Elements of Mathematical Quantum Mechanics

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1 The GNS representation

GNS = Gelfand–Naimark–Segal $\Longrightarrow$ Hilbert space picture = choice of coordinates on a C* algebra.

Representations of C* algebras in Hilbert space (this will allow us to do computations).

Analogy: coordinates on a manifold

The Gelfand–Naimark–Segal theorem allows to construct representations of C^* algebras on an Hilbert space starting from any state ω (i.e. normalized positive linear functional).

A representation: a map $\varphi: \mathscr{A} \longrightarrow \mathscr{L}(H)$ for some complex Hilbert space H such that φ is linear, $\varphi(1) = 1$, $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ where on the r.h.s. the involution is understood as the adjoint in the Hilbert space. This is also called a $*$-homomorphism.

More interesting: Any positive multiplicative functional $\varphi: \mathscr{A} \longrightarrow \mathbb{C}$ give a one-dimensional representation on the Hilbert space $H = \mathbb{C}$. [Can we give up positivity?] Recall that positive linear functionals ω satisfy $\omega(a^*) = \overline{\omega(a)}$.

Lemma 1. *Any representation is a contraction.*

Proof. Let's assume that a is normal. Then Note that if $\lambda - a$ is invertible in $\mathscr{A}$ then exists $c \in \mathscr{A}$ s.t. $c(\lambda - a) = 1$ that implies $\varphi(c)(\lambda - \varphi(a)) = 1$ so $\lambda - \varphi(a)$ is also invertible, that is $\sigma_{\mathscr{L}(H)}(\varphi(a)) \subseteq \sigma_{\mathscr{A}}(a)$. So for C^*-algebras

$$\|\varphi(a)\|_{\mathscr{L}(H)} \underset{C^*}{=} \varrho_{\mathscr{L}(H)}(\varphi(a)) \leqslant \varrho_{\mathscr{A}}(a) \underset{C^*}{=} \|a\|.$$

Now if b is a generic element then $a = b^* b$ is self-adjoint. And using the C* property twice we have

$$\|\varphi(b)\|_{\mathscr{L}(H)}^2 = \|\varphi(b)^* \varphi(b)\|_{\mathscr{L}(H)} = \|\varphi(a)\|_{\mathscr{L}(H)} \leqslant \|a\| = \|b\|^2.$$

□

We want to construct representations.

Assume ω is a state and define the Hermitean form on $\mathscr{A}$:

$$\langle a, b \rangle_\omega = \omega(a^* b).$$

The linear space $\mathscr{A}$ with this scalar product is a pre-Hilbert space. Let

$$\mathscr{N}_\omega = \{a \in \mathscr{A} : \langle a, a\rangle_\omega = 0\}$$

the set of zero elements and define the Hilbert space $H_\omega = \overline{\mathscr{A} \backslash \mathscr{N}_\omega}$ where the bar denotes the completion wrt. the topology generated by the scalar product $\langle , \rangle_\omega$. Denotes $\|a\|_\omega = \langle a, a\rangle_\omega^{1/2}$ the corresponding norm.

Observe that since $\|b^* b\| a^* a - a^* b^* b a \geqslant 0$ then

$$\langle ba, ba\rangle_\omega = \omega(a^* b^* b a) \leqslant \|b^* b\| \omega(a^* a) = \|b\|^2 \omega(a^* a) = \|b\|^2 \langle a, a\rangle_\omega$$

so the operator $L_b: H_\omega \longrightarrow H_\omega$ defined by $L_b a = ba$ on the dense subset $\mathscr{A}$ is bounded with norm $\|L_b\| \leqslant \|b\|$. Note that it is well defined, since $L_b a = 0$ if $a \in \mathscr{N}$.

Moreover $L_b L_c = L_{bc}$ and $L_b^* = L_{b^*}$ as can be easily checked. Therefore $a \mapsto L_a$ is an homomorphism of C^* algebras (since $\{L_a : a \in \mathscr{A}\}$ is a C^* subalgebra of $\mathscr{L}(H_\omega)$), indeed recall that $\|L_b\|^2_{\mathscr{L}(H_\omega)} = \|L_b^* L_b\|_{\mathscr{L}(H_\omega)}$. So $\varphi_\omega(a) = L_a$ is a representation of $\mathscr{A}$ on H_ω and if we denote by $\Omega_\omega = [1] \in \mathscr{H}_\omega$ we have

$$\omega(a) = \langle \Omega_\omega, L_a \Omega_\omega\rangle.$$

This is the GNS construction.

Note that the set $\{L_a \Omega_\omega : a \in \mathscr{A}\} \subseteq H_\omega$ is dense in H_ω. Then one says that Ω_ω is a cyclic vector for the representation φ_ω and that the representation is cyclic.

If K is another Hilbert space supporting a cyclic representation $\pi : \mathscr{A} \longrightarrow \mathscr{L}(K)$ with cyclic vector $\psi \in K$ such that $\omega(a) = \langle \psi, \pi(a) \psi\rangle_K$ then the map $a \in \mathscr{A} \mapsto \pi(a) \psi \in K$ is an densely defined isometry from H_ω to K since

$$\langle a, a\rangle_{H_\omega} = \omega(a^* a) = \langle \pi(a) \psi, \pi(a) \psi\rangle_K.$$

Therefore the cyclic representations of $\mathscr{A}$ associated to a state ω are unique up to isomorphism.

In general one call it the GNS representation associated to the state ω.

Example 2. Consider the commutative setting and let $H = L^2(\Omega, \mathscr{F}, \mu)$ for some probability space $(\Omega, \mathscr{F}, \mu)$ then on this space there are three different C^* algebras acting with pointwise multiplication on the elements of H: that of the continuous functions (taking $\mathscr{F}$ to be the Borel σ-algebra on some compact space K), that of the bounded measurable functions and that of the $L^\infty(\mu)$ functions (i.e. equivalence classes modulo μ-null sets).

Remark 3. The space H_ω of the GNS construction can be thought as a non-commutative version of the commutative $L^2(\Omega, \mathscr{F}, \mu)$. However here right multiplication $R_b a = ab$ is not in general given by a bounded operator.

Theorem 4. *(Gelfand–Naimark) The exists a faithful representation of $\mathscr{A}$ in Hilbert space H*

Faithful means that $\varphi(a) = 0 \Longrightarrow a = 0$. (i.e. the representation is injective)

Then GN theorem shows that there is no loss of generality to consider representations of physical systems in Hilbert space.

Remark 5. Consider a state ω and a self-adjoint a such that ω is dispersion-free wrt. a, i.e. $0 = \Delta_\omega(a) = [\omega((a - \omega(a))^2)]^{1/2}$ then in the corresponding GNS representation we have

$$\omega((a - \omega(a))^2) = \langle \Omega_\omega, (\varphi(a) - \omega(a))^2 \Omega_\omega \rangle = \|(\varphi(a) - \omega(a))\Omega_\omega\|^2$$

so $(\varphi(a) - \omega(a))\Omega_\omega = 0$ and $\omega(a)$ is an eigenvalue of $\varphi(a)$ with eigenvector Ω_ω. In particular $\omega(a)$ should be in $\sigma(\varphi(a)) \subseteq \sigma(a)$.

1.1 Pure states and irreducible representations

Definition 6. *A representation φ on the Hilbert space H is* ***irreducible*** *if the only invariant subspaces of the family $(\varphi(a))_{a \in \mathscr{A}}$ are $\{0\}$ and H.*

For any family $\mathscr{B} \subseteq \mathscr{L}(H)$ we denote by $\mathscr{B}'$ the commutant of $\mathscr{B}$, that is the set

$$\mathscr{B}' = \{C \in \mathscr{L}(H) : [C, B] = 0, \forall B \in \mathscr{B}\},$$

where $[C, B] = CB - BC$. Note that $\mathscr{B} \subseteq \mathscr{B}''$ and that $\mathscr{B}' \supseteq \mathbb{C} = \{\lambda \mathbb{1} : \lambda \in \mathbb{C}\}$.

Lemma 7. *The representation $\varphi: \mathscr{A} \longrightarrow \mathscr{L}(H)$ is irreducible iff $\varphi(\mathscr{A})' = \mathbb{C} = \{\lambda \mathbb{1} : \lambda \in \mathbb{C}\}$.*

Proof. If φ is reducible then let P the orthogonal projection on a non-trivial invariant subspace. Let $v \in PH$ then we have $\varphi(a) v \in PH$ and $\varphi(a) P v = \varphi(a)\, v = P \varphi(a) v$. If $v \notin PH$ then $v \in QH$ with $Q = 1 - P$ and then for any w

$$\langle w, \varphi(a) Q v \rangle = \langle \varphi(a)^* w, Q v \rangle = \langle \varphi(a)^* (P + Q) w, Q v \rangle$$

$$= \langle P \varphi(a^*)\, w, Q v \rangle + \langle \varphi(a)^* Q w, Q v \rangle = \langle Q w, \varphi(a) v \rangle = \langle\, w, Q \varphi(a) v \rangle$$

so $[\varphi(a), Q] = 0$. Then is clear that $Q \in \varphi(\mathscr{A})'$ so finished.

Reciprocally if $R \in \varphi(\mathscr{A})'$ is a nontrivial self-adjoint element of $\mathscr{L}(H)$, by spectral calculus we can produce a projection $P \in \varphi(\mathscr{A})'$ by setting $P = \chi(R)$ with $\chi: \mathbb{R} \longrightarrow \mathbb{R}$ some characteristic function of a subset of $\mathbb{R}$, then $P^2 = P$ so P is indeed a projection and the associated subspace is invariant under $\varphi(\mathscr{A})$ since P commute with any $\varphi(a)$. □

Proposition 8. *The GNS representation φ_ω is irreducible iff ω is extremal in the set of states, i.e. a pure state for $\mathscr{A}$.*
