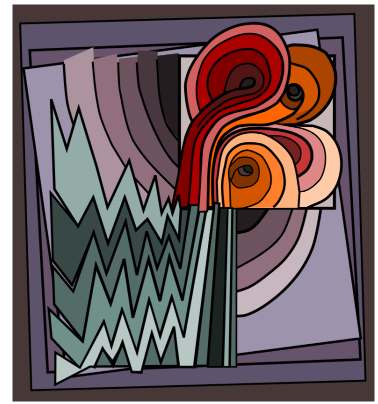
OOPS 2021 · September 1st 2021

Facets of stochastic quantisation

lecture 3



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[made with TeXmacs]

done

- euclidean quantum fields
- what is stochastic quantisation?
- varieties of stochastic quantisation
- apriori bounds in weighted spaces
- infinite volume limit ($L \to \infty$) & uniqueness

todo

- renormalization and small scale limit ($\varepsilon \to 0$)
- properties of stochastically quantised measures

reference material

https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021

the small scale limit $\varepsilon \to 0$ for L fixed and $d = 2$

▹ let's go back to the shifted SQ equation

$$\frac{\partial}{\partial t} Z_t = (\Delta_\varepsilon - m^2) Z_t - V'(Y_t + Z_t), \qquad \text{with } V'(\varphi) = \lambda \varphi^3 + \beta \varphi.$$

▹ the main problem is the following: ($L = 1$) as $\varepsilon \to 0$

$$\mathbb{E}[Y_t(x)^2] = (m^2 - \Delta_\varepsilon)^{-1}(x,x) = \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{1}{(m^2 + \sum_i (2\varepsilon^{-1} \sin(\pi \varepsilon k_i))^2)}$$

$$\approx \sum_{k \in [\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1})]^d} \frac{1}{(m^2 + 2\pi |k|^2)} \propto \begin{cases} \varepsilon^{2-d} & d > 2 \\ \log(\varepsilon^{-1}) & d = 2 \end{cases}$$

▹ this tells us that the typical size of $Y_t(x)$ is $\varepsilon^{2-d} \to \infty$.

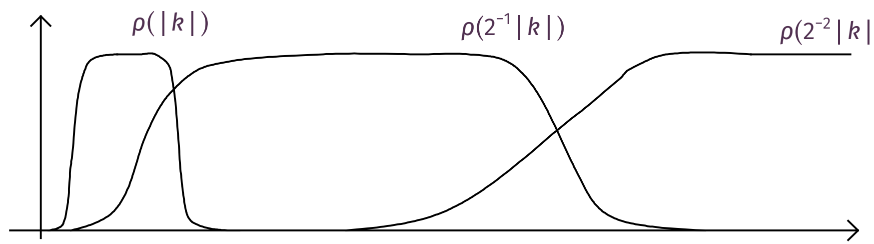
▹ previous apriori estimates depending on $\|Y_0\|_{L^4}$ useless in this limit. problem of small scales. Y^ε is not converging to a function on $\mathbb{T}^d \approx [0,1]^d$, not even locally.

Littlewood–Paley decomposition

$$f(x) = \sum_{i \geqslant -1} (\Delta_i f)(x),$$

where for $i \geqslant 0$,

$$\Delta_i f(x) := \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \rho(2^{-i} |k|) \hat{f}(k) e^{2\pi i k \cdot x} \qquad \Delta_{-1} f(x) := \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \chi(|k|) \hat{f}(k) e^{2\pi i k \cdot x}$$



$$\chi(|k|) + \sum_{i \geqslant 0} \rho(2^{-i} |k|) = 1$$

- ▷ for $\varepsilon > 0$ we have $\Delta_i f = 0$ if $2^i \gtrsim \varepsilon^{-1}$, so we sum over i up to $N_\varepsilon \approx \log_2 \varepsilon^{-1}$: $f = \sum_{i=-1}^{N_\varepsilon} (\Delta_i f)$, with our new tool let's compute again

$$\mathbb{E}[(\Delta_i Y_t(x))^2] = \sum_{k\in(\mathbb{Z}\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d} \frac{\rho(2^{-i}k)^2}{(m^2+\sum_i (2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2)}$$

$$\approx \sum_{k\in 2^i\mathscr{A}\subseteq(\mathbb{Z}\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d} \frac{\rho(2^{-i}k)^2}{(m^2+|k|^2)} \approx \sum_{k\in 2^i\mathscr{A}\subseteq(\mathbb{Z}\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d} \frac{\rho(2^{-i}k)^2}{\underbrace{|k|^2}_{\approx(2^i)^2}} \lesssim (2^i)^{d-2}$$

Which says that $\Delta_i Y_t \approx (2^i)^{(d-2)/2}$ which is uniform in ε! (but of course not in i, and i can be large)

One can prove actually that $\Delta_i Y^\varepsilon$ converges to a nice C^∞ random function on $\mathbb{T}^d$ as $\varepsilon \to 0$.

Besov spaces

▹ let $\alpha \in \mathbb{R}$ and $p, q \in [1, +\infty]$. we say that $f \in B^{\alpha}_{p,q}$ (a Besov space) iff

$$\| f \|_{B^{\alpha}_{p,q}} := \| i \geqslant -1 \mapsto 2^{i\alpha} \| \Delta_i f \|_{L^p} \|_{\ell^q} = \left[\sum_i (2^{i\alpha} \| \Delta_i f \|_{L^p})^q \right]^{1/q} < \infty.$$

▹ Banach spaces in $\mathscr{S}'$. when $\alpha > 0$ these are spaces of regular functions, when $\alpha < 0$ these are just distributions (of course when $\varepsilon = 0$). e.g. $\delta_0 \in B^{-d}_{\infty,\infty} \cap B^0_{1,\infty}$

▹ products of Besov functions are well defined (continuous) provided the regularities sum up to a striclty positive quantity, e.g. $B^{\alpha}_{1,1} \times B^{\beta}_{\infty,\infty} \rightarrow B^{\alpha \wedge \beta}_{1,1}$ for $\alpha + \beta > 0$

▹ $\mathscr{C}^{\alpha} = B^{\alpha}_{\infty,\infty}$ and $H^{\alpha} = B^{\alpha}_{2,2}$:

$$\| \Delta_i f \|_{L^\infty} \leqslant 2^{-\alpha i} \| f \|_{\mathscr{C}^{\alpha}}$$

$$\| f \|^2_{B^{\alpha}_{2,2}} = \sum_{i \geqslant -1} 2^{2i\alpha} \| \Delta_i f \|^2_{L^2(\mathbb{T}^d_{\varepsilon})} = \varepsilon^d \sum_{x \in \mathbb{T}^d_{\varepsilon}} \sum_{i \geqslant -1} 2^{2i\alpha} | \Delta_i f(x) |^2 \approx \varepsilon^d \sum_{x \in \mathbb{T}^d_{\varepsilon}} | (1 - \Delta_{\varepsilon})^{\alpha/2} f |^2 =: \| f \|^2_{H^{\alpha}}.$$

▹ $Y^{\varepsilon}_t \in \mathscr{C}^{-(d-2)/2-\kappa}$ for any small $\kappa > 0$ uniformly in ε and $Y^{\varepsilon} \in C(\mathbb{R}; \mathscr{C}^{-(d-2)/2-\kappa})$ uniformly in ε and almost surely (for a suitable coupling of the $(Y^{\varepsilon})_{\varepsilon}$).

Theorem. *There exists a constant* c_ε *such that the random field (renormalized square)*

$$\mathbb{Y}_t^{\varepsilon,2}(x) := (Y_t^\varepsilon(x))^2 - c_\varepsilon,$$

converges (in law) as $\varepsilon \to 0$ *to a random field* $\mathbb{Y}^2$ *in* $C(\mathbb{R}; \mathscr{C}^{2\alpha})$ *with* $\alpha = (2-d)/2 - \kappa < 0$ *(if* $d \geqslant 2$*). Similarly if* $d = 2$ *the renormalized cube*

$$\mathbb{Y}_t^{\varepsilon,3}(x) := (Y_t^\varepsilon(x))^3 - 3\, c_\varepsilon\, Y_t^\varepsilon(x),$$

converges as $\varepsilon \to 0$ *to a random field in* $C(\mathbb{R}; \mathscr{C}^{3\alpha})$ *while if* $d = 3$ *the convergence holds* $C^{-\kappa}(\mathbb{R}; \mathscr{C}^{3\alpha})$ *(where* $C^{-\kappa}$ *is a space of distributions in the time variable with negative regularity). One can take*

$$c_\varepsilon := \mathbb{E}[(Y_t^\varepsilon(x))^2] \approx \varepsilon^{(2-d)}.$$

With this choice the renormalization corresponds to “Wick ordering”.

renormalization

- take $\beta = -3\lambda c_\varepsilon + \beta'$ and $\mathbb{Y}^1 = Y$ in

$$V'(Y+Z) - \lambda Z^3 = \lambda Y^3 + 3\lambda Y^2 Z + 3\lambda Y Z^2 + \beta Y + \beta Z$$

- renormalized drift term:

$$V'(Y+Z) - \lambda Z^3 = \lambda \mathbb{Y}^3 + 3\lambda \mathbb{Y}^2 Z + 3\lambda \mathbb{Y}^1 Z^2 + \beta' Y + \beta' Z$$

- Besov estimates ($\int_{\mathbb{T}_\varepsilon^d} \mathrm{d}x := \varepsilon^d \sum_{x \in \mathbb{T}_\varepsilon^d}$)

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^3 Z \right| \lesssim \| \mathbb{Y}^3 \|_{\mathscr{C}^{3\alpha}} \| Z \|_{B_{1,1}^{4\kappa}} \lesssim C_\delta \| \mathbb{Y}^3 \|_{\mathscr{C}^{3\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^2}^2,$$

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^2 Z^2 \right| \lesssim \| \mathbb{Y}^2 \|_{\mathscr{C}^{2\alpha}} \| Z^2 \|_{B_{1,1}^{3\kappa}} \lesssim C_\delta \| \mathbb{Y}^2 \|_{\mathscr{C}^{2\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^4}^4,$$

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^1 Z^3 \right| \lesssim \| \mathbb{Y}^1 \|_{\mathscr{C}^{\alpha}} \| Z^3 \|_{B_{1,1}^{2\kappa}} \lesssim C_\delta \| \mathbb{Y}^1 \|_{\mathscr{C}^{\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^4}^4,$$

renormalized bounds

- pde estimates

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^d} Z_t^2 + (1-\delta)\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right] \leqslant Q(\mathbb{Y}_t^\varepsilon),$$

where

$$Q(\mathbb{Y}_t^\varepsilon) := 1 + C \sum_{k=1,2,3} \|\mathbb{Y}_t^k\|_{\mathscr{C}^{k\alpha}}^K,$$

- stationary coupling

$$\underbrace{\frac{1}{2}\frac{\partial}{\partial t}\mathbb{E}\int_{\mathbb{T}_\varepsilon^d} Z_t^2}_{=0 \text{ (by stationarity)}} + \mathbb{E}\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right] \lesssim \mathbb{E}\, Q(\mathbb{Y}_t^\varepsilon) = \mathbb{E}\, Q(\mathbb{Y}_0^\varepsilon),$$

- probabilistic estimates

$$\sup_\varepsilon \mathbb{E}\, Q(\mathbb{Y}_0^\varepsilon) < \infty,$$

tightness for $\varepsilon \to 0$ at L fixed

$$\sup_{\varepsilon} \int (\|\psi\|^2_{\mathscr{C}^\alpha} + \|\nabla\zeta\|^2_{L^2} + m^2 \|\zeta\|^2_{L^2} + \lambda \|\zeta\|^4_{L^4}) \gamma^{\varepsilon}(\mathrm{d}\psi \times \mathrm{d}\zeta)$$

$$= \sup_{\varepsilon} \mathbb{E}_{\mathbb{P}^\varepsilon}[\|Y_0\|^2_{\mathscr{C}^\alpha} + \|\nabla Z_0\|^2_{L^2} + m^2 \|Z_0\|^2_{L^2} + \lambda \|Z_0\|^4_{L^4}] \lesssim \sup_{\varepsilon} \mathbb{E}\, Q(\mathbb{Y}_0^{\varepsilon}) < C_L < \infty,$$

note that $\|Y_0\|^2_{\mathscr{C}^\alpha} \lesssim Q_L(\mathbb{Y}_0^{\varepsilon})$ if K large enough (estimates not uniform in L)

- ▹ we have tightness of $(\gamma^{\varepsilon})_{\varepsilon}$ for example in $\mathscr{C}^{2\alpha} \times H^{1-\kappa}$ and that any limit γ is supported on $\mathscr{C}^{\alpha} \times (H^1 \cap L^4)$
- ▹ projecting down to $(\nu^{\varepsilon})_{\varepsilon}$ one get tightness of $(\nu^{\varepsilon})_{\varepsilon}$ in $H^{2\alpha} = B^{2\alpha}_{2,2}$:

$$\int \|\varphi\|^2_{B^{\alpha}_{2,2}} \nu^{\varepsilon}(\mathrm{d}\varphi) = \int \|\psi + \zeta\|^2_{B^{\alpha}_{2,2}} \gamma^{\varepsilon}(\mathrm{d}\psi \times \mathrm{d}\zeta) \leqslant 2 \int (\|\psi\|^2_{B^{\alpha}_{2,2}} + \|\zeta\|^2_{B^{\alpha}_{2,2}}) \gamma^{\varepsilon}(\mathrm{d}\psi \times \mathrm{d}\zeta)$$

$$\leqslant 2 \int (\|\psi\|^2_{\mathscr{C}^\alpha} + \|\zeta\|^2_{H^1}) \gamma^{\varepsilon}(\mathrm{d}\psi \times \mathrm{d}\zeta) \leqslant C < +\infty \quad \text{(uniformly in } \varepsilon \to 0)$$

putting all together

- improving the renormalized apriori estimates with a spatial weight ρ (+ some results on weighted Besov spaces) one can prove the same estimates in weighted spaces:

$$\sup_{\varepsilon,L} \int (\|\rho\psi\|_{\mathscr{C}^{\alpha}}^2 + \|\rho\nabla\zeta\|_{L^2}^2 + m^2\|\rho\zeta\|_{L^2}^2 + \lambda\|\rho^{1/2}\zeta\|_{L^4}^4)\,\gamma^{\varepsilon,L}(\mathrm{d}\psi\times\mathrm{d}\zeta)$$

$$\leqslant 1 + C\sup_{\varepsilon,L}\sum_{k=1,2,3}\mathbb{E}\|\rho^{\sigma}\mathbb{Y}_0^{\varepsilon,L,k}\|_{\mathscr{C}^{k\alpha}}^{K} < +\infty$$

(some $\sigma > 0$)

- using Hairer/Steele kind of arguments ad explained yesterday also have uniform exponential bounds of the form

$$\sup_{\varepsilon,L}\int e^{\eta\|\rho(\psi+\zeta)\|_{B_{4,4}^{\alpha}}^4}\gamma^{\varepsilon,L}(\mathrm{d}\psi\times\mathrm{d}\zeta) < \infty$$

for some small η.

Theorem. *Provided* $d=2$ *and we take* $\beta=-3\lambda c_\varepsilon+\beta'$ *for some constant* $\beta'\in\mathbb{R}$ *and* $c_\varepsilon=\mathbb{E}[Y_t^\varepsilon(x)^2]$ *then the family* $(\nu^{\varepsilon,L})_{\varepsilon,L}$ *is tight in* $\mathscr{S}'(\mathbb{R}^2)$.

Any accumulation point ν *is regular, RP and translation invariant and satisfies*

$$\int e^{\eta\|\rho\varphi\|^4_{B^\alpha_{4,4}}}\nu(\mathrm{d}\varphi)<\infty \tag{1}$$

for small $\eta>0$. *(no rotation invariance)*

- any limiting measure ν is non-Gaussian due to (1) (cfr. Hairer/Steele for $d=3$).
- we actually construct a stationary coupling (Y,Z) which solves the system

$$\frac{\partial}{\partial t}Z_t=(\Delta-m^2)Z_t-\lambda Z_t^3-\lambda\mathbb{Y}_t^3+3\lambda\mathbb{Y}_t^2Z_t+3\lambda\mathbb{Y}_t^1Z_t^2+\beta' Y_t+\beta' Z_t$$

$$\frac{\partial}{\partial t}Y_t=(\Delta-m^2)Y_t+\xi(t,\cdot)$$

and for which $X_t=Y_t+Z_t\sim\nu$.

[Detailed construction of the $d=3$ case in Hofmanova/G. - CMP 2021]

uniqueness?

▹ the SQ proof of uniqueness sketched on the lattice in the last lecture fails for the renomalized equation since we do not have anymore convexity (we subtracted an infinite 2nd order polynomial):

$$H = Z^{(1)} - Z^{(2)}, \qquad Y^{(1)} = Y^{(2)} = Y$$

$$\frac{\partial}{\partial t} H_t = (\Delta - m^2) H_t - \lambda \int_0^1 \mathrm{d}\tau \left\{ 3[Z_t^{(2)} + \tau H_t]^2 + 6\, \mathbb{Y}_t^1 [Z_t^{(2)} + \tau H_t] + 3\, \mathbb{Y}_t^2 \right\} H_t + \beta' H_t$$

OPEN PROBLEM

▹ the “standard” approach to uniqueness of the limit (in certain conditions) is via correlation inequalities [see Glimm-Jaffe's book] or cluster expansion.

▹ uniqueness in finite volume via Markovian techniques (irreducibility, see e.g. Hairer-Steele)

▹ any limit coupling $\gamma(\mathrm{d}X \times \mathrm{d}\psi)$ is supported on

$$\mathscr{C}^{\alpha}(\rho) \times (H^1(\rho) \cap L^4(\rho^{1/2}))$$

more regularity of the second component can be obtained by using parabolic estimates on the equation, essentially one can arrive to $2+\alpha$ spatial regularity.

▹ under the measure $\gamma(\mathrm{d}X \times \mathrm{d}\psi)$ we have $\varphi = X + \psi \sim \nu$ and

$$[(\theta_\varepsilon * \varphi)^3 - 3c_\varepsilon(\theta_\varepsilon * \varphi)] = \underbrace{[(\theta_\varepsilon * X)^3 - 3c_\varepsilon(\theta_\varepsilon * X)]}_{\to [\![X^3]\!] \text{ in } \mathscr{C}^{3\alpha}(\rho)} + 3\underbrace{[(\theta_\varepsilon * X)^2 - c_\varepsilon]}_{\to [\![X^2]\!] \text{ in } \mathscr{C}^{2\alpha}(\rho)} \underbrace{(\theta_\varepsilon * \psi)}_{\to \psi \text{ in } H^{1-\kappa}(\rho)}$$

$$+3 \underbrace{(\theta_\varepsilon * X)}_{\to X \text{ in } \mathscr{C}^{\alpha}(\rho)} \underbrace{(\theta_\varepsilon * \psi)^2}_{\to \psi^2 \text{ in } B^{1-\kappa}_{1,1}(\rho)} + (\theta_\varepsilon * \psi)^3$$

$$\xrightarrow[\varepsilon \to 0]{} [\![X^3]\!] + \left\{ [\![X^2]\!]\psi + X\psi^2 + \psi^3 \right\} =: [\![\varphi^3]\!]$$

the terms in the r.h.s are under control as products of Besov functions

integration by parts formula

- at the discrete level we have

$$\int \nabla_\varphi F(\varphi) \nu^{\varepsilon,L}(\varphi) = \int F(\varphi)\left\{(\Delta_\varepsilon - m^2)\varphi - \lambda(\varphi^3 - c_\varepsilon \varphi)\right\}\nu^{\varepsilon,L}(\varphi)$$

- estimates and tightness allow to pass to the limit in this equation and obtain an IBP formula for any accumulation point ν

$$\int \nabla_{\varphi(f)} F(\varphi)\nu(\varphi) = \int F(\varphi)\left\{\varphi((\Delta - m^2)f) - \lambda[\![\varphi^3]\!](f)\right\}\nu(\varphi)$$

where appears the renormalized square $[\![\varphi^3]\!]$ which is well defined under ν as

$$[\![\varphi^3]\!](f) = \lim_{\varepsilon\to 0}\left[\int (\theta_\varepsilon * \varphi)^3 f - 3c_\varepsilon \int (\theta_\varepsilon * \varphi) f\right]$$

- Dyson–Schwinger equations for correlation functions by taking $F(\varphi) = \varphi(f_1)\cdots\varphi(f_n)$:

$$\sum_k \int \varphi(f_1)\cdots\varphi(\not{f}_k)\cdots\varphi(f_n)\nu(\varphi) = \int \varphi(f_1)\cdots\varphi(f_n)\left\{\varphi((\Delta - m^2)f) - \lambda[\![\varphi^3]\!](f)\right\}\nu(\varphi)$$

Goal: develop a stochastic analysis of EQFs
(at least for superrenormalizable models)

- identify "building blocks" and describe EQFs (non-perturbatively) in terms of these simpler objects
- small scales behaviour/renormalization: well understood in most models in some of the approaches (see e.g. recent results of Hairer et al. on Yang-Mills fields)
- coercivity (large fields problem) plays a key role for global control and infinite volume limit. So far, not understood at all for YM
- uniqueness (high or low temp)? still open in most models, especially $\Phi^4_{2,3}$

some recent literature on SQ for EQF

[I list here some results which apply to the $\varepsilon \to 0$ and $L \to \infty$. More results are available on a finite box]

- construction of Φ_3^4 by G./Hofmanova (CMP 2021) and IBP formula
- construction of Sinh-Gordon $d = 2$ (all axioms) by Barashkov/de Vecchi via elliptic SQ (arXiv:2108.12664)
- construction of the $(\exp(\beta\varphi))_2$ model via elliptic SQ (arXiv:1906.11187)
- optimal bounds by Hairer/Steele (arXiv:2102.11685)
- some results on phase transition by Chandra/Gunaratnam/Weber (arXiv:2006.15933)
- ongoing work on control of correlations by Gubinelli/Hofmanova/Rana
- recent paper on perturbation theory for Φ_2^4 by Shen/Zhu/Zhu (arXiv:2108.11312)
- work on the $N \to \infty$ limit of the $O(N)$ model by Shen/Zhu/Zhu

open problems

- How to apply these ideas to gauge theories/geometric models? Higgs model, Yang-Mills? [Hairer/Zambotti/Chandra/Chevyrev/Shen/...] Large field problem not well understood.
- Grassmann fields? [partial progress in Albeverio/Borasi/De Vecchi/G., no renorm yet]
- Small coupling regime? (proof of Borel-summability?)
- Decay of correlations at high temperature? [some results Rana/Hofmanova/G.]
- Dyson-Schwinger (IBP) eq. determine the measure?
- Use the approach for lattice unbounded spin systems?
- What about mass-less models on the lattice: $\nabla\varphi$ models?
- Weak-universality and triviality of models above the critical dimension?
- ...

…end of lecture 3