Elements of Mathematical Quantum Mechanics

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Lecture 6 [May 29th 2024]

In the last lecture we understood that to any strongly continuous family $(U(t))_{t\in\mathbb{R}}$ of operators in $\mathscr{L}(H)$ we can associate in essentially a unique way an $*$-homomorphism $Q: C(\mathbb{R}) \to \mathscr{L}(H)$ such that $Q(e^{i\alpha\cdot}) = U(\alpha)$ and similarly for V: $P: C(\mathbb{R}) \to \mathscr{L}(H)$ such that $P(e^{i\beta\cdot}) = V(\beta)$, then we can write $f(Q) := Q(f)$ and have that Weyl relations have the form

$$e^{i\alpha Q}e^{i\beta P} = e^{i\beta P}e^{i\alpha Q}e^{i\alpha\beta}.$$

Recall that $Q(f)$ is defined by polarisation of the relation $\langle v, Q(f)v\rangle = \ell^v(f)$ where v is any unit vector in H and ℓ^v is the unique positive functional on $C(\mathbb{R})$ which has appropriate locality properties (and therefore corresponds to a unique Borel probability measure μ^v on $\mathbb{R}$) and such that $\ell^v(e^{i\alpha\cdot}) = \langle v, U(\alpha)v\rangle$.

From the point of the of C^*algebraic approach the homomorphism Q, P represents families of observables which are then given by choosing a particular way f to measure the quantity Q so that we have a definite observable $Q(f)$, i.e. self-adjoint element of C^*. Let's call them *extended observables*.

For the moment we understand a Weyl C^*-algebra as given by the concrete realisation in $L^2(\mathbb{R})$ (in particular regular):

$$U(\alpha)f(x) = e^{i\alpha x}f(x), \qquad V(\beta)f(x) = f(x-\beta).$$

Note that we can form the Weyl operators $(W(z))_{z\in\mathbb{C}}$ defined for $z = \alpha + i\beta \in \mathbb{C}$ as

$$W(\alpha + i\beta) := e^{i\alpha\beta/2}e^{i\alpha Q}e^{i\beta P}.$$

One can check that $W(z)$ is unitary for any $z \in \mathbb{C}$ and that

$$W(z)W(z') = e^{i\mathrm{Im}\langle z, z'\rangle}W(z + z'), \qquad z, z' \in \mathbb{C} \tag{1}$$

where $\langle z, z'\rangle = \bar{z}z'$ is the Hermitian scalar product of $\mathbb{C}$ (a one dimesional complex Hilbert space).

Remark that $\omega(z, z') = \mathrm{Im}\langle z, z'\rangle$ is antisymmetric i.e. $\omega(z, z') = -\omega(z', z)$ and that $\omega(z, z') = 0$ for all z implies $z' = 0$ (i.e. ω is non-degenerate).

Let

$$\tilde{W}(z,\lambda) := e^{i\lambda} W(z)$$

for $\lambda \in \mathbb{R}$ then

$$\tilde{W}(z,\lambda)\,\tilde{W}(z',\lambda') = \tilde{W}(z+z',\lambda+\lambda'+\mathrm{Im}\langle z,z'\rangle),$$

which means that the $(\tilde{W}(z,\lambda))_{z,\lambda}$ give a unitary representation of the *Heisenberg group* $\mathbb{H} \approx \mathbb{C}\times\mathbb{R}$ with composition $(z,\lambda)(z',\lambda') = (z+z',\lambda+\lambda'+\mathrm{Im}\langle z,z'\rangle)$. It a non-commutative group since ω is not symmetric.

Theorem 1. *(Von Neumann) Regular irreducible representations of the (finite dimensional, i.e. where instead of* $\mathbb{C}$ *we consider* $\mathbb{C}^n$*) Weyl relations are all unitarily equivalent, i.e there is only one up to isomorphism.*

This is quite stricking. Think about having only one $(U(t) = e^{itQ})_{t\in\mathbb{R}}$ strongly continuous group representation of $\mathbb{R}$. Then any multiplicative state of the form

$$\varphi_\alpha(U(t)) = e^{it\alpha}.$$

Remark 2. Regular representation means that

$$t \longmapsto \pi(W(tz))$$

(where we understand we are considering the representation π) is weakly continuous in the Hilbert space for any $z\in\mathbb{C}$. This is equivalent to strong continuous.

Proof. (one dimensional case) Let us introduce the operator

$$P := \int_{\mathbb{R}^2} \mathrm{d}\alpha\,\mathrm{d}\beta\, e^{-(|\alpha|^2+|\beta|^2)/4} e^{i\alpha\beta/2} e^{i\alpha Q} e^{i\beta P} = \int_{\mathbb{C}} e^{-|z|^2/4} W(z)\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

which is well defined as a strong integral, i.e when computed on vectors $\psi\in H$ (regularity is needed here, at least).

We can check that $P\neq 0$ by observing that

$$W(-w)\,W(z)\,W(w) = e^{i\mathrm{Im}\langle z,w\rangle} W(-w)\,W(z+w) = e^{i\mathrm{Im}\langle z,w\rangle} e^{i\mathrm{Im}\langle -w,z+w\rangle} W(z) = e^{i2\mathrm{Im}\langle z,w\rangle} W(z)$$

and looking at

$$W(-w)PW(w) = \int_{\mathbb{C}} e^{-|z|^2/4} W(-w)\,W(z)\,W(w)\,\mathrm{d}z\,\mathrm{d}\bar{z} = \int_{\mathbb{C}} e^{-|z|^2/4} e^{i2\mathrm{Im}\langle z,w\rangle} W(z)\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

Assume that $P=0$, so we have $W(-w)PW(w)=0$ and for any vector $\psi\in H$ we will have for any $w\in\mathbb{C}$

$$0=\int_{\mathbb{C}} e^{-|z|^2/4}e^{i2\mathrm{Im}\langle z,w\rangle}\langle\psi, W(z)\psi\rangle \mathrm{d}z\,\mathrm{d}\bar{z}$$

by Fourier transform with respect to both real and imaginary part of w we deduce that

$$e^{-|z|^2/4}\langle\psi, W(z)\psi\rangle=0$$

for almost all $z\in\mathbb{C}$ and by continuity of this function we have that $\langle\psi, W(z)\psi\rangle=0$ for all z, and ψ but this is in contradiction with $W(0)=1$.

With a tedious but elementary computartion with Fubini theorem and Gaussian integrals one can check that (exercise)

$$PW(w)P=e^{-|w|^2/4}P, \qquad w\in\mathbb{C}$$

so in particular this says that $P^2=P$ and since is clear by definition that $P^*=P$ we have that that P is a non-trivial projection (it cannot be $P=1$).

So let ψ_0 be a unit vector in $\mathrm{Im}(P)$ so that $P\psi_0=\psi_0$.

By irreducibility the linear space $\mathscr{D}:=\mathrm{span}\{W(z)\psi_0: z\in\mathbb{C}\}$ is dense in H since any element of the C^*-algebra generated by $(W(z))_{z\in\mathbb{C}}$ can be approximated by linear combination of $W(z)$s.

We have also that ψ_0 is the only eigenvector of P since if φ is another one orthogonal to ψ_0 we have

$$\langle\varphi, W(z)\psi_0\rangle=\langle P\varphi, W(z)P\psi_0\rangle=\langle\varphi, PW(z)P\psi_0\rangle=e^{-|z|^2/4}\langle\varphi,\psi_0\rangle=0$$

so we learn that $\langle\varphi, W(z)\psi_0\rangle=0$ for all z but then $\langle\varphi,\psi\rangle=0$ for all $\psi\in\mathscr{D}$ and this implies that $\varphi=0$.

We learned also that there is a state ω such that

$$\omega_0(W(z))=\langle\psi_0, W(z)\psi_0\rangle=e^{-|z|^2/4}.$$

(this relation define ω_0 on the full C^*-algebra, because any element can be approx. by linear comb of Ws).

Now if $(H,(W(z))_{z\in\mathbb{C}})$ and $(H',(W'(z))_{z\in\mathbb{C}})$ are two irreducible regular representations of the Weyl algebra we can construct a unitary operator $U: H\longrightarrow H'$ by extending by linearity the equality

$$UW(z)\psi_0=W'(z)\psi_0'$$

to the full $\mathscr{D}$ and observe that U is unitary since

$$\langle UW(z)\psi_0, UW(w)\psi_0\rangle = \langle W'(z)\psi_0', W'(w)\psi_0'\rangle = \langle \psi_0', PW'(-z)W'(w)P\psi_0'\rangle$$

$$= e^{-i\mathrm{Im}\langle z,w\rangle}\langle \psi_0', PW'(w-z)P\psi_0'\rangle = e^{-i\mathrm{Im}\langle z,w\rangle}e^{-|w-z|^2/4} = \langle W(z)\psi_0, W(w)\psi_0\rangle$$

therefore is bounded and can be extended to a unitary operator on the whole H. This show that the two representations of the Weyl relations are unitarily equivalent.

□

The regular state ω_0 such that

$$\omega_0(W(z)) = e^{-|z|^2/4}$$

is called Fock vacuum or vacuum state for the Weyl representation.

Since the representation of the Weyl relation is essentially unique we could think to use the one we like (or the one more convenient).

One of them is the Schrödinger representation which is given on $H = L^2(\mathbb{R})$ by taking

$$U(t)f(x) = e^{itx}f(x), \qquad V(s)f(x) = f(x-s), \qquad f\in H, t, s\in\mathbb{R}.$$

Is this irreducible?

If it is not irreducible then there exists two unit vectors $f, g\in L^2(\mathbb{R})$ such that for all t, $s\in\mathbb{R}$

$$0 = \langle f, U(t)V(s)g\rangle = \int_{\mathbb{R}}\bar{f}(x)e^{itx}g(x-s)\mathrm{d}x.$$

But then if this is true for any t we have that (by Fourier transform)

$$|\bar{f}(x)g(x-s)| = 0$$

for almost every s and x.

But then if this is true for any t we have that (by Fourier transform) $|\bar{f}(x)g(x-s)| = 0$ for almost every s and x, by squaring and integrating in x, s we have

$$0 = \int \mathrm{d}x\int \mathrm{d}s|\bar{f}(x)g(x-s)|^2 = \|f\|_{L^2}^2\|g\|_{L^2}^2 = 1$$

so we have a contradiction and this proves that the Schrödinger representation is irreducible.

Therefore there must exist a vector $\psi_0 \in L^2(\mathbb{R})$ such that

$$\langle \psi_0, e^{-its/2} U(t) V(s) \psi_0 \rangle = \exp\left(-\frac{1}{4}(s^2+t^2)\right), \qquad s, t \in \mathbb{R}$$

and by taking $s = 0$ we have

$$\int |\psi_0(x)|^2 e^{itx} \mathrm{d}x = \exp\left(-\frac{t^2}{4}\right)$$

which means that $|\psi_0(x)|^2$ is a Gaussian function (actually the density of a $\mathcal{N}(0, 1/2)$ random variable), namely

$$|\psi_0(x)|^2 = \frac{1}{(\pi)^{1/2}} e^{-x^2}$$

this determines ψ_0 up to a phase factor:

$$\psi_0(x) = e^{if(x)} \frac{1}{(\pi)^{1/4}} e^{-x^2/2}.$$

However

$$\exp\left(-\frac{s^2+t^2}{4}\right) = \langle \psi_0, e^{-its/2} U(t) V(s) \psi_0 \rangle$$

$$= e^{-its/2} \int \mathrm{d}x e^{itx} e^{-if(x)} \frac{1}{(\pi)^{1/4}} e^{-x^2/2} e^{if(x-s)} \frac{1}{(\pi)^{1/4}} e^{-(x-s)^2/2}$$

$$= \frac{e^{-its/2}}{(\pi)^{1/2}} \int \mathrm{d}x e^{it(x+s/2)} e^{-if(x+s/2)} e^{-(x+s/2)^2/2} e^{if(x-s/2)} e^{-(x-s/2)^2/2}$$

$$= \frac{e^{-s^2/4}}{(\pi)^{1/2}} \int \mathrm{d}x e^{-x^2} e^{itx} e^{i(f(x-s/2)-f(x+s/2))}$$

so we have

$$\frac{1}{(\pi)^{1/2}} \int \mathrm{d}x e^{itx} e^{i(f(x-s/2)-f(x+s/2))} e^{-x^2} = \exp\left(-\frac{t^2}{4}\right)$$

Now is better because this is saying that the function

$$\frac{1}{(\pi)^{1/2}} e^{i(f(x-s/2)-f(x+s/2))} e^{-x^2}$$

is the density of a Gaussian $\mathcal{N}(0, 1/2)$ so it is equal to $\frac{1}{(\pi)^{1/2}} e^{-x^2}$ and we conclude that $f = 0$, so we have proven that, in the Schrödinger representation we have

$$\psi_0(x) = \frac{e^{-x^2/2}}{\pi^{1/4}}.$$

▸ **Gaussian representation.** We can introduce the unitary transformation (ground state transformation)

$$\mathcal{J}: L^2(\mathbb{R}) \to L^2(\gamma)$$

where γ is the Gaussian measure with mean zero and variance $1/2$ by letting

$$(\mathcal{J}\psi)(x) = \psi(x) / \psi_0(x), \qquad x \in \mathbb{R}.$$

Then we have the images U', V' of the Weyl pair U, V given by (for $f \in L^2(\gamma)$)

$$U'(t) f(x) = (\mathcal{J} U(t) \mathcal{J}^{-1} f)(x) = \psi_0(x)^{-1} U(t)(\psi_0 f)(x) = e^{itx} f(x)$$

$$V'(s) f(x) = (\mathcal{J} V(s) \mathcal{J}^{-1} f)(x) = \psi_0(x)^{-1} V(s)(\psi_0 f)(x) = \psi_0(x)^{-1} \psi_0(x-s) f(x-s)$$

$$= e^{xs - s^2/2} f(x-s)$$

One can check directly that this gives indeed a strongly continuous representation of the Weyl relation on $L^2(\gamma)$.

This is called the Gaussian representation and is useful because there is a nice basis for $L^2(\gamma)$ given by polynomial functions, the Hermite basis $(h_n(x))_{n \geqslant 0}$ (indeed note that polynomials are in $L^2(\gamma)$ and that one can perform a Gram–Schmidt ortogonalisation procedure of the family $(x^n)_{n \geqslant 0}$ which is a separating family for $L^2(\gamma)$ by Stone-Weierstrass) and every $h_n(x)$ has monomial of highest degree n.

▸ **Reducible (regular) representations of Weyl relations.**

Assume now that $(W(z))_{z \in \mathbb{C}}$ does not act irreducibly on H then the range of P is not one dimensional.

Corollary 3. *Any regular representation $((W(z))_{z \in \mathbb{C}}, H)$ of the Weyl relations is unitarily equivalent to the representation $((W^\#(z))_{z \in \mathbb{C}}, L^2(\mathbb{R}) \otimes K)$ where $K = PH$ and $W^\#(z)$ acts trivially on K and as the Schrödinger representation on $L^2(\mathbb{R})$, i.e.*

$$W^\#(z)(\psi^\# \otimes \psi^\$) = (W_{\text{Schrödinger}}(z) \psi^\#) \otimes \psi^\$, \qquad z \in \mathbb{C}, \psi^\# \in L^2(\mathbb{R}), \psi^\$ \in K.$$

Theorem 4. *For any $Q \geqslant 1/2$ there exists a state ω_Q on the Weyl algebra such that*

$$\omega_Q(W(z)) = e^{-Q|z|^2/2}.$$

Moreover we know that for $Q = 1/2$ is pure (because it corresponds to the Schrödinger model) and for $Q > 1/2$ it is not.

Question 1. It is a fact that there not exists states on the Weyl algebra for which

$$\omega_Q(W(z)) = e^{-Q|z|^2/2},$$

with $Q < 1/2$. How to prove it? (one possible attempt is to prove that ω_Q is dominated by $\omega_{1/2}$, in the sense that $\omega_{1/2}$ could be written as a linear combination of ω_Q and other states which is impossible by irreducibility, maybe use product of two representations).

Sketch of proof of Theorem 4.

The easiest way to come up with a reducible representation is to that two copies $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^2)$ of the Schrödinger representation and define Weyl operators

$$(\tilde{W}(s+it)f)(x_1, x_2) = (e^{its/2}\tilde{U}(s)\tilde{V}(t)f)(x_1, x_2)$$

$$= e^{its/2}e^{is(ax_1+bx_2)}f(x_1 - at, x_2 + bt) = e^{its/2}U_1(as)U_2(bs)V_1(at)V_2(-bt)$$

where (U_1, V_1) and (U_2, V_2) are Weyl pairs acting independenlty on the two factors of $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, so they commute among them.

We check that

$$\tilde{W}(s+it)\tilde{W}(s'+it') = e^{-i(b^2-a^2)\mathrm{Im}[(\overline{s+it})(s'+it')]}\tilde{W}(s'+it')\tilde{W}(s+it)$$

so it is a representation if $a^2 - b^2 = 1$. In this way we can construct a family of Weyl pairs. Let $\Psi_0 = \psi_0 \otimes \psi_0$ the tensor product of the two vacuum states, then

$$\langle \psi_0 \otimes \psi_0, \tilde{W}(s+it)(\psi_0 \otimes \psi_0)\rangle_{L^2(\mathbb{R}^2)} = e^{-(1+2b^2)|s+it|^2/4}.$$

Let us show concretely that the representation given by $\tilde{W}$ on $L^2(\mathbb{R}^2)$ is not irreducible.

Consider the operators

$$(W^\sharp(s+it)f)(x_1, x_2) = e^{its/2}U_1(bs)U_2(as)V_1(-bt)V_2(at) = W_1(bs - ibt)W_2(as + iat)$$

and note that

$$\tilde{W}(s'+it')W^\sharp(s+it) = W^\sharp(s+it)\tilde{W}(s'+it')$$

so the two families commute. In particular the Stone–von Neumann projector $P^\sharp$ associated to the Weyl system $W^\sharp$ satisfy

$$P^\sharp\tilde{W}(z) = \tilde{W}(z)P^\sharp$$

and therefore $(W^\sharp(z))_{z\in\mathbb{C}}$ is not an irreducible representation since $P^\sharp$ is a non-trivial self-adjoint operator.

Moreover if $\psi_0^\sharp \in L^2(\mathbb{R}^2)$ is a unit vector such that $P^\sharp \psi_0^\sharp = \psi_0^\sharp$ then the space $K = \overline{\{W^\sharp(z)\psi_0^\sharp : z \in \mathbb{C}\}}^{L^2(\mathbb{R}^2)}$ is invariant under the action of $\tilde{W}(z)$ and we have that $\{\tilde{W}(z)K: z \in \mathbb{C}\}$ is dense in $L^2(\mathbb{R}^2)$. □