Elements of Mathematical Quantum Mechanics

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Lecture 1 [April 24rd 2024]

Ideal Plan

1. Introduction to quantum mechanics (Strocchi)

    a. Motivation for quantum mechanics

    b. Axioms (C^* algebras, GNS representation, Hilbert space setting)

    c. Heisenberg group and its representation, Von Neumann theorem, Schrödinger representation

    d. Dynamics and the Hamiltonian (time $t \in \mathbb{R}$) H self-adjoint operator (matrix) Unitary transformation on an Hilbert space $U(t) = e^{iHt}$. $U(t)U(s) = U(t+s)$. $H \geqslant 0$.

    e. Examples: harmonic oscillator, particle in a potential

2. **Euclidean quantum mechanics** ($t \longrightarrow -it = \tau$ imaginary time) $\Longrightarrow$ Probability ('70-'80) Nelson/Symanzik/…

    a. Wick rotation ($t \longrightarrow -it = \tau$ imaginary time) and Feynmann–Kac's formula, Wiener measure and connection with free particles.

    $$U(t) \longrightarrow e^{-H\tau}$$

    b. Eucledian axioms (with *reflection positivity*) and reconstruction theorem

    c. *Nelson's positivity*, uniqueness of ground state and stochastic processes

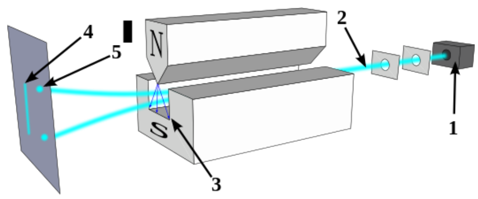
    d. Particle in a potential and symmetric-stationary measure of SDEs with additive noise

    e. Semiclassical limit ($\hbar \longrightarrow 0$) and asymptotic expansion

    f. *Introduction to Euclidean quantum **field** theory.* (special relativity)

The Stern–Gerlach experiment

(1922)



1. Oven. 2. Beam of atoms out of it. 3. Magnet (create a magnetic field) 5. Actual result. 4. Prediction of classical mechanics.

Result:

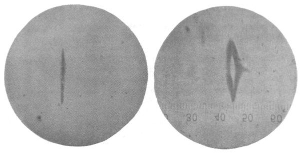
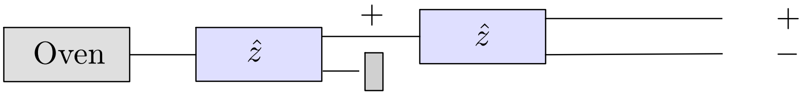
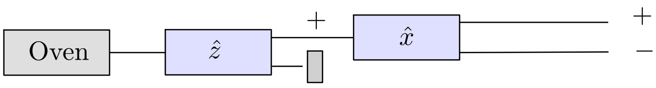


Fig. 2. Fig. 3.

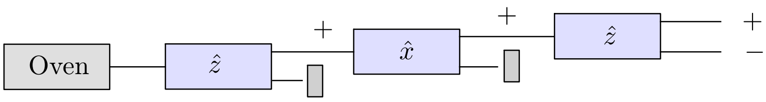
Quantization of spin. $m_{\text{electron}} = \pm M$. Even weirder:



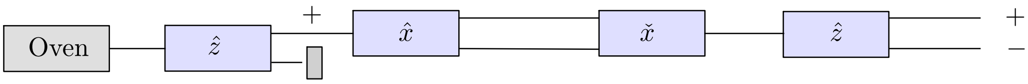
We measure twice $\hat{z}$ and all the atoms go in the + direction.



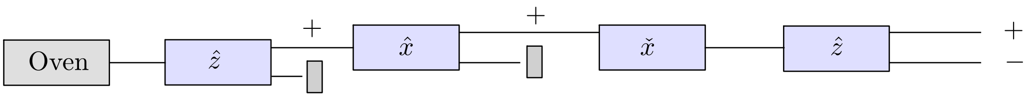
We measure $\hat{z}$ and then $\hat{x}$ and the atoms go in the + direction the 50% of the times.



Now 50%/50%!



100%/0%!



50%/50%!

This is a manifestation of quantum mechanical *interference* effects.

A mathematical model for a measurement process

Main references:

F. Strocchi. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians*. World Scientific Publishing Company, New Jersey, 2 edition edition, oct 2008.

I. E. Segal. Postulates for General Quantum Mechanics. *The Annals of Mathematics*, 48(4):930, oct 1947.

We have two basic players in this game: observables and states.

Observables. An observable is a physical quantity which we can measure (e.g. components of magnetic moment, position, speed/momentum, energy). Connected with some measuring apparatus which has a scale where you read a real number. We write $\mathcal{O}$ for the set of all observables. Given an observable $A \in \mathcal{O}$ more observables can be constructed from A by elementary procedures (i.e. relabeling the scale of the apparatus) E.g. $\lambda A, A^n \in \mathcal{O}$ $\lambda \in \mathbb{R}$. $A^n A^m = A^{n+m}$. In general we could imagine to define in a similar way $f(A)$ for any $f : \mathbb{R} \to \mathbb{R}$. An observable is *positive* if gives only positive results, in symbols we can reformulate this property as $A \geqslant 0 \Longleftrightarrow \exists B \in \mathcal{O} : A \equiv B^2$ (there with $\equiv$ we just mean that operationally the two observables A and B^2 gives the same values).

States. We imagine that a certain physical object under study can be *prepared* in such a way that it is meaningful to speak about repeated experiments on the *same* entity. This entity is the *state* $\omega \in \mathcal{S}$ of the system under consideration. E.g. the state of the atoms in the Stern–Gerlach experiment beam, the state of a particle in motion in a particle accellerator. (And what about "the state of world"?) There is a relation between measurements on states and values of observables and it is "statistical" in the sense that $\omega(A) = \langle \omega, A\rangle \in \mathbb{R}$ represent the measuring of A on the state ω, has to be considered as an average over "experiences". Operationally we measure an observable A in a given state ω by perfoming a sequence of repeated experiments and taking the average

$$\omega(A) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} m_{\omega}^{(i)}(A),$$

where each $m_{\omega}^{(i)}(A)$ is the i-th measurament of A in the state ω. A state is a map $\omega: \mathcal{O} \longrightarrow \mathbb{R}$ understood as all the values it takes on every possible observable $\omega \equiv \{\omega(A) : A \in \mathcal{O}\}$.

You know that different states exists because when we measure an observable we get different numbers:

$$\omega(A) = \omega'(A), \forall A \in \mathcal{O} \Longleftrightarrow \omega = \omega'.$$

You know that two observables are different because there is a state where they give different values:

$$\omega(A) = \omega(B), \forall \omega \in \mathcal{S} \Longleftrightarrow A = B.$$

With respect to the operations we defined on observable we obtain the followin relations:

$$\omega(\lambda A) = \lambda\omega(A), \qquad \omega(A^n + A^m) = \omega(A^n) + \omega(A^m).$$

$$\omega(A^0) = 1 \Longrightarrow A^0 = 1, \omega(1) = 1.$$

$$\omega(A+B) = \omega(A) + \omega(B) \qquad \omega \in \mathcal{S}$$

Functions of an observable:

$$\omega(f(A)) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(m_{\omega}^{(i)}(A)),$$

An observable is positive iff its value on any state is positive:

$$A \geqslant 0 \Longleftrightarrow A = B^2 \Longleftrightarrow \forall \omega : \omega(A) = \omega(B^2) \geqslant 0.$$

States form a convex set: ω_1, ω_2 then for $\lambda \in [0,1]$ (for positivity and normalization):

$$\omega(A) = \lambda\omega_1(A) + (1-\lambda)\omega_2(A)$$

and [I think] this forbids to have (exercise)

$$\omega(AB) = \omega(A)\omega(B).$$

We introduce a norm on $\mathcal{O}$ which measure the size of an observable $A \in \mathcal{O}$ via the largest possible value of a state on it:

$$\|A\| = \sup_{\omega \in \mathcal{S}} |\omega(A)|$$

$$\|\lambda A\| = |\lambda| \|A\|, \quad \|A\| = 0 \Longrightarrow A = 0.$$

We have also (in the notes)

$$\|A^2\| = \|A\|^2.$$

The states induce a linear structure over $\mathcal{O}$: we can define a new observable C by doing

$$\omega(C) = \omega(A) + \omega(B),$$

for given $A, B \in \mathcal{O}$. We can extend $\mathcal{O}$ to a linear space and

$$\|A + B\| \leq \|A\| + \|B\|.$$

The observables form a (pre-)Banach space.

The observables form a Jordan algebra:

$$A \circ B = \frac{1}{2}[(A+B)^2 - A^2 - B^2].$$

At this point we make a leap (of faith) and **assume** that $\mathcal{O}$ are the self-adjoint elements of a C^*-algebra $\mathscr{A}$ over $\mathbb{C}$. [WHY??? I do not know] and

$$A \circ B = \frac{AB + BA}{2}, \qquad A, B \in \mathcal{O}.$$

Crucial techinical assumption. $\mathcal{O} \subseteq \mathscr{A}$ where $\mathscr{A}$ is a (non-commutative) algebra over $\mathbb{C}$ with involution $A \longmapsto A^*$ and such that the following properties are true

$$(\lambda A + \beta B)^* = \bar{\lambda} A^* + \bar{\beta} B^*, \qquad (AB)^* = B^* A^*$$

$$\forall A \in \mathscr{A}, \quad A^* A \geq 0, \qquad \omega(A^* A) \geq 0 \qquad \omega \in \mathscr{S}$$

$$\|AB\| := \sup_{\omega \in \mathscr{S}} |\omega(AB)| \leq \|A\| \, \|B\|. \qquad \boldsymbol{\|A^* A\| = \|A\| \|A^*\|.}$$

Mathematical model for a physical system.

A physical system is the given of observables and states,

- Observables form a C^**-algebra* $\mathscr{A}$ *with unity*.
- States $\mathscr{S}$ are normalized positive linear functionals on $\mathscr{A}$. We assume the set of states to be *full* (i.e. it separates the observables). Moreover observables should separate states (but this is by definition). Usually $\mathscr{S}$ is only a subset of all the positive linear functionals.

Example. Classical mechanical system $(q, p) \in \Gamma \subseteq T^* \mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$ where q is position and p momentum. The set of observables are the (continuous) functions $\mathscr{A} = C(\Gamma, \mathbb{C})$ $f^*(q, p) = \overline{f(q, p)}$. The states are (a subset of) the probability measures on Γ:

$$\omega(f) = \int_\Gamma f(q, p) \omega(\mathrm{d}q \times \mathrm{d}p).$$

$$\|f\| = \sup_{\omega \in \mathscr{S}} |\omega(f)|.$$

The algebra $\mathscr{A}$ is Abelian or commutative: $AB = BA$.

In classical physics one assume that states of the form $\omega = \delta_{(q_0, p_0)}$ are possible, these states are characterised by the fact that the *dispersion* (which can be operationally realized)

$$\Delta_\omega(f) = [\omega(f^2 - \omega(f)^2)]^{1/2} \geqslant 0,$$

is zero for all f.

C^*-algebras

References:

M. A. Naimark. *Normed Algebras.* Springer, Dordrecht, 1972 edition edition, dec 2011.

F. Strocchi. *An Introduction to the Mathematical Structure of Quantum Mechanics: A Short Course for Mathematicians.* World Scientific Publishing Company, New Jersey, 2 edition edition, oct 2008.

Definition 1. *A C^*-algebra $\mathscr{A}$ is an associative algebra over $\mathbb{C}$ which is endowed with the following additional structures: a norm $\|\cdot\|$ for which $\mathscr{A}$ is complete and which satisfy $\|ab\| \leqslant \|a\|\,\|b\|$ for all $a, b \in \mathscr{A}$ and an antilinear involution $*: \mathscr{A} \longrightarrow \mathscr{A}$ for which $(ab)^* = b^* a^*$. These structures satisfy the following compatibility condition (C^* condition)*

$$\|a^* a\| = \|a\|^2, \qquad a \in \mathscr{A}.$$

Example 2. The algebra of all continuous complex-valued functions $C(X)$ on a compact space topological Hausdorff space X wrt. the pointwise product and endowed with the supremum norm

$$\|f\| = \sup_{x \in X} |f(x)|, \qquad f \in C(X)$$

is a C^*-algebra which is commutative or Abelian.

Example 3. Let $\mathscr{H}$ be an Hilbert space. The set of all bounded linear operators $\mathscr{L}(\mathscr{H})$ on $\mathscr{H}$ together with the operator norm

$$\|A\| = \sup_{\varphi \neq 0} \frac{\|A\varphi\|}{\|\varphi\|}, \qquad A \in \mathscr{L}(\mathscr{H}),$$

and the involution given by the adjuction wrt. the scalar product of $\mathscr{H}$, is a C^* algebra, indeed by the property of the Hilbert space norm we have

$$\|A^* A\| = \sup_{\|\varphi\| = 1} \|A^* A\varphi\| = \sup_{\|\varphi\| = \|\psi\| = 1} \langle \psi, A^* A\varphi \rangle = \sup_{\|\varphi\| = \|\psi\| = 1} \langle A\psi, A\varphi \rangle \leqslant \|A\|^2$$

and

$$\|A\|^2 = \sup_{\|\varphi\| = 1} \|A\varphi\|^2 = \sup_{\|\varphi\| = 1} \langle A\varphi, A\varphi \rangle = \sup_{\|\varphi\| = 1} \langle \varphi, A^* A\varphi \rangle \leqslant \|A^* A\|.$$

Any norm-closed subalgebra $\mathscr{B}$ of $\mathscr{L}(\mathscr{H})$ which is self-adjoint, i.e. $\mathscr{B} = \mathscr{B}^*$ is a *concrete* C^*-algebra. For example, the compact operators form such a subalgebra or the C^*-algebra $C(T)$ generated by a single bounded self-adjoint operator T, i.e. the closure of all the polynomials in T, T^*, I.

Example 4. The subalgebra $C^*(a) \subseteq \mathscr{A}$ generated by $a \in \mathscr{A}$ and the unity is a C^*-algebra with the restiction of the norm and the involutions of $\mathscr{A}$. The Banach algebra generated by a set of elements $a_1, \ldots, a_n$ is just the closure of all the polynomials in $a_1, \ldots, a_n$ and in their adjoints.

We call a self-adjoint iff $a = a^*$, a is *normal* if $a a^* = a^* a$. Any a can be decomposed into $a = b + ic$ with b, c self-adjoint. If a is normal then $C^*(a)$ is Abelian (i.e. commutative).

Keep in mind that, for us, the observables of a physical system will be self-adjoints elements of an (abstract) C^* algebra.

Definition 5. *A Banach algebra $\mathscr{B}$ is a Banach space with a product such that $\|ab\| \leq \|a\| \, \|b\|$.*

In any (unital) Banach algebra $\mathscr{B}$ we can define the *spectrum* $\sigma(a) = \sigma_{\mathscr{B}}(a)$ of an element $a \in \mathscr{B}$ to be the set of $\lambda \in \mathbb{C}$ for which $(\lambda - a)$ is not invertible in $\mathscr{B}$. The complement of the spectrum is called the *resolvent set* and

$$R_a(\lambda) = (\lambda - a)^{-1}$$

is the resolvent function.

Theorem 6. *For any $a \in \mathscr{B}$, the spectrum $\sigma(a)$ is a non-empty compact set and the resolvent function is analytic in $\mathbb{C} \backslash \sigma(a)$.*

Proposition 7. *(Spectral radius formula) For any $a \in \mathscr{B}$ we have*

$$\rho(a) := \sup_{\lambda \in \sigma(a)} |\lambda| = \lim_{n \to \infty} \|a^n\|^{1/n} \leq \|a\|$$

with equality in case of a normal element of a C^*-algebra.

In the C^* case we have

$$\|a^2\|^2 = \|a^* a^* a a\| = \|a a^* a^* a\| = \|a^* a\|^2 = \|a\|^4.$$
