Elements of Mathematical Quantum Mechanics

Massimiliano Gubinelli · *TCC course* · *Oxford, Hilary 2024*

Lecture 2 [May 1st 2024]

Some remarks on topics of the last lecture

- Sum of observables: https://physics.stackexchange.com/questions/498675/what-is-the-physical-meaning-of-the-sum-of-two-non-commuting-observables

- N.Drago, S. Mazzucchi, V.Moretti: *An operational construction of the sum of two non-commuting observables in quantum theory and related constructions* **Lett. Math. Phys**, 110 (2020) 3197–3242 DOI: 10.1007/s11005-020-01332-7 *arxiv.org/abs/1909.10974*

- Jordan algebras: https://ncatlab.org/nlab/show/Jordan-Lie-Banach+algebra

1 Spectral theory of C^* algebra

Definition 1. *A Banach algebra $\mathscr{B}$ is a Banach space with a product such that $\|ab\| \leqslant \|a\| \, \|b\|$.*

We will always assume that $1 \in \mathscr{B}$.

C^* algebra $\Longleftrightarrow \|a^* a\| = \|a\|^2$.

Resolvent

$$R_a(\lambda) = (\lambda - a)^{-1}, \qquad \lambda \notin \sigma(a)$$

$\sigma(a)$ spectrum of $a \in \mathscr{B}$. $\lambda \in \sigma(a) \Longleftrightarrow (\lambda - a)$ is not invertible.

Theorem 2. *For any $a \in \mathscr{B}$, the spectrum $\sigma(a)$ is a non-empty compact set and the resolvent function is analytic in $\mathbb{C} \backslash \sigma(a)$.*

Proposition 3. *(Spectral radius formula) For any $a \in \mathscr{B}$ we have*

$$\varrho(a) := \sup_{\lambda \in \sigma(a)} |\lambda| = \lim_{n \rightarrow \infty} \|a^n\|^{1/n} \leqslant \|a\|$$

with equality in case of a normal element of a C^*-algebra.

Remark 4. This shows that C^* are quite rigid, in the sense that the algebraic data defines the norm. The quantity $\varrho(a)$ is called the spectral radius of a.

The space $\mathscr{B}^*$ of linear functionals on $\mathscr{B}$ is a Banach space with the norm $\|\varphi\| = \sup_{a \in \mathscr{B}, \|a\| \leqslant 1} |\varphi(a)|$.

We consider the the coarsest topology for which the maps $\varphi \in \mathscr{B}^* \mapsto \hat{a}(\varphi) = \varphi(a) \in \mathbb{C}$ are continuous for all $a \in \mathscr{B}$.

The Banach-Alaoglu theorem ensure that the closed unit ball of $\mathscr{B}^*$ is compact for the weak-$*$ topology.

A linear functional $\varphi : \mathscr{B} \rightarrow \mathbb{C}$ is multiplicative if $\varphi(ab) = \varphi(a)\varphi(b)$.

[(**Maybe** we have to assume that $\varphi \neq 0$)]

Lemma 5. *The space $\Sigma(\mathscr{B})$ of all the multiplicative linear functionals on $\mathscr{B}$ is a compact Hausdorff space when endowed with the weak-$*$ topology.*

For any $a \in \mathscr{B}$ we can define a continuous function $\hat{a}: \Sigma(\mathscr{B}) \longrightarrow \mathbb{C}$ as $\hat{a}(\varphi) = \varphi(a)$, it is called the Gelfand transform of a.

Note that $\widehat{(ab)}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \hat{a}(\varphi)\hat{b}(\varphi)$.

Theorem 6. *The Gelfand transform is a contractive algebra homomorphism from $\mathscr{B}$ to $C(\Sigma(\mathscr{B}))$. The image algebra separates the points of $\Sigma(\mathscr{B})$.*

For commutative Banach algebra $\mathscr{B}$ any proper maximal ideal is closed and any proper ideal is contained in a proper maximal ideal.

Corollary 7. *Assume $\mathscr{B}$ is commutative. If $a \in \mathscr{B}$ is invertible iff $\hat{a} \in \Sigma(\mathscr{B})$ is invertible, that is $\hat{a}(\varphi) \neq 0$ for all $\varphi \in \Sigma(\mathscr{B})$. Therefore $\sigma(a) = \sigma(\hat{a}) = \{\varphi(a) : \varphi \in \Sigma(B)\}$ and $\sup\{|\lambda| : \lambda \in \sigma(a)\} = \|\hat{a}\|_\infty$.*

Example 8. For $L^1(\mathbb{R};\mathbb{C})$ with product given by convolution the Gelfand transform is the Fourier transform. For $L^1(\mathbb{R}_+;\mathbb{C})$ with half-line convolution the Gelfand transform is the Laplace transform.

[what about the unit in these examples. Possibility is to add a unit and check that the result is about a particular ideal]

In the case of the convolution algebra let

$$\varphi_\omega(a) := \int_{\mathbb{R}} e^{it\omega} a(t)\mathrm{d}t, \qquad \omega \in \mathbb{R}$$

$$\varphi_\omega(ab) = \int_{\mathbb{R}} e^{it\omega}(a * b)(t)\mathrm{d}t = \varphi_\omega(a)\varphi_\omega(b)$$

so $\varphi_\omega \in \Sigma(\mathscr{B})$ for any $\omega \in \mathbb{R}$. [Question: how to prove that these are all the multiplicative functionals?]

In the case of C^* algebras we have an isomorphism $\mathscr{A} \approx C(\Sigma(\mathscr{A}))$ of C^* algebras.

Theorem 9. (Gelfand–Naimark) *Any abelian C^*-algebra $\mathscr{A}$ is isometrically isomorphic to $C(\Sigma(\mathscr{A}))$.*

The proof requires to check that $\varphi(a^*) = \overline{\varphi(a)}$ and uses then the equality between the spectral radius and the norm in a C^* algebra.

Remark 10. Multiplicative linear functionals in $\mathscr{B}$ corresponds to maximal proper ideals. See Strocchi for the details.

In the $C(\Sigma(\mathscr{B}))$ case these maximal proper ideals corresponds to functions which are zero in a single point.

Exercise 1. Take the commutative C^* algebra $\mathscr{A}$ of diagonal $n \times n$ matrices. Prove that it is a C^*-algebra with the structure inherited from the space of all matrices, i.e. norm is the operator norm, involution is the adjoint, product is product of matrices. Try to work out the space $\Sigma(\mathscr{A})$.

If $a \in \mathscr{A}$ is normal, then the C^* algebra $C^*(a)$ (generated by $1, a, a^*$) is Abelian and therefore isomorphic to $C(\Sigma(C^*(a)))$ but $\varphi \in \Sigma(C^*(a))$ is uniquely determined by the value of $\varphi(a) \in \mathbb{C}$ since for any polynomial $p(a, a^*)$ we have $\varphi(p(a, a^*)) = p(\varphi(a), \overline{\varphi(a)})$.

In particular

$$f(a) = \lim_{n\to\infty} p_n(a, a^*) \qquad \text{where} \sup_{x \in \sigma(a)} |p_n(x, x^*) - f(x)| \longrightarrow 0.$$

Note that $\sigma_{\mathscr{A}}(a) = \sigma_{C^*(a)}(a)$.

Then $\sigma(a) = \sigma(\hat{a}) = \{\varphi(a) : \varphi \in \Sigma(C^*(a))\}$ and $\Sigma(C^*(a)) = \sigma(a)$. This means that for any $f \in C(\sigma(a))$ there exists a unique $h \in C^*(a)$ such that $\hat{h} = f$ under the Gelfand transform map. In this case we write $h = f(a)$.

And we have $\sigma(f(a)) = f(\sigma(a))$ (spectral mapping principle). This holds for continuous functions f and normal elements a or for any element $a \in \mathscr{A}$ and f holomorphic by defining $f(a)$ by power series.

Moreover $\sigma(ab)$ and $\sigma(ba)$ differ at most by $\{0\}$. (proof in the notes)

If a is unitary (i.e $aa^* = 1 = a^* a$) then $\sigma(a) \subseteq \{z : |z| = 1\}$.

1.1 Positive elements

Definition 11. *We call $a \in \mathscr{A}$ positive if a is self-adjoint and $\sigma(a) \subseteq \mathbb{R}_{\geqslant 0}$ and we denote with $\mathscr{A}_+$ the set of positive elements of $\mathscr{A}$ and also write $a \geqslant 0$.*

Some properties

- If $a, b \in \mathscr{A}_+$ and $a + b = 0$ then $a = b = 0$.
- If a is self-adjoint and $\|a\| \leqslant 1$ then $a \in \mathscr{A}_+$ iff $\|1 - a\| \leqslant 1$.
- $\mathscr{A}_+$ is a closed convex cone.
- By functional calculus every positive element has a positive square root $a^{1/2}$.
- By functional calculus one can decompose any self-adjoint element into the difference of two positive elements $a = a_+ - a_-$.
- Any element $a \in \mathscr{A}$ is the sum of four unitaries. Indeed write $a = p + iq$ with self-adjoint elements p, q and then assuming $\|p\|, \|q\| \leqslant 1$ consider the unitaties $p \pm i(1 - p^2)^{1/2}$ and $q \pm i(1 - q^2)^{1/2}$.
- Note that in $\mathscr{L}(\mathscr{H})$ any operator in the form $A^* A$ is positive (i.e. it has positive spectrum).

Theorem 12. *In a C^* algebra the following properties are equivalent:*

1) $a \in \mathscr{A}_+$, *2) $a = b^2, b = b^*$,* *3) $\|1 - a / \|a\|\| \leqslant 1$,* *4) $a = c^* c$.*

Remark 13. If $a, b \geqslant 0$ then $a + b \geqslant 0$ however positivity is tricky due to non-commutativity. For example even if $0 \leqslant a \leqslant b$ it does not follow in general that $a^2 \leqslant b^2$ unless a, b commute. If we try to define

$$|a| = (a^* a)^{1/2}$$

then is not true that $|a + b| \leqslant |a| + |b|$.

Let us give some true inequalities.

- We have $a \leqslant \|a\|$ and $a^2 \leqslant \|a\| a$ as easily seen from spectral consideations.
- $a \geqslant 0$ implies $cac^* \geqslant 0$ and by difference $a \geqslant b \Longrightarrow cac^* \geqslant cbc^*$.
- $a \geqslant b \geqslant 0$ then $(\lambda - a)^{-1} \leqslant (\lambda - b)^{-1}$ for $\lambda \geqslant 0$. (see Meyer for a proof)
- $a \geqslant b \Longrightarrow f(a) \geqslant f(b)$ for functions of the form $f(x) = x^\alpha$ with $\alpha \in (0, 1)$.

Proposition 14. *Let ω be a continuous linear functional such that $\|\omega\| = \omega(1) = 1$ then $\omega(a^*) = \overline{\omega(a)}$.*

Proof. We can assume that a is s.a. since then is easy to conclude. Assume that $\omega(a) = f + ig$ with $f, g \in \mathbb{R}$ I need to prove that $g = 0$. Take $a + ic$ with $c \in \mathbb{R}$ and observe that $(a + ic)^*(a + ic) = a^2 + c^2$ then $\omega(a + ic) = f + i(g + c)$ so

$$f^2 + (g + c)^2 = |\omega(a + ic)|^2 \leq \|a + ic\|^2 \underset{C^*}{=} \|(a + ic)^*(a + ic)\| = \|a^2 + c^2\| \leq \|a^2\| + c^2.$$

Now c is arbitrary so we get that $g^2 + 2gc \leq \|a\|^2$ which is impossible unless $g = 0$. □

1.2 States on C^* algebras

A linear functional on $\mathscr{A}$ is positive if $\omega(a) \geq 0$ for all $a \in \mathscr{A}_+$.

For positive linear functionals Cauchy-Schwarz inequality holds true:

$$|\omega(a^* b)|^2 \leq \omega(a^* a)\omega(b^* b).$$

Proposition 15. *A linear functional $\omega \in \mathscr{A}^*$ is positive iff $\|\omega\| = \omega(1)$.*

Proposition 16. *Positive linear functionals separate $\mathscr{A}$ and $a \in \mathscr{A}_+$ iff $\omega(a) \geq 0$ for all positive linear functionals ω.*

Recall that a *state* is a normalized ($\omega(1) = 1$) positive linear functional on $\mathscr{A}$.

The set of positive linear functionals of norm ≤ 1 is a compact convex closed set (in the weak-$*$ topology). By a theorem of Krein–Milman it is the closed convex hull of its extreme points which are called *pure states*. Recall that an extreme point of a convex set is a point which cannot be written as the convex combination of other points. Pure states separate points in $\mathscr{A}$.

Example 17. On $\mathscr{L}(H)$ the states given by $\omega(A) = \langle x, Ax \rangle$ for some normalized $x \in H$ are pure states. While on $C(X)$ for a compact Hausdorff space X all the pure state are of the form $\omega_x(f) = f(x)$ with $x \in X$.
