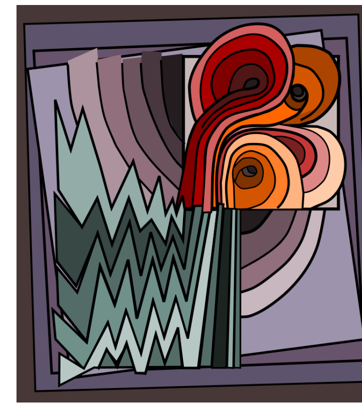
OOPS 2021 · September 1st 2021

Facets of stochastic quantisation

lecture 2



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done

- euclidean quantum fields
- what is stochastic quantisation?
- varieties of stochastic quantisation

todo

- infinite volume limit ($L \to \infty$)
- renormalization and small scale limit ($\varepsilon \to 0$)
- properties of stochastically quantised measures
- elliptic stochastic quantisation (?) & supersymmetry

reference material

https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021

coupling to the GFF

we work on $\Lambda_{\varepsilon,L} = \mathbb{Z}^d_{\varepsilon,L}$. The solution $X: \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{\Lambda_{\varepsilon,L}}$ to

$$\mathrm{d}X_t(x) = -(AX_t)(x)\mathrm{d}t - \frac{1}{2}V'_\varepsilon(X_t(x))\mathrm{d}t + 2^{1/2}\mathrm{d}B_t(x) \qquad x \in \Lambda_{\varepsilon,L}$$

with $A = m^2 - \Delta$ (discrete Laplacian) leaves the measure

$$\nu^{\varepsilon,L}(\mathrm{d}\varphi) = Z^{-1} e^{-\sum_{x \in \Lambda_{\varepsilon,L}} V_\varepsilon(\varphi(x))} \mu^{\varepsilon,L}(\mathrm{d}\varphi), \qquad V_\varepsilon(\xi) = \lambda \xi^4 - \beta_\varepsilon \xi^2$$

invariant. Here $(B_t(x))_{t \geqslant 0, x \in \Lambda_{\varepsilon,L}}$ are iid BM and $\mu^{\varepsilon,L}$ is the GFF (i.e. $\mathscr{N}(0, A^{-1})$).

Let Y be the solution of the linear equation (dynamic GFF):

$$\mathrm{d}Y_t = -AY_t\mathrm{d}t + 2^{1/2}\mathrm{d}B_t,$$

with invariant measure $\mu^{\varepsilon,L}$. Define $Z = X - Y$ which solves a RDE:

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = -AZ_t - V'_\varepsilon(Y_t + Z_t).$$

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = -AZ_t - V_\varepsilon'(Y_t + Z_t)$$

$$V_\varepsilon'(\varphi) = \lambda\varphi^3 - \beta\varphi$$

test this equation with Z:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_x |Z_t(x)|^2 + G(Z_t) = -\lambda\sum_x (Y_t(x)^3 Z_t(x) + 3Y_t(x)^2 Z_t(x)^2 + 3Y_t(x)Z_t(x)^3)$$

$$+\beta\sum_x (Z_t(x)Y_t(x) + Z_t(x)^2)$$

where

$$G(\varphi) = \|\nabla\varphi\|_{L^2}^2 + m^2\|\varphi\|_{L^2}^2 + \lambda\|\varphi\|_{L^4}^4$$

with the natural Lebesgue spaces on $\Lambda = \Lambda_{\varepsilon,M}$ (with counting measure).

the key property being that in the r.h.s. we have all terms which we can bound via Hölder inequality as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_x |Z_t(x)|^2 + G(Z_t) \leqslant C_\delta[\|Y_t\|_{L^4}^4 + \|Y_t\|_{L^2}^2] + \delta G(Z_t),$$

for $\delta > 0$ small as we wish, e.g. $\delta = 1/2$. We conclude that

$$\|Z_t\|_{L^2}^2 + \frac{1}{2}\int_0^t G(Z_s)\mathrm{d}s \leqslant \|Z_0\|_{L^2}^2 + C\int_0^t [\|Y_t\|_{L^4}^4 + \|Y_t\|_{L^2}^2]\mathrm{d}s. \qquad (1)$$

This bound implies that solutions cannot explode and we have an explicit bound on its growth in term of Y and Z_0.

However we do not know Z_0… how to close the argument???

stationary coupling

- one possible approach: construct a stationary coupling of Y and Z via Krylov-Bogoliubov argument.

- we can construct a measure γ_T on a pair of fields $(\varphi,\psi)\in\mathbb{R}^\Lambda\times\mathbb{R}^\Lambda$ by the formula

$$\int f(\varphi,\psi)\mathrm{d}\gamma_T(\varphi,\psi) := \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s,Z_s)]\mathrm{d}s,$$

for any bounded function f of the pair $(\varphi,\psi)\in\mathbb{R}^\Lambda\times\mathbb{R}^\Lambda$ where (Y,Z) are started from $\mu\times\nu$.

- note also that

$$\int f(\varphi+\psi)\mathrm{d}\gamma_T(\varphi,\psi) = \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s+Z_s)]\mathrm{d}s = \frac{1}{T}\int_0^T \mathbb{E}[f(X_s)]\mathrm{d}s = \mathbb{E}[f(X_0)] = \int f(\varphi)\nu(\mathrm{d}\phi)$$

therefore the law of $\varphi+\psi$ under γ_T is always given by ν for any T and also

$$\int f(\varphi)\mathrm{d}\gamma_T(\varphi,\psi) = \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s)]\mathrm{d}s = \mathbb{E}[f(Y_0)] = \int f(\varphi)\mu(\mathrm{d}\phi)$$

▹ we have that

$$\int [G(\psi) + \|\varphi\|_{L^4}^4] \mathrm{d}\gamma_T(\varphi,\psi) = \frac{1}{T}\int_0^T \mathbb{E}[G(Z_s) + \|Y_s\|_{L^4}^4] \mathrm{d}s \leqslant \frac{2}{T}\left(\mathbb{E}\|Z_0\|_{L^2}^2 + C'\int_0^T \mathbb{E}\|Y_s\|_{L^4}^4 \mathrm{d}s\right),$$

$$\leqslant \left(\frac{2}{T}\mathbb{E}\|Z_0\|_{L^2}^2\right) + 2C'\mathbb{E}\|Y_0\|_{L^4}^4,$$

which is uniformly bounded in T.

▹ this implies that the family $(\gamma_T)_T$ is tight on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ and one can extract a weakly convergent subsequence to a limit γ.

▹ as a consequence the law of $\varphi + \psi$ under γ is ν.

▹ the measure γ is stationary under the joint dynamics of (Z, Y), i.e. if $(Z_0, Y_0) \sim \gamma$ then $(Z_t, Y_t) \sim \gamma$.

what happens when we want to take the limit $M \to \infty$?

modify our apriori estimate introducing a polynomial weight $\rho : \Lambda = (\varepsilon\mathbb{Z})^d \to \mathbb{R}$

$$\rho(x) = (1 + \ell |x|)^{-\sigma}, \qquad \sigma > 0, \ell > 0,$$

and test the equation for Z with $\rho^2 Z$ summing over the full lattice Λ and we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x \in \Lambda_\varepsilon} |\rho(x) Z_t(x)|^2 + G(Z_t) \leqslant -\lambda \sum_{x \in \Lambda_\varepsilon} \rho(x)(Y_t(x)^3 Z_t(x) + 3Y_t(x)^2 Z_t(x)^2 + 3Y_t(x) Z_t(x)^3)$$

$$+\beta \sum_{x \in \Lambda_\varepsilon} \rho(x)(Z_t(x) Y_t(x) + Z_t(x)^2) + C_{\rho,\ell} \sum_{x \in \Lambda_\varepsilon} \rho(x) Z_t(x)^2$$

$$G(\varphi) = \|\rho \nabla \varphi\|^2_{L^2(\Lambda_\varepsilon)} + m^2 \|\rho\varphi\|^2_{L^2(\Lambda_\varepsilon)} + \lambda \|\rho^{1/2}\varphi\|^4_{L^4(\Lambda_\varepsilon)}.$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho Z_t\|_{L^2(\Lambda_\varepsilon)}^2 + G(Z_t) \leqslant C_\delta \|\rho^{1/2} Y_t\|_{L^4(\Lambda_\varepsilon)}^4 + \delta G(Z_t)$$

indeed the interaction terms can be estimated as

$$\lambda\left|\sum_{x\in\Lambda_\varepsilon} \rho(x) Y_t(x)^3 Z_t(x)\right| \leqslant \lambda\left|\sum_{x\in\Lambda_\varepsilon} (\rho(x)^{3/2} Y_t(x)^3)(\rho(x)^{1/2} Z_t(x))\right|$$

$$\leqslant \lambda\frac{C}{\delta}\|\rho^{1/2}Y_t\|_{L^4}^4 + \delta\lambda\|\rho^{1/2}Z_t\|_{L^4}^4 \leqslant \lambda\frac{C}{\delta}\|\rho^{1/2}Y_t\|_{L^4}^4 + \delta G(Z_t)$$

for any small $\delta > 0$.

$$\boxed{\|\rho Z_t\|_{L^2(\Lambda)}^2 + \frac{1}{2}\int_0^t G(Z_s)\mathrm{d}s \leqslant \|\rho Z_0\|_{L^2(\Lambda)}^2 + C\int_0^t \|\rho^{1/2} Y_s\|_{L^4(\Lambda)}^4 \mathrm{d}s}$$

use the stationary coupling:

$$\mathbb{E}\|\rho Z_t\|_{L^2(\Lambda)}^2 = \mathbb{E}\|\rho Z_0\|_{L^2(\Lambda)}^2$$

so

$$\mathbb{E}G(Z_0) = \frac{1}{t}\int_0^t \mathbb{E}G(Z_s)\mathrm{d}s \leqslant \frac{2C}{t}\int_0^t \mathbb{E}\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4 \mathrm{d}s = \mathbb{E}\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4$$

$$\mathbb{E}\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4 = \mathbb{E}\sum_{x\in\Lambda_\varepsilon}\rho(x)^2|Y_0(x)|^4 = \sum_{x\in\Lambda_\varepsilon}\rho(x)^2\mathbb{E}|Y_0(x)|^4 = C\sum_{x\in\Lambda_\varepsilon}\rho(x)^2 < \infty$$

uniformly in L. Namely from this estimate one can deduce that

$$\sup_L \int \|\rho^{1/2}\varphi\|_{L^4(\Lambda_\varepsilon)}^4 \nu^{\varepsilon,L}(\mathrm{d}\varphi) < \infty$$

This is a key estimate to take the infinite volume limit since it allows to use tightness on the family $(\nu^{\varepsilon,L})_L$ in the topology of local convergence.

It gives also a stationary infinite volume limit coupling to the GFF.

bounds

- the local (or weighted) $L^p(\Lambda_\varepsilon)$ norms of $\varphi : \mathbb{R}^{\Lambda_\varepsilon} \to \mathbb{R}$ under the measure $\nu^{\varepsilon,M}$ have finite moments:

$$\sup_L \int \|\rho\varphi\|_{L^p}^p \nu^{\varepsilon,L}(\mathrm{d}\varphi) < \infty$$

for any $p > 1$.

- by working a bit harder one can prove uniform integrability of functions like $\exp(\|\rho\varphi\|_{L^2})$. (see Gubinelli-Hofmanova CMP 2021)

- another approach is to use the “coming down from infinity” to remove dependence on the initial condition (see Mourrat-Weber CMP 2017, Gubinelli-Hofmanova CMP 2020, Moinat-Weber CPAM 2020)

coming down from infinity (in one slide)

consider the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = -y(t)^3 + f(t)$$

take $\rho(t) = t^{\alpha}$ with $\alpha > 0$. If $t_* > 0$ is a maximum point of $\rho(t)y(t)$ for $t \in [0, T]$ then:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho(t)y(t)]\Big|_{t=t_*} = \frac{\alpha}{t_*}\rho(t_*)y(t_*) - \rho(t_*)y(t_*)^3 + \rho(t_*)f(t_*) \geqslant 0$$

that is

$$[\rho(t_*)^{1/3}y(t_*)]^3 \leqslant \frac{\alpha\rho(t_*)^{2/3}}{t_*}\rho(t_*)^{1/3}y(t_*) + \rho(t_*)f(t_*)$$

take $\alpha = 3/2$ so that

$$\sup_{t\in[0,T]}[\rho(t)^{1/3}y(t)] = \rho(t_*)^{1/3}y(t_*) \leqslant C\left\{1 + \sup_{t\in[0,T]}[\rho(t)f(t)]^{1/3}\right\} \qquad \Rightarrow y(t) \leqslant C_{f,T}t^{-1/2}$$

independently of $y(0)$. Extendable to parabolic PDEs (maximum principle).

optimal bounds: Hairer/Steele argument

- we want to bound $Z_H = \int e^{H(\varphi)} \nu^{\varepsilon,L}(\mathrm{d}\varphi)$ for some nice function $H(\varphi) \geqslant 0$.
- the idea of Hairer/Steele (slightly revisited here) is to consider the new measure

$$\rho^H(\mathrm{d}\varphi) = \frac{e^{H(\varphi)} \nu^{\varepsilon,L}(\mathrm{d}\varphi)}{Z_H} = \frac{e^{H(\varphi) - V_\varepsilon(\varphi)}}{Z_H Z_{\varepsilon,L}} \mu^{\varepsilon,L}(d\varphi)$$

and observe that by Jensen's:

$$1 = \int e^{-H(\varphi)} e^{H(\varphi)} \nu^{\varepsilon,L}(\mathrm{d}\varphi) = Z_H \int e^{-H(\varphi)} \rho^H(\mathrm{d}\varphi) \geqslant Z_H \exp\left(- \int H(\varphi) \rho^H(\mathrm{d}\varphi) \right)$$

so

$$\log Z_H \leqslant \int H(\varphi) \rho^H(\mathrm{d}\varphi).$$

- the SQ of ρ^H can be used as before to obtain bounds which depends only on the GFF provided (e.g.)

$$\left| \sum_\Lambda \rho^2 \varphi H'(\psi + \varphi) \right| \leqslant Q(\psi) + \delta G(\varphi), \qquad | H(\psi + \varphi) | \leqslant Q(\psi) + G(\varphi)$$

$$G(\varphi) = \| \rho \nabla \varphi \|^2_{L^2(\Lambda_\varepsilon)} + m^2 \| \rho \varphi \|^2_{L^2(\Lambda_\varepsilon)} + \lambda \| \rho^{1/2} \varphi \|^4_{L^4(\Lambda_\varepsilon)}.$$

▹ shifted SQ equation

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = -AZ_t - V_\varepsilon'(Y_t + Z_t) + H'(Y_t + Z_t)$$

▹ bounds

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho Z_t\|_{L^2(\Lambda_\varepsilon)}^2 + G(Z_t) \leqslant C_\delta \|\rho^{1/2} Y_t\|_{L^4(\Lambda_\varepsilon)}^4 + Q(Y_t) + 2\delta G(Z_t)$$

▹ use stationary coupling

$$\mathbb{E}G(Z_0) = \frac{1}{t}\int_0^t \mathbb{E}G(Z_s)\mathrm{d}s \leqslant \frac{2C}{t}\int_0^t \mathbb{E}\{\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4 + Q(Y_s)\}\mathrm{d}s = \mathbb{E}\{\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4 + Q(Y_0)\}$$

therefore

$$\int H(\varphi)\rho^H(\mathrm{d}\varphi) = \mathbb{E}[H(X_0)] = \mathbb{E}[H(Y_0 + Z_0)] \leqslant C[\mathbb{E}\{\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4 + Q(Y_0)\}] < \infty.$$

example: $H(\varphi) = \eta\|\rho\varphi\|_{L^4}^4$ for $\eta > 0$ small gives the optimal bound

$$\sup_L \int e^{\eta\|\rho\varphi\|_{L^4}^4} \nu^{\varepsilon,L}(\mathrm{d}\varphi) < \infty.$$

what about uniqueness of the accumulation points?

▹ using essentially a similar approach one can prove that provided

$$V_\varepsilon''(\varphi) \geqslant -\chi,$$

for some $\chi > 0$ then for m large enough (depending on χ) we have also uniqueness of the limit measure ν^ε.

▹ this is natural because we do not expect in general that the limit is unique (there could be phase transitions in the model, in $d \geqslant 2$ since it is a model of ferromagnetic unbounded spin).

▹ the idea to prove uniquess is to compare two solutions Z^1, Z^2 driven by two Gaussian processes Y^1, Y^2 and use a coupling approach.

▹ the same idea can be used to control correlations.

▹ let (Z^1, Y^1) and (Z^2, Y^2) be two solutions of the shifted SQ equation. then $H = Z^1 - Z^2$ solves

$$\partial_t H - AH = Q := -[V'(X^1) - V'(X^1 + H + K)] = -\underbrace{\int_0^1 \mathrm{d}\tau\, V''(X^1 + \tau(H+K))}_{=:G \geqslant -\chi}(H+K)$$

with $K := Y^1 - Y^2$ and $X^1 = Y^1 + Z^1$ as usual. assume that $V''(\varphi) \geqslant -\chi$ for some $\chi > 0$.

▹ test the equation with $\rho^2 H$ for some weight ρ. RHS:

$$\sum_\Lambda \rho^2 HQ = \sum_\Lambda \rho^2 GKH - \sum_\Lambda \rho^2 GH^2 \leqslant C_\delta \|\rho GK\|_{L^2}^2 + \delta \|\rho H\|_{L^2}^2 - \underbrace{\sum_\Lambda \rho^2 GH^2}_{\geqslant \chi \|\rho H\|_{L^2}^2}$$

$$\leqslant C_\delta \|\rho GK\|_{L^2}^2 + (\chi + \delta) \|\rho H\|_{L^2}^2$$

▹ consider

$$\frac{1}{2}\partial_t(e^{ct}\|\rho H\|_{L^2}^2) = \frac{c}{2}(e^{ct}\|\rho H\|_{L^2}^2) + \frac{e^{ct}}{2}\partial_t\|\rho H\|_{L^2}^2$$

$$\leqslant -e^{ct}\sum_{\Lambda}\rho^2 H\left(m^2 - 2\chi - \frac{c}{2} - \Delta\right)H + e^{ct}C\|\rho G K\|_{L^2}^2.$$

▹ for $\rho(x) = e^{-\theta|x|}$ we have

$$\sum_{\Lambda}\rho^2 H(-\Delta)H \geqslant \frac{1}{2}\sum_{\Lambda}\sum_{i}\rho^2|\nabla_i H|^2 - C\theta^2\sum_{\Lambda}\rho^2|H|^2.$$

▹ putting all together:

$$\frac{1}{2}\partial_t(e^{ct}\|\rho H\|_{L^2}^2) + e^{ct}\underbrace{\left(m^2 - 2\chi - \frac{c}{2} - C\theta^2\right)}_{\geqslant 0}\sum_{\Lambda}\rho^2 H^2 + e^{ct}\frac{1}{2}\sum_{\Lambda}\sum_{i}\rho^2|\nabla_i H|^2$$

$$\leqslant e^{ct}C\underbrace{\|\rho G(t)K(t)\|_{L^2}^2}_{=:Q(t)}.$$

▹ integrating

$$\|\rho H_t\|_{L^2}^2 \leqslant e^{-ct} \|\rho H_0\|_{L^2}^2 + 2\int_0^t e^{-c(t-s)} \|\rho G(s) K(s)\|_{L^2}^2 \mathrm{d}s$$

▹ from which one can deduce bounds of the form

$$\mathbb{E}\|\rho H_t\|_{L^2}^2 \leqslant e^{-ct} \mathbb{E}\|\rho H_0\|_{L^2}^2 + C \sum_{x\in\Lambda} \rho^2(x) (\mathbb{E} K_0^4(x))^{1/2}$$

e.g. when $K = Y^1 - Y^2$ is stationary.

▹ many informations:

- by coupling two different invariant measures via a common dynamics ($K = 0$) one can show that the two measures are equal. This gives uniqueness.
- one can use noises which coincide in a bounded region Ω to drive two different dynamics, e.g. started from the same invariant measure. in this case $K = 0$ in Ω and this shows that the two solutions X^1 and X^2 are near inside $\Omega' \subseteq \Omega$.
- one can modify this setup to obtain decay of correlations in SQ (work in progress with Hofmanova and Rana)

features of stochastic quantisation

the interacting field X is expressed as a function of the (dynamic) Gaussian free field Y:

$$X(t) = F(Y), \qquad \nu = \mathrm{Law}(X(t)) = F_* \mathrm{Law}(Y) = F_* \mathrm{GFF}$$

- estimates on X obtained via two ingredients:
  - pathwise PDE (weigthed) estimates for the map F
  - probabilistic estimates for the GFF Y
- coupling (X, Y)

$$X = Y + Z$$

  where Z is a random field which is more regular (i.e. smaller at small scale) than Y (link with asymptotic freedom / perturbation theory)

…end of lecture 2