \circ K_T) K_T - \int (\mathbb{W}_T^2 \succ K_T) K_T \Big). \end{aligned}$$

The first two terms will require renormalisations which we put in place below. All the other terms will be well behaved and we collect them in $\Upsilon_T^{(1)}$. In particular we observe that the last one can be rewritten as

$$\frac{\lambda}{2} \Big(\int (\mathbb{W}_T^2 \circ K_T) K_T - \int (\mathbb{W}_T^2 \succ K_T) K_T \Big) = -\frac{\lambda}{2} \mathfrak{K}_2(\mathbb{W}_T^2, K_T, K_T)$$

introducing the trilinear form $\mathfrak{K}_2$ whose properties are detailed in Proposition ? below.

As we anticipated, the resonant term $\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}$ requires renormalisation. In the expression of F_T in (18.3) we have the counterterm $-2\lambda^2\gamma_T \fint W_T Z_T$ available, which we put now in use writing

$$-\lambda^2 \fint (\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}) K_T - 2\lambda^2\gamma_T \fint W_T Z_T = -\lambda^2 \fint \underbrace{(\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} + 2\gamma_T W_T)}_{\mathbb{W}_T^{2\circ[3]}} K_T + 2\lambda^3\gamma_T \fint W_T \mathbb{W}_T^{[3]}.$$

The first contribution is collected in $\Upsilon_T^{(6)}$ and the expectation of the second will contribute to δ_T.

As far as the term $\lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T$ is concerned, we want to absorb it into $\int \|w_s\|^2 \mathrm{d}s$. Before we can do this we must be sure that, after applying Itô's formula, it will be still possible to use $\fint Z_T^4$ to control some of the growth of this term. Indeed the quadratic dependence in K_T (via Z_T) cannot be fully taken care of by the quadratic cost $\int \|w_s\|^2 \mathrm{d}s$.

We need to introduce a regularisation which make compatible Fourier cutoffs with L^4 estimates. To introduce such a regularisation fix smooth functions $\tilde{\theta}, \eta : \mathbb{R}^d \to \mathbb{R}_+$ such that $\tilde{\theta}(\xi) = 1$ if $|\xi| \leqslant 1/4$ and $\tilde{\theta}(\xi) = 0$ if $|\xi| \geqslant 1/3$, $\eta(\xi) = 1$ if $|\xi| \leqslant 1$ and $\eta(\xi) = 0$ if $|\xi| \geqslant 2$. Set $\tilde{\theta}_t(\xi) := \tilde{\theta}(\xi/t)$, then define

$$\theta_t(\xi) = (1 - \eta(\xi))\tilde{\theta}_t(\xi) + \zeta(t)\eta(\xi)\tilde{\theta}_t(\xi)$$

where $\zeta(t) : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function such that $\zeta(t) = 0$ for $t \leqslant 10$ and $\zeta(t) = 1$ for $t \geqslant 11$.

$$\begin{array}{lcl} \theta_t(\xi)\sigma_s(\xi) & = & 0 \text{ for } s \geqslant t, \\ \theta_t(\xi) & = & 1 \text{ for } |\xi| \leqslant ct \text{ for some } c > 0 \text{ provided that } t \geqslant 11. \end{array} \tag{18.6}$$

By the Mihlin-Hörmander theorem, the operator $\theta_t = \theta_t(\mathrm{D})$ is bounded on L^p for any $1 < p < \infty$.

In the following, for any $f \in C([0,\infty], \mathscr{S}'(\Lambda))$ we define $f_t^\flat := \theta_t f_t$ then

$$Z_t^\flat = \theta_t Z_t = \int_0^t \theta_t \langle \mathrm{D} \rangle^{-1} \sigma_s(\mathrm{D}) u_s \, \mathrm{d}s = \int_0^T \theta_t \langle \mathrm{D} \rangle^{-1} \sigma_s(\mathrm{D}) u_s \, \mathrm{d}s = \theta_t Z_T.$$

In this way we have $\|Z_t^\flat\|_{L^p} \lesssim \|Z_T\|_{L^p}$ for all $t \leqslant T$. In the sequel we will always assume $T \gg 1$.

Now decompose

$$\lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T = \lambda \fint (\mathbb{W}_T^2 \succ Z_T^\flat) K_T + \lambda \fint (\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat)) K_T$$

and use the fact that the functions $Z_T - Z_T^\flat$ and $K_T - K_T^\flat$ are spectrally supported outside of a ball or radius cT to show that the second term is nice enough as $T \to \infty$ to not require further analysis and we collect it in $\Upsilon_T^{(2)}$.

For the first we apply Itô's formula to decompose it along the flow of scales as

$$\lambda \fint (\mathbb{W}_T^2 \succ Z_T^\flat) K_T = \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t + \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat) K_t \mathrm{d}t + \text{martingale}.$$

The second term will be fine and we collect it in $\Upsilon_T^{(3)}$.

We are left with the singular term $\int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t$. We again absorb it by completing the square together with the cost $\frac{1}{2}\|w\|_{\mathcal{H}}^2$:

$$\begin{aligned} \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t + \frac{1}{2}\int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t &= -\frac{\lambda^2}{2}\int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))^2 \mathrm{d}t + \frac{1}{2}\int_0^\infty \|l_t\|_{L^2}^2 \mathrm{d}t \\ &= -\frac{\lambda^2}{2}\int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))(J_t(\mathbb{W}_t^2 \succ Z_t^\flat)) \mathrm{d}t + \frac{1}{2}\|l\|_{\mathcal{H}}^2 \end{aligned} \tag{18.7}$$

where we introduced

$$l_t := u_t + \lambda \mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle} + \lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ Z_t^\flat). \tag{18.8}$$

To renormalise the first term on the r.h.s. of (18.7) we observe that the remaining counterterm can be rewritten as

$$-\lambda^2\gamma_T \fint Z_T^2 = -\lambda^2\gamma_T \fint (Z_T^\flat)^2 - 2\lambda^2\gamma_T \fint Z_T^\flat (Z_T - Z_T^\flat) - \lambda^2\gamma_T \fint (Z_T - Z_T^\flat)^2. \tag{18.9}$$

Differentiating in T the first term in the r.h.s. of eq. (18.9) we get

$$-\lambda^2\gamma_T \fint (Z_T^\flat)^2 = -\lambda^2 \int_0^T \fint \dot\gamma_t (Z_t^\flat)^2 \mathrm{d}t - 2\lambda^2 \int_0^T \fint \gamma_t Z_t^\flat \dot Z_t^\flat \mathrm{d}t. \tag{18.10}$$

The last term in eq. (18.10) and the last two contributions in (18.9) are collected in $\Upsilon_T^{(5)}$. The first contribution in eq. (18.10) has the right form to be used as a counterterm for the resonant product in (18.7).

Using the commutator $\mathfrak{K}_{3,t}$ described in Proposition ? below we have

$$\begin{aligned} &-\frac{\lambda^2}{2}\int_0^T \fint [(J_t(\mathbb{W}_t^2 \succ Z_t^\flat))(J_t(\mathbb{W}_t^2 \succ Z_t^\flat)) + 2\dot\gamma_t (Z_t^\flat)^2] \mathrm{d}t \\ =&-\frac{\lambda^2}{2}\int_0^T \fint \underbrace{[(J_t\mathbb{W}_t^2)\circ(J_t\mathbb{W}_t^2) + 2\dot\gamma_t]}_{\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}} (Z_t^\flat)^2 \mathrm{d}t - \frac{\lambda^2}{2}\int_0^T \mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, Z_t^\flat)\mathrm{d}t \end{aligned}$$

and we collect both terms in $\Upsilon_T^{(6)}$.

Finally, we are left with the cubic term which we rewrite as

$$4\lambda \fint W_T Z_T^3 = -4\lambda^4 \fint W_T (\mathbb{W}_T^{[3]})^3 + 12\lambda^3 \fint W_T (\mathbb{W}_T^{[3]})^2 K_T - 12\lambda^2 \fint W_T \mathbb{W}_T^{[3]} K_T^2 + 4\lambda \fint W_T K_T^3.$$

The average of the first term is collected in δ_T while all the remaining terms in $\Upsilon_T^{(4)}$. At last we have established the claimed decomposition since the residual cost functional, from eq. (18.7) is indeed $\|l\|_{\mathcal{H}}^2/2$.

Collecting all the terms, we proved the following

Lemma 18.5. *The functional $F_T(u)$ defined in eq. (18.3) takes the form*

$$F_T(u) \;=\; \mathbb{E}\Big[\Phi_T(\mathbb{W}, Z, K) + \lambda \fint (Z_T)^4 + \frac{1}{2}\|l\|_{\mathcal{H}}^2\Big],$$

where

$$w_t := u_t + \lambda \mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle}, \quad Z_t = I_t(u), \quad K_t = I_t(w),$$

$$l_t := u_t + \lambda \mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle} + \lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ Z_t^\flat),$$

while the stochastic objects are collected in the vector $\mathbb{W}$ defined as

$$\mathbb{W} := (\mathbb{W}^1, \mathbb{W}^2, \mathbb{W}^{\langle 3\rangle}, \mathbb{W}^{[3]\circ 1}, \mathbb{W}^{2\diamond[3]}, \mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}),$$

with $\mathbb{W}^1 := W$,

$$\mathbb{W}_t^{\langle 3\rangle} := J_t \mathbb{W}_t^3, \quad \mathbb{W}_t^{[3]} := \int_0^t J_s \mathbb{W}_s^{\langle 3\rangle} \mathrm{d}s, \quad \mathbb{W}_t^{[3]\circ 1} := \mathbb{W}_t^1 \circ \mathbb{W}_t^{[3]},$$

$$\mathbb{W}_t^{2\diamond[3]} := \mathbb{W}_t^2 \circ \mathbb{W}_t^{[3]} + 2\gamma_t \mathbb{W}_t^1, \quad \mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle} := (J_t\mathbb{W}_t^2)\circ(J_t\mathbb{W}_t^2) + 2\dot\gamma_t.$$

and moreover the potential has the explicit form

$$\Phi_T(\mathbb{W}, Z, K) := f(W_T + Z_T) + \sum_{i=1}^6 \Upsilon^{(i)},$$

with

$$\begin{aligned}
\Upsilon_T^{(1)} &:= -\frac{\lambda}{2}\mathfrak{K}_2(\mathbb{W}_T^2, K_T, K_T) + \frac{\lambda}{2}\fint(\mathbb{W}_T^2 \prec K_T)K_T - \lambda^2\fint(\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]})K_T,\\
\Upsilon_T^{(2)} &:= \lambda\fint(\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat))K_T,\\
\Upsilon_T^{(3)} &:= \lambda\int_0^T\fint(\mathbb{W}_t^2 \succ \dot{Z}_t^\flat)K_t\mathrm{d}t,\\
\Upsilon_T^{(4)} &:= 4\lambda\fint W_T K_T^3 - 12\lambda^2\fint W_T\mathbb{W}_T^{[3]}K_T^2 + 12\lambda^3\fint W_T(\mathbb{W}_T^{[3]})^2K_T,\\
\Upsilon_T^{(5)} &:= -2\lambda^2\fint\gamma_T Z_T^\flat(Z_T - Z_T^\flat) - \lambda^2\fint\gamma_T(Z_T - Z_T^\flat)^2 - 2\lambda^2\int_0^T\fint\gamma_t Z_t^\flat\dot{Z}_t^\flat\mathrm{d}t,\\
\Upsilon_T^{(6)} &:= -\lambda^2\fint\mathbb{W}_T^{2\diamond[3]}K_T - \frac{\lambda^2}{2}\int_0^T\fint\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}(Z_t^\flat)^2\mathrm{d}t - \frac{\lambda^2}{2}\int_0^T\mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, Z_t^\flat)\mathrm{d}t,
\end{aligned}$$

for a suitable value of the renormalisation constant δ_T.

We do not need to include $\mathbb{W}^{[3]}$ in the vector $\mathbb{W}$ since it can be obtained as a function of $\mathbb{W}^{\langle 3\rangle}$ thanks to the bound

$$\begin{aligned}
\|\mathbb{W}_t^{[3]} - \mathbb{W}_s^{[3]}\|_{\mathscr{C}^{1/2-2\kappa}} &\leqslant \int_s^t \|J_r\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{1/2-2\kappa}}\mathrm{d}r \leqslant \Big[\int_0^T \|J_r\mathbb{W}_r^{\langle 3\rangle}\|^2_{\mathscr{C}^{1/2-2\kappa}}\mathrm{d}r\Big]^{1/2}|t-s|^{1/2}\\
&\leqslant \Big[\int_0^T \|\mathbb{W}_r^{\langle 3\rangle}\|^2_{\mathscr{C}^{-1/2-\kappa}}\frac{\mathrm{d}r}{\langle r\rangle^{1+2\kappa}}\Big]^{1/2}|t-s|^{1/2} \lesssim \sup_{r\in[0,T]}\|\mathbb{W}_r^{\langle 3\rangle}\|^2_{\mathscr{C}^{-1/2-\kappa}}|t-s|^{1/2},
\end{aligned}$$

valid for all $0 \leqslant s \leqslant t \leqslant T$ which shows that the deterministic linear map $\mathbb{W}^{\langle 3\rangle} \mapsto \mathbb{W}^{[3]}$ is continuous from $C([0,\infty], \mathscr{C}^{-1/2-\kappa})$ to $C^{1/2}([0,\infty], \mathscr{C}^{1/2-2\kappa})$.

The path-wise regularity of all the other stochastic objects follows from the next lemma, provided the function γ is chosen appropriately.

Lemma 18.6. *There exists a function $\gamma_t \in C^1(\mathbb{R}_+, \mathbb{R})$ such that*

$$|\gamma_t| + \langle t\rangle|\dot{\gamma}_t| \lesssim \log\langle t\rangle, \qquad t \geqslant 0. \tag{18.11}$$

and such that the vector $\mathbb{W}$ is almost surely in $\mathfrak{S}$ where $\mathfrak{S}$ is the Banach space

$$\mathfrak{S} = C([0,\infty], \mathfrak{W}) \cap \{\mathbb{W}^{\langle 3\rangle} \in L^2(\mathbb{R}_+, \mathscr{C}^{-1/2-\kappa}), \mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle} \in L^1(\mathbb{R}_+, \mathscr{C}^{-\kappa})\}$$

with

$$\mathfrak{W} = \mathfrak{W}_\kappa := \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-1-\kappa} \times \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-\kappa} \times \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-\kappa},$$

and equipped with the norm

$$\|\mathbb{W}\|_{\mathfrak{S}} := \|\mathbb{W}\|_{C([0,\infty],\mathfrak{W})} + \|\mathbb{W}^{\langle 3\rangle}\|_{L^2(\mathbb{R}_+,\mathscr{C}^{-1/2-\kappa})} + \|\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}\|_{L^1(\mathbb{R}_+,\mathscr{C}^{-\kappa})}.$$

The norm $\|\mathbb{W}\|_{\mathfrak{S}}$ belongs to all $L^p(\mathbb{P})$ spaces. Moreover the averages of the Besov norms $B^\alpha_{q,r}$ of the components of $\mathbb{W}$ of regularity α are uniformly bounded in the volume $|\Lambda|$ if $r < \infty$.

Proof. The proof is based on the observation that one can choose γ in such a way that every component $\mathbb{W}^{(i)}$ of the vector $\mathbb{W}$ is such that $(\Delta_q\mathbb{W}_t^{(i)}(x))_{t\geqslant 0}$ for $q \geqslant -1$ and $x \in \Lambda$ is a martingale wrt. the Brownian filtration (possibly modulo a deterministic term we can control). This can be seen by writing these terms as iterated stochastic integrals. For example, introducing the notation $\mathrm{d}w_s(k) = \langle k\rangle^{-1}\sigma_s(k)\mathrm{d}B_s^k$ we can write

$$\mathbb{W}_T^2(x) \;=\; 24\sum_{k_1,k_2} e^{i(k_1+k_2)\cdot x}\int_0^T\int_0^{s_2}\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)$$

so, recalling the definition of Littlewood-Paley kernels ϱ_i, we have

$$\Delta_i \mathbb{W}_T^2(x) = 24 \sum_{k_1,k_2} e^{i(k_1+k_2)\cdot x} \varrho_i(k_1+k_2) \int_0^T \int_0^{s_2} \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2).$$

By Burkholder's inequality and Fubini's theorem

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\leqslant T} \|\Delta_i \mathbb{W}_t^2\|_{L^p}^p\Big] &\lesssim \left(\sum_{k_1,k_2} \varrho_i(k_1+k_2) \int_0^T \int_0^{s_2} \frac{\sigma_{s_1}^2(k_1)}{\langle k_1\rangle^2} \frac{\sigma_{s_2}^2(k_2)}{\langle k_2\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2\right)^{p/2} \\ &\lesssim 2^{p(2+\kappa)i/2}, \end{aligned}$$

uniformly in T and so

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\leqslant T} \|\mathbb{W}_T^2\|_{B^{-1-\kappa}_{p,p}}^p\Big] &\leqslant \mathbb{E}\Big[\Big(\sum_i 2^{p(-1-\kappa)i} \sup_{t\leqslant T} \|\Delta_i \mathbb{W}_t^2\|_{L^p}^p\Big)\Big] \\ &\lesssim \sum_i 2^{p(-1-\kappa)i}\, \mathbb{E}\Big[\sup_{t\leqslant T} \|\Delta_i \mathbb{W}_t^2\|_{L^p}^p\Big] \\ &\lesssim \sum_i 2^{p(-1-\kappa)i} 2^{p(1+\kappa/2)i} \lesssim \sum_i 2^{-pi\kappa/2} < +\infty \end{aligned}$$

By Besov embedding this implies that $\mathbb{E}[\sup_{T<\infty} \|\mathbb{W}_T^2\|_{B^{-1-\kappa}_{p,q}}^p]$ is finite for any $p, q < \infty$ uniformly in the volume and $\mathbb{E}[\|\mathbb{W}_T^2\|_{C\mathscr{C}^{-1-}}^p]$ is finite. Since $\mathbb{W}_T^2$ is a continuous, L^2-bounded martingale, it converges and therefore it belongs to $C([0,\infty], \mathscr{C}^{-1-})$. The same reasoning can be carried out for the more complicated terms $\mathbb{W}^{\langle 3\rangle}, \mathbb{W}^{[3]\circ 1}, \mathbb{W}^{2\diamond[3]}, \mathbb{W}^{\langle 2\rangle \diamond \langle 2\rangle}$. The details can be found in Section ?. □

For convenience of the reader we summarise the probabilistic estimates in Table ?.

$\mathbb{W}^1$	$\mathbb{W}^2$	$\mathbb{W}^{\langle 3\rangle}$	$\mathbb{W}^{[3]}$	$\mathbb{W}^{[3]\circ 1}$	$\mathbb{W}^{2\diamond[3]}$	$\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}$
$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{-1-}$	$C\mathscr{C}^{-1/2-} \cap L^2\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{1/2-}$	$C\mathscr{C}^{0-}$	$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{0-} \cap L^1\mathscr{C}^{0-}$

Table 18.1. Regularities of the various stochastic objects, the domain of the time variable is understood to be $[0,\infty]$. Estimates in these norms hold a.s. and in $L^p(\mathbb{P})$ for all $p \geqslant 1$ (see Lemma 18.6).

Remark 18.7. The requirement that $\mathbb{W}^{\langle 3\rangle} \in L^2\mathscr{C}^{-1/2-}$ will be used in Section ? to establish equicoercivity and to relax the variational problem to a suitable space of measures.

18.1 Commutators

We collect here the analysis of the various commutators introduced above. The properties of these commutators are crucial to the analysis of the resonant products in the paracontrolled analysis above. As such they have an important role in the argument. Their proof however, is a routine application of basic ideas of harmonic analysis and LP decompositions. We decided to report the proof as it can be useful for the reader to study them in order to adapt the ideas to other contexts.

Proposition 18.8. *Let $\alpha \in (0,1)$ $\beta, \gamma \in \mathbb{R}$ such that $\beta+\gamma<0$, $\alpha+\beta+\gamma>0$ and $p_1, p_2, p_3, p \in [1,\infty]$ such that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{p}$. Then there exists a trilinear form $\mathfrak{K}_1(f,g,h)$ such that ,*

$$\|\mathfrak{K}_1(f,g,h)\|_{B^{\alpha+\beta+\gamma}_{p,\infty}} \lesssim \|g\|_{B^{\alpha}_{p_1,\infty}} \|f\|_{B^{\beta}_{p_2,\infty}} \|h\|_{B^{\gamma}_{p_3,\infty}},$$

and when $f, g, h \in \mathscr{S}$ it has the form

$$\mathfrak{K}_1(f,g,h) = (f \succ g) \circ h - g(f \circ h).$$

Proof. The proof is a slight modification of the one given in [?]. Lemma 2.97 from [?] and an interpolation imply that $\|\Delta_j fg - \Delta_j(fg)\|_{L^p} \leq 2^{-j\alpha}\|f\|_{W^{\alpha,p_1}}\|g\|_{L^{p_2}}$. This in turn gives after some algebraic computation (see [?] for details) that

$$\Delta_j(f \succ g) = (\Delta_j f) \succ g + R_j(f,g),$$

with $\|R_j(f,g)\|_{L^p} \lesssim 2^{-j(\alpha+\beta)}\|f\|_{B^\alpha_{p_1,\infty}}\|g\|_{B^\beta_{p_2,\infty}}$. Now to prove the statement of the proposition observe that for smooth f,g,h we have

$$\mathfrak{K}_1(f,g,h) = \sum_{j,k\geqslant -1}\sum_{|i-j|\leqslant 1} \Delta_j(f \succ \Delta_k g)\Delta_i h - \Delta_k g \Delta_j f \Delta_i h.$$

Now observe that the term $f \succ \Delta_k g$ has Fourier transform outside of $2^k B$ for some ball B independent of k, so choosing N large enough we can rewrite the sum as

$$\mathfrak{K}_1(f,g,h) = \sum_{j,k\geqslant -1}\sum_{|i-j|\leqslant 1} \mathbb{1}_{k\leqslant i+N}(\Delta_j f \Delta_k g \Delta_i h + R_j(f,\Delta_k g)) - \Delta_k g \Delta_j f \Delta_i h$$

$$\sum_{j,k\geqslant -1}\sum_{|i-j|\leqslant 1} \mathbb{1}_{k\leqslant i+N} R_j(f,\Delta_k g)\Delta_i h - \mathbb{1}_{k\geqslant i+N}\Delta_k g \Delta_j f \Delta_i h.$$

Now we estimate the norm of the two terms separately. First note that for fixed j

$$\sum_{k\geqslant -1}\sum_{|i-j|\leqslant 1} \mathbb{1}_{k\leqslant i+N} R_j(f,\Delta_k g)$$

has a Fourier transform supported in $2^j B$. By Lemma 2.69 from [?] it is enough to get an estimate on

$$\sup_k \left\| 2^{(\alpha+\beta+\gamma)j} \sum_{j\geqslant -1}\sum_{|i-j|\leqslant 1} \mathbb{1}_{k\leqslant i+N} R_j(f,\Delta_k g)\Delta_i h \right\|_{L^p}$$

to bound it in $B^{\alpha+\beta+\gamma}_{p,\infty}$, so by Hölder inequality,

$$\begin{aligned}\left\| \sum_{|i-j|\leqslant 1} R_j\left(f, \sum_{k\geqslant -1}^{i+N} \Delta_k g\right)\Delta_i h \right\|_{L^p} &\lesssim \sum_{|i-j|\leqslant 1} 2^{-j(\alpha+\beta)}2^{-i\gamma}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,q_1}}\|h\|_{B^\gamma_{p_3,q_2}} \\ &\lesssim 2^{-j(\alpha+\beta+\gamma)}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,q_1}}\|h\|_{B^\gamma_{p_3,q_2}}.\end{aligned}$$

For the second term observe that for fixed k the Fourier transform of

$$\sum_{j\geqslant -1}\sum_{|i-j|\leqslant 1} \mathbb{1}_{k\geqslant i+N}\Delta_k g \Delta_j f \Delta_i h$$

is supported in $2^k B$. Now we can estimate again by Hölder inequality

$$\begin{aligned}&\lesssim \left\| \sum_{j\geqslant -1}\sum_{|i-j|\leqslant 1} \mathbb{1}_{k\geqslant i+N}\Delta_k g \Delta_j f \Delta_i h \right\|_{L^p} \\ &\lesssim 2^{-\alpha k}\sum_{j\geqslant -1}^{k+N} 2^{-(\beta+\gamma)k}\mathbb{1}_{k\geqslant i+N}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,\infty}}\|h\|_{B^\gamma_{p_1,\infty}} \\ &\lesssim 2^{-j(\alpha+\beta+\gamma)}\|g\|_{B^\alpha_{p_1,\infty}}\|f\|_{B^\beta_{p_2,q_1}}\|h\|_{B^\gamma_{p_3,q_2}}.\end{aligned}$$

□

Proposition 18.9. *Assume $\beta \in (0,1)$, $\alpha,\gamma \in \mathbb{R}$ such that $\alpha+\gamma<0$, and $\alpha+\beta+\gamma=0$, $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$ and $\frac{1}{q_1}+\frac{1}{q_2}=1$. Then there exists a trilinear form $\mathfrak{K}_2(f,g,h)$ for which*

$$|\mathfrak{K}_2(f,g,h)| \lesssim \|f\|_{B^\alpha_{p_1,\infty}}\|g\|_{B^\beta_{p_2,q_1}}\|h\|_{B^\gamma_{p_3,q_2}},$$

and on smooth functions

$$\mathfrak{K}_2(f,g,h) = \fint [(f \succ g)h - g(f \circ h)].$$

Proof. This is modification of the proof of Lemma A.6 in [?]. Repeating an algebraic computation given in the proof of that lemma, we get that for smooth f, g, h we have

$$\mathfrak{K}_2(f,g,h) = \left(\sum_{j \leqslant i-1, |i-k| \leqslant L} - \sum_{i \sim k, j < i+L} \right) \fint (\Delta_i f \Delta_j g \Delta_k h),$$

for some $L \geqslant 1$. Then we estimate

$$\begin{aligned} |\mathfrak{K}_2(f,g,h)| &\lesssim \sum_{i \sim j \sim k} \|\Delta_i f \Delta_j g \Delta_k h\|_{L^1} \\ &\lesssim \sum_{i \sim j \sim k} \|\Delta_i f\|_{L^{p_1}} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}} \\ &\lesssim \sup_i (2^{\alpha i} \|\Delta_i f\|_{L^{p_1}}) \sum_{j \sim k} 2^{(\beta+\gamma)k} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}} \\ &\lesssim \|f\|_{B^{\alpha}_{p_1,\infty}} \|g\|_{B^{\beta}_{p_2,q_1}} \|h\|_{B^{\gamma}_{p_3,q_2}}. \end{aligned}$$

□

Proposition 18.10. *There exists a family $(\mathfrak{K}_{3,t})_{t \geqslant 0}$ of bounded multilinear forms on $\mathscr{C}^{-1-\kappa} \times \mathscr{C}^{-1-\kappa} \times H^{1/2-\kappa} \times H^{1/2-\kappa}$ such that for smooth $\varphi, \psi, g^{(1)}, g^{(2)}$ it holds*

$$\mathfrak{K}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)}) = \fint [J_t(\varphi \succ g^{(1)}) J_t(\psi \succ g^{(2)}) - (J_t \varphi \circ J_t \psi) g^{(1)} g^{(2)}],$$

and

$$|\mathfrak{K}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)})| \lesssim \frac{1}{\langle t \rangle^{1+\delta}} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}}.$$

Proof. Note that $\langle t$ satisfies the assumptions of Proposition 18.4 and with $m = -1$, therefore using also Proposition 13.7 □

$$\|J_t(\varphi \succ g^{(1)}) - J_t \varphi \succ g^{(1)}\|_{H^{1/2-3\kappa}} \lesssim \langle t \rangle^{-1/2} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}}.$$

Therefore

$$\begin{aligned} &\left| \fint [J_t(\varphi \succ g^{(1)}) - (J_t \varphi \succ g^{(1)})] J_t(\psi \succ g^{(2)}) \right| \\ \lesssim\ & \|J_t(\varphi \succ g^{(1)}) - J_t \varphi \succ g^{(1)}\|_{H^{1/2-3\kappa}} \|J_t(\psi \succ g^{(2)})\|_{H^{-1/2+3\kappa}} \\ \lesssim\ & \langle t \rangle^{-1/2} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \langle t \rangle^{-1/2-\delta} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \end{aligned}$$

and by symmetry also

$$\begin{aligned} &\left| \fint [J_t(\varphi \succ g^{(1)}) J_t(\psi \succ g^{(2)}) - (J_t \varphi \succ g^{(1)})(J_t \psi \succ g^{(2)})] \right| \\ \lesssim\ & \langle t \rangle^{-1-\kappa} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \end{aligned}$$

Furthermore from Proposition 18.9 and for sufficiently small $\kappa > 0$,

$$\begin{aligned} &\left| \fint (J_t \varphi \succ g^{(1)})(J_t \psi \succ g^{(2)}) - \fint ((J_t \varphi \succ g^{(1)}) \circ J_t \psi) g_t^{(2)} \right| \\ \lesssim\ & \|J_t \varphi\|_{\mathscr{C}^{-2\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|J_t \psi\|_{\mathscr{C}^{-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \\ \lesssim\ & \langle t \rangle^{-1-\kappa} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}}. \end{aligned}$$

Applying Proposition – 18.8 we get

$$\begin{aligned} &\|(J_t \varphi^{(1)} \succ g^{(1)}) \circ J_t \psi_t - (J_t \varphi_t \circ J_t \psi_t)(g^{(1)})\|_{H^{-1/2+\kappa}} \\ \lesssim\ & \|J_t \varphi_t\|_{\mathscr{C}^{-2\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|J_t \psi_t\|_{\mathscr{C}^{-\kappa}} \\ \lesssim\ & \langle t \rangle^{-1-\delta} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \end{aligned}$$

and putting things together gives the required estimate.