Elements of Mathematical Quantum Mechanics
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Lecture 8 [June 26th 2024]

Wightman and Schwinger functions

We work now with the data $(\mathscr{H}, \mathscr{A}, Q_0, U(t))$ where $(U(t))_t$ is a positive energy strongly continuous unitary group, or equivalently $(K(t))_t$ a self-adjoint, strongly continuous, contraction semigroup. We saw that the given of U is equivalent to the given of K.

Definition 1. *We say that $h_0 \in \mathscr{H}$ is a ground state for U iff $U(t) h_0 = h_0$.*

Theorem 2. *h_0 is a ground state for U iff one of the following equivalent conditions hold:*

1. $\mu^{h_0}(\mathrm{d}x) = \delta_0(\mathrm{d}x)$
2. $K(t) h_0 = h_0$
3. $h_0 \in \mathscr{D}_H$ *and* $H h_0 = 0$
4. $h_0 \in \mathscr{D}_H$ *and* $\mathscr{E}(h_0, h_0) = 0$

Remark 3. The name ground state comes from the fact that h_0 is the state of minimal energy of the system (i.e. the zero energy, in our normalization).

Definition 4. *h_0 a cyclic ground state if*

$$\mathrm{span}\{U(t_1) Q_0(a_1) U(t_2) Q_0(a_2) \cdots h_0\}$$

is dense in $\mathscr{H}$.

Indeed any $\omega^h(Q_t(a))$ can then be approximated by linear combinations of expressions of the form

$$\langle Q_{t_1}(a_1) \cdots Q_{t_n}(a_n) h_0, h_0 \rangle$$

for suitable $t_1, \ldots, t_n$ since we used the fact that h_0 is invariant under U.

Recall that (up to a sign)

$$Q_{t_1}(a_1) = U(t_1) Q_0(a_1) U(-t_1)$$

Assume now that we are given a cyclic ground state.

Definition 5. *Wightman functions are defined as*

$$\mathbb{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_k) = \langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)h_0, h_0\rangle$$

where $\mathbb{A}_k = (a_1,\ldots,a_k)\in\mathscr{A}^k$.

Lemma 6. $\mathbb{W}_{k,\mathbb{A}_k}$ *is invariant wrt. to time translations, namely*

$$\mathbb{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_k) = \mathbb{W}_{k,\mathbb{A}_k}(t_1+s,\ldots,t_k+s)$$

for all $s\in\mathbb{R}$.

We observe also that we can define the (reduced) function

$$W_{k,\mathbb{A}_k}(\xi_1,\ldots,\xi_{k-1}) = \mathbb{W}_{k,\mathbb{A}_k}(t, t+\xi_1,\ldots,t_k+\xi_{k-1}) = \langle Q_0(a_1)U(\xi_1)Q_0(a_2)U(\xi_2)\cdots Q_0(a_k)h_0, h_0\rangle$$

for $\xi_1,\ldots,\xi_{k_1}\in\mathbb{R}$. We have the property that

$$\mathbb{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_k) = W_{k,\mathbb{A}_k}(t_2-t_1,\ldots,t_k-t_{k-1}).$$

Definition 7. *We consider a set of functions* $\tilde{W}_{k,\cdot}(\cdot)\colon\mathscr{A}^k\times\mathbb{R}^{k-1}\longrightarrow\mathbb{C}$. *We say that* $\tilde{W}_{k,\mathbb{A}_k}$ *satisfy Axiom W1 (compatibility conditions) if the following properties hold*

1. $\mathbb{A}_k = (a_1,\ldots,a_k)\in\mathscr{A}^k$ *and* $(t_1,\ldots,t_{k-1})\in\mathbb{R}^{k-1}$, *we have*

   $$\tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{i-1},0,t_{i+1},\ldots,t_{k-1}) = \tilde{W}_{k-1,\tilde{\mathbb{A}}_{k-1}}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{k-1})$$

   where $\tilde{\mathbb{A}}_{k-1} = (a_1,\ldots a_{i-1},a_i a_{i+1},a_{i+2},\ldots,a_k)\in\mathscr{A}^{k-1}$.

2. $\mathbb{A}_{k-1} = (a_1,\ldots,a_{k-1})\in\mathscr{A}^{k-1}$ *and* $T_{k-1} = (t_1,\ldots,t_{k-1})\in\mathbb{R}^{k-1}$, *we have*

   $$\tilde{W}_{k,(a_1,\ldots,a_{i-1},1_{\mathscr{A}},a_i,\ldots,a_{k-1})}(t_1,\ldots,t_k) = \tilde{W}_{k-1,\mathbb{A}_{k-1}}(t_1,\ldots,t_{i-2},t_{i-1}+t_i,t_{i+1},\ldots,t_{k-1}).$$

3. $\mathbb{A}_k = (a_1,\ldots,a_k)\in\mathscr{A}^k$ *and* $T_{k-1} = (t_1,\ldots,t_{k-1})\in\mathbb{R}^{k-1}$ *we have*

   $$\overline{\tilde{W}_{k,\mathbb{A}_k}(T_{k-1})} = \tilde{W}_{k,\theta\mathbb{A}_k}(\bar{\theta}(T_{k-1}))$$

   where $\theta(\mathbb{A}_k) = (a_k^*, a_{k-1}^*,\ldots,a_1^*)$ *and* $\bar{\theta}(T_{k-1}) = (-t_{k-1},-t_{k-2},\ldots,-t_2,-t_1)$.

Lemma 8. *Reduced Wightman functions satisfy these compatibility conditions (i.e. Axiom W1).*

Definition 9. $\tilde{W}_{k,\cdot}(\cdot)$ *satisfy Axiom W2 (i.e. it is a Fourier transform of a distribution with support in* $\mathbb{R}_+^{k-1}$*) if* $\tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1})$ *is continuous in* $t_1,\ldots,t_{k-1}$ *and* $\tilde{W}_{k,\mathbb{A}_k} = \mathscr{F}(T_{k,\mathbb{A}_k})$ *for some* $T_{k,\mathbb{A}_k} \in \mathcal{S}'$ *such that*

$$|T_{k,\mathbb{A}_k}(f_1 \otimes \cdots \otimes f_{k-1})| \leqslant C_k \prod_{\ell=1}^{k-1} \|f_\ell\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k} \|a_k\|_{\mathscr{A}}. \tag{1}$$

Remark 10. The equation (1) is equivalent to

$$\left|\int_{\mathbb{R}^{k-1}} \tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1}) \overline{g_1(t_1)} \cdots \overline{g_{k-1}(t_{k-1})} \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}\right| \lesssim \tilde{C}_k \prod_{\ell=1}^{k-1} \|\mathscr{F}^{-1}(g_\ell)\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k} \|a_k\|_{\mathscr{A}}.$$

for $g_1,\ldots,g_{k-1} \in \mathcal{S}(\mathbb{R})$. Indeed recall that $\mathscr{F}(T_{k,\mathbb{A}_k}) = \tilde{W}_{k,\mathbb{A}_k}$ and

$$\mathscr{F}(T_{k,\mathbb{A}_k})(g) = \langle \tilde{W}_{k,\mathbb{A}_k}, g \rangle = \int_{\mathbb{R}^{k-1}} \tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1}) \overline{g(t_1,\ldots,t_k)} \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}$$

but $\mathscr{F}(T_{k,\mathbb{A}_k})(g) = T_{k,\mathbb{A}_k}(\mathscr{F}^{-1}(g))$ and calling $\mathscr{F}^{-1}(g) = f$ and from this one can conclude.

Lemma 11. *The Wightman functions satisfy Axiom W2.*

Let us consider now our last axiom. Recall that we defined $\theta(\mathbb{A}_k) = (a_k^*, a_{k-1}^*, \ldots, a_1^*)$ and $\bar{\theta}(T_{k-1}) = (-t_{k-1}, -t_{k-2}, \ldots, -t_2, -t_1)$.

For $\mathbb{A}_{k_1} = (a_1,\ldots,a_{k_1})$ and $\mathbb{A}'_{k_1} = (a'_1,\ldots,a'_{k_2})$ then we let

$$\mathbb{A}_{k_1}\mathbb{A}'_{k_2} = (a_1, a_2, \ldots, a_{k_1}a'_1, \ldots, a'_{k_2}) \in \mathscr{A}^{k_1+k_2-1}$$

Definition 12. *The functions* $\tilde{W}_{k,\cdot}(\cdot) \colon \mathscr{A}^k \times \mathbb{R}^{k-1} \longrightarrow \mathbb{C}$ *satisfy Axiom W3 (Hilbert-space positivity) if for any* $k \in \mathbb{N}_0$ *and any* $j_1,\ldots,j_k \in \mathbb{N}$, *any* $T_{n-1,j} = (t_{1,(n-1,j)}, \ldots t_{n-1,(n-1,j)})$ *and* $\lambda_{n,j} \in \mathbb{C}$ *and* $\mathbb{A}_{n,j} = (a_{1,(n,j)}, \ldots a_{n,(n,j)}) \in \mathscr{A}^n$ *where* $j \leqslant j_n$ *and* $n \leqslant k$ *we have*

$$\sum_{n_1+n_2=1}^{k} \sum_{h_1=1}^{j_{n_1}} \sum_{h_2=1}^{j_{n_2}} \lambda_{n_1,h_1} \overline{\lambda_{n_2,h_2}} \tilde{W}_{n_1+n_2-1,\theta(\mathbb{A}_{n_2,h_2})\mathbb{A}_{n_1,h_1}}(\bar{\theta}(T_{n_2-1,h_2}), T_{n_1-1,h_1}) \geqslant 0.$$

Lemma 13. *Wightman functions satisfy Axiom W3.*

Summarizing, we have shown that the reduced Wightman functions $(W_{k,\mathbb{A}_k})_k$ statisfy three basic properties

a) W1 – compatibility condition (encodes the fact that Q_0 is a C^*-representation and that U is a unitary group)

b) W2 – tempered distribution axiom (encodes the fact that U is strongly continuous with positive energy)

c) W3 – Hilbert space positivity (encodes the fact that the scalar product is Hermitian and positive)

Definition 14. *Schwinger functions, $k\in\mathbb{N}$ and $\mathbb{A}_k=(a_1,\ldots,a_k)\in\mathscr{A}^k$, $t_1,\ldots,t_{k-1}\geqslant 0$ and let*

$$S_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1})=\langle Q_0(a_1)K(t_1)Q_0(a_2)K(t_2)\cdots K(t_{k-1})Q_0(a_k)h_0,h_0\rangle.$$

Recall that $\theta(\mathbb{A}_k)=(a_k^*,\ldots,a_1^*)$ and $\bar{\theta}(T_{k-1})=(-t_{k-1},-t_{k-2},\ldots,-t_1)$. We introduce now also another map on times as $\hat{\theta}(T_{k-1})=(t_{k-1},t_{k-2},\ldots,t_1)$. We will need also the composition $\mathbb{A}_k\cdot\mathbb{A}'_{k'}=(a_1,\ldots,a_ka'_1,\ldots a'_{k'})$.

Definition 15. *We say that the set of functions $(\tilde{S}_k:\mathscr{A}^k\times\mathbb{R}_+^{k-1}\longrightarrow\mathbb{C})_k$ satisfy the axiom S1 (or compatibility condition)*

1. $\tilde{S}_{k,\mathbb{A}_k}(t_1,\ldots,t_{i-1},0,t_{i+1},\ldots,t_{k-1})=\tilde{S}_{k-1,\tilde{\mathbb{A}}_k}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{k-1})$ *where* $\tilde{\mathbb{A}}_k=(a_1,\ldots,a_ia_{i+1},\ldots,a_k)$ *and*

   $\tilde{S}_{k,(a_1,\ldots,\lambda a_i+\mu b_i,\ldots,a_k)}(t_1,\ldots,t_{k-1})=\lambda\tilde{S}_{k,(a_1,\ldots,a_i,\ldots,a_k)}(t_1,\ldots,t_{k-1})+\mu\tilde{S}_{k,(a_1,\ldots,b_i,\ldots,a_k)}(t_1,\ldots,t_{k-1})$

2. $\tilde{S}_{k,(a_1,\ldots a_{i-1},1,a_i,\ldots,a_k)}(t_1,\ldots,t_{k-1})=\tilde{S}_{k-1,\mathbb{A}_{k-1}}(t_1,\ldots,t_{i-1}+t_i,\ldots,t_{k-1})$

3. $\overline{\tilde{S}_{k,\mathbb{A}_k}(T_{k-1})}=\tilde{S}_{k,\theta(\mathbb{A}_k)}(\hat{\theta}(T_k))$ *which is due to the fact that $K(t)^*=K(t)$.*

Lemma 16. *The Schwinger functions satisfy Axiom S1*

We define the Laplace transform $\mathscr{L}(T)=G(t_1,\ldots,t_{k-1})$ as

$$G(t_1,\ldots,t_{k-1})=T((s_1,\ldots,s_{k-1})\longmapsto e^{-t_1s_1\cdots-t_{k-1}s_{k-1}}).$$

Definition 17. *Let $(\tilde{S}_k)_k$ as before. We say that they satisfy Axiom S2 (or that they are Laplace transform of a tempered distribution) if $\exists T_{k,\mathbb{A}_k}$ such that $\tilde{S}_{k,\mathbb{A}_k} = \mathscr{L}(T_{k,\mathbb{A}_k})$ and for all $g_1, \ldots, g_{k-1} \in \mathscr{S}(\mathbb{R}_+)$*

$$\left| \int_{\mathbb{R}_+^{k-1}} \tilde{S}_{k,\mathbb{A}_k}(t_1, \ldots, t_{k-1}) g(t_1) \cdots g_{k-1}(t_{k-1}) \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1} \right| \leq \prod_{\ell=1}^{k-1} \|\mathscr{L} g_\ell\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k-1} \|a_k\|_{\mathscr{A}}. \tag{2}$$

Theorem 18. *The inequality (2) implies that $\tilde{S}_{k,\mathbb{A}_k}$ is the Laplace transform of a distribution.*

Lemma 19. *The Schwinger functions, satisfy Axiom S2.*

Definition 20. *Let $(\tilde{S}_k)_k$ as before. They satisfy Axiom S3 (or reflection positivity) if for $k \in \mathbb{N}$, $j_1, \ldots, j_k \in \mathbb{N}$ and $T_{n-1,j} = (t_{1,(n-1,j)}, \ldots, t_{n-1,(n-1,j)}) \in \mathbb{R}_+^{n-1}$ and $\lambda_{n,j} \in \mathbb{C}$ ($n \leq k$ on $j \leq j_n$)*

$$\sum_{n_1,n_2=1}^{k} \sum_{h_1=1}^{j_{n_1}} \sum_{h_2=1}^{j_{n_2}} \lambda_{n_1,h_1} \overline{\lambda_{n_2,h_2}} S_{n_1+n_2-1, \theta(\mathbb{A}_{n_2,h_2}) \cdot \mathbb{A}_{n_1,h_1}}(\hat{\theta}(T_{n_2-1,h_2}), T_{n_1-1,h_1}) \geq 0$$

Lemma 21. *The Schwinger functions satisfy Axiom S3.*

Definition 22. *We say that $\tilde{S}_{k,\cdot} \cdot : \mathscr{A}^k \times \mathbb{R}_+^{k-1} \longrightarrow \mathbb{C}$ is linear in $\mathscr{A}$ (or satisfies Axiom S0) if for all $a_1, \ldots, a_k \in \mathscr{A}$ and $t_1, \ldots t_{k-1} \in \mathbb{R}_+$ if the map*

$$a \longmapsto \tilde{S}_{k,(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_k)}(t_1, \ldots, t_{k-1})$$

is linear in $a \in \mathscr{A}$.

Theorem 23. *If $\tilde{S}_{k,\cdot} \cdot : \mathscr{A}^k \times \mathbb{R}_+^{k-1} \longrightarrow \mathbb{C}$ satisfy Axioms S0,S1,S2,S3 we have that there exists an Hilbert space $\mathscr{H}$, a representation Q_0 of $\mathscr{A}$ in $\mathscr{B}(\mathscr{H})$ and a self-adjoint, strongly continuous semigroup $(K(t))_{t \geq 0}$ on $\mathscr{H}$ and a vector $h_0 \in \mathscr{H}$ cyclic wrt. Q_0, K and invariant, i.e. $K(t) h_0 = h_0$ (in other words, h_0 is a ground state), such that $\tilde{S}_{k,\mathbb{A}_k}(T_{k-1})$ are the Schwinger functions generated by $(\mathscr{H}, Q_0, K, h_0)$.*

Remark 24. An analogous theorem holds for families $(\tilde{W}_{k,\mathbb{A}_k})_k$ satisfying W0,W1,W2,W3, from which one can construct data $(\mathscr{H}, Q_0, (U(t))_t, h_0)$ for which they are the Wightman functions.

Proof. Let $\mathscr{F}$ be the free algebra generated by the symbols $\tilde{Q}_0(a)$ and $\tilde{K}(t)$ where $a \in \mathscr{A}$ and $t \in \mathbb{R}_+$ equipped with the relations

i. $\tilde{Q}_0(a) \tilde{Q}_0(b) = \tilde{Q}_0(ab)$, $\lambda \tilde{Q}_0(a) + \mu \tilde{Q}_0(b) = \tilde{Q}_0(\lambda a + \mu b)$ for $a, b \in \mathscr{A}$ and $\lambda, \mu \in \mathbb{C}$

ii. $\tilde{Q}_0(1_{\mathscr{A}}) = 1_{\mathscr{F}}$

iii. $\tilde{K}(t_1)\tilde{K}(t_2) = \tilde{K}(t_1 + t_2)$

iv. $\tilde{K}(0) = 1_{\mathscr{F}}$

By definition $\mathscr{F}$ is the complex vector space generated by the words of the form $\tilde{Q}_0(a)\tilde{Q}_0(b)\tilde{K}(t)\cdots\tilde{Q}_0(c)\tilde{K}(t')$ which then is extended to an algebra by justapposition of the linear generators and then we take the quotient wrt. the relations listed above. Introduce a useful notation: if $T_{k-1} = (t_1, \dots, t_{k-1}) \in \mathbb{R}_+^{k-1}$ and $\mathbb{A}_k = (a_1, \dots, a_k) \in \mathscr{A}^k$, we call $\mathbb{F}_k(T_{k-1}, \mathbb{A}_k) = \tilde{Q}_0(a_1)\tilde{K}(t_1)\cdots\tilde{K}(t_{k-1})\tilde{Q}_0(a_k) \in \mathscr{F}$. Using the previous relations we have that if $A \in \mathscr{F}$ then

$$A = \sum_{n=1}^{k} \sum_{h=1}^{j_n} \lambda_{n,h} \mathbb{F}_n(T_{n-1,h}, \mathbb{A}_{n-1,h})$$

for some $\lambda_{n,h}, T_{n-1,h}, \mathbb{A}_{n-1,h}$ (in general not in a unique way). On $\mathscr{F}$ we define the scalar product $\langle *, * \rangle_{\mathscr{F}}$ by

$$\langle \mathbb{F}_k(T_{k-1}, \mathbb{A}_k), \mathbb{F}_{k'}(T'_{k'-1}, \mathbb{A}'_k) \rangle_{\mathscr{F}} = \tilde{S}_{k+k'-1, \theta(\mathbb{A}'_{k'})\cdot\mathbb{A}_k}(\hat{\theta}(T'_{k'-1}), T_{k-1})$$

and extend it by linearity to all $\mathscr{F}$ in the first component and by antilinearity in the second component. This definition is well posed since $(\tilde{S}_{k,\mathbb{A}_k})_k$ satisfy the compatibility conditions of Axiom S1 and moreover by the last of the property in Axiom S1 we have that the form $\langle *, * \rangle_{\mathscr{F}}$ is Hermitian and for Axiom S3 that this scalar product is positive definite. We define the linear subspace $\mathscr{N} = \{A \in \mathscr{F}, \langle A, A \rangle_{\mathscr{F}} = 0\}$ and we define $\mathscr{H}_0 = \mathscr{F} \backslash \mathscr{N}$ as a vector space. On $\mathscr{H}_0$ we define $\langle [A], [B] \rangle_{\mathscr{H}} = \langle A, B \rangle_{\mathscr{F}}$ which is well defined by the Cauchy–Schwartz inequality and where $[A] \in \mathscr{H}_0$ denotes the class of $A \in \mathscr{F}$. Moreover we let $\mathscr{H}$ the completion of $\mathscr{H}_0$ with respect to this non-degenerate scalar product $\langle , \rangle_{\mathscr{H}}$ (which is stricly positive on $\mathscr{H}_0 - \{0\}$). We let $h_0 = [1_{\mathscr{F}}]$. We define $\mathbb{K}(t): \mathscr{F} \longrightarrow \mathscr{F}$ linear such that

$$\mathbb{K}(t)(\mathbb{F}_k(T_{k-1}, \mathbb{A}_k)) := \tilde{K}(t)\mathbb{F}_k(T_{k-1}, \mathbb{A}_k) = \mathbb{F}_{k+1}((t, T_{k-1}), (1_{\mathscr{A}}, \mathbb{A}_k)).$$

We have that $\mathbb{K}(t)\mathbb{K}(s) = \mathbb{K}(t+s)$ and $\mathbb{K}(0) = 1$. Moreover $\mathbb{K}_t$ is symmetric wrt. the scalar product on $\mathscr{F}$ (this is a consequence of Axiom S1), indeed

$$\langle \mathbb{K}(t)(\mathbb{F}_k(\mathbb{T}_{k-1}, \mathbb{A}_k)), \mathbb{F}_h(T'_{h-1}, \mathbb{A}_h) \rangle = \langle \mathbb{F}_{k+1}((t, T_{k-1}), (1_{\mathscr{A}}, \mathbb{A}_k)), \mathbb{F}_h(T'_{h-1}, \mathbb{A}_h) \rangle$$

$$= S_{k+h-1, \theta(\mathbb{A}'_h)(1_{\mathscr{A}}, \mathbb{A}_k)}(\hat{\theta}(T'_{h-1}), (t, T_{k-1})) = S_{k+h-1, \theta((1_{\mathscr{A}}, \mathbb{A}'_h))\mathbb{A}_k}(\hat{\theta}(t, T'_{h-1}), T_{k-1})$$

$$= \langle \mathbb{F}_k(\mathbb{T}_{k-1}, \mathbb{A}_k), \mathbb{K}(t)\mathbb{F}_h(T'_{h-1}, \mathbb{A}_h) \rangle$$

and this extends by linearity to deduce the symmetry for $\mathbb{K}(t)$.

Next, we have that

$$\langle \mathbb{K}(t)A, A\rangle = \langle \mathbb{K}(t/2)A, \mathbb{K}(t/2)A\rangle \geqslant 0$$

moreover by repeated use of Cauchy–Schwartz we also have

$$\langle \mathbb{K}(t)A, A\rangle \leqslant (\langle \mathbb{K}(2t)A, A\rangle)^{1/2}(\langle A, A\rangle)^{1/2} < \cdots < (\langle \mathbb{K}(2^n t)A, A\rangle)^{1/2^n}(\langle A, A\rangle)^{1-1/2^n}$$

By Axiom S2 we know that $\langle \mathbb{K}(2^n t)A, A\rangle$ can be written as a sum of the form

$$\langle \mathbb{K}(2^n t)A, A\rangle = \sum_{k, \mathbb{A}_k} S_{k, \mathbb{A}_k}(2^n t, t_1, \ldots, t_{k-2})$$

where everything does not depends on n and is uniformly bounded so the quantity $\langle \mathbb{K}(2^n t)A, A\rangle$ is bounded uniformly in n. So

$$\langle \mathbb{K}(t)A, A\rangle \leqslant C^{1/2^n}(\langle A, A\rangle)^{1-1/2^n}$$

and taking $n \longrightarrow \infty$ we have

$$\langle \mathbb{K}(t)A, A\rangle \leqslant \langle A, A\rangle$$

so $\mathbb{K}(t)\mathcal{N} \subset \mathcal{N}$ and $\mathbb{K}(t)$ is well defined on $\hat{\mathscr{H}}$ and we let $K_0(t)[A] = [\mathbb{K}(t)A]$. We have that for all $t \geqslant 0$

$$\langle K_0(t/2)[A], K_0(t/2)[A]\rangle = \langle K_0(t)[A], [A]\rangle \leqslant \langle [A], [A]\rangle$$

so $K_0(t)$ is a contraction for all $t \geqslant 0$ so it extends to $\mathscr{H}$ as K. It is also self-adjoint and a $(K(t))_{t \geqslant 0}$ is a semigroup. For the strong continuity of the family $(K(t))_{t \geqslant 0}$ we observe that the Schwinger functions are continuous at least when considered as a functions of one of the time variables (fixing all the other parameters). This is enough to prove that $t \mapsto K(t)$ is weakly continuous and then strong continuity follows since it is a contraction.

We define a linear map $\mathbb{Q}(a): \mathscr{F} \longrightarrow \mathscr{F}$ as $\mathbb{Q}(a)A = \tilde{Q}_0(a)A$. It is a representation of $\mathscr{A}$ on $\mathscr{F}$ (this follows from the relations we imposed on the algebra $\mathscr{F}$). We have that is a $*$-represenation:

$$\langle \mathbb{Q}(a)A, B\rangle_{\mathscr{F}} = \langle A, \mathbb{Q}(a^*)B\rangle_{\mathscr{F}}$$

this can be proved by looking at the definition of the Hermitian form. Moreover on can show $\mathbb{Q}(a)\mathcal{N} \subset \mathcal{N}$ so that we can define the operator on $\hat{\mathscr{H}}$. Define the linear functional on $\mathscr{A}$: $L_A(a) = \langle \mathbb{Q}(a)A, A\rangle_{\mathscr{F}}$. It is positive since

$$L_A(bb^*) = \langle \mathbb{Q}(bb^*)A, A\rangle_{\mathscr{F}} = \langle \mathbb{Q}(b)\mathbb{Q}(b^*)A, A\rangle_{\mathscr{F}} = \langle \mathbb{Q}(b^*)A, \mathbb{Q}(b^*)A\rangle_{\mathscr{F}} \geqslant 0.$$

Therefore it is continuous and its norm on $\mathscr{A}^*$ is given by $L_A(1_{\mathscr{A}}) = \langle A, A\rangle_{\mathscr{F}}$ so if $A \in \mathscr{N}$ then $L_A = 0$. From this, in particular we have $0 = L_A(b^* b) = \langle \mathbb{Q}(b)A, \mathbb{Q}(b)A\rangle_{\mathscr{F}}$ so $Q(b)A \in \mathscr{N}$ for any $b \in \mathscr{A}$. We can then pass to the quotient and define $Q_{00}(a)[A] = [\mathbb{Q}(a)A]$. We have also $\|Q_{00}(a)[A]\|_{\mathscr{F}} \leqslant \|a\|_{\mathscr{A}} \|[A]\|_{\mathscr{F}}$ so Q_{00} is bounded and can be extended to $\mathscr{H}$ as a C^*-homomorphism. We let $h_0 = [1_{\mathscr{F}}]$ and by S1 prove that it is invariant. □

1 The Ornstein–Uhlenbeck process

We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and consider a Gaussian process $X: \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$, that is such that for all $\xi_1, \ldots, \xi_k \in \mathbb{R}$ we have that $(X_{\xi_1}, \ldots, X_{\xi_k})$ is a k-dimensional Gaussian. A Gaussian process is characterised by its mean and covariance function. We let $\mathbb{E}[X_\xi] = 0$ for all $\xi \in \mathbb{R}$ and

$$\mathrm{Cov}(X_\xi, X_{\xi'}) = \mathbb{E}[X_\xi X_{\xi'}] = \frac{1}{2\theta} e^{-\theta|\xi - \xi'|}, \qquad \xi, \xi' \in \mathbb{R}.$$

If $\tilde{S}_{k,\cdot} : \mathscr{A}^k \times \mathbb{R}_+^{k-1} \longrightarrow \mathbb{C}$ then we define extended functions $\tilde{\mathcal{S}}_{k,\cdot} : \mathscr{A}^k \times \mathbb{R}_+^k \longrightarrow \mathbb{C}$ such that, if $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_k$ we let

$$\tilde{\mathcal{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \tilde{S}_{k,\mathbb{A}_k}(\xi_2 - \xi_1, \xi_3 - \xi_2, \ldots, \xi_k - \xi_{k-1})$$

and if $\xi_1, \ldots, \xi_k$ are general then we let

$$\tilde{\mathcal{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \tilde{\mathcal{S}}_{k,\mathbb{A}_k}(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)})$$

where $\sigma \in S_n$ is the permutation such that $\xi_{\sigma(1)} < \cdots < \xi_{\sigma(k)}$. Note that to a family $\tilde{\mathcal{S}}$ invariant under translation and permutation of the time variables it associated a unique family $\tilde{S}$ and viceversa.

We choose now $\mathscr{A} = C_b^0(\mathbb{R})$ and let

$$\tilde{\mathcal{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \mathbb{E}[a_1(X_{\xi_1}) \cdots a_k(X_{\xi_k})]$$

Theorem 25. *$(\tilde{S}_k)_k$ satisfy Axiom S3 iff for any $F \in \mathscr{C}^0_{\mathrm{cyl}}(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$ we have that*

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}X)}] \geqslant 0.$$

Theorem 26. *The OU process X is reflection positive.*

In this case we can prove that the free algebra $\mathscr{F}$ introduced in the reconstruction is isomorphic to the algebra $\mathscr{F}_X \subseteq C_c^0(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$ by identifying

$$\tilde{Q}_0(a_0)\tilde{K}(t_1)\tilde{Q}_0(a_1) \cdots \tilde{K}(t_{k-1})\tilde{Q}_0(a_k)$$

with

$$a_0(X_0)a_1(X_{t_1})\cdots a_k(X_{t_1+\cdots+t_{k-1}})$$

and extending this map by linearity. We leave as an exercise to prove the isomorphism (as algebras). Under this ispomorphis if $F, G \in \mathscr{F}_X$ then we also have that the Hermitian form $\langle\rangle_{\mathscr{F}}$ can be represented probabilistically as

$$\langle F, G\rangle_{\mathscr{F}_X} = \mathbb{E}[F(X)\overline{G(\mathbb{R}(X))}]$$

which we know to be non-negative and Hermitian. Let $\mathscr{N}_X := \{F \in \mathscr{F}_X | \langle F, F\rangle_{\mathscr{F}_X} = 0\} \subseteq \mathscr{F}_X$

Remark 27. If $F \in \mathscr{F}_X$ then there exists a version $\mathbb{E}[F|X_0]$ which belongs to $\mathscr{F}_X$, indeed the conditional expectation can be written as $\mathbb{E}[F|X_0] = F(L_F X_0)$ for soem linear map L_F depending on F

Lemma 28. *We have (if F is supported on positive times)*

$$F - \mathbb{E}[F|X_0] \in \mathscr{N}_X$$

Proof. Observe that

$$\mathbb{E}[(F - \mathbb{E}[F|X_0])\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}]$$

$$= \mathbb{E}[\mathbb{E}[(F - \mathbb{E}[F|X_0])\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}|X_0]]$$

$$= \mathbb{E}[\mathbb{E}[(F - \mathbb{E}[F|X_0])|X_0][\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}|X_0]] = 0$$

since clearly $\mathbb{E}[(F - \mathbb{E}[F|X_0])|X_0] = 0$. □

So from an algebraic point of view we have that $\hat{\mathscr{H}} = \mathscr{F}_X \backslash \mathscr{N}_X$ is just $C_b^0(\mathbb{R}, \mathbb{C})$ where the map $\mathscr{F}_X \longrightarrow \hat{\mathscr{H}}$ is just the conditional expectation $F \mapsto \mathbb{E}[F|X_0]$. That $\hat{\mathscr{H}} = C_b^0(\mathbb{R}, \mathbb{C})$ is clear since $\mathbb{E}[a_0(X_0)|X_0] = a_0(X_0)$ so it is a surjective mapping. Moreover the scalar product can be written

$$\langle f, g\rangle_{\hat{\mathscr{H}}} = \mathbb{E}[f(X_0)\overline{g(X_0)}] = \int_{\mathbb{R}} f(z)\overline{g(z)}\underbrace{\frac{e^{-\theta z^2/2}}{(2\pi/\theta)^{1/2}}\mathrm{d}z}_{\mu_\theta(\mathrm{d}z)}$$

and as a consequence $\mathscr{H} = L^2(\mathbb{R}, \mathbb{C}, \mu_\theta)$ moreover $(Q_0(a)f)(z) = a(z)f(z)$. Recall now that $\mathbb{K}(t)F = \tilde{K}(t)A$ which under our isomorphism it is send to a translation of the time variable:

$$\mathbb{K}(t)F(X) = F(X_{t+\cdot}).$$

In particular $\mathbb{K}(t)f(X_0) = f(X_t)$ and we have

$$(K(t)f)(X_0) = \mathbb{E}[\mathbb{K}(t)f(X_0)|X_0] = \mathbb{E}[f(X_t)|X_0]$$

This is not what is done usually in quantum mechanics since the usual space there is taken to be $L^2(\mathbb{R}, \lambda)$ where λ is the Lebesgue measure, not μ_θ. The map connecting the two representations is

$$f \in \hat{\mathscr{H}} \longrightarrow \tilde{f}(z) = f(z)\frac{e^{-\theta z^2/4}}{(2\pi/\theta)^{1/4}} \in \tilde{\mathscr{H}} = L^2(\mathbb{R}, \lambda)$$

Let's compute the generator H of $K(t)$:

$$-Hf(z) = \lim_{t\longrightarrow 0}\frac{K(t)f(z) - f(z)}{t} = \lim_{t\longrightarrow 0}\int_{\mathbb{R}}\frac{f(e^{-\theta t}z + (1-e^{-2\theta t})^{1/2}y) - f(z)}{t}\mu_\theta(\mathrm{d}y)$$

By Taylor expansion:

$$=\lim_{t\longrightarrow 0}\int_{\mathbb{R}}\frac{f'(z)((e^{-\theta t}-1)z + (1-e^{-2\theta t})^{1/2}y) + \frac{1}{2}f''(z)((e^{-\theta t}-1)z + (1-e^{-2\theta t})^{1/2}y)^2 + O(t^{3/2})}{t}\mu_\theta(\mathrm{d}y)$$

and since μ_θ has zero first moment we have

$$=\lim_{t\longrightarrow 0}\int_{\mathbb{R}}\frac{f'(z)(e^{-\theta t}-1)z + \frac{1}{2}f''(z)((1-e^{-2\theta t})^{1/2}y)^2 + O(t^{3/2})}{t}\mu_\theta(\mathrm{d}y)$$

$$=\lim_{t\longrightarrow 0}\frac{f'(z)(-\theta t)z + \frac{1}{2}f''(z)(1-e^{-2\theta t})(1/2\theta) + O(t^{3/2})}{t} = -\theta f'(z) + \frac{1}{4}f''(z)$$

so on $\hat{\mathscr{H}}$ we have

$$Hf(z) = \theta f'(z) - \frac{1}{4}f''(z)$$

and the same operator on $\tilde{\mathscr{H}}$ has the form

$$\tilde{H}f(z) = -\theta z^2\tilde{f}(z) - \frac{1}{4}\Delta\tilde{f}(z)$$

and this is usually called the Schrödinger representation of the harmonic oscillator, indeed note that

$$\tilde{H} = \frac{1}{4}P^2 + Q^2\frac{\theta^2}{2}$$

which if interpreted classically is the Hamiltonian of the harmonic oscillator.

Therefore we have proven that the quantum mechanical harmonic oscillator is related via the reconstruction theorem with the Ornstein–Uhlenbeck process.

The course ends here.