Elements of Mathematical Quantum Mechanics
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The quantum particle

Table of contents

1 Non-commutativity and probability 3

2 Complementary observables in finite quantum system 4

3 Quantum degrees of freedom 6

4 Unitary representations of $\mathbb{R}$ and observables as homorphisms 10

5 The Weyl algebra 14

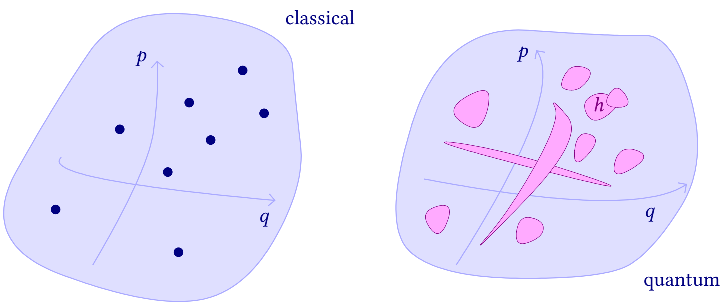
6 Dynamics on a canonical pair 21

7 Through the mirror 24

8 Wightman and Schwinger functions 35

9 The Ornstein–Uhlenbeck process 42

10 Euclidean processes 48



We want now to construct a physical system (observables+states) that encodes Heisenberg's indetermination principle

$$\Delta_\omega(q)\Delta_\omega(p) \geqslant \frac{\hbar}{2} \qquad (1)$$

for the position q and momentum p of a particle and other experimental observations.

The C^*-algebra of observables $\mathscr{A}$ should contain the C^*-algebra $\mathscr{Q}$ of all the bounded functions $f(q)$ of the position q and the C^*-algebra $\mathscr{P}$ of all the bounded functions $g(p)$ of the velocity (or momentum) p but I need to rule out that q, p commutes otherwise I violate Heisenberg principle unless I restrict the set of states. But restricting the set of states is mode difficult than dealing with a non-commutative algebra because we have more structure on $\mathscr{A}$ than on $\mathscr{S}$.

So we postulate that $[a, b] \neq 0$ at least for some $a \in \mathscr{Q}$ and $b \in \mathscr{P}$ and we let $\mathscr{A}$ to be the smallest C^* algebra containing the abelian subalgebras $\mathscr{Q}$, $\mathscr{P}$. In order for this to describe a single degree of freedom we require that $\Sigma(\mathscr{Q}) \approx \mathbb{R}$ and $\Sigma(\mathscr{P}) \approx \mathbb{R}$.

We want to explore how non-commutativity is related to the indetermination principle (1) and also to the notion of "complementarity". Complementary observables are somehow observables which do not allow simultaneous measurement, that is if we are able to have states in which one of the is completely detemined, then the other has to be completely "undetermined". Think about the Stern-Gerlach experiment and the measurement of the magnetic moment in two orthogonal directions.

Let us see what we can get from (1). Observe that if $a, b \in \mathscr{A}$ and self-adjoint then

$$(a + i\lambda b)^*(a + i\lambda b) \geq 0$$

for any $\lambda \in \mathbb{R}$ and if ω is a state we have

$$0 \leq \omega((a + i\lambda b)^*(a + i\lambda b)) = \omega(a^2) + \lambda^2\omega(b^2) + i\lambda\omega(ab - ba),$$

therefore we need to have, letting $[a, b] = ab - ba$,

$$|\omega(i[a, b])| \leq 2(\omega(a^2))^{1/2}(\omega(b^2))^{1/2}.$$

Therefore in any C^* algebra we have the (Schrödinger–Robertons) relation

$$\Delta_\omega(a)\Delta_\omega(b) \geq \frac{1}{2}|\omega(i[a, b])|.$$

If we want to implement Heisenberg's principle for a pair of complementary observables q, p a way is to require that $i[p, q]$ is constant element of $\mathscr{A}$:

$$[q, p] = i\hbar, \tag{2}$$

These are called canonical commutation relations Heisenberg's matrix mechanics consist in a model where q, p are matrices satisfying the above relation. First problem: these cannot be finite dimensional matrices, indeed if they were we could take the trace over the vector space $\mathbb{C}^n$ they acts on and get

$$\mathrm{Tr}([q, p]) = \sum_n \langle e_n, [q, p]e_n\rangle = 0, \qquad \mathrm{Tr}(i\hbar) = i\hbar n\ldots$$

not very nice. Moreover they cannot implemented even in an abstract C^* algebra, indeed if q, $p \in \mathscr{A}_{sa}$ then

$$[q^n, p] = i\hbar n q^{n-1}$$

and therefore by the C^* condition

$$n\hbar\|q\|^{n-1} = n\hbar\|q^{n-1}\| = \|i\hbar\, n q^{n-1}\| = \|[q^n, p]\| \leq 2\|p\|\,\|q\|^n$$

which implies $\|p\|\,\|q\| \geq n\hbar/2$ if $\|q\| \neq 0$. This is true for any n and so either $\|p\|$ or $\|q\|$ has to be infinite.

This somehow is to be expected because "the position" is not really a bounded observable. We cannot really talk about the position of the particle as an element of a C^*-algebra but it is ok if the think to any bounded function of q and an element of the C^* algebra. So we need to avoid to talk about q and talk instead of a C^* algebra $\mathcal{Q}$ which plays the role of the algebra of functions of the position, that is has to be a commutative C^* algebra without unit (in order to allow for non-compact spectrum).

At this point is not clear how to single out an algebra of observables which satisfies something like the indetermination principle.

The discussion in this part is inspired by the following papers:

- Accardi, Luigi. "Some Trends and Problems in Quantum Probability." In *Quantum Probability and Applications to the Quantum Theory of Irreversible Processes*, edited by Luigi Accardi, Alberto Frigerio, and Vittorio Gorini, 1055:1–19. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1984. https://doi.org/10.1007/BFb0071706.

- Ohya, Masanori, and Dénes Petz. *Quantum Entropy and Its Use*. Texts and Monographs in Physics. Berlin; New York: Springer-Verlag, 1993.

- Schwinger, Julian. "Unitary Operator Bases." *Proceedings of the National Academy of Sciences* 46, no. 4 (April 1, 1960): 570–79. https://doi.org/10.1073/pnas.46.4.570.

1 Non-commutativity and probability

To start simpler we consider first system which possess "finitely many" pure states. Think about the two states in the Stern–Gerlach experiment.

Let us assume we have two observables a, b which generates $\mathscr{A}$ and such that $\sigma(a), \sigma(b)$ are finite.

We would like to inquire about the "most indeterminate" relative position of a and b inside the C^*-algebra $\mathscr{A} = C^*(a, b)$ they generate. First of all it is clear that since $\sigma(a)$ is finite, let's say with n elements, we can find function $(\rho_k \in C(\mathbb{R}))_{k=1,\ldots,n}$ such that $\rho_k(x) \in [0,1]$ and $\sum_{k=1}^n \rho_k(x) = 1$ for all $x \in \mathbb{R}$ and $\rho_k(x)\rho_\ell(x) = \delta_{k,\ell}$ for all $x \in \sigma(a)$. Let $\pi_k^a := \rho_k(a)$ and observe that by construction

$$\sum_{k=1}^n \pi_k^a = 1, \quad \pi_k^a \pi_\ell^a = \delta_{k,\ell}, \qquad k, \ell = 1, \ldots, n,$$

i.e. $(\pi_k^a)_k$ form a partition of unity in self-adjoint projections. We let $(\pi_k^b)_{k=1,\ldots,m}$ the analogous objects associated to b where m is the size of $\sigma(b)$. Clearly there exists constants $(a_k)_k$ such that $f(a) = \sum_k f(a_k)\pi_k^a$ for any $f \in C(\mathbb{R})$ and similarly for b so we need that $[\pi_k^a, \pi_\ell^b] \neq 0$ for some $k = \ell$ in order to have a non-commutative algebra. We have $\omega(f(a)) = \sum_k f(a_k)\omega(\pi_k^a)$ for any state ω.

Let us assume that $C^*(a)$ and $C^*(b)$ are maximal abelian subalgebras in $\mathscr{A}$. Then observe that the observable $\sum_k \pi_k^a \pi_\ell^b \pi_k^a$ commutes with any element in $C^*(a)$ and therefore it should belong to it. As a consequence there exist complex numbers $(p_{\ell,k}^{b|a})_{k,\ell}$ such that

$$\sum_k \pi_k^a \pi_\ell^b \pi_k^a = \sum_k p_{\ell,k}^{b|a} \pi_k^a.$$

Since the l.h.s. is positive on any state and there exist states (ω_k^a) such that $\omega_k^a(\pi_\ell^a) = \delta_{k,\ell}$ we have that $(\pi_k^a)_k$ is a basis of $C^*(a)$, that $(p_{\ell,k}^{b|a})_{k,\ell}$ are uniquely determined and that

$$p_{\ell,k}^{b|a} \geq 0, \qquad \sum_k p_{\ell,k}^{b|a} = \sum_\ell p_{\ell,k}^{b|a} = 1.$$

Therefore we have a set of probabilities $(p_{\ell,k}^{b|a})_{k,\ell}$ which are generated intrinsically by the non-commutativity of the algebra, even before we consider the states on that algebra.

That is the matrix $(p_{\ell,k}^{b|a})_{k,\ell}$ is bistochastic. This shows that, as soon as we allow for non-commutativity, some "randomness" is already built into our algebra of observables.

For any state ω we can construct a new state

$$\omega^a(h) = \sum_k \omega(\pi_k^a h \pi_k^a)$$

and now observe that

$$\omega(f(a)) = \omega^a(f(a)), \qquad \omega^a(f(b)) = \sum_{k,\ell} f(b_\ell)\omega(\pi_k^a \pi_\ell^b \pi_k^a) = \sum_{\ell,k} f(b_\ell) p_{\ell,k}^{b|a} \omega(\pi_k^a)$$

so $\omega^a(\pi_\ell^b) = \sum_k p_{\ell,k}^{b|a} \omega(\pi_k^a)$.

We can attempt the following interpretation of this formula: the matrix $(p_{\ell,k}^{b|a})_{k,\ell}$ gives the probability of observing given values of b under the condition that we have preliminarily measured a specific (but unspecified) value for a and therefore changed the state ω into a new state ω^a in which a has a specific value, i.e. is a convex combination of states multiplicative on $C^*(a)$.

2 Complementary observables in finite quantum system

We want now to devise observables a, b for which the matrix $p_{\ell,k}^{b|a}$ is as uniform as possible, meaning that if we have measured a then there is no particular knowledge on b. We call these observables "complementary". We require also that either a or b provides an as complete as possible description of the physical system, i.e. that $C^*(a)$ and $C^*(b)$ are maximally abelian. Without loss of generality we can assume that $\sigma(a) = \{0, \ldots, n-1\}$ for some integer $n \geq 2$.

Consider still systems with finitely many pure states. All the observables have to take only finitely many values, let say n. So we can assume that they have all the same spectrum with n points and to be given by

$$\Gamma = \{\gamma_k = e^{2\pi i k/n}\}_{k=0,\ldots,n-1}.$$

We want to construct an algebra of two non-commuting observables u, v where both have the same spectrum (as above) and they are complementary, and for that we mean here that we are trying to impose that $p_{\ell,k}^{v|u} = 1/n$ for any k, ℓ.

There is no loss of generality to restrict to operators in Hilbert space, they have to be unitary because $\Gamma \subset \mathbb{S} = \{z \in \mathbb{C}: |z| = 1\}$ and is clear we need at least a space of dimensions n otherwise we cannot accomodate the n different eigenvalues Γ. By abuse of language u, v the representatives of u, v in the space $\mathscr{L}(\mathbb{C}^n)$. Let $(\varphi_k)_k$ be the eigenvectors of u, i.e.

$$u\varphi_k = \gamma_k \varphi_k$$

and then take

$$v\varphi_k := \varphi_{k+1}$$

with $k+1$ understood modulus n. Now observe that $uv\varphi_k = u\varphi_{k+1} = \gamma_{k+1}\varphi_{k+1} = \gamma_{k+1} v\varphi_k = (\gamma_{k+1}/\gamma_k) vu\varphi_k$ for any $k = 0, \ldots, n-1$ so

$$uv = e^{2\pi i/n} vu. \tag{3}$$

If we assume that u, v generate the algebra of observables then this fixes the full algebraic structure. Observe also that $u^n = v^n = 1$.

Remark 1. Note that we could have defined $v\varphi_k = \alpha\varphi_{k+1}$ for some $\alpha \in \mathbb{S}$ and then we would have $v^n = \alpha^n$ and we could have let also $u\varphi_k = \beta\gamma_k\varphi_k$ for some $\beta \in \mathbb{S}$ and then we would have $u^n = \beta^n$. This preserves the commutation relation (3) but changes the spectra of u, v.

Remark 2. Observe also that (3) implies that $u^n v = vu^n$ and also $v^n u = uv^n$ so the elements u^n, v^n belongs to the center (i.e. the elements which commutes with all the others) of the algebra generated by u, v. If we assume that u, v generate each of them a maximally abelian subalgebra then we can conclude from the commutation relation only that $u^n, v^n \in \mathbb{C}$. From this one can see that any irreducible representation of the commutation relation is n dimensional.

In particular

$$0 = (\gamma_k^{-1} u)^n - 1 = (\gamma_k^{-1} u - 1) \sum_{\ell=0}^{n-1} (\gamma_k^{-1} u)^\ell$$

and from this we deduce that $\pi_k^u := n^{-1} \sum_{\ell=0}^{n-1} (\gamma_k^{-1} u)^\ell$ satisfies $u\pi_k^u = \gamma_k \pi_k^u$ so π_k^u is the orthogonal projection on the span of φ_k, indeed one can check that $(\pi_k^u)^* = \pi_k^u$ and $\pi_k^u \pi_\ell^u = \delta_{k,\ell} \pi_k^u$. So we have also $u = \sum_{k=0}^{n-1} \gamma_k \pi_k^u$. For v we can proceed in the same way and define π_k^v. Now let's compute $\sum_k \pi_k^u \pi_\ell^v \pi_k^u$ using (3) and get

$$\sum_k p_{\ell,k}^{v|u} \pi_k^u = \sum_k \pi_k^u \pi_\ell^v \pi_k^u = \frac{1}{n}, \qquad \ell = 1, \ldots, n-1$$

so as required we have $p_{\ell,k}^{v|u} = 1/n$. So we confirm that our choice of algebraic structure give indeed a maximally complementary pair of observables.

We want now to argue that u, v are sufficient to generate all $\mathscr{L}(\mathbb{C}^n)$ (i.e. all the $n \times n$ complex matrices). Let $X \in \mathscr{L}(\mathbb{C}^n)$ and observe that the operator

$$Y = \frac{1}{n^2} \sum_{k,\ell} u^{-k} v^{-\ell} X v^\ell u^k,$$

satisfy $uY = Yu$ and $vY = Yv$ so Y commutes with all the algebra generated by u, v (this actually depends only on the commutation relation (3)). Then this means that Y is a multiple of the identity, because since it commutes with u we must have $Y = \sum_k y_k \pi_k^u$ but then $Y = vYv^* = \sum_k y_k v \pi_k^u v^* = \sum_k y_k \pi_{k+1}^u$ and this implies that $y_k = y_{k+1}$ that is $Y = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$ so we can construct a linear functional ρ such that $\rho(X) = \lambda$ and by thinking a bit is clear that $\rho: \mathscr{L}(\mathbb{C}^n) \longrightarrow \mathbb{C}$ is a actually a positive linear functional (think about it, is clear from the definition of Y) and $\rho(\mathbb{1}) = 1$. The definition of Y implies easily that for any $X \in \mathscr{L}(\mathbb{C}^n)$

$$X = \sum_{k,\ell} u^k v^\ell \rho((u^k v^\ell)^* X)$$

that is $(u^k v^\ell)_{k,\ell}$ is an orthornomal basis of $\mathscr{L}(\mathbb{C}^n)$ with respect to the non-degenerate scalar product $\langle X, Y \rangle = \rho(X^* Y)$. So in particular the algebra generated by u, v span all the $n \times n$ complex matrices.

This proves that the representation we gave is irreducible and therefore the pure states of this algebra are exactly the vector states of this representation. So to describe all the possible states is enough to restrict to states of the form

$$\omega(X) = \mathrm{Tr}_{\mathbb{C}^n}[\rho \pi(X)],$$

where $\rho \in \mathscr{L}(\mathbb{C}^n)$ is a density matrix (i.e. $\rho \geqslant 0$, $\mathrm{Tr}_{\mathbb{C}^n}(\rho) = 1$) and π is the concrete representation of this algebra that we have analyzed. So the pure states are those for which $\omega(X) = \langle \psi, \pi(X) \psi \rangle$ for some unit vector in $\mathbb{C}^n$, i.e. ρ has to be of rank one. All the pure states of this quantum system are described by a ray in $\mathbb{C}^n$ i.e the set $\{e^{i\theta} \psi : \theta \in \mathbb{C}, \|\psi\| = 1\}$. This is very different from the commutative case where two observables u, v with each n different values have has possible pure states the n^2 different values of the pair.

The ray ψ is called the wave-function of the system and it provides a complete description as we saw. However it is so only because it parametrizes the set of all pure states. Irreducible representations are like “charts” that we use to compute over the manifold of all the possible states of a physical (quantum) system.

We have completely classified this quantum system.

3 Quantum degrees of freedom

Assume that $n = n_1 n_2$ for two integers n_1, n_2 then there exist an alternative way to construct two complementary set of observables which each of them is maximally abelian. For $\alpha = 1, 2$, make the same construction above with n_α and obtain $u_\alpha, v_\alpha \in \mathscr{L}(H_\alpha)$ on the space $H_\alpha = \mathbb{C}^{n_\alpha}$ and consider the Hilbert space product $H = \mathbb{C}^n = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ and let u_α, v_α act on this product in the natural way so that u_1 and v_1 commutes with u_2, v_2. The operator u_1, u_2 together generate an abelian subalgebra and is maximal. Same for v_1, v_2 moreover the monomials $u_1^{k_1} u_2^{k_2} v_1^{\ell_1} v_2^{\ell_2}$ generates all $\mathscr{L}(H)$, so this representation is irreducible. And by the same reasoning as above we can show that $p_{(k_1,k_2),(\ell_1,\ell_2)}^{(v_1,v_2)|(u_1,u_2)} = 1/n$, so these pairs of maximally commutative observables are complementary.

So the full system $\mathscr{L}(H)$ splits into two subsystems $\mathscr{L}(H_1)$ and $\mathscr{L}(H_2)$ which do not interefere with each other. They represent two physically *kinematically* independent quantum systems $\mathscr{A}_1, \mathscr{A}_2$ whose observable algebras are generated resp. by (u_1, v_1) and (u_2, v_2). They could be not really independent because like in classical probability independence is a notion linked to a state.

We can proceed this way for any n by factorising into prime factors. So we could think of as this construction when n is prime as giving very basic quantum systems.

Example, when $n=2$ we have u,v satisfying $u^2=v^2=1$ and $uv=-vu$. Let $\sigma_x=u$, $\sigma_y=v$, $\sigma_z=(-i)uv$ unitary and hermitian matrices for which we can check that they satisfy the commutation relations

$$[\sigma_1,\sigma_2]=2i\sigma_3, \qquad [\sigma_2,\sigma_3]=2i\sigma_1, \quad [\sigma_3,\sigma_1]=2i\sigma_2,$$

and moreover any 2×2 complex matrix X can be written $X=\alpha+\beta\sigma_x+\gamma\sigma_y+\delta\sigma_z$. The operators $(\sigma_x,\sigma_y,\sigma_z)$ are called Pauli matrices and describe a quantum degrees of freedom with only two possible values, i.e. the abelian subalgebras have a spectrum with two points. This is the kind of model suitable to model the Stern-Gerlach experiment.

These C^*algebras (let's call them discrete canonical pairs) gives examples of very simple and discrete quantum observables. In particular we could take a state on which u has a given value, meaning that there exist states ω_k such that $\omega_k(u^\ell)=e^{2\pi i\ell k/n}$ for all $\ell=1,...,n-1$ (recall that $u^n=1$). These states are just given by

$$\omega_k(a)=\langle\varphi_k,a\varphi_k\rangle$$

where φ_k are the eigenfunctions of u. This means that ω_k is multiplicative on $C^*(u)$.

However we have also that it cannot be multiplicative on v (because u,v do not commue) and actually

$$\omega_k(v^\ell)=\langle\varphi_k,v^\ell\varphi_k\rangle=0, \qquad \ell=1,\ldots,n-1.$$

This means that they are uniformly distributed on the set $\{\exp(2\pi ik/n):k=0,\ldots,n-1\}$.

Here their maximal complementary shows up in the fact that while one is completely determined, the other is uniformly distributed. So in some sense they can be considered the quantum equivalent of discrete uniform random variables.

We would like now to take some limit $n\longrightarrow\infty$ in order to produce in this way continuous analogs of these algebras. This would give us an example of non-commutative C^* algebra generated by two abelian subalgebras with continuous spectrum.

The intuition we want to carry on is how we go from discrete uniform r.v. to continuous ones. In particular imagine that X is a r.v. with continuous distribution described by a density $p(x)$ on $\mathbb{R}$. I can imagine to approximate it in law by taking a discrete r.v. X_L such that $X_L=[X]_L$ for $L\in\mathbb{N}$ where $[x]_L=\lfloor Lx\rfloor/L$. Then we have for any continuous and bounded function $f:\mathbb{R}\longrightarrow\mathbb{R}$

$$\mathbb{E}[f(X_L)]=\int_{\mathbb{R}}f([x]_L)p(x)\mathrm{d}x\longrightarrow\int_{\mathbb{R}}f(x)p(x)\mathrm{d}x=\mathbb{E}[f(X)].$$

Let's try to implement the same procedure for a C^*-algebra. The first observation is that if we denote (u_n,v_n) a discrete canonical pair of degree n we have the following. We can take $L^2(\mathbb{T})$ as Hilbert space where $\mathbb{T}=\mathbb{R}\backslash\mathbb{Z}$ and represent each u_n and v_n as

$$u_nf(x)=\exp(2\pi i\,[x]_n)f(x), \qquad v_nf(x)=f(x-1/n), \qquad x\in\mathbb{T}.$$

One can check that u_n, v_n is a representation of the algebra we constructed above. In this way we can embed all the operators $(u_n, v_n)_{n\geqslant 0}$ into $\mathscr{L}(L^2(\mathbb{T}))$.

We have to understand what plays the role of "continuous functions" in this context. We just take monomials of the form $u_n^k v_n^\ell$ (they suffice to determine any other element of $C^*(u_n, v_n)$ due to their commutation relation). However is easy to see that $u_n^k v_n^\ell \to 1$ in the weak topology of $L^2(\mathbb{T})$. Somehow we need to look at high powers of u_n, v_n to see something interesting. We take $\ell_n = n^{1/2}[s]_{n^{1/2}}$ and $k_n = n^{1/2}[t]_{n^{1/2}}$ and now consider

$$\langle f_n, u_n^{k_n} v_n^{\ell_n} g_n \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \overline{f_n(x)} \exp(2\pi i k_n [x]_n) g_n(x - \ell_n / n) \mathrm{d}x.$$

Note that we choose ℓ_n, k_n in that particular way since the commutation relations reads

$$u_n^{k_n} v_n^{\ell_n} = e^{2\pi i k_n \ell_n / n} v_n^{\ell_n} u_n^{k_n} = e^{2\pi i [s]_{n^{1/2}} [t]_{n^{1/2}}} v_n^{\ell_n} u_n^{k_n}$$

so the choice of the factor $n^{1/2}$ was due to the nice cancellation in the phase factor here. By rescaling we have, for functions f_n, g_n supported on $(-\pi, \pi)$ and letting $x = y / n^{1/2}$.

$$\langle f_n, u_n^{k_n} v_n^{\ell_n} g_n \rangle_{L^2(\mathbb{T})} = \int_{(-\pi,\pi)} \overline{f_n(x)} \exp(2\pi i [t]_{n^{1/2}} [x]_n n^{1/2}) g_n(x - [s]_{n^{1/2}} / n^{1/2}) \mathrm{d}x$$

$$= n^{-1} \int_{(-\pi n^{1/2}, \pi n^{1/2})} \overline{f_n(y / n^{1/2})} \exp(2\pi i [t]_{n^{1/2}} n^{1/2} [y / n^{1/2}]_n) g_n((y - [s]_{n^{1/2}}) / n^{1/2}) \mathrm{d}y$$

so to have a well defined limit we can take $f_n(x) = n^{1/4} f(n^{1/2} x)$ and $g_n(x) = n^{1/4} g(n^{1/2} x)$ with f, $g \in C_0^\infty(\mathbb{R})$ so that for n large enough we have

$$\langle f_n, u_n^{k_n} v_n^{\ell_n} g_n \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{R}} \overline{f(y)} \exp(2\pi i [t]_{n^{1/2}} n^{1/2} [y / n^{1/2}]_n) g(y - [s]_{n^{1/2}}) \mathrm{d}y$$

so here now we can take the limit and obtain that

$$\lim_n \langle f_n, u_n^{k_n} v_n^{\ell_n} g_n \rangle_{L^2(\mathbb{T})} = \langle f, U(t) V(s) g \rangle_{L^2(\mathbb{R})} \tag{4}$$

where (U, V) are two **unitary groups** acting on $L^2(\mathbb{R})$ as

$$U(t) f(y) = \exp(2\pi i t y) f(y), \qquad V(s) f(y) = f(y - s).$$

Unitary group means that $U(t)^* = U(-t)$, $U(t)U(s) = U(t+s)$ for all $t, s \in \mathbb{R}$ and $U(0) = 1$. These relations come from the formula for the convergence in law above.

Exercise 1. Justify that U, V are unitary groups. Actually try to prove it using only (4) and not the explicit form of the operators.

Moreover they are weakly continuous, i.e. $t \mapsto \langle f, U(t) g \rangle$ is continuous for all $f, g \in L^2(\mathbb{R})$. Since they are unitary they are also strongly continuous.

They satisfying the commutation relations

$$U(t) V(s) = e^{2\pi i s t} V(s) U(t), \qquad t, s \in \mathbb{R}. \tag{5}$$

These commutation relations are called the Weyl form of the canonical commutation relations and they are the implementation of the Heisenberg's commutation relations

$$[P, Q] = i\hbar,$$

within the C^*-framework (i.e. working only with bounded operators). The link between these formulas comes from interpreting the two unitary groups as being generated by the self-adjoint operators P, Q i.e. as

$$U(t) = \exp(iQt), \qquad V(s) = \exp(iPs),$$

and recalling that Baker-Campbell-Hausdorff forumula gives (under suitable conditions for unbounded self-adjoint operators A, B with $[A, B]$ given by a scalar that)

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

Applying it formally to P, Q we have

$$e^{iQt} e^{iPs} = e^{i(Ps+Qt)+\frac{1}{2}[P,Q]} = e^{\frac{1}{2}[P,Q]} e^{i(Ps+Qt)}, \qquad e^{iPs} e^{iQt} = e^{i(Ps+Qt)-\frac{1}{2}[P,Q]} = e^{\frac{1}{2}[P,Q]} e^{i(Ps+Qt)}$$

so that

$$e^{iQt} e^{iPs} = e^{i\hbar st} e^{iPs} e^{iQt},$$

so in my notations $\hbar = 2\pi$.

Putting aside for the moment unbounded operators we obtained a pair of commutative C^* algebras $\mathscr{Q}, \mathscr{P}$ given by $\mathscr{Q} = C^*((U(t))_{t\in\mathbb{R}})$, $\mathscr{P} = C^*((V(s))_{s\in\mathbb{R}})$ which are concrete C^* algebras on $L^2(\mathbb{R})$. We denote $\mathscr{A} = C^*(\mathscr{Q}, \mathscr{P})$.

The spectrum of $\mathscr{Q}$ and $\mathscr{P}$ can be identified with a subset of $\mathbb{S} \subset \mathbb{C}$. So they are like random variables taking values on $\mathbb{S}$ and they can be easily parametrized by real number. In particular if ω is a state on $\mathscr{A}$ then the function $t \mapsto \omega(U(t))$ is continuous on $\mathbb{R}$ and positive definite and normalized so it corresponds to probability measure on $\mathbb{R}$, which we denote by $\mu^{\mathscr{Q},\omega}$ this is the law of $\mathscr{Q}$ on ω. Similarly for $\mathscr{P}$. However $\mathscr{Q}$ and $\mathscr{P}$ do not commute.

The C^*-algebra $\mathscr{A}$ is called the Weyl algebra. It is the fundamental example of two continuous observables which do not commute and in some sense they show complementarity.

4 Unitary representations of $\mathbb{R}$ and generalized observables

Let us concentrate only on one of the families of unitaries, let's say $(U(t))_{t\in\mathbb{R}}$. I want to look at it at some kind of non-commutative Fourier transfrom (or characteristic function). It is giving me information about an observable very much like the characteristic function give informations about a random variable.

Assume that we the family $(U(t))_{t\in\mathbb{R}}$ is a family of bounded operators on an Hilbert space H (giving a representation of $\mathbb{R}$ on H).

For any unit vector $v \in H$ we can form the function $\varphi^v(t) = \langle v, U(t)v\rangle$, it is easy to show that $\varphi^v(0) = 1$, and φ^v is positive definite, i.e.

$$\sum_{i,j} \bar{\lambda}_i \lambda_j \varphi^v(t_j - t_i) \geq 0 \qquad (\lambda_i)_i \subseteq \mathbb{C}, (t_i)_i \subseteq \mathbb{R}.$$

This are the same properties of the characteristic function of a measure, so by Bochner's theorem, there exist a measure μ^v on $\mathbb{R}$ so that

$$\varphi^v(t) = \int_{\mathbb{R}} e^{itx} \mu^v(\mathrm{d}x), \qquad t \in \mathbb{R}.$$

In particular this defines a linear functional ℓ^v on $C(\mathbb{R})$ by

As soon as we have extended ℓ^v continously we can define a $*$-representation Q of $C_0(\mathbb{R})$ on $\mathscr{L}(H)$. For any $f \in C_0(\mathbb{R})$ define the operator $Q(f)$ by the relation

$$\langle v, Q(f)v\rangle = \ell^v(f)$$

and its polarization. This define a bounded operator such that $\|Q(f)\|_{\mathscr{L}(H)} \leq \|f\|_\infty$ and $Q(f)^* = Q(\bar{f})$ and Q is linear in f and $Q(f)Q(g) = Q(fg)$ (by continuity is enough to check there relations of $f \in \mathscr{S}(\mathbb{R})$ and this case we have the more precise relation

$$Q(f) = \int_{\mathbb{R}} U(t)\hat{f}(t)\mathrm{d}t$$

(remember that the r.h.s is defined as a weak integral). I would like to use $f(x) = e^{isx}$, in order to do this observe that for any $v \in H$

$$\langle v, Q(f)v\rangle = \int_{\mathbb{R}} \varphi^v(t)\hat{f}(t)\mathrm{d}t,$$

looking at this formula is clear that if $f_n \longrightarrow f$ in such a way that the r.h.s. converges, so we can take $f_n(x) = e^{isx}e^{-x^2/(2n)}$ so that

$$\langle v, Q(f_n)v\rangle = \int_{\mathbb{R}} \varphi^v(t)\hat{f}_n(t)\mathrm{d}t = (2\pi n^{-1})^{-1/2}\int_{\mathbb{R}} \varphi^v(t)e^{-n(t-s)^2/2}\mathrm{d}t \longrightarrow \varphi^v(s)$$

by continuity of φ^v. So this suggest that we can define $Q(e^{is\cdot}) = U(s)$.

Note also that if $f_n \uparrow f$ then the sequence $(\langle v, Q(f_n)v\rangle)_n$ is monotone increasing since if $f \geq 0$ then $\langle v, Q(f)v\rangle \geq 0$ so we can extend Q to all $C_b(\mathbb{R})$. To check that the extension is unique the following argument works.

Take now the family $(h_n(x) = \exp(-nx^2))_n$ then by continuity of φ^v it is easy to prove that

$$Q(h_n) \longrightarrow 1_{\mathscr{L}(H)}.$$

Observe that if $f \in C_b(\mathbb{R})$ then $h_n f \in C_0(\mathbb{R})$ and it follows that for any extension Q' of Q to $C_b(\mathbb{R})$ we have

$$Q'(h_n)Q'(f) = Q'(h_n f) = Q(h_n f) = Q(h_n)Q(f)$$

and taking limits we have $Q'(f) = Q(f)$.

This argument proves the following theorem

Theorem 3. *Any weakly-continuous one-parameter unitary group $(U(t))_t$ in $\mathscr{L}(H)$ corresponds to a C^* algebra representation Q of $C_b(\mathbb{R})$ on $\mathscr{L}(H)$.*

It is suggestive to write $f(Q) = Q(f)$ and think to $f(Q)$ as a function computed on an operator Q in such a way that the formula

$$U(t) = \exp(itQ),$$

has now a sense.

We could of course associate to Q an unbounded linear operator $\hat{Q}$ on a dense domain within H in such a way that by Stone theorem $\hat{Q}$ is the generator of the group $(U(t))_{t\in\mathbb{R}}$.

From the operational point of view such an homomorphism Q represent an observable in the sense that we can measure its expectation value on any state ω and also we can see it as a random variable with a law given by the linear functional

$$f \mapsto \omega(f(Q)).$$

If we go back to the Weyl relation we now understand that they describe two observables P, Q which satisfy the commutation relations

$$\exp(itQ)\exp(isP) = \exp(2\pi ist)\exp(isP)\exp(itQ).$$

Combining unbouded operators is a task of the same difficulty of combining two homomorphism or two unitary representations of $\mathbb{R}$. There is no simple way to understand, for example, the sum $P + Q$.

Tentantively in this course we take the attitude that an observable is really a *-homomorphism of $C_b(\mathbb{R})$ into either some abstract C^*-algebra or into a C^*-algebra of operators. This extends to the non-commutative/quantum context the probabilistic notion of real random variable.

This is coherent with our modelisation which sees observables as self-adoint elements of a C^*-algebra in that if $f: \mathbb{R} \longrightarrow \mathbb{R}$ then $f(Q)$ is a self-adjoint operator.

From the point of the of C^*algebraic approach the homomorphism Q, P represents families of observables which are then given by choosing a particular way f to measure the quantity Q so that we have a definite observable $Q(f)$, i.e. self-adjoint element of C^*. Let's call them *extended observables.*

If a is a self-adjoint element of a C^*-algebra $\mathscr{A}$ we can always via continuous functional calculus associate to it an observable A in this extended sense by letting $A(f) := f(a)$ and therefore have that $A \in \mathrm{Hom}(C(\mathbb{R}), \mathscr{A})$.

Extendend observables allows to handle quantities which are not naturally bounded and therefore cannot be represented by elements of the C^*-algebra.

Think for example to a Gaussian random variable X. A Gaussian random variable is not an element of a C^*-algebra since X can take arbitrarily large values. However if we look at X has a $*$-homomorphism by letting $X(f) := f(X)$ for any $f \in C(\mathbb{R})$ then X is a well defined observable. In this case it has a concrete realisation on $L^2(\mathbb{P})$ and if we take $v(\omega) = 1$ we have that

$$\langle v, X(f)v\rangle = \mathbb{E}[f(X)].$$

Commutative setting: representation Q_0 of an abelian C^* algebra $\mathscr{A}$ on an Hilbert space $\mathscr{H}$.

$$\mathscr{A} = C^0_b(\mathbb{R}^n, \mathbb{C}), \qquad \mathscr{H} = L^2(\mathbb{R}^n, \mathbb{C})$$

$$a(x) \in \mathscr{A}, \qquad h(x) \in \mathscr{H}$$

$$Q_0(a)h = a(x)h(x)$$

$$Q_0(e^{i\alpha x})h = e^{i\alpha x}h(x)$$

Norm on $\mathscr{A}$ is the uniform norm on $C^0_b(\mathbb{R}^n, \mathbb{C})$. This representation is faithful $\ker(Q_0) = 0$.

Suppose that we have a cyclic vector $h_0 \in \mathscr{H}$.

$$\mathscr{H}_0 = \{Q_0(a)h_0, a \in \mathscr{A}\}, \qquad \overline{\mathscr{H}_0} = \mathscr{H}.$$

Theorem 4. *Under the hypothesis $\overline{\mathscr{H}_0} = \mathscr{H}$ the system $(\mathscr{H}, \mathscr{A}, Q_0)$ is isomorphic to $(L^2(X, \mathbb{C}, \mu)$, $C^0_\infty(X, \mathbb{C}), m)$ where X is a locally compact Hausdorff space, μ is a measure on X and C^0_∞ is the set of continuous functions going to zero at infinity and m is the multiplication operator.*

Proof. By Gelfand–Naimark $\mathscr{A} \approx C^0_\infty(X, \mathbb{C})$ where X is the space of characters (i.e. pure, positive states on $\mathscr{A}$) equipped with the weak-* topology. □

Remark 5. In the case where $1 \in \mathscr{A}$ then X is compact, so $\mathscr{C}^0_\infty(X) = \mathscr{C}^0(X)$.

We can take $\mathscr{H} = \mathscr{H}^{\mathrm{GNS}}$ where the state generating the GNS construction is $\omega^{h_0}(a) = \langle h_0, Q_0(a)h_0\rangle$. Here ω^{h_0} is a positive functional on $\mathscr{A}$. ω^{h_0} is continuous wrt. the $\|\cdot\|_\infty$ norm where we identify $\mathscr{A} \approx C^0_\infty(X, \mathbb{C})$. So ω^{h_0} defines a measure on X since is in $(C^0_\infty(X, \mathbb{C}))^*$ (the dual space, i.e. the space of bounded measures). Moreover it is a non-negative measure. We call it μ and have that

$$\mathscr{H}^{\mathrm{GNS}} \longrightarrow L^2(X, \mu)$$

$$U(Q_0(a)h_0) = a(x) \in L^2(X, \mu)$$

This is an isomorphism where Q_0 corresponds to the multiplication m.

Let us note that we have that $\mathbb{R}^n \hookrightarrow X$ and actually X is a compactification of $\mathbb{R}^n$ which we are not able to work with explicitly.

5 The Weyl algebra

Let's go back to a representation of the canonical commutation relations in Weyl form, i.e. to a pair of two unitary representations U, V on an Hilbert space H of the additive group of the reals, i.e.

$$U(t)U(s) = U(t+s), \qquad U(t)^* = U(-t),$$

and similarly for V, satisfying

$$U(t)V(s) = V(s)U(t)\exp(ist), \qquad s, t \in \mathbb{R}. \tag{6}$$

Note that we can form the Weyl operators $(W(z))_{z\in\mathbb{C}}$ defined for $z=\alpha+i\beta\in\mathbb{C}$ as

$$W(\alpha+i\beta)=e^{i\alpha\beta/2}e^{i\alpha Q}e^{i\beta P}.$$

One can check that $W(z)$ is unitary for any $z\in\mathbb{C}$ and that

$$W(z)W(z')=e^{i\mathrm{Im}\langle z,z'\rangle}W(z+z'),\qquad z,z'\in\mathbb{C} \tag{7}$$

where $\langle z,z'\rangle=\bar{z}z'$ is the Hermitian scalar product of $\mathbb{C}$ (a one dimesional complex Hilbert space). All we are going to say generalises easily to $(W(z))_{z\in K}$ for finite dimensional Hilbert spaces K and strongly continuous unitary operators $(W(z))_{z\in K}$ such that (7) is satisfied.

Remark that $\omega(z,z')=\mathrm{Im}\langle z,z'\rangle$ is antisymmetric i.e. $\omega(z,z')=-\omega(z',z)$ and that $\omega(z,z')=0$ for all z implies $z'=0$ (i.e. ω is non-degenerate).

Let $\tilde{W}(z,\lambda)=e^{i\lambda}W(z)$ for $\lambda\in\mathbb{R}$ then

$$\tilde{W}(z,\lambda)\tilde{W}(z',\lambda')=\tilde{W}(z+z',\lambda+\lambda'+\mathrm{Im}\langle z,z'\rangle),$$

which means that the $(\tilde{W}(z,\lambda))_{z,\lambda}$ give a unitary representation of the *Heisenberg group* $\mathbb{H}\approx\mathbb{C}\times\mathbb{R}$ with composition $(z,\lambda)(z',\lambda')=(z+z',\lambda+\lambda'+\mathrm{Im}\langle z,z'\rangle)$. It a non-commutative group since ω is not symmetric.

Theorem 6. *(Von Neumann) Regular irreducible representations of the (finite dimensional) Weyl relations are all unitarily equivalent, i.e there is only one up to isomorphism.*

Remark 7. This theorem is fundamental because allows to use the most convenient representation to study the QM of finitely many quantum degrees of freedom (given by Weyl relations). Historically QM was developed independently by Schrödinger and Heisenberg (with Born and Jordan), then Dirac ('20) showed (formally) that the two approaches were unitarily equivalent. And later on Von Neumann ('30-'40) closed the matter by showing that there are no other possible representations. The theorem is false in infinite dimensions (and for physically motivated reasons).

Proof. (one dimensional case) Let us introduce the operator

$$P:=\int_{\mathbb{R}^2}\mathrm{d}\alpha\,\mathrm{d}\beta\, e^{-(|\alpha|^2+|\beta|^2)/4}e^{i\alpha\beta/2}e^{i\alpha Q}e^{i\beta P}=\int_{\mathbb{C}}e^{-|z|^2/4}W(z)\mathrm{d}z\mathrm{d}\bar{z}$$

which is well defined as a strong integral, i.e when computed on vectors $\psi\in H$ (regularity is needed here, at least). We can check that $P\neq 0$ by observing that

$$W(-w)W(z)W(w)=e^{i\mathrm{Im}\langle z,w\rangle}W(-w)W(z+w)=e^{i\mathrm{Im}\langle z,w\rangle}e^{i\mathrm{Im}\langle -w,z+w\rangle}W(z)=e^{i2\mathrm{Im}\langle z,w\rangle}W(z)$$

and looking at

$$W(-w)PW(w)=\int_{\mathbb{C}}e^{-|z|^2/4}W(-w)W(z)W(w)\mathrm{d}z\mathrm{d}\bar{z}=\int_{\mathbb{C}}e^{-|z|^2/4}e^{i2\mathrm{Im}\langle z,w\rangle}W(z)\mathrm{d}z\mathrm{d}\bar{z}$$

Assume that $P=0$, so we have $W(-w)PW(w)=0$ and for any vector $\psi\in H$ we will have for any $w\in\mathbb{C}$

$$0=\int_{\mathbb{C}}e^{-|z|^2/4}e^{i2\mathrm{Im}\langle z,w\rangle}\langle\psi,W(z)\psi\rangle\mathrm{d}z\mathrm{d}\bar{z}$$

by Fourier transform with respect to both real and imaginary part of w we deduce that $e^{-|z|^2/4}\langle\psi, W(z)\psi\rangle = 0$ for almost all $z\in\mathbb{C}$ and by continuity of this function we have that $\langle\psi, W(z)\psi\rangle = 0$ for all z, and ψ but this is in contradiction with $W(0) = 1$. So $P \neq 0$.

With a tedious but elementary computartion with Fubini theorem and Gaussian integrals one can check that (exercise)

$$PW(w)P = e^{-|w|^2/4}P, \qquad w\in\mathbb{C}$$

so in particular this says that $P^2 = P$ and since is clear by definition that $P^* = P$ we have that that P is a non-trivial projection (it cannot be $P = 1$). So let ψ_0 be a unit vector in $\mathrm{Im}(P)$ so that $P\psi_0 = \psi_0$.

By irreducibility the linear space $\mathscr{D} := \mathrm{span}\{W(z)\psi_0 : z\in\mathbb{C}\}$ is dense in H since any element of the C^*-algebra generated by $(W(z))_{z\in\mathbb{C}}$ can be approximated by linear combination of $W(z)$s. We have also that ψ_0 is the only eigenvector of P since if φ is another one orthogonal to ψ_0 we have

$$\langle\varphi, W(z)\psi_0\rangle = \langle P\varphi, W(z)P\psi_0\rangle = \langle\varphi, PW(z)P\psi_0\rangle = e^{-|w|^2/4}\langle\varphi, \psi_0\rangle$$

so we learn that $\langle\varphi, W(z)\psi_0\rangle = 0$ for all z but then $\langle\varphi, \psi\rangle = 0$ for all $\psi\in\mathscr{D}$ and this implies that $\varphi = 0$. We learned also that there is a state ω such that

$$\omega_0(W(z)) = \langle\psi_0, W(z)\psi_0\rangle = e^{-|w|^2/4}.$$

Therefore we conclude that on any irreducible Weyl system there is a state ω such that

$$\omega_0(W(z)) = e^{-|z|^2/4}$$

(this relation define ω_0 on the full C^*-algebra, because any element can be approx. by linear comb of Ws).

Now if $(H, (W(z))_{z\in\mathbb{C}})$ and $(H', (W'(z))_{z\in\mathbb{C}})$ are two irreducible regular representations of the Weyl algebra we can construct a unitary operator $U: H \longrightarrow H'$ by extending by linearity the equality

$$UW(z)\psi_0 = W'(z)\psi'_0$$

to the full $\mathscr{D}$ and observe that U is unitary since

$$\langle UW(z)\psi_0, UW(w)\psi_0\rangle = \langle W'(z)\psi'_0, W'(w)\psi'_0\rangle = \langle\psi'_0, PW'(-z)W'(w)P\psi'_0\rangle$$

$$= e^{-i\mathrm{Im}\langle z,w\rangle}\langle\psi'_0, PW'(w-z)P\psi'_0\rangle = e^{-i\mathrm{Im}\langle z,w\rangle}e^{-|w-z|^2/4} = \langle W(z)\psi_0, W(w)\psi_0\rangle$$

therefore is bounded and can be extended to a unitary operator on the whole H. This show that the two representations of the Weyl relations are unitarily equivalent. □

The regular state ω_0 such that

$$\omega_0(W(z)) = e^{-|z|^2/4}$$

is called Fock vacuum or vacuum state for the Weyl representation.

Since the representation of the Weyl relation is essentially unique we could think to use the one we like (or the one more convenient).

Remark 8. All we are saying generalises easily to $(W(z))_{z\in K}$ for finite dimensional Hilbert spaces K and strongly continuous unitary operators $(W(z))_{z\in K}$ such that (7) is satisfied.

▸ **Schrödinger representation.** This is given on $H = L^2(\mathbb{R})$ by taking

$$U(t)f(x) = e^{itx}f(x), \qquad V(s)f(x) = f(x-s), \qquad f\in H, t, s\in\mathbb{R}.$$

Is this irreducible? If it is not irreducible then there exists two unit vectors $f, g\in L^2(\mathbb{R})$ such that for all $t, s\in\mathbb{R}$

$$0 = \langle f, U(t)V(s)g\rangle = \int_{\mathbb{R}} \bar{f}(x)e^{itx}g(x-s)\mathrm{d}x.$$

But then if this is true for any t we have that (by Fourier transform) $|\bar{f}(x)g(x-s)| = 0$ for almost every s and x, by squaring and integrating in x, s we have

$$0 = \int \mathrm{d}x\int \mathrm{d}s|\bar{f}(x)g(x-s)|^2 = \|f\|_{L^2}^2\|g\|_{L^2}^2 = 1$$

so we have a contradiction and this proves that the Schrödinger representation is irreducible.

Therefore there must exist a vector $\psi_0\in L^2(\mathbb{R})$ such that

$$\langle\psi_0, e^{-its/2}U(t)V(s)\psi_0\rangle = \exp\left(-\frac{1}{4}(s^2+t^2)\right), \qquad s, t\in\mathbb{R}$$

and by taking $s = 0$ we have

$$\int |\psi_0(x)|^2 e^{itx}\mathrm{d}x = \exp\left(-\frac{t^2}{4}\right)$$

which means that $|\psi_0(x)|^2$ is a Gaussian function (actually the density of a $\mathcal{N}(0, 1/2)$ random variable), namely

$$|\psi_0(x)|^2 = \frac{1}{(\pi)^{1/2}}e^{-x^2}$$

this determines ψ_0 up to a phase factor: $\psi_0(x) = e^{if(x)}\frac{1}{(\pi)^{1/4}}e^{-x^2/2}$. However

$$\exp\left(-\frac{s^2+t^2}{4}\right) = \langle\psi_0, e^{-its/2}U(t)V(s)\psi_0\rangle = e^{-its/2}\int \mathrm{d}x e^{itx}e^{-if(x)}\frac{1}{(\pi)^{1/4}}e^{-x^2/2}e^{if(x-s)}\frac{1}{(\pi)^{1/4}}e^{-(x-s)^2/2}$$

$$= \frac{e^{-its/2}}{(\pi)^{1/2}}\int \mathrm{d}x e^{it(x+s/2)}e^{-if(x+s/2)}e^{-(x+s/2)^2/2}e^{if(x-s/2)}e^{-(x-s/2)^2/2}$$

$$= \frac{e^{-s^2/4}}{(\pi)^{1/2}}\int \mathrm{d}x e^{-x^2}e^{itx}e^{i(f(x-s/2)-f(x+s/2))}$$

so we have

$$\frac{1}{(\pi)^{1/2}}\int \mathrm{d}x e^{itx}e^{i(f(x-s/2)-f(x+s/2))}e^{-x^2} = \exp\left(-\frac{t^2}{4}\right)$$

Now is better because this is saying that the function

$$\frac{1}{(\pi)^{1/2}} e^{i(f(x-s/2)-f(x+s/2))} e^{-x^2}$$

is the density of a Gaussian $\mathcal{N}(0,1/2)$ so it is equal to $\frac{1}{(\pi)^{1/2}} e^{-x^2}$ and we conclude that $f=0$, so we have proven that, in the Schrödinger representation we have

$$\psi_0(x) = \frac{e^{-x^2/2}}{\pi^{1/4}}.$$

▸ **Gaussian representation.** We can introduce the unitary transformation (ground state transformation)

$$\mathcal{J}: L^2(\mathbb{R}) \longrightarrow L^2(\gamma)$$

where γ is the Gaussian measure with mean zero and variance $1/2$ by letting

$$(\mathcal{J}\psi)(x) = \psi(x) / \psi_0(x), \qquad x \in \mathbb{R}.$$

Then we have the images U', V' of the Weyl pair U, V given by (for $f \in L^2(\gamma)$)

$$U'(t) f(x) = (\mathcal{J} U(t) \mathcal{J}^{-1} f)(x) = \psi_0(x)^{-1} U(t)(\psi_0 f)(x) = e^{itx} f(x)$$

$$V'(s) f(x) = (\mathcal{J} V(s) \mathcal{J}^{-1} f)(x) = \psi_0(x)^{-1} V(s)(\psi_0 f)(x) = \psi_0(x)^{-1} \psi_0(x-s) f(x-s)$$

$$= e^{xs - s^2/2} f(x-s)$$

One can check directly that this gives indeed a strongly continuous representation of the Weyl relation on $L^2(\gamma)$. This is called the Gaussian representation and is useful because there is a nice basis for $L^2(\gamma)$ given by polynomial functions, the Hermite basis $(h_n(x))_{n \geqslant 0}$ (indeed note that polynomials are in $L^2(\gamma)$ and that one can perform a Gram-Schmidt ortogonalisation procedure of the family $(x^n)_{n \geqslant 0}$ which is a separating family for $L^2(\gamma)$ by Stone-Weierstrass) and every $h_n(x)$ has monomial of highest degree n.

▸ **Reducible (regular) representations of Weyl relations.**

Assume now that $(W(z))_{z \in \mathbb{C}}$ does not act irreducibly on H then the range of P is not one dimensional. However in general we have that for any $\psi, \varphi \in H$

$$\langle W(z) P\varphi, W(z') P\psi \rangle = \langle \varphi, P W(z)^* W(z') P\psi \rangle = f(z, z') \langle P\varphi, P\psi \rangle$$

where we used that there exists a function f such that $f(z, z') P = P W(z)^* W(z') P$ and that does not depend on the specific representation. This means that I can compute it in any representation, in particular if we denote $\psi_0^\#$ the vacuum vector of the Schrödinger representation and by $(W^\#(z))_{z \in \mathbb{C}}$ the Weyl operators in the Schrödinger representation we have $\langle \psi_0^\#, P^\# W^\#(z)^* W^\#(z') P^\# \psi_0^\# \rangle_{L^2(\mathbb{R})} = f(z, z')$ and

$$\langle W(z) P\varphi, W(z') P\psi \rangle_H = \langle W^\#(z) P^\# \psi_0^\#, W^\#(z') P^\# \psi_0^\# \rangle_{L^2(\mathbb{R})} \langle P\varphi, P\psi \rangle_{\mathrm{Im}(P)} \tag{8}$$

therefore we can introduce a unitary operator $\mathcal{J}: L^2(\mathbb{R}) \otimes \mathrm{Im}(P) \longrightarrow H$ defined by

$$\mathcal{J}(W^\sharp(z)\psi_0^\sharp \otimes P\varphi) = W(z)P\varphi.$$

Remark 9. Let us recall that if K_1, K_2 are two Hilbert spaces there is a canonical notion of product of them, which is the Hilbert space $K_1 \otimes K_2$ obtained by completing the span of all the monomials of the form $\{v \otimes w: v \in K_1, w \in K_2\}$ with respect to the Hermitian scalar product define on monomials by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{K_1 \otimes K_2} := \langle v_1, v_2 \rangle_{K_1} \langle w_1, w_2 \rangle_{K_2},$$

and extended by linearity (one has to check that this is a positive definite quantity, but for general results the product of positive definite kernels is a positive definite kernel).

Since $\{W^\sharp(z)\psi_0^\sharp\}_{z\in\mathbb{C}}$ span a dense subset of $L^2(\mathbb{R})$ (by irreducibility of the Schrödinger rep.) and $\{P\varphi\}_{\varphi\in H} = \mathrm{Im}(P)$ as Hilbert space, then $\mathcal{J}$ is well defined on all $L^2(\mathbb{R}) \otimes \mathrm{Im}(P)$ and by construction it is isometric on H by (8). It remains to check that it is surjective. Let $\varphi \notin \mathrm{Im}(\mathcal{J})$ then we must have for any vector of the form $W(z)PW(-z)\psi$ since these are surely in the image of $\mathcal{J}$, so for any $z\in\mathbb{C}$ and $\psi \in H$ we have

$$0 = \langle \varphi, \mathcal{J}(W^\sharp(z)\psi_0^\sharp \otimes PW(-z)\psi)\rangle = \langle \varphi, W(z)PW(-z)\psi\rangle$$

recalling the definition of P we have

$$0 = \int_{\mathbb{C}} e^{-|w|^2/4} \langle \varphi, W(z)W(w)\, W(-z)\psi\rangle \mathrm{d}w\mathrm{d}\bar{w} = \int_{\mathbb{C}} e^{-|w|^2/4} e^{-2\mathrm{Im}\langle z,w\rangle} \langle \varphi, W(w)\psi\rangle \mathrm{d}w\mathrm{d}\bar{w}$$

since this has to be zero for any $z\in\mathbb{C}$ we deduce by Fourier transform that $\langle \varphi, W(w)\psi\rangle = 0$ for a.e. w but is also continuous in w so it is zero for all $w\in\mathbb{C}$ and then also for any $\psi\in H$. By taking $w = 0$ and $\psi = \varphi$ we deduce that $\|\varphi\| = 0$ so $\varphi = 0$. In this way we proved that $\mathcal{J}$ is surjective and therefore that it is unitary.

Corollary 10. *Any regular representation $((W(z))_{z\in\mathbb{C}}, H)$ of the Weyl relations is unitarily equivalent to the representation $((W^\sharp(z))_{z\in\mathbb{C}}, L^2(\mathbb{R}) \otimes K)$ where $K = PH$ and $W^\sharp(z)$ acts trivially on K and as the Schrödinger representation on $L^2(\mathbb{R})$, i.e.*

$$W^\sharp(z)(\psi^\sharp \otimes \psi^\natural) = (W^\sharp(z)\psi^\sharp) \otimes \psi^\natural, \qquad z\in\mathbb{C}, \psi^\sharp \in L^2(\mathbb{R}), \psi^\natural \in K.$$

We know that the only regular irreducible representation on a Hilbert space H of the Weyl relations is given by a state such that

$$\omega(W(z)) = e^{-|z|^2/4}.$$

This state corresponds to a cyclic vector $\psi_0 \in H$ by means of the relation $\omega(a) = \langle \psi_0, a\psi_0\rangle$ which defines a state on $\mathscr{L}(H)$, we have also that the weak closure of the Weyl algebra $(W(z))_{z\in\mathbb{C}}$ is the whole $\mathscr{L}(H)$.

Moreover any state with the same expectation of the Weyl operators give rise to a representation (via GNS construction) which is unitarily equivalent with the Schrödinger representation, in particular it is irreducible.

How do reducible representations looks like? I want to give an example. The easiest way to come up with a reducible representation is to that two copies $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^2)$ of the Schrödinger representation and define Weyl operators

$$(\tilde{W}(s+it)f)(x_1, x_2) = (e^{its/2}\tilde{U}(s)\tilde{V}(t)f)(x_1, x_2)$$

$$= e^{its/2}e^{is(ax_1+bx_2)}f(x_1 - at, x_2 + bt) = e^{its/2}U_1(as)U_2(bs)V_1(at)V_2(-bt)$$

where (U_1, V_1) and (U_2, V_2) are Weyl pairs acting independenlty on the two factors of $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, so they commute among them. We can check that they satisfy the Weyl relations

$$\tilde{W}(s+it)\tilde{W}(s'+it') = e^{its/2}U_1(as)U_2(bs)V_1(at)V_2(-bt)e^{it's'/2}U_1(as')U_2(bs')V_1(at')V_2(-bt')$$

$$= e^{its/2}e^{it's'/2}(U_1(as')U_2(bs')V_1(at')V_2(-bt'))(U_1(as)U_2(bs)V_1(at)V_2(-bt)) \times$$

$$\times e^{-i(-bt)(bs') - i(at)(as') + i(bs)(-bt') + i(as)(at')}$$

$$= e^{i(b^2s't - a^2s't - b^2st' + a^2st')}\tilde{W}(s+it)\tilde{W}(s'+it')$$

$$= e^{-i(b^2-a^2)\mathrm{Im}[(\overline{s+it})(s'+it')]}\tilde{W}(s'+it')\tilde{W}(s+it)$$

iff $a^2 - b^2 = 1$. This also implies that the operators $\tilde{W}$ are unitary, indeed

$$(e^{its/2}U_1(as)U_2(bs)V_1(at)V_2(-bt))^* = e^{-its/2}V_2(bt)V_1(-at)U_2(-bs)U_1(-as)$$

$$= e^{-its/2}e^{i(-as(-at))}e^{i(-bs(bt))}U_1(-as)U_2(-bs)V_1(-at)V_2(bt)$$

$$= e^{-its/2}e^{i((a^2-b^2)st)}U_1(-as)U_2(-bs)V_1(-at)V_2(bt) = \tilde{W}(-s-it).$$

In this way we can construct a family of Weyl pairs. Let $\Psi_0 = \psi_0 \otimes \psi_0$ the tensor product of the two vacuum states, then

$$\langle \psi_0 \otimes \psi_0, \tilde{W}(s+it)(\psi_0 \otimes \psi_0)\rangle_{L^2(\mathbb{R}^2)} = e^{i(a^2-b^2)ts/2}\langle \psi_0, U_1(as)V_1(at)\psi_0\rangle_{L^2(\mathbb{R})}\langle \psi_0, U_2(bs)V_2(-bt)\psi_0\rangle_{L^2(\mathbb{R})}$$

$$= \langle \psi_0, e^{ia^2ts/2}U_1(as)V_1(at)\psi_0\rangle_{L^2(\mathbb{R})}\langle \psi_0, e^{-ib^2ts/2}U_2(bs)V_2(-bt)\psi_0\rangle_{L^2(\mathbb{R})}$$

$$= \langle \psi_0, W(as+iat)\psi_0\rangle_{L^2(\mathbb{R})}\langle \psi_0, W(bs-ibt)\psi_0\rangle_{L^2(\mathbb{R})}$$

$$= e^{-|as+ait|^2/4}e^{-|-bt+bis|^2/4} = e^{-(a^2+b^2)|s+it|^2/4} = e^{-(1+2b^2)|s+it|^2/4}.$$

We have proven the following:

Theorem 11. *For any $Q \geq 1/2$ there exists a state ω_Q on the Weyl algebra such that*

$$\omega_Q(W(z)) = e^{-Q|z|^2/2}.$$

Moreover we know that for $Q = 1/2$ is pure (because it corresponds to the Schrödinger model) and for $Q > 1/2$ it is not.

Let us show concretely that the representation given by $\tilde{W}$ on $L^2(\mathbb{R}^2)$ is not irreducible. Consider the operators

$$(W^{\#}(s+it)f)(x_1,x_2) = e^{its/2}U_1(bs)U_2(as)V_1(-bt)V_2(at) = W_1(bs-ibt)W_2(as+iat)$$

and note that

$$\tilde{W}(s'+it')W^{\#}(s+it) = W_1(as+iat)W_2(bs-ibt)W_1(bs-ibt)W_2(as+iat)$$

$$= \underbrace{e^{i\mathrm{Im}\langle as+iat, bs-ibt\rangle}e^{i\mathrm{Im}\langle bs-ibt, as+iat\rangle}}_{=1} W_1(bs-ibt)W_2(as+iat)W_1(as+iat)W_2(bs-ibt)$$

$$= W^{\#}(s+it)\tilde{W}(s'+it')$$

so the two families commute. In particular the Stone–von Neumann projector $P^{\#}$ associated to the Weyl system $W^{\#}$ satisfies

$$P^{\#}\tilde{W}(z) = \tilde{W}(z)P^{\#}$$

and therefore $(W^{\#}(z))_{z\in\mathbb{C}}$ is not an irreducible representation since $P^{\#}$ is a non-trivial self-adjoint operator.

Moreover if $\psi_0^{\#} \in L^2(\mathbb{R}^2)$ is a unit vector such that $P^{\#}\psi_0^{\#} = \psi_0^{\#}$ then the space $K = \overline{\{W^{\#}(z)\psi_0^{\#}: z\in\mathbb{C}\}}^{L^2(\mathbb{R}^2)}$ is invariant under the action of $\tilde{W}(z)$ and we have that $\{\tilde{W}(z)K: z\in\mathbb{C}\}$ is dense in $L^2(\mathbb{R}^2)$.

Question 1. It is a fact that there not exists states on the Weyl algebra for which

$$\omega_Q(W(z)) = e^{-Q|z|^2/2},$$

with $Q < 1/2$. How to prove it?

Remark 12. After the lecture, Jaka came up with the following idea to prove it. Unfortunately, it uses unbounded operators and analytic vectors, which we haven't discussed. The idea is to prove that if $Q < 1/2$ then this clashes with the Heisenberg commutation relations. It goes as follows.

Consider

$$\omega_c(W(z)) = e^{-\frac{1}{4}c|z|^2}, \qquad c \in [0,1).$$

This is an analytic state so we can work in its GNS representation without worrying about any unbonudedness problems since the vacuum will be an analytic vector for Q, P. Using the definitions

$$Q = \frac{1}{i}\frac{d}{dt}\bigg|_{t=0} W(t), \qquad P = \frac{1}{i}\frac{d}{dt}\bigg|_{t=0} W(it),$$

we can use the two-point function

$$\omega_c(W(z)W(w)) = e^{-i\mathrm{Im}(\bar{z}w)}e^{-\frac{1}{4}c|z+w|^2}$$

to compute the dispersions Δ_Q, Δ_P: obviously the expectation values will be 0, so we just need

$$\begin{aligned}\omega_c(Q^2) &= -\frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} e^{-\frac{1}{4}c(t+s)^2} = \frac{c}{2},\\ \omega_c(P^2) &= -\frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} e^{-\frac{1}{4}c(t+s)^2} = \frac{c}{2}.\end{aligned}$$

Thus $\Delta_Q\Delta_P = \frac{c}{2}$, violating Heisenberg's uncertainty principle.

6 Dynamics on a canonical pair

So far we described the kinematics, that is the structure of the space of observables which holds at a specific time, because we imagine to perform a measurement described by $\mathscr{A}$ on a state ω.

In order to do predictions one has to correlate the measurements on the same system at different times: we have a model which given information from the past allows us to predict the future. (that's one of the basic goals of physics).

Example 13. Let us start from an example. Note that if $(W(z))_{z\in\mathbb{C}}$ is an irreducible Weyl system on some Hilbert space H then also

$$(\tilde{W}_t(z) = W(e^{it}z))_{z\in\mathbb{C}}$$

is a Weyl system for any $t\in\mathbb{R}$. Then it must be that there exists a unitary operator U_t such that

$$U_t\tilde{W}_t(z)U_t^* = W(z), \qquad t\in\mathbb{R}, z\in\mathbb{C}.$$

Moreover we can define an automorphism of the Weyl algebra by letting $\alpha_t(W(z)) = W(e^{it}z)$ (i.e. a map of the Weyl algebra in itself which respects the $*$-operation and the algebraic relations in the C^*-algebra, and as a consequence is an isometry). This is an example of *dynamics*, i.e. the introduction of a time evolution in our description of a physical system.

Let us obseve that $\alpha_{2\pi}(W(z)) = W(z)$ so $\alpha_{2\pi} = \mathrm{id}$. So the dynamics is periodic of period 2π, we will see that it corresponds to the quantum motion of an harmonic oscillator.

The time and dynamics enters into the model via a group $(\alpha_t)_t$ of ($*$-)automorphisms of $\mathscr{A}$, which have the following meaning $\omega(\alpha_t(a))$ is the measurement of the observable a at the time t. $\alpha_0 = \mathrm{id}$. $\alpha_{t+s} = \alpha_t \circ \alpha_s$, i.e. is a representation of the additive group of $\mathbb{R}$ onto automorphisms of the C^*-algebra $\mathscr{A}$.

We can let α act on the linear functional by duality: $(\alpha_t^*\varphi)(a) := \varphi(\alpha_t(a))$ and then this gives a group of linear transformations on linear functionals on $\mathscr{A}$ and is easy to see that it preserves the states of $\mathscr{A}$.

Suppose that $\alpha_t^*\omega$ is not pure, then it can be decomposed into two states $\alpha_t^*\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ but then $\omega = \alpha_{-t}^*\alpha_t^*\omega = \lambda\alpha_{-t}^*\omega_1 + (1-\lambda)\alpha_{-t}^*\omega_2$ so ω is not pure either. Therefore the dynamics preseves pure states.

To proceed we need some assumption on the automorphism, essentially its compatibility with the representation space under consideration.

Fix a specific setting $(\mathscr{H},\tilde{\mathscr{A}},\tilde{Q}_0)$ where $\tilde{\mathscr{A}}$ is a general C^*-algebra and $\tilde{Q}_0$ is a representation in $\mathscr{H}$.

Definition 14. *Let $(\alpha_t)_{t\in\mathbb{R}}$ a set of C^*-automorphisms of $\tilde{\mathscr{A}}$. We call α a regular dynamics, if*

i. $(\alpha_t)_{t\in\mathbb{R}}$ is a group wrt. t, i.e. $\alpha_0=\mathrm{id}$ and $\alpha_t\circ\alpha_s=\alpha_{t+s}$ for any $t,s\in\mathbb{R}$

ii. the map $t\mapsto\alpha_t$ is weakly continuous, i.e. for any state ω and for any $a\in\tilde{\mathscr{A}}$ the map $t\mapsto\omega(\alpha_t(a))$ is continuous.

Define $\tilde{Q}_t(a):=\tilde{Q}(\alpha_t(a))$ for $a\in\tilde{\mathscr{A}}$.

Definition 15. *The set $\{U(t)\}_{t\in\mathbb{R}}\subset\mathscr{B}(\mathscr{H})$ is a unitary group of strongly continuous operators, if $U(t)U(s)=U(t+s)$ and $U(t)^*=U(-t)$ and if the map $t\mapsto U(t)$ is weakly (and thus strongly) continuous.*

Theorem 16. *Assume that there exists a state $\omega^{h_0}(\alpha_t(a))=\omega^{h_0}(a)$ for all $t\in\mathbb{R}$ and $a\in\tilde{\mathscr{A}}$ and $(\alpha_t)_t$ is a regular dynamics of $\tilde{\mathscr{A}}$, then if $\mathscr{H}$ is the GNS representation space associated with ω^{h_0} and $h_0\in\mathscr{H}$ is the corresponding cyclic vector, then there exists a unitary strongly continuous group $(U(t))_{t\in\mathbb{R}}$ on $\mathscr{H}$ such that*

$$\tilde{Q}_t(\cdot)=U(t)\tilde{Q}_0(\cdot)U(-t)$$

and also $U(t)h_0=h_0$.

Lemma 17. *Suppose that we have a contraction $V(t)$, i.e. $\|V(t)h\|\leq\|h\|$, such that $V(0)=1$ and $V(t)$ is weakly continuous in t at zero, then it is strongly continuous at zero.*

Proof. We have

$$0\leq\|V(t)h-h\|_{\mathscr{H}}^2=\|V(t)h\|_{\mathscr{H}}^2+\|h\|_{\mathscr{H}}^2-2\mathrm{Re}\langle V(t)h,h\rangle_{\mathscr{H}}\leq2\|h\|_{\mathscr{H}}^2-2\mathrm{Re}\langle V(t)h,h\rangle_{\mathscr{H}}$$

so weak continuity at zero is enough for strong continuity at zero. □

Proof. (of the Theorem 16)

$$\mathscr{H}_0=\{\widetilde{Q}_0(a)h_0|a\in\tilde{\mathscr{A}}\},\qquad\overline{\mathscr{H}_0}=\mathscr{H},$$

Let's define

$$U_0(t)(\widetilde{Q}_0(a)h_0)=\widetilde{Q}_t(a)h_0=\widetilde{Q}_0(\alpha_t(a))h_0$$

We first prove that $U_0(t)$ is an isometry

$$\langle U_0(t)(\widetilde{Q}_0(a_1)h_0),U_0(t)(\widetilde{Q}_0(a_2)h_0)\rangle=\langle\widetilde{Q}_0(\alpha_t(a_1))h_0,\widetilde{Q}_0(\alpha_t(a_2))h_0\rangle$$

$$=\langle h_0,\widetilde{Q}_0(\alpha_t(a_1))^*\widetilde{Q}_0(\alpha_t(a_2))h_0\rangle=\langle h_0,\widetilde{Q}_0(\alpha_t(a_1^*a_2))h_0\rangle=\omega^{h_0}(\alpha_t(a_1^*a_2))$$

$$=\omega^{h_0}(a_1^*a_2)=\langle h_0,\widetilde{Q}_0(a_1^*a_2)h_0\rangle=\langle\widetilde{Q}_0(a_1)h_0h_0,\widetilde{Q}_0(a_2)h_0\rangle$$

So $U_0(t)$ is an isometry on $\mathscr{H}_0$ so it is bounded on $\mathscr{H}_0$ and can be extended by continuity to $\overline{\mathscr{H}_0} = \mathscr{H}$. It remains to prove that it form a group. $\alpha_0 = 1 \Longrightarrow U_0(0) = I_{\mathscr{H}}$ and

$U_0(t)U_0(s)(\widetilde{Q}_0(a)h_0) = U_0(t)(\widetilde{Q}_0(\alpha_s(a))h_0) = (\widetilde{Q}_0(\alpha_t(\alpha_s(a)))h_0) = (\widetilde{Q}_0(\alpha_{t+s}(a))h_0) = U_0(t+s)(\widetilde{Q}_0(a)h_0)$

so $U_0(t)U_0(s) = U_0(t+s)$ on $\mathscr{H}_0$ and therefore on all $\mathscr{H}$. It remains to prove that h_0 is invariant, but of course $U_0(t)h_0 = U_0(t)(\widetilde{Q}_0(1)h_0) = \widetilde{Q}_0(\alpha_t(1))h_0 = h_0$. We also have that it is weakly continuous

$$\langle(\widetilde{Q}_0(a)h_0), U_0(t)(\widetilde{Q}_0(b)h_0)\rangle = \langle h_0, \widetilde{Q}_0(a^*\alpha_t(b))h_0\rangle = \omega^{h_0}(a^*\alpha_t(b))$$

and $\omega^{h_0}(a^*\cdot)$ is a continuous functional on $\mathscr{A}$ and therefore $t \mapsto \omega^{h_0}(a^*\alpha_t(b))$ is continuous, which proves that $U_0(t)$ is weaky continuous in $\mathscr{H}_0$ and then strongly continuous and can be extended as a strongly continuous group in $\mathscr{H}$. Note finally that

$$\tilde{Q}_t(a)\tilde{Q}_0(b)h_0 = \tilde{Q}_t(a\alpha_{-t}(b))h_0 = U_0(t)(\widetilde{Q}_0(a\alpha_{-t}(b))h_0) = U_0(t)(\widetilde{Q}_0(a)\widetilde{Q}_0(\alpha_{-t}(b))h_0)$$

$$= U_0(t)\widetilde{Q}_0(a)U_0(-t)\widetilde{Q}_0(b)h_0$$

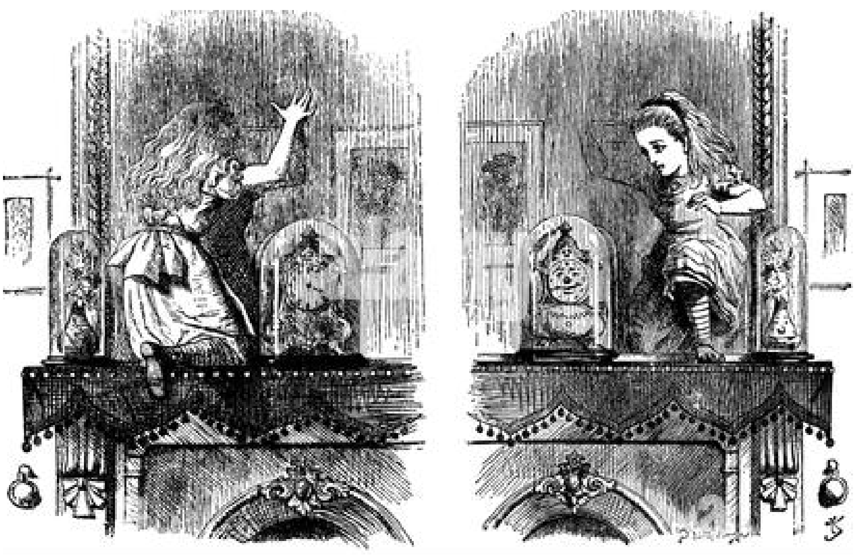
so this proves that $\tilde{Q}_t(a) = U_0(t)\widetilde{Q}_0(a)U_0(-t)$. □

Remark 18. Without the hypothesis that the state is invariant, then this construction is not true in general anymore. Take for example $\mathscr{A}$ commutative, i..e $C^0_\infty(\mathbb{R}^2)$ and consider an Hilbert space $L^2(\mathbb{R}^2, \mu)$ where

$$\mu(\mathrm{d}x) = e^{-x^2/2}\mathrm{d}x + \delta_0(\mathrm{d}x)$$

and the usual moltiplication and take $\alpha_t(f(x)) = f(x-t)$. But here there is no unitary group associated to α. Indeed take the state $\omega^\mu(a) = \int a(x)\mu(\mathrm{d}x)$. Consider the translated state $\omega^\mu(\alpha_t(\cdot))$, then GNS representation of it lives on $L^2(\mathbb{R}^n, \mu_t)$ where $\mu_t = T_t^*\mu$ the pull forward of μ by the translation operator. In order to have a unitary transformation we need that μ_t has to be absolutely continuous wrt. μ, but this is not the case.

7 Through the mirror



In this lectures we will require always to have a unitary implementation of the dynamics $(\alpha_t)_{t\in\mathbb{R}}$ for $(\mathscr{H},\mathscr{A},Q_0)$, i.e. to have a strongly continuous group of unitary operators $(U(t))_{t\in\mathbb{R}}$ so that $Q_t(\cdot)=Q_0(\alpha_t(\cdot))=U(t)Q_0(\cdot)U(-t)$.

Recall that we have proven the following link between unitary groups and representations of $C(\mathbb{R})$:

Theorem 19. *Consider an Hilbert space $\mathscr{H}$, a strongly continuous unitary group $(U(t))_{t\in\mathbb{R}}$ on $\mathscr{H}$, then there exists a unique C^*-representation X of $C_b^0(\mathbb{R},\mathbb{C})$ on $\mathscr{H}$ such that*

i. $X(e^{it\cdot})=U(t)$

ii. If $f_n\longrightarrow f$ *pointwise and* $\sup_n\|f_n\|<\infty$ *then* $X(f_n)\longrightarrow X(f)$ *weakly.*

Which could be considered a C* version of the Fourier transform. We want now to do the same for certain semigroups. This essentially is the C* analogon of the Laplace transform.

Definition 20. $\{K(t)\}_{t\in\mathbb{R}_+}\subseteq\mathscr{B}(\mathscr{H})$. *We say that* $K(t)$ *is a strongly continuous semigroup of self-adjoint contractions if*

i. $K(0)=1$, $K(t)K(s)=K(t+s)$, *for* $t,s\geq 0$.

ii. $K(t)=K(t)^*$,

iii. $t\longmapsto K(t)$ *is strongly continuous*

iv. $\|K(t)h\|\leq\|h\|, t\geq 0$.

Theorem 21. *Assume that K is a strongly continuous semigroup of self-adjoint contractions then there exists a unique representation X of $C_b^0(\mathbb{R}_+)$ on $\mathscr{H}$ such that*

i. $X(e^{-t\cdot})=K(t)$

ii. If $f_n\longrightarrow f$ *pointwise and* $\sup_n\|f_n\|<\infty$ *then* $X(f_n)\longrightarrow X(f)$ *weakly.*

To prove this theorem we need few more definitions.

Definition 22. *If* $G:\mathbb{R}\longrightarrow\mathbb{C}$ *we call G positive definite if for any* $\lambda_1,\ldots,\lambda_k\in\mathbb{C}$ *and* $t_1,\ldots,t_k\in\mathbb{R}$ *we have*

$$\sum_{i,j=1}^{k}\lambda_i\bar{\lambda}_jG(t_i-t_j)\geq 0$$

Definition 23. *We say that* $F:\mathbb{R}_+\longrightarrow\mathbb{C}$ *is totally monotone if for any* $\lambda_1,\ldots,\lambda_k\in\mathbb{C}$ *and* $t_1,\ldots,t_k\in\mathbb{R}_+$ *we have*

$$\sum_{i,j=1}^{k}\lambda_i\bar{\lambda}_jF(t_i+t_j)\geq 0.$$

Take U a unitary group on $\mathscr{H}$. For any $h \in \mathscr{H}$ we define $F_U(t,h) = \langle U(t)h, h\rangle$. If K is a self-adjoint contration semigroup we define $F_K(t,h) = \langle K(t)h, h\rangle$.

Theorem 24. *Let U and K as before, then F_U is positive definite and F_K is totally monotone.*

Proof. Consider K, the case of U is similar. Take $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $t_1, \ldots, t_k \in \mathbb{R}_+$ and just compute

$$0 \leqslant \left\langle \sum_i \lambda_i K(t_i) h, \sum_i \lambda_i K(t_i) h \right\rangle = \sum_{i,j=1}^{k} \lambda_i \bar{\lambda}_j F_K(t_i + t_j)$$

using the fact that K is self-adjoint and a semigroup. □

Theorem 25. *(Bochner) G is a continuous positive definite function iff there exists a bounded positive measure μ on $\mathbb{R}$ such that*

$$G(t) = \int_{\mathbb{R}} e^{itx} \mu(\mathrm{d}x).$$

Theorem 26. *(Bernstein) F is a bounded totally monotone function iff there exists a bounded positive measure μ on $\mathbb{R}_+$ and a constant $C \geqslant 0$ such that*

$$F(t) = C \int_{\mathbb{R}_+} e^{-tx} \mu(\mathrm{d}x).$$

Remark 27. These results can be generalises in a more abstract setting by replacing $\mathbb{R}$ and $\mathbb{R}_+$ with other topological groups/semigroups and exponentials with characters.

Lemma 28. *Assume that F is a bounded, totally monotone function, then*

a) For any $a > 0$, $-\Delta_a F$ is bounded totally monotone with $\Delta_a F(t) = F(t+a) - F(t)$.

Proof. $F \geqslant 0$, $a, t \geqslant 0$

$$\begin{pmatrix} F(2t) & F(t+a) \\ F(t+a) & F(2a) \end{pmatrix}$$

is positive definite, so its determinant is positive and

$$F(t+a) \leqslant \sqrt{F(2t)F(2a)}$$

Then (starting with $a = 0$)

$$F(t) \leqslant F(0)^{1/2} F(2t)^{1/2} \leqslant F(0)^{3/4} F(4t)^{1/4} \leqslant \cdots \leqslant F(0)^{(2^n-1)/2^n} F(2^n t)^{1/2^n} \leqslant F(0)^{(2^n-1)/2^n} C^{1/2^n}$$

and so we conclude that $F(t) \leqslant F(0)$. Take $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $t_1, \ldots, t_k \in \mathbb{R}_+$ and define

$$G(a) = \sum_{i,j}^{n} F(a + t_i + t_j) \lambda_i \bar{\lambda}_j$$

and consider other points $\sigma_1, \ldots, \sigma_n \in \mathbb{C}$ and $a_1, \ldots, a_n \in \mathbb{R}_+$ then

$$\sum_{i,j}^{k} G(a_i + a_j)\sigma_i\bar{\sigma}_j = \sum_{i,j}^{n} \sum_{r,s}^{k} F(a_i + a_j + t_r + t_s)\lambda_r\bar{\lambda}_s\sigma_i\bar{\sigma}_j \geq 0$$

using the fact that F is totally monotone. So G is also totally monotone and as a consequence $G(a) \leq G(0)$ and $G(0) - G(a) \geq 0$ or otherwise

$$\sum_{i,j}^{n} (-\Delta_a F(t_i + t_j))\lambda_i\bar{\lambda}_j = \sum_{i,j}^{n} (F(t_i + t_j) - F(a + t_i + t_j))\lambda_i\bar{\lambda}_j \geq 0$$

so $-\Delta_a F$ is bounded and totally monotone. □

Corollary 29. *If F is bounded and totally monotone, for any $a_1, \ldots, a_n \in \mathbb{R}_+$*

$$(-1)^n \Delta_{a_1} \cdots \Delta_{a_n} F$$

is totally monotone and therefore $(-1)^n \Delta_{a_1} \cdots \Delta_{a_n} F \geq 0$.

Theorem 30. *(Krein–Milman) Let X be a locally convex Hausdorff topological vector space and let $K \subseteq X$ be a compact convex subset, then the set $E(K)$ of extreme points of K is non-void and for any $y \in K$ there exists a probability measure ν^y on $E(K)$ such that*

$$y = \int_{E(K)} x \nu^y(\mathrm{d}x)$$

where the integral is understood in the weak sense, i.e. for any $\lambda \in X^*$ we have (Pettis integral)

$$\lambda(y) = \int_{E(K)} \lambda(x)\, \nu^y(\mathrm{d}x).$$

Recall that locally convex means that there is a base of the topology composed by convex sets. For example $\mathbb{R}^{(0,+\infty)}$ with the product topology is a locally convex and Hausdorff.

Proof. (of Bernstein theorem) We prove now that if F is bounded and totally monotone there exists a positive measure μ on $\mathbb{R}_+$ such that $F(t) = \int_{\mathbb{R}_+} e^{-tx}\mu(\mathrm{d}x)$. The rest of the claim is left as an exercise. Consider the space $\mathscr{C} \subseteq \mathbb{R}^{(0,\infty)}$ such that

$$\mathscr{C} = \{F \in \mathbb{R}^{(0,\infty)}, F \geq 0 \text{: for all } a_1, \ldots, a_n \in \mathbb{R}_+ \ (-1)^n \Delta_{a_1} \cdots \Delta_{a_n} F \geq 0\}$$

Note that $\mathscr{C}$ is closed for the pointwise convergence and it is convex, but not compact. In particular this means that for $F \in \mathscr{C}$ we have $F(t_1) - F(t_2) \geq 0$ if $t_1 \leq t_2$ and we let $F(0+) = \lim_{t \downarrow 0} F(t)$ by monotone limit. In principle we could have $F(0+) = +\infty$. F is bounded iff $F(0+) < \infty$. Since $\Delta_a \Delta_a F \geq 0$ we have

$$\frac{1}{2}F(t) + \frac{1}{2}F(t + 2a) \geq F(t + a)$$

and this means that F is midpoint convext. On the other hand, for any $0<c<d$ we have that $0\leqslant F(d)\leqslant F(c)$ so F is bounded in $[c,d]$. It is left as an exercise to prove that if F is midpoint convex and bounded then F is continuous in (c,d) (Hint: show that $F:[-\delta,\delta]\longrightarrow\mathbb{R}$ midpoint convex and if F has a discontinuity in 0 then it is unbounded). By this result, F is continuous on $\mathbb{R}_+$. Consider a subset $K\subseteq\mathscr{C}$ as follows $K=\{F\in\mathscr{C}:F(0+)=1\}$. This is now a closed convex set and $K\subset[0,1]^{\mathbb{R}_+}$ which is a compact space (always wrt. to the pointwise convergence). By Krein–Milman this means that for any $y\in K$ we can write it as a convex combination of extreme points. What are these extreme points $E(K)$ of K? For any $F\in K$ we have that exists $a\in\mathbb{R}_+$ such that $F(a)>0$ and $1=F(0)>F(a)>0$ unless $F=1$ everywhere. In the second case $1\in E(K)$ since it is the biggest element of K and therefore cannot be decomposed in a convex combination of other elements. In the other case

$$F(t)=\frac{F(t+a)}{F(a)}F(a)+\frac{-\Delta_a F(t)}{1-F(a)}(1-F(a))$$

so $F(t+a)/F(a)\in K\subseteq\mathscr{C}$ so this implies that if $F\in E(K)$ we need to have $F(t+a)=F(t)F(a)$. This is true to all a for which $1>F(a)>0$. Since F is continuous and a solution of that functional equation, but all these solutions are of the form $F(t)=\exp(-st)$ for some $s\in\mathbb{R}_+$. Then if $F\in K$ there exists a probability measure μ on $\mathbb{R}_+$ such that

$$F(t)=\int_{\mathbb{R}_+}e^{-st}\mu(\mathrm{d}s).$$

This proves the key claim in the theorem if F is bounded and $F\in K$. However is clear that if F is totally monotone, then $F\in\mathscr{C}$ and if $0<F(0+)<\infty$ we have that $F(t)/F(0+)$ is bounded and >0 and in K. □

Lemma 31. *For any $h\in\mathscr{H}$ and $t\geqslant 0$,*

$$F_K(t,h)=\int_{\mathbb{R}_+}e^{-tx}\mu^h(\mathrm{d}x)$$

where $\mu^h(\mathbb{R}_+)=\|h\|^2$.

Proof. F_K is bounded because $|F_K(t,h)|\leqslant\|Kh\|\,\|h\|\leqslant\|h\|^2$ and totally monotone, so it has this representation note that $F(0,h)=\|h\|^2$. □

Lemma 32. *There is only one C^* representation X_0 of $C^0_\infty(\mathbb{R}_+,\mathbb{C})$ such that*

$$X_0(e^{-t\cdot})=K(t)$$

Proof. Consider the set $\mathscr{E}=\operatorname{span}_{\mathbb{C}}\{e^{-tx},t\geqslant 0\}\subset C^0_\infty$. Moreover $\mathscr{E}$ is a $*$-subalgebra on C^0_∞ and we define

$$X_{00}:\mathscr{E}\longrightarrow\mathscr{B}(\mathscr{H})$$

as $X_{00}(e^{-tx})=K(t)$ and then extend by linearity to all $\mathscr{E}$. X_{00} is a $*$-homomorphism since K is a semigroup. Moreover for $f=\sum_i\lambda_i e^{-t_j x}$ we have

$$\langle h,X_{00}(f)\,h\rangle=\sum_i\lambda_i F_K(t_i,h)=\sum_i\lambda_i\int_{\mathbb{R}_+}e^{-t_i x}\mu^h(\mathrm{d}x)=\int_{\mathbb{R}_+}f(x)\mu^h(\mathrm{d}x)$$

so by using that $X_{00}(f)$ is self-adjoint

$$|\langle X_{00}(f)h, X_{00}(f)\,h\rangle| = |\langle h, X_{00}(f^2)\,h\rangle| \leqslant \|f^2\|_\infty \|h\|^2 = \|f\|_\infty^2 \|h\|^2,$$

and , we have that $\|X_{00}(f)\| \leqslant \|f\|_\infty$. As exercise we leave to prove that $\mathscr{E}$ is dense in $C_\infty^0(\mathbb{R}_+, \mathbb{C})$ (Stone-Weierstrass and a localization argument). Then we can extend X_{00} from $\mathscr{E}$ to C_∞^0 by continuity with the operator norm. □

Up to now we proved that:

1. There exists a unique $*$-homomorphism $X: C_\infty^0(\mathbb{R}_+, \mathbb{C}) \longrightarrow \mathscr{B}(\mathscr{H})$ where $C_\infty^0(\mathbb{R}_+, \mathbb{C})$ is the set of continuous functions going to zero at infinity.
2. For any $h \in \mathscr{H}$ there exists a unique positive measure μ^h on $\mathbb{R}_+$ such that $\mu^h(\mathbb{R}_+) = \|h\|^2$ and

$$\langle K(t)h, h\rangle = \int_{\mathbb{R}_+} e^{-tx} \mu^h(\mathrm{d}x).$$

3. For any $f \in C_\infty^0(\mathbb{R}_+, \mathbb{C})$ we have

$$\langle X(f)h, h\rangle = \int_{\mathbb{R}_+} f(x) \mu^h(\mathrm{d}x).$$

We introduce a measure

$$\mu^{h_1, h_2} := \frac{1}{4} \sum_{k=0}^{3} i^k \mu^{h_1 + (i)^k h_2}$$

by polarisation and we have

$$\langle X(f)h_1, h_2\rangle = \int_{\mathbb{R}_+} f(x) \mu^{h_1, h_2}(\mathrm{d}x).$$

Lemma 33. *We have that*

$$\frac{\mathrm{d}\mu^{X(f)h_1, h_2}}{\mathrm{d}\mu^{h_1, h_2}} = f(x)$$

Proof. The measure μ^{h_1, h_2} can be characterised by

$$\langle K(t)h_1, h_2\rangle = \int_{\mathbb{R}_+} e^{-tx} \mu^{h_1, h_2}(\mathrm{d}x)$$

and we have

$$\langle K(t)X(f)h_1, h_2\rangle = \int_{\mathbb{R}_+} e^{-tx} \mu^{X(f)h_1, h_2}(\mathrm{d}x) = \int_{\mathbb{R}_+} e^{-tx} f(x) \mu^{h_1, h_2}(\mathrm{d}x)$$

so by identification of Laplace transforms we have the claim. □

Proof. (of Theorem 19) Define the linear operator $\tilde{X}(f)$ by

$$\langle \tilde{X}(f)h_1, h_2\rangle = \int_{\mathbb{R}_+} f(x) \mu^{h_1, h_2} \mathrm{d}x$$

for all $h_1, h_2 \in \mathscr{H}$. We have

$$\|\tilde{X}(f)\|_{\mathscr{B}(\mathscr{H})} = \sup_{\|h_1\|=\|h_2\|=1} \left|\int_{\mathbb{R}_+} f(x)\mu^{h_1,h_2}(\mathrm{d}x)\right| \leqslant \|f\|_\infty \sup_{\|h_1\|=\|h_2\|=1} |\mu^{h_1,h_2}(\mathbb{R}_+)| \lesssim \|f\|_\infty$$

so $\tilde{X}(f)$ is bounded. Moreover one can show easily that $\langle \tilde{X}(f)h_1, h_2\rangle = \langle h_1, \tilde{X}(f^*)h_2\rangle$. The approximation property is quite easy to prove since if $f_n \longrightarrow f$ pointwise and the family is bounded then by dominated convergence

$$\langle \tilde{X}(f_n)h_1, h_2\rangle = \int_{\mathbb{R}_+} f_n(x)\mu^{h_1,h_2}\mathrm{d}x \longrightarrow \int_{\mathbb{R}_+} f(x)\mu^{h_1,h_2}\mathrm{d}x = \langle \tilde{X}(f)h_1, h_2\rangle$$

so we have weak convergence. Moreover if $f \in C_b^0(\mathbb{R}_+)$ then there exists $(f_n)_{n\geqslant 0} \subset C_\infty^0(\mathbb{R}_+)$ such that $f_n \longrightarrow f$ pointwise and $\sup_n \|f_n\| < \infty$ (simply by multiplying f with a sequence of dilations of a given bounded functions of compact support). So there can be only one such operator which extends X from C_∞^0. We have to prove that $\tilde{X}$ is an homomorphism. Take $f, g \in C_b^0(\mathbb{R}_+, \mathbb{C})$ and consider two approximating sequences $(f_n)_n, (g_n)_n \subseteq C_\infty^0(\mathbb{R}_+)$ then taking $n \longrightarrow \infty$

$$\langle \tilde{X}(fg_m)h_1, h_2\rangle \longleftarrow \langle \tilde{X}(f_n g_m)h_1, h_2\rangle = \langle \tilde{X}(f_n)\tilde{X}(g_m)h_1, h_2\rangle \longrightarrow \langle \tilde{X}(f)\tilde{X}(g_m)h_1, h_2\rangle$$

so taking $m \longrightarrow \infty$ we get $\langle \tilde{X}(fg)h_1, h_2\rangle = \langle \tilde{X}(f)\tilde{X}(g)h_1, h_2\rangle$. This concludes the proof by taking $X = \tilde{X}$. □

Now we have seen that if $(U(t))_{t\in\mathbb{R}}$ is a strongly continuous unitary group this is equivalent to have an representation X_U of $C_b^0(\mathbb{R}, \mathbb{C})$ in $\mathscr{B}(\mathscr{H})$ and if $(K(t))_{t\geqslant 0}$ is a self-adjoint, strongly continuous contraction semigroup, then we have a representation X_K of $C_b^0(\mathbb{R}_+, \mathbb{C})$ on $\mathscr{B}(\mathscr{H})$. We want to look into the relation between these two objects.

Definition 34. *We say that $(U(t))_{t\in\mathbb{R}}$ (as before) has positive energy for each $f \in C_b^0(\mathbb{R}, \mathbb{C})$ such that $\mathrm{supp}(f) \subseteq (-\infty, 0)$ we have that $X_U(f) = 0$.*

Remark 35. Assume that $f_1, f_2 \in C_b^0(\mathbb{R}, \mathbb{C})$ such that $f_1 = f_2$ on $[0, \infty)$ then if U has positive energy then $X_U(f_1) = X_U(f_2)$.

Lemma 36. *U has positive energy iff for any $h \in \mathscr{H}$ μ_U^h is supported on $\mathbb{R}_+ = [0, \infty)$.*

Proof. $\langle X_U(f)h_1, h_2\rangle = \int_{\mathbb{R}} f(x)\mu^h(\mathrm{d}x)$ if the measure is supported on $\mathbb{R}_+$ then $X(f) = 0$ if $\mathrm{supp}(f) \subseteq \mathbb{R}_{<0}$. On the other hand if $\mathrm{supp}(f) = (-\infty, 0)$ then $\int_{\mathbb{R}} f(x)\mu^h(\mathrm{d}x) = 0$ from which we get that $\mathrm{supp}(\mu^h) \subseteq \mathbb{R}_+$. □

Remark 37. If $(U(t))_{t\in\mathbb{R}}$ has positive energy and $g \in C_b^0(\mathbb{R}_+, \mathbb{C})$ then we can define $X_U(g)$ in a unique way as follows: we take $\tilde{g} \in C_b^0(\mathbb{R}, \mathbb{C})$ such that $\tilde{g} = g$ on $\mathbb{R}_+$ and we define $X_U(g) = X_U(\tilde{g})$. This definition is a good one since the value do not depends on the extension $\tilde{g}$, indeed if $\hat{g}$ is another extension then $\tilde{g} - \hat{g}$ is supported on $(-\infty, 0)$ and $X_U(\hat{g}) = X_U(\tilde{g})$.

Theorem 38. *Assume $(U(t))_{t\in\mathbb{R}}$ is a strongly continuous unitary group with positive energy, then $K(t) = X_U(e^{-t\cdot})$ is a strongly continuous self-adjoint contraction semigroup and also $X_U = X_K$ on $C_b^0(\mathbb{R}_+, \mathbb{C})$. The converse is true, i.e. if we have K and we define $U(t) = X_K(e^{it\cdot})$, then $(U(t))_{t\in\mathbb{R}}$ is a strongly continuous unitary group with positive energy and $X_K = X_U$.*

Proof. From $e^{-t_1 s}e^{-t_2 s}=e^{-(t_1+t_2)s}$ we have $K(t_1)K(t_2)=K(t_1+t_2)$ and the other properties follows easily, moreover by dominated convergence $\langle h_1, K(t)h_2\rangle \longrightarrow \langle h_1, K(s)h_2\rangle$ if $t\longrightarrow s$ and strong continuity follows since K is a contraction, i.e. $\|K(t)h\|^2=\langle h, K(2t)h\rangle \leqslant \|e^{-2t\cdot}\|_{C_b^0(\mathbb{R}_+)}\mu^h(\mathbb{R}_+)=\|h\|^2$. The reverse implication is left as exercise. □

We want to justify now the name of "positive energy". This is not fundamental in the following but will give a better grasp of the connection with standard physical intuition.

Let $\mathscr{D}_H$ be a subspace of $\mathscr{H}$ such that $h\in\mathscr{D}_H$ iff $t\mapsto U(t)h$ is strongly differentiable in 0. For any $h\in\mathscr{D}_H$ we define

$$Hh=\frac{1}{i}\lim_{t\longrightarrow 0}\frac{U(t)h-h}{t}\in\mathscr{H}.$$

Is simple to prove that H is a linear operator $H:\mathscr{D}_H\longrightarrow\mathscr{H}$. For generic U, the operator H is not bounded, which implies that H cannot be extended as a continuous operator on all $\mathscr{H}$. H is an *unbounded operator* and $\mathscr{D}_H$ is called the domain of H.

Lemma 39. *$h\in\mathscr{D}_H$ iff*

$$\int_{\mathbb{R}}x^2\mu^{h,U}(\mathrm{d}x)<\infty, \quad \textit{and then} \qquad \|Hh\|^2=\int_{\mathbb{R}}x^2\mu^{h_1,h_2,U}(\mathrm{d}x).$$

If $h_1\in\mathscr{D}_H$ and $h_2\in\mathscr{H}$ then

$$\int_{\mathbb{R}}|x||\mu^{h_1,h_2,U}|(\mathrm{d}x)<\infty, \quad \textit{and} \qquad \langle Hh_1,h_2\rangle=\int_{\mathbb{R}}x\mu^{h_1,h_2,U}(\mathrm{d}x).$$

Proof. Step 1. For any $h_1\in\mathscr{D}_H$ and $h_2\in H$

$$\int_{\mathbb{R}}|x||\mu^{h_1,h_2,U}|(\mathrm{d}x)=\sup_{f\in C_c^0(\mathbb{R},\mathbb{C}),\|f\|\leqslant 1}\int_{\mathbb{R}}xf(x)\mu^{h_1,h_2,U}(\mathrm{d}x)=\sup_{f\in C_c^0(\mathbb{R},\mathbb{C}),\|f\|\leqslant 1}\langle X(xf(x))h_1,h_2\rangle$$

$$\leqslant\|h_2\|_H\left(\sup_{f\in C_c^0(\mathbb{R},\mathbb{C}),\|f\|\leqslant 1}\|X(xf(x))h_1\|\right)^{1/2}\leqslant\|h_2\|_H\left(\sup_{f\in C_c^0(\mathbb{R},\mathbb{C}),\|f\|\leqslant 1}\int_{\mathbb{R}}(xf(x))^2\mu^{h_1,h_1,U}(\mathrm{d}x)\right)^{1/2}\leqslant$$
$$C_{h_1}\|h_2\|_H$$

But this implies that there exists h_1' such that $\langle h_1',h_2\rangle=\int_{\mathbb{R}}x\mu^{h_1,h_2,U}(\mathrm{d}x)$. Now we want to prove that $h_1'=Hh_1$

$$\left\langle\frac{1}{it}(U(t)-1)h-h_1',\frac{1}{it}(U(t)-1)h-h_1'\right\rangle=\left\|\frac{1}{it}(U(t)-1)h\right\|^2+\|h_1'\|^2-2\mathrm{Re}\left\langle\frac{1}{it}(U(t)-1)h,h_1'\right\rangle$$

$$=\int_{\mathbb{R}}\underbrace{\left(2\frac{1-\cos(tx)}{t^2}+x^2-2\frac{\sin(tx)}{t}x\right)}_{G(t,x)}\mu^{h_1}(\mathrm{d}x)$$

Now $|G(t,x)|\leqslant Cx^2$ is uniformly bounded and pointwise converge to zero as $t\longrightarrow 0$, so by Lebesgue dominated convergence we conclude that this quantity goes not zero. So we have that if $\int x^2\mu^h(\mathrm{d}x)<\infty$ we have that $U(t)h$ is strongly differentiable in zero. On the other hand, if $U(t)h$ is strongly differentiable then

$$\sup_{t\in(-1,1)}\left\|\frac{1}{it}(U(t)-1)h\right\|^2=C<\infty$$

and in particular

$$\int x^2\mu^h(\mathrm{d}x) = 2\int \liminf_{t\longrightarrow 0}\frac{1-\cos(tx)}{t^2}\mu^h(\mathrm{d}x) \leqslant \liminf_{t\longrightarrow 0} 2\int \frac{1-\cos(tx)}{t^2}\mu^h(\mathrm{d}x) = \liminf_{t\longrightarrow 0}\left\|\frac{1}{it}(U(t)-1)h\right\|^2 < C.$$

The rest of the proof is left as exercise. □

Theorem 40. *$\mathscr{D}_H$ is dense in $\mathscr{H}$ and $h_1, h_2 \in \mathscr{D}(H)$ we have $\langle Hh_1, h_2\rangle = \langle h_1, Hh_2\rangle$, so H is symmetric*

Proof. If $h\in\mathscr{H}$ define $h_\ell = \int_0^\ell U(s)h\mathrm{d}s$ we prove that $h_\ell\in\mathscr{D}_H$: indeed

$$\frac{\mathrm{d}\mu^{h_\ell}}{\mathrm{d}\mu^h} = \frac{1}{x^2}(e^{ihx}-1)(e^{-ihx}-1)$$

then

$$\int x^2\mu^{h_\ell}(\mathrm{d}x) \leqslant C\int \mu^h(\mathrm{d}x) < \infty$$

and $h_\ell\in\mathscr{D}_H$.

$$\int e^{itx}\mu^{h_\ell}(\mathrm{d}x) = \left\langle U(t)\int_0^\ell U(s_1)h\mathrm{d}s_1, \int_0^\ell U(s_2)h\mathrm{d}s_2\right\rangle = \int_{[0,\ell]^2}\int_{\mathbb{R}} e^{i(t+s_1+s_2)x}\mu^h(\mathrm{d}x)\mathrm{d}s_1\mathrm{d}_2$$

and by Fubini we can exchange the integrals and obtain

$$\int e^{itx}\mu^{h_\ell}(\mathrm{d}x) = \int e^{itx}\frac{1}{x^2}(e^{ihx}-1)(e^{-ihx}-1)\mu^h(\mathrm{d}x)$$

and by identification of Fourier transforms. We have $\|h_\ell/\ell - h\| \longrightarrow 0$ as $\ell\longrightarrow 0$, we have

$$\|h_\ell/\ell - h\|^2 = \left\|\frac{1}{\ell}\int_0^\ell (U(s)-1)h\mathrm{d}s\right\|^2 \leqslant \sup_{s\in[0,\ell]}\|(U(s)-1)h\| = o(\ell)$$

by strong continuity. The symmetry is quite simple since

$$\langle Hh_1, h_2\rangle = \lim_{t\longrightarrow 0}\left\langle \frac{U(t)-1}{it}h_1, h_2\right\rangle = \lim_{t\longrightarrow 0}\left\langle h_1, \frac{U(-t)-1}{-it}h_2\right\rangle = \langle h_1, Hh_2\rangle.$$

□

Remark 41. Is possible to prove that $(H, \mathscr{D}_H)$ is self-adjoint, i.e. $H^* = H$. (given the natural definition of the adjoint of a densely defined unbounded operator)

If $h_1, h_2 \in \mathscr{D}_H$ we define $\mathscr{E}(h_1, h_2) = \langle Hh_1, h_2\rangle$. If $h_1\in\mathscr{D}_H$ and $\|h\|_{\mathscr{H}} = 1$ then we define $\mathscr{E}(h,h)$ to be the energy of the state $h\in\mathscr{H}$.

Recall that $(\mathscr{H}, \mathscr{A}, Q_0)$ is our quantum space and if $h\in\mathscr{H}$ gives the vector state $\omega^h(a) = \langle Q_0(a)h, h\rangle$. So the energy is an extension of this formula for the unbounded operator H which formally is the derivative of the time-evolution group U. We had $Q_t(a) = U(-t)Q_0(a)U(t)$. If it is possible to take the derivative wrt. to t then we obtain

$$\partial_t Q_t(a) = \frac{1}{i}[H, Q_t(a)]$$

(this has to justified).

We have that $(h_1,h_2)\mapsto\mathscr{E}(h_1,h_2)$ is an Hermitian form (i.e. linear in the first and antilinear in the second variable).

Theorem 42. *U has positive energy iff $\mathscr{E}(h,h)\geqslant 0$ for all $h\in\mathscr{D}_H$.*

Proof. If U has positive energy, we saw that μ^h is supported in $\mathbb{R}_+$ and we have

$$\mathscr{E}(h,h)=\int_{\mathbb{R}}x\mu^{h,U}(\mathrm{d}x)=\int_{\mathbb{R}_+}x\mu^{h,U}(\mathrm{d}x)\geqslant 0.$$

Assume now that $\mathscr{E}$ is non-negative definite and assume that U has not positive energy, therefore there exists $h\in\mathscr{H}$ such that μ^h has some support on $(-\infty,0)$. We can assume that $\mathrm{supp}(\mu^h)\subset(-\infty,-c)$ for some $c>0$ since we can consider the vector $X_U(f)h$ with $\mathrm{supp}(f)\subset(-\infty,-c)$ and $\mathrm{d}\mu^{X(f)h}=f\mathrm{d}\mu^h$. So now taking $h_\ell=\int_0^\ell U(s)h\mathrm{d}s$ and

$$\mu^{h_\ell}(\mathrm{d}x)=\frac{1}{x^2}|e^{i\ell x}-1|^2\mu^h(\mathrm{d}x).$$

Let $d>c$ such that $\mu([-d,-c])>0$. Note that $h_\ell\in\mathscr{D}_H$ and

$$\mathscr{E}(h_\ell,h_\ell)=\int_{\mathbb{R}}x\mu^{h_\ell}(\mathrm{d}x)=\int_{\mathbb{R}}x\frac{1}{x^2}|e^{i\ell x}-1|^2\mu^h(\mathrm{d}x)<\int_{[-d,-c]}\frac{1}{x}|e^{i\ell x}-1|^2\mu^h(\mathrm{d}x)$$

and if ℓ is small enough this quantity is negative. □

Recall the definitions

$$F_U(t,h)=\langle U(t)h,h\rangle=\int_{\mathbb{R}}e^{itx}\mu^{h,U}(\mathrm{d}x),$$

$$F_K(t,h)=\langle K(t)h,h\rangle=\int_{\mathbb{R}_+}e^{-tx}\mu^{h,K}(\mathrm{d}x).$$

Theorem 43. *The function F_K is holomorphic when $t\in\mathbb{C}$ and* $\mathrm{Re}(t)>0$ *and it is continuous when* $\mathrm{Re}(t)\geqslant 0$. *Moreover, we have that*

$$F_U(s,h)=F_K(is,h)=\lim_{y\downarrow 0}F_K(is+y,h).$$

Proof. If $\mathrm{Re}(t_1)>0$ take $\varepsilon\in\mathbb{C}$ with $|\varepsilon|<\mathrm{Re}(t_1)$ then

$$|F(t_1+\varepsilon,h)|=\left|\int_{\mathbb{R}_+}e^{-t_1x}e^{-\varepsilon s}\mu^{h,K}(\mathrm{d}x)\right|\leqslant\int_{\mathbb{R}_+}e^{-\mathrm{Re}(t_1)x}e^{-|\varepsilon|s}\mu^{h,K}(\mathrm{d}x)<\infty,$$

and by monotone convergence the series

$$\sum_n|\varepsilon|^n\int_{\mathbb{R}_+}e^{-t_1x}\frac{x^n}{n!}\mu^{h,K}(\mathrm{d}x)$$

is convergent and so F has a convergent power series expansion in the claimed domain and continuity derives from the dominated convergence theorem. Moreover

$$\lim_{y\downarrow 0} F_K(is+y,h) = \int_{\mathbb{R}_+} e^{isx}\mu^{h,K}(\mathrm{d}x) = F_U(s,h)$$

when U is defined so that $\mu^{h,K} = \mu^{h,U}$. □

Remark 44. We can define the generator H' of K similarly as we defined the generator H of U. Namely $\mathscr{D}_{H'}$ is defined as the set of vectors $h \in \mathscr{H}$ such that $K(t)h$ is strongly differentiable in zero and define

$$H'h = -\lim_{t\downarrow 0} \frac{K(t)h - h}{t}.$$

But if U and K are related so that $X_U = X_K$ then $H' = H$ and $\mathscr{D}_H = \mathscr{D}_{H'}$.

Consider now $\mathscr{H} = L^2(\mathbb{R}^n, \mathrm{d}x)$. $\mathscr{A} = C_b^0(\mathbb{R}^n, \mathbb{C})$ and $(Q_0(a)h)(x) = a(x)h(x)$. Define

$$K(t)h = \rho_t * h = \frac{1}{(2\pi t)^{n/2}} \int e^{-|x-y|^2/(2t)} h(y)\mathrm{d}y.$$

Theorem 45. *$(K(t))_{t\geqslant 0}$ is a strongy continuous, self-adjoint contraction semigroup.*

Proof. Let $\mathscr{F}(h) = \int_{\mathbb{R}^n} e^{ikx} h(x)\mathrm{d}x$ the Fourier transform. Recall Plancherel's theorem

$$\int_{\mathbb{R}^n} h_1(x)\overline{h_2(x)}\mathrm{d}x = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathscr{F}(h_1)(k)\overline{\mathscr{F}(h_2)}(k)\mathrm{d}k$$

and that $\mathscr{F}(a * b) = (\mathscr{F}a)(\mathscr{F}b)$. Moreover $\mathscr{F}(\rho_t)(k) = \exp(-t|k|^2/2)$. Now

$$\|K(t)h\|_{L^2}^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\mathscr{F}(\rho_t * h)(k)|^2 \mathrm{d}k = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp(-t|k|^2)|\mathscr{F}(h)(k)|^2\mathrm{d}k$$

$$\leqslant \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\mathscr{F}(h)(k)|^2\mathrm{d}k = \|h\|_{L^2}^2$$

so K is a contraction. Moreover

$$\|K(t)h - h\|_{L^2}^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (1 - \exp(-t|k|^2/2))^2 |\mathscr{F}(h)(k)|^2 \mathrm{d}k \longrightarrow 0$$

as $t \longrightarrow 0$, so it is strongly continuous. Additionally it is self-adjoint since

$$\langle K(t)h_1, h_2\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp(-t|k|^2/2)^2 \mathscr{F}(h_1)(k)\overline{\mathscr{F}(h_2)(k)}\mathrm{d}k = \langle h_1, K(t)h_2\rangle$$

and the semigroup property derives from

$\mathscr{F}(K(t)K(s)h)(k) = \exp(-t|k|^2/2)\exp(-s|k|^2/2)\mathscr{F}(h)(k) = \exp(-(t+s)|k|^2/2)\mathscr{F}(h)(k) = \mathscr{F}(K(t+s)h)(k).$

□

Take $f \in C^\infty \cap L^p$ for any $p \geq 1$. Then in $L^2(\mathbb{R}^n)$ we have

$$\lim_{t\downarrow 0} \mathscr{F}\left(\frac{K(t)f - f}{t}\right)(k) = \lim_{t\downarrow 0} \frac{e^{-tk^2/2} - 1}{t}\mathscr{F}(f)(k) = -k^2\mathscr{F}(f)(k) = \mathscr{F}(\Delta f)(k)$$

so $H = -\Delta$ and one can prove that $\mathscr{D}_H = H^2$. Moreover $\mathscr{E}(h,h) = \int_{\mathbb{R}^n} |\nabla h|^2 \mathrm{d}x \geq 0$. So the semigroup has positive energy (it was already clear from the fact that it is a contraction).

So now

$$F_K(t,h) = \int_{\mathbb{R}^{2n}} \frac{e^{-|x-y|^2/2t}}{(2\pi t)^{n/2}} h(x)\overline{h(y)}\mathrm{d}x\mathrm{d}y$$

and for $h \in L^2 \cap L^1$ we have the explicit representation

$$F_U(s,h) = F_K(is,h) = \int_{\mathbb{R}^{2n}} \frac{e^{-|x-y|^2/2(is)}}{(2\pi is)^{n/2}} h(x)\overline{h(y)}\mathrm{d}x\mathrm{d}y$$

where $(i)^{n/2} = e^{\pi i n/4}$ given the kind of limit we had to perform. We conclude therefore that for $h \in L^2 \cap L^1$

$$(U(s)h)(x) = \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/2(is)}}{(2\pi is)^{n/2}} h(y)\mathrm{d}y.$$

This is the model of the free particle in $\mathbb{R}^n$, i.e. a particle not interacting with any external system. In this case $(U(t))_{t\in\mathbb{R}}$ is a unitary group on $L^2(\mathbb{R}^n)$ and the expectation of any observable $Q_t(a)$ on the state ω^h evolves according to the equation

$$\omega_t^h(a) = \langle Q_t(a)h, h\rangle = \langle U(-t)Q_0(a)U(t)h, h\rangle = \langle Q_0(a)U(t)h, U(t)h\rangle.$$

8 Wightman and Schwinger functions

We work now with the data $(\mathscr{H}, \mathscr{A}, Q_0, U(t))$ where $(U(t))_t$ is a positive energy strongly continuous unitary group, or equivalently $(K(t))_t$ a self-adjoint, strongly continuous, contraction semigroup. We saw that the given of U is equivalent to the given of K.

Definition 46. *We say that $h_0 \in \mathscr{H}$ is a ground state for U iff $U(t)h_0 = h_0$.*

Theorem 47. *h_0 is a ground state for U iff one of the following equivalent conditions hold:*

1. $\mu^{h_0}(\mathrm{d}x) = \delta_0(\mathrm{d}x)$

2. $K(t)h_0 = h_0$

3. $h_0 \in \mathscr{D}_H$ *and* $Hh_0 = 0$

4. $h_0 \in \mathscr{D}_H$ *and* $\mathscr{E}(h_0, h_0) = 0$

Proof. Exercise. □

Remark 48. The name ground state comes from the fact that h_0 is the state of minimal energy of the system (i.e. the zero energy, in our normalization).

Definition 49. h_0 *a cyclic ground state if*

$$\mathrm{span}\{U(t_1)Q_0(a_1)U(t_2)Q_0(a_2)\cdots h_0\}$$

is dense in $\mathscr{H}$.

A cyclic ground state allows to reconstruct all the Hilbert space from expectations of time evolutions of observables.

Indeed any $\omega^h(Q_t(a))$ can then be approximated by linear combinations of expressions of the form

$$\langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)h_0, h_0\rangle$$

for suitable $t_1, \ldots, t_n$ since we used the fact that h_0 is invariant under U.

Assume now that we are given a cyclic ground state.

Definition 50. *Wightman functions are defined as*

$$\mathbb{W}_{k,\mathbb{A}_k}(t_1, \ldots, t_k) = \langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)h_0, h_0\rangle$$

where $\mathbb{A}_k = (a_1, \ldots, a_k) \in \mathscr{A}^k$.

Lemma 51. $\mathbb{W}_{k,\mathbb{A}_k}$ *is invariant wrt. to time translations, namely*

$$\mathbb{W}_{k,\mathbb{A}_k}(t_1, \ldots, t_k) = \mathbb{W}_{k,\mathbb{A}_k}(t_1 + s, \ldots, t_k + s)$$

for all $s \in \mathbb{R}$.

Proof. By invariance of the ground state we have

$$\mathbb{W}_{k,\mathbb{A}_k}(t_1, \ldots, t_k) = \langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)h_0, h_0\rangle$$

$$= \langle Q_{t_1}(a_1)\cdots Q_{t_n}(a_n)U(s)h_0, U(s)h_0\rangle$$

$$= \langle U(-s)Q_{t_1}(a_1)U(s)U(-s)\cdots U(-s)Q_{t_n}(a_n)U(s)h_0, h_0\rangle$$

and since $U(-s)Q_{t_1}(a)U(s)=Q_{t_1+s}(a)$ we have the result. □

We observe also that we can define the (reduced) function

$$W_{k,\mathbb{A}_k}(\xi_1,\ldots,\xi_{k-1})=\mathbb{W}_{k,\mathbb{A}_k}(t,t+\xi_1,\ldots,t_k+\xi_{k-1})=\langle Q_0(a_1)U(\xi_1)Q_0(a_2)U(\xi_2)\cdots Q_0(a_k)h_0,h_0\rangle$$

for $\xi_1,\ldots,\xi_{k_1}\in\mathbb{R}$. We have the property that

$$\mathscr{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_k)=W_{k,\mathbb{A}_k}(t_2-t_1,\ldots,t_k-t_{k-1}).$$

Definition 52. *We consider a set of functions $\tilde{W}_{k,\cdot}(\cdot):\mathscr{A}^k\times\mathbb{R}^{k-1}\longrightarrow\mathbb{C}$. We say that $\tilde{W}_{k,\mathbb{A}_k}$ satisfy Axiom W1 (compatibility conditions) if the following properties hold*

1. *$\mathbb{A}_k=(a_1,\ldots,a_k)\in\mathscr{A}^k$ and $(t_1,\ldots,t_{k-1})\in\mathbb{R}^{k-1}$, we have*

$$\tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{i-1},0,t_{i+1},\ldots,t_{k-1})=\tilde{W}_{k-1,\tilde{\mathbb{A}}_{k-1}}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{k-1})$$

   where $\tilde{\mathbb{A}}_{k-1}=(a_1,\ldots a_{i-1},a_i a_{i+1},a_{i+2},\ldots,a_k)\in\mathscr{A}^{k-1}$.

2. *$\mathbb{A}_{k-1}=(a_1,\ldots,a_{k-1})\in\mathscr{A}^{k-1}$ and $T_{k-1}=(t_1,\ldots,t_{k-1})\in\mathbb{R}^{k-1}$, we have*

$$\tilde{W}_{k,(a_1,\ldots,a_{i-1},1_{\mathscr{A}},a_i,\ldots,a_{k-1})}(t_1,\ldots,t_k)=\tilde{W}_{k-1,\mathbb{A}_{k-1}}(t_1,\ldots,t_{i-2},t_{i-1}+t_i,t_{i+1},\ldots,t_{k-1}).$$

3. *$\mathbb{A}_k=(a_1,\ldots,a_k)\in\mathscr{A}^k$ and $T_{k-1}=(t_1,\ldots,t_{k-1})\in\mathbb{R}^{k-1}$ we have*

$$\overline{\tilde{W}_{k,\mathbb{A}_k}(T_{k-1})}=\tilde{W}_{k,\theta\mathbb{A}_k}(\bar{\theta}(T_{k-1}))$$

   where $\theta(\mathbb{A}_k)=(a_k^*,a_{k-1}^*,\ldots,a_1^*)$ and $\bar{\theta}(T_{k-1})=(-t_{k-1},-t_{k-2},\ldots,-t_2,-t_1)$.

Lemma 53. *Reduced Wightman functions satisfy these compatibility conditions (i.e. Axiom W1).*

Proof. Easy exercise. □

Let now introduce the Fréchet space $\mathcal{S}(\mathbb{R}^k)$ (locally convex topological vector space) such that $f\in\mathcal{S}(\mathbb{R}^k)$ iff $f\in C^\infty(\mathbb{R}^k)$ and $\|f\|_{n,\alpha}=\sup_{x\in\mathbb{R}^k}|(1+|x|)^n\mathrm{D}^\alpha f(x)|<\infty$ where $n\geqslant 0$ and $\alpha=(\alpha_1,\ldots,\alpha_k)\in\mathbb{N}_0^k$ with $\mathrm{D}^\alpha f=\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}$. We can consider the dual $\mathcal{S}'(\mathbb{R}^k)=(\mathcal{S}(\mathbb{R}^k))^*$, that is the space of linear functionals $T\in\mathcal{S}(\mathbb{R}^k)\longrightarrow\mathbb{C}$ such that there exists n,α for which $|T(f)|\leqslant C_T\|f\|_{n,\alpha}$. Recall also that the Fourier transform $\mathscr{F}:L^1(\mathbb{R}^k)\longrightarrow C^0(\mathbb{R}^k)$ is defined by

$$\mathscr{F}f(y)=\int_{\mathbb{R}^k}e^{ik\cdot y}f(x)\mathrm{d}x$$

and such that $\mathscr{F}:\mathcal{S}(\mathbb{R}^k)\longrightarrow\mathcal{S}(\mathbb{R}^k)$ and the map is continuous wrt. to the topology $\mathcal{S}(\mathbb{R}^k)$ and invertible with

$$\mathscr{F}^{-1}f(y)=\frac{1}{(2\pi)^k}\int_{\mathbb{R}^k}e^{-ik\cdot y}f(x)\mathrm{d}x.$$

Then if $T \in \mathcal{S}'(\mathbb{R}^k)$ we can define $\mathcal{F}(T) = T \circ \mathcal{F}^{-1}$.

Definition 54. *$\tilde{W}_{k,\cdot}(\cdot)$ satisfy Axiom W2 (i.e. it is a Fourier transform of a distribution with support in $\mathbb{R}_+^{k-1}$) if $\tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1})$ is continuous in $t_1,\ldots,t_{k-1}$ and $\tilde{W}_{k,\mathbb{A}_k} = \mathcal{F}(T_{k,\mathbb{A}_k})$ for some $T_{k,\mathbb{A}_k} \in \mathcal{S}'$ such that*

$$|T_{k,\mathbb{A}_k}(f_1 \otimes \cdots \otimes f_{k-1})| \leq C_k \prod_{\ell=1}^{k-1} \|f_\ell\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k} \|a_k\|_{\mathcal{A}}. \tag{9}$$

What means that T has support on $\mathbb{R}_+^k$? This means that if $f \in \mathcal{S}(\mathbb{R}^k)$ and $\mathrm{supp}(f) \subset \mathbb{R}^{k-1} \backslash (\mathbb{R}_+)^{k-1}$ then $T(f) = 0$.

Remark 55. The equation (9) is equivalent to

$$\left| \int_{\mathbb{R}^{k-1}} \tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1}) \overline{g_1(t_1)} \cdots \overline{g_{k-1}(t_{k-1})} \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1} \right| \lesssim \tilde{C}_k \prod_{\ell=1}^{k-1} \|\mathcal{F}^{-1}(g_\ell)\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k} \|a_k\|_{\mathcal{A}}.$$

for $g_1,\ldots,g_{k-1} \in \mathcal{S}(\mathbb{R})$. Indeed recall that $\mathcal{F}(T_{k,\mathbb{A}_k}) = \tilde{W}_{k,\mathbb{A}_k}$ and

$$\mathcal{F}(T_{k,\mathbb{A}_k})(g) = \langle \tilde{W}_{k,\mathbb{A}_k}, g \rangle = \int_{\mathbb{R}^{k-1}} \tilde{W}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1}) \overline{g(t_1,\ldots,t_k)} \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}$$

but $\mathcal{F}(T_{k,\mathbb{A}_k})(g) = T_{k,\mathbb{A}_k}(\mathcal{F}^{-1}(g))$ and calling $\mathcal{F}^{-1}(g) = f$ and from this one can conclude.

Lemma 56. *The Wightman functions satisfy Axiom W2.*

Proof. Recall that from U we can construct homomorphisms $X_U: C_b^0(\mathbb{R},\mathbb{C}) \longrightarrow \mathcal{B}(\mathcal{H})$ such that $X_U(e^{it\cdot}) = U(t)$ and which is strongly continuous with respect to pointwise sequential convergence in bounded sets. So for any $g \in \mathcal{S}(\mathbb{R})$ we can define $U_g = \int_{\mathbb{R}} g(t) U(t) \mathrm{d}t = X_U(\mathcal{F}g)$. Indeed

$$\langle X_U(\mathcal{F}g) h_1, h_2 \rangle = \int \mathcal{F}g(x) \mu^{h_1,h_2}(\mathrm{d}x) = \int \int g(t) e^{itx} \mathrm{d}t \mu^{h_1,h_2}(\mathrm{d}x) = \int g(t) \int e^{itx} \mu^{h_1,h_2}(\mathrm{d}x) \mathrm{d}t$$

$$= \int g(t) \langle U(t) h_1, h_2 \rangle \mathrm{d}t.$$

Now

$\int W_{k,\mathbb{A}_k}(t_1, \ldots, t_k) g_1(t_1) \cdots g_k(t_k) = \langle Q_0(a_1) U(t_1) Q_0(a_2) U(t_2) \cdots Q_0(a_k) h_0,$ $h_0 \rangle g_1(t_1) \cdots g_{k-1}(t_{k-1}) \mathrm{d}t_1 \cdots \mathrm{d}t_k$

$$= \langle Q_0(a_1) U(g_1) Q_0(a_2) U(t_2) \cdots U(g_{k-1}) Q_0(a_k) h_0, h_0 \rangle$$

$$= \langle Q_0(a_1) X_U(\mathcal{F}(g_1)) Q_0(a_2) U(t_2) \cdots X_U(\mathcal{F}(g_{k-1})) Q_0(a_k) h_0, h_0 \rangle$$

which can be bounded by

$$\|Q_0(a_1)\| \cdots \|Q_0(a_k)\| \|X_U(\mathcal{F}(g_1))\| \cdots \|X_U(\mathcal{F}(g_{k-1}))\|$$

which then gives readily the result using the fact that U has positive energy so

$$\|X_U(f)\| \leq \|f\|_{L^\infty(\mathbb{R}_+)}.$$

□

Let us consider now our last axiom. Recall that we defined $\theta(\mathbb{A}_k) = (a_k^*, a_{k-1}^*, ..., a_1^*)$ and $\bar{\theta}(T_{k-1}) = (-t_{k-1}, -t_{k-2}, \dots, -t_2, -t_1)$.

For $\mathbb{A}_{k_1} = (a_1, \dots, a_{k_1})$ and $\mathbb{A}'_{k_1} = (a'_1, \dots, a'_{k_2})$ then we let

$$\mathbb{A}_{k_1}\mathbb{A}'_{k_2} = (a_1, a_2, \dots, a_{k_1}a'_1, \dots, a'_{k_2}) \in \mathscr{A}^{k_1+k_2-1}$$

Definition 57. *The functions $\tilde{W}_{k,\cdot}(\cdot) : \mathscr{A}^k \times \mathbb{R}^{k-1} \to \mathbb{C}$ satisfy Axiom W3 (Hilbert-space positivity) if for any $k \in \mathbb{N}_0$ and any $j_1, \dots, j_k \in \mathbb{N}$, any $T_{n-1,j} = (t_{1,(n-1,j)}, \dots t_{n-1,(n-1,j)})$ and $\lambda_{n,j} \in \mathbb{C}$ and $\mathbb{A}_{n,j} = (a_{1,(n,j)}, \dots a_{n,(n,j)}) \in \mathscr{A}^n$ where $j \leq j_n$ and $n \leq k$ we have*

$$\sum_{n_1+n_2=1}^{k} \sum_{h_1=1}^{j_{n_1}} \sum_{h_2=1}^{j_{n_2}} \lambda_{n_1,h_1}\overline{\lambda_{n_2,h_2}}\tilde{W}_{n_1+n_2-1,\theta(\mathbb{A}_{n_2,h_2})\mathbb{A}_{n_1,h_1}}(\bar{\theta}(T_{n_2-1,h_2}), T_{n_1-1,h_1}) \geq 0.$$

Example: if $k = 1$ we have only

$$\sum_{h_1=1}^{j_1} \sum_{h_2=1}^{j_2} \lambda_{h_1}\bar{\lambda}_{h_2}\tilde{W}_{1,a_{h_2}^* a_{h_1}} = \sum_{h_1=1}^{j_1} \sum_{h_2=1}^{j_2} \lambda_{h_1}\bar{\lambda}_{h_2}\langle Q_0(a_{h_2}^*)Q_0(a_{h_1})h_0, h_0\rangle$$

$$= \left\langle \sum_{h_1=1}^{j_1} \lambda_{h_1}Q_0(a_{h_1})h_0, \sum_{h_2=1}^{j_2} \lambda_{h_2}Q_0(a_{h_2})h_0 \right\rangle \geq 0.$$

Another example gives

$$0 \leq \lambda\bar{\lambda}\tilde{W}_{2,(a_2^*, a_1^* a_1, a_2)}(t_1, -t_1) = \langle Q_0(a_2^*)U(t_1)Q_0(a_1^* a_1)U(-t_1)Q_0(a_2)h_0, h_0\rangle = \|Q_0(a_1)U(-t_1)Q_0(a_2)h_0\|^2$$

Lemma 58. *Wightman functions satisfy Axiom W3.*

Proof. Let

$$H = \sum_{n_1=1}^{k} \sum_{h_1=1}^{j_{n_1}} \lambda_{n_1,h_1}Q_0(a_{1,(n_1,h_1)})U(t_{1,(n_1-1,h_1)})\cdots U(t_{n-1,(n_1-1,h_1)})Q_0(a_{n_1,(n_1,h_1)})h_0$$

and using $\langle H, H\rangle \geq 0$ and the definition of Wightman functions we get the claim. □

Summarizing, we have shown that the reduced Wightman functions $(W_{k,\mathbb{A}_k})_k$ statisfy three basic properties

a) W1 – compatibility condition (encodes the fact that Q_0 is a C^*-representation and that U is a unitary group)

b) W2 – tempered distribution axiom (encodes the fact that U is strongly continuous with positive energy)

c) W3 – Hilbert space positivity (encodes the fact that the scalar product is Hermitian and positive)

The next set is to give the idea of the proof of equivalent properties $S1, S2, S3$ for the Schwinger functions (which are like Wightman functions but with K in place of U) and then we prove that if we are given functions $W1, W2, W3$ or $S1, S2, S3$ then we can come back and obtain the data of $(\mathscr{H}, Q_0, U)$ or $(\mathscr{H}, Q_0, K)$ of the Hilbert space, representations Q_0, K or U.

Definition 59. *Schwinger functions, $k \in \mathbb{N}$ and $\mathbb{A}_k = (a_1, \ldots, a_k) \in \mathscr{A}^k$, $t_1, \ldots, t_{k-1} \geq 0$ and let*

$$S_{k,\mathbb{A}_k}(t_1, \ldots, t_{k-1}) = \langle Q_0(a_1) K(t_1) Q_0(a_2) K(t_2) \cdots K(t_{k-1}) Q_0(a_k) h_0, h_0 \rangle.$$

Recall that $\theta(\mathbb{A}_k) = (a_k^*, \ldots, a_1^*)$ and $\bar{\theta}(T_{k-1}) = (-t_{k-1}, -t_{k-2}, \ldots, -t_1)$. We introduce now also another map on times as $\hat{\theta}(T_{k-1}) = (t_{k-1}, t_{k-2}, \ldots, t_1)$. We will need also the composition $\mathbb{A}_k \cdot \mathbb{A}'_{k'} = (a_1, \ldots, a_k a'_1, \ldots a'_{k'})$.

Definition 60. *We say that the set of functions $(\tilde{S}_k : \mathscr{A}^k \times \mathbb{R}_+^{k-1} \longrightarrow \mathbb{C})_k$ satisfy the axiom S1 (or compatibility condition)*

1. $\tilde{S}_{k,\mathbb{A}_k}(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{k-1}) = \tilde{S}_{k-1,\tilde{\mathbb{A}}_k}(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{k-1})$ *where* $\tilde{\mathbb{A}}_k = (a_1, \ldots, a_i a_{i+1}, \ldots, a_k)$ *and*

$$\tilde{S}_{k,(a_1, \ldots, \lambda a_i + \mu b_i, \ldots, a_k)}(t_1, \ldots, t_{k-1}) = \lambda \tilde{S}_{k,(a_1, \ldots, a_i, \ldots, a_k)}(t_1, \ldots, t_{k-1}) + \mu \tilde{S}_{k,(a_1, \ldots, b_i, \ldots, a_k)}(t_1, \ldots, t_{k-1})$$

2. $\tilde{S}_{k,(a_1, \ldots a_{i-1}, 1, a_i, \ldots, a_k)}(t_1, \ldots, t_{k-1}) = \tilde{S}_{k-1,\mathbb{A}_{k-1}}(t_1, \ldots, t_{k-1})$

3. $\overline{\tilde{S}_{k,\mathbb{A}_k}(T_{k-1})} = \tilde{S}_{k,\theta(\mathbb{A}_k)}(\hat{\theta}(T_k))$ *which is due to the fact that* $K(t)^* = K(t)$.

Lemma 61. *The Schwinger functions satisfy Axiom S1*

Let $T \in \mathcal{S}'(\mathbb{R}^{k-1})$ supported in $\mathbb{R}_+^{k-1} = (\mathbb{R}_+)^{k-1}$, i.e. $T(f) = T(f')$ when $f = f'$ on $\mathbb{R}_+^{k-1}$, i.e. functions which behave on $\mathbb{R}_+^{k-1}$ but arbitrarily elsewhere. For example $s \mapsto e^{-ts}$ belongs to $\mathcal{S}(\mathbb{R}_+)$ and

$$(s_1, \ldots, s_{k-1}) \mapsto e^{-t_1 s_1 \cdots - t_{k-1} s_{k-1}}$$

is in $\mathcal{S}(\mathbb{R}_+^{k-1})$. We define the Laplace transform $\mathscr{L}(T) = G(t_1, \ldots, t_{k-1})$ as

$$G(t_1, \ldots, t_{k-1}) = T((s_1, \ldots, s_{k-1}) \mapsto e^{-t_1 s_1 \cdots - t_{k-1} s_{k-1}}).$$

If $f \in L^1$

$$\mathscr{L}(f)(t) = \int_0^\infty e^{-ts} f(s) \mathrm{d}s.$$

Definition 62. *Let $(\tilde{S}_k)_k$ as before. We say that they satisfy Axiom S2 (or that they are Laplace transform of a tempered distribution) if $\exists T_{k,\mathbb{A}_k}$ such that $\tilde{S}_{k,\mathbb{A}_k} = \mathscr{L}(T_{k,\mathbb{A}_k})$ and for all $g_1, \dots, g_{k-1} \in \mathcal{S}(\mathbb{R}_+)$*

$$\left| \int_{\mathbb{R}_+^{k-1}} \tilde{S}_{k,\mathbb{A}_k}(t_1, \dots, t_{k-1}) g(t_1) \cdots g_{k-1}(t_{k-1}) \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1} \right| \leqslant \prod_{\ell=1}^{k-1} \|\mathscr{L} g_\ell\|_{L^\infty(\mathbb{R}_+)} \prod_{\ell=1}^{k-1} \|a_k\|_{\mathscr{A}}. \tag{10}$$

Theorem 63. *The inequality (10) implies that $\tilde{S}_{k,\mathbb{A}_k}$ is the Laplace transform of a distribution.*

Proof. For a proof see the book of B. Simon "The $P(\varphi)_2$ Euclidean Quantum Field Theory", Chap. 2 Sect. 2.2. □

Lemma 64. *The Schwinger functions, satisfy Axiom S2.*

Proof. Similar to the analogous statement for Wightman functions. The essential step is to observe that

$$\int_{\mathbb{R}_+^{k-1}} S_{k,\mathbb{A}_k}(t_1, \dots, t_{k-1}) g(t_1) \cdots g_{k-1}(t_{k-1}) \mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}$$

$$= \langle Q_0(a_1) X_K(\mathscr{L} g_1) \cdots X_K(\mathscr{L} g_{k-1}) Q_0(a_k) h_0, h_0 \rangle$$

where X_K is the homomorphism generated by K as we introduced few lectures ago. □

Remark 65. We proved that $S_{k,\mathbb{A}_k} = \mathscr{L}(T_{k,\mathbb{A}_k})$, moreover $T_{k,\mathbb{A}_k}$ for $k=2$ is a measure (easy to see) from the definition. For $k>2$ is not a measure but a *poly-measure* (i.e. is a measure in each components, but not jointly).

Remark 66. We have that the reduced Schwinger functions $S_{k,\mathbb{A}_k}$ are holomorphic in $\{\mathrm{Re}(t_i) > 0 : i = 1, \dots, k\} \subset \mathbb{C}^k$ and continuous in $\{\mathrm{Re}(t_i) \geqslant 0 : i = 1, \dots, k\}$, moreover we have

$$W_{k,\mathbb{A}_k}(s_1, \dots, s_{k-1}) = S_{k,\mathbb{A}_k}(i s_1, \dots, i s_{k-1})$$

where the r.h.s is defined as the limit

$$S_{k,\mathbb{A}_k}(i s_1, \dots, \mathrm{i} s_{k-1}) = \lim_{\lambda_1, \dots, \lambda_{k-1} \longrightarrow 0+} S_{k,\mathbb{A}_k}(\lambda_1 + i s_1, \dots, \lambda_{k-1} + i s_{k-1}).$$

This follows directly from the fact that $S_{k,\mathbb{A}_k}$ is the Laplace transform of a tempered distribution supported on $\mathbb{R}_+^{k-1}$.

Definition 67. *Let $(\tilde{S}_k)_k$ as before. They satisfy Axiom S3 (or reflection positivity) if for $k \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}$ and $T_{n-1,j} = (t_{1,(n-1,j)}, \dots, t_{n-1,(n-1,j)}) \in \mathbb{R}_+^{n-1}$ and $\lambda_{n,j} \in \mathbb{C}$ ($n \leqslant k$ on $j \leqslant j_n$)*

$$\sum_{n_1,n_2=1}^{k} \sum_{h_1=1}^{j_{n_1}} \sum_{h_2=1}^{j_{n_2}} \lambda_{n_1,h_1} \overline{\lambda_{n_2,h_2}} S_{n_1+n_2-1, \theta(\mathbb{A}_{n_2,h_2}) \cdot \mathbb{A}_{n_1,h_1}}(\hat{\theta}(T_{n_2-1,h_2}), T_{n_1-1,h_1}) \geqslant 0$$

This property derives from the fact that the Hilbert scalar product is Hermitian and positive definite, that Q_0 is a representation and that K is self-adjoint (which is linked with the form of $\hat{\theta}$).

Lemma 68. *The Schwinger functions satisfy Axiom S3.*

Now the important result, the reconstruction theorem.

Definition 69. *We say that $\tilde{S}_{k,\cdot}\cdot : \mathscr{A}^k \times \mathbb{R}_+^{k-1} \longrightarrow \mathbb{C}$ is linear in $\mathscr{A}$ (or satisfies Axiom S0) if for all $a_1, \ldots, a_k \in \mathscr{A}$ and $t_1, \ldots t_{k-1} \in \mathbb{R}_+$ if the map*

$$a \longmapsto \tilde{S}_{k,(a_1,\ldots,a_{i-1},a,a_{i+1},\ldots,a_k)}(t_1,\ldots,t_{k-1})$$

is linear in $a \in \mathscr{A}$.

Theorem 70. *If $\tilde{S}_{k,\cdot}\cdot : \mathscr{A}^k \times \mathbb{R}_+^{k-1} \longrightarrow \mathbb{C}$ satisfy Axioms S0,S1,S2,S3 we have that there exists an Hilbert space $\mathscr{H}$, a representation Q_0 of $\mathscr{A}$ in $\mathscr{B}(\mathscr{H})$ and a self-adjoint, strongly continuous semi-group $(K(t))_{t\geqslant 0}$ on $\mathscr{H}$ and a vector $h_0 \in \mathscr{H}$ cyclic wrt. Q_0, K and invariant, i.e. $K(t)h_0 = h_0$ (in other words, h_0 is a ground state), such that $\tilde{S}_{k,\mathbb{A}_k}(T_{k-1})$ are the Schwinger functions generated by $(\mathscr{H}, Q_0, K, h_0)$.*

Remark 71. An analogous theorem holds for families $(\tilde{W}_{k,\mathbb{A}_k})_k$ satisfying W1,W2,W3, from which one can construct data $(\mathscr{H}, Q_0, (U(t))_t, h_0)$ for which they are the Wightman functions.

Proof. Let $\mathscr{F}$ be the free algebra generated by the symbols $\tilde{Q}_0(a)$ and $\tilde{K}(t)$ where $a \in \mathscr{A}$ and $t \in \mathbb{R}_+$ equipped with the relations

i. $\tilde{Q}_0(a)\tilde{Q}_0(b) = \tilde{Q}_0(ab)$, $\lambda\tilde{Q}_0(a) + \mu\tilde{Q}_0(b) = \tilde{Q}_0(\lambda a + \mu b)$ for $a, b \in \mathscr{A}$ and $\lambda, \mu \in \mathbb{C}$

ii. $\tilde{Q}_0(1_{\mathscr{A}}) = 1_{\mathscr{F}}$

iii. $\tilde{K}(t_1)\tilde{K}(t_2) = \tilde{K}(t_1 + t_2)$

iv. $\tilde{K}(0) = 1_{\mathscr{F}}$

By definition $\mathscr{F}$ is the complex vector space generated by the words of the form $\tilde{Q}_0(a)\tilde{Q}_0(b)\tilde{K}(t)\cdots\tilde{Q}_0(c)\tilde{K}(t')$ which then is extended to an algebra by justapposition of the linear generators and then we take the quotient wrt. the relations listed above. Introduce a useful notation: if $T_{k-1} = (t_1, \ldots, t_{k-1}) \in \mathbb{R}_+^{k-1}$ and $\mathbb{A}_k = (a_1, \ldots, a_k) \in \mathscr{A}^k$, we call $\mathbb{F}_k(T_{k-1}, \mathbb{A}_k) = \tilde{Q}_0(a_1)\tilde{K}(t_1)\cdots\tilde{K}(t_{k-1})\tilde{Q}_0(a_k) \in \mathscr{F}$. Using the previous relations we have that if $A \in \mathscr{F}$ then

$$A = \sum_{n=1}^{k}\sum_{h=1}^{j_n} \lambda_{n,h}\mathbb{F}_n(T_{n-1,h}, \mathbb{A}_{n-1,h})$$

for some $\lambda_{n,h}, T_{n-1,h}, \mathbb{A}_{n-1,h}$ (in general not in a unique way). On $\mathscr{F}$ we define the scalar product $\langle *, * \rangle_{\mathscr{F}}$ by

$$\langle \mathbb{F}_k(T_{k-1}, \mathbb{A}_k), \mathbb{F}_{k'}(T'_{k'-1}, \mathbb{A}'_k) \rangle_{\mathscr{F}} = \tilde{S}_{k+k'-1,\theta(\mathbb{A}'_{k'})\cdot\mathbb{A}_k}(\hat{\theta}(T'_{k'-1}), T_{k-1})$$

and extend it by linearity to all $\mathscr{F}$ in the first component and by antilinearity in the second component. This definition is well posed since $(\tilde{S}_{k,\mathbb{A}_k})_k$ satisfy the compatibility conditions of Axiom S1 and moreover by the last of the property in Axiom S1 we have that the form $\langle *, *\rangle_{\mathscr{F}}$ is Hermitian and for Axiom S3 that this scalar product is positive definite. We define the linear subspace $\mathscr{N} = \{A \in \mathscr{F}, \langle A, A\rangle_{\mathscr{F}} = 0\}$ and we define $\mathscr{H}_0 = \mathscr{F} \backslash \mathscr{N}$ as a vector space. On $\mathscr{H}_0$ we define $\langle [A], [B]\rangle_{\mathscr{H}} = \langle A, B\rangle_{\mathscr{F}}$ which is well defined by the Cauchy–Schwartz inequality and where $[A] \in \mathscr{H}_0$ denotes the class of $A \in \mathscr{F}$. Moreover we let $\mathscr{H}$ the completion of $\mathscr{H}_0$ with respect to this non-degenerate scalar product $\langle , \rangle_{\mathscr{H}}$ (which is stricly positive on $\mathscr{H}_0 - \{0\}$). We let $h_0 = [1_{\mathscr{F}}]$. We define $\mathbb{K}(t): \mathscr{F} \longrightarrow \mathscr{F}$ linear such that

$$\mathbb{K}(t)(\mathbb{F}_k(T_{k-1}, \mathbb{A}_k)) := \tilde{K}(t)\mathbb{F}_k(T_{k-1}, \mathbb{A}_k) = \mathbb{F}_{k+1}((t, T_{k-1}), (1_{\mathscr{A}}, \mathbb{A}_k)).$$

We have that $\mathbb{K}(t)\mathbb{K}(s) = \mathbb{K}(t+s)$ and $\mathbb{K}(0) = 1$. Moreover $\mathbb{K}_t$ is symmetric wrt. the scalar product on $\mathscr{F}$ (this is a consequence of Axiom S1), indeed

$$\langle \mathbb{K}(t)(\mathbb{F}_k(\mathbb{T}_{k-1}, \mathbb{A}_k)), \mathbb{F}_h(T'_{h-1}, \mathbb{A}_h)\rangle = \langle \mathbb{F}_{k+1}((t, T_{k-1}), (1_{\mathscr{A}}, \mathbb{A}_k)), \mathbb{F}_h(T'_{h-1}, \mathbb{A}_h)\rangle$$

$$= S_{k+h-1, \theta(\mathbb{A}'_h)(1_{\mathscr{A}}, \mathbb{A}_k)}(\hat{\theta}(T'_{h-1}), (t, T_{k-1})) = S_{k+h-1, \theta((1_{\mathscr{A}}, \mathbb{A}'_h))\mathbb{A}_k}(\hat{\theta}(t, T'_{h-1}), T_{k-1})$$

$$= \langle \mathbb{F}_k(\mathbb{T}_{k-1}, \mathbb{A}_k), \mathbb{K}(t)\mathbb{F}_h(T'_{h-1}, \mathbb{A}_h)\rangle$$

and this extends by linearity to deduce the symmetry for $\mathbb{K}(t)$.

Next, we have that

$$\langle \mathbb{K}(t)A, A\rangle = \langle \mathbb{K}(t/2)A, \mathbb{K}(t/2)A\rangle \geqslant 0$$

moreover by repeated use of Cauchy–Schwartz we also have

$$\langle \mathbb{K}(t)A, A\rangle \leqslant (\langle \mathbb{K}(2t)A, A\rangle)^{1/2}(\langle A, A\rangle)^{1/2} < \cdots < (\langle \mathbb{K}(2^n t)A, A\rangle)^{1/2^n}(\langle A, A\rangle)^{1-1/2^n}$$

By Axiom S2 we know that $\langle \mathbb{K}(2^n t)A, A\rangle$ can be written as a sum of the form

$$\langle \mathbb{K}(2^n t)A, A\rangle = \sum_{k, \mathbb{A}_k} S_{k, \mathbb{A}_k}(2^n t, t_1, \ldots, t_{k-2})$$

where everything does not depends on n and is uniformly bounded so the quantity $\langle \mathbb{K}(2^n t)A, A\rangle$ is bounded uniformly in n. So

$$\langle \mathbb{K}(t)A, A\rangle \leqslant C^{1/2^n}(\langle A, A\rangle)^{1-1/2^n}$$

and taking $n \longrightarrow \infty$ we have

$$\langle \mathbb{K}(t)A, A\rangle \leqslant \langle A, A\rangle$$

so $\mathbb{K}(t)\mathscr{N} \subset \mathscr{N}$ and $\mathbb{K}(t)$ is well defined on $\hat{\mathscr{H}}$ and we let $K_0(t)[A] = [\mathbb{K}(t)A]$. We have that for all $t \geqslant 0$

$$\langle K_0(t/2)[A], K_0(t/2)[A]\rangle = \langle K_0(t)[A], [A]\rangle \leqslant \langle [A], [A]\rangle$$

so $K_0(t)$ is a contraction for all $t \geqslant 0$ so it extends to $\mathscr{H}$ as K. It is also self-adjoint and a $(K(t))_{t\geqslant 0}$ is a semigroup. For the strong continuity of the family $(K(t))_{t\geqslant 0}$ we observe that the Schwinger functions are continuous at least when considered as a functions of one of the time variables (fixing all the other parameters). This is enough to prove that $t \mapsto K(t)$ is weakly continuous and then strong continuity follows since it is a contraction.

We define a linear map $\mathbb{Q}(a): \mathscr{F} \longrightarrow \mathscr{F}$ as $\mathbb{Q}(a)A = \tilde{Q}_0(a)A$. It is a representation of $\mathscr{A}$ on $\mathscr{F}$ (this follows from the relations we imposed on the algebra $\mathscr{F}$). We have that is a $*$-represenation:

$$\langle \mathbb{Q}(a)A, B\rangle_{\mathscr{F}} = \langle A, \mathbb{Q}(a^*)B\rangle_{\mathscr{F}}$$

this can be proved by looking at the definition of the Hermitian form. Moreover on can show $\mathbb{Q}(a)\mathscr{N} \subset \mathscr{N}$ so that we can define the operator on $\hat{\mathscr{H}}$. Define the linear functional on $\mathscr{A}$: $L_A(a) = \langle \mathbb{Q}(a)A, A\rangle_{\mathscr{F}}$. It is positive since

$$L_A(bb^*) = \langle \mathbb{Q}(bb^*)A, A\rangle_{\mathscr{F}} = \langle \mathbb{Q}(b)\mathbb{Q}(b^*)A, A\rangle_{\mathscr{F}} = \langle \mathbb{Q}(b^*)A, \mathbb{Q}(b^*)A\rangle_{\mathscr{F}} \geqslant 0.$$

Therefore it is continuous and its norm on $\mathscr{A}^*$ is given by $L_A(1_{\mathscr{A}}) = \langle A, A\rangle_{\mathscr{F}}$ so if $A \in \mathscr{N}$ then $L_A = 0$. From this, in particular we have $0 = L_A(b^* b) = \langle \mathbb{Q}(b)A, \mathbb{Q}(b)A\rangle_{\mathscr{F}}$ so $Q(b)A \in \mathscr{N}$ for any $b \in \mathscr{A}$. We can then pass to the quotient and define $Q_{00}(a)[A] = [\mathbb{Q}(a)A]$. We have also $\|Q_{00}(a)[A]\|_{\mathscr{F}} \leqslant \|a\|_{\mathscr{A}} \|[A]\|_{\mathscr{F}}$ so Q_{00} is bounded and can be extended to $\mathscr{H}$ as a C^*-homomorphism. We let $h_0 = [1_{\mathscr{F}}]$ and by S1 prove that it is invariant. □

Remark 72. We can replace S2 by S2' which is the property that $\tilde{S}_{k,\cdot}\cdot: \mathscr{A}^k \times \mathbb{R}_+^{k-1} \longrightarrow \mathbb{C}$ are bounded and continuous in each of the time variables separately. This implies that all together (S0,S1,S2,S3) are equivalent to (S0,S1,S2',S3). (Of course S2 is not equivalent to S2').

9 The Ornstein–Uhlenbeck process

We want now to construct Schwinger functions starting from a stochastic process.

We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and consider a Gaussian process $X: \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$, that is such that for all $\xi_1, ..., \xi_k \in \mathbb{R}$ we have that $(X_{\xi_1}, ..., X_{\xi_k})$ is a k-dimensional Gaussian. A Gaussian process is characterised by its mean and covariance function. We let $\mathbb{E}[X_\xi] = 0$ for all $\xi \in \mathbb{R}$ and

$$\mathrm{Cov}(X_\xi, X_{\xi'}) = \mathbb{E}[X_\xi X_{\xi'}] = \frac{1}{2\theta} e^{-\theta|\xi - \xi'|}, \qquad \xi, \xi' \in \mathbb{R}.$$

If $\tilde{S}_{k,\cdot}\cdot: \mathscr{A}^k \times \mathbb{R}_+^{k-1} \longrightarrow \mathbb{C}$ then we define extended functions $\tilde{\mathcal{S}}_{k,\cdot}\cdot: \mathscr{A}^k \times \mathbb{R}_+^k \longrightarrow \mathbb{C}$ such that, if $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_k$ we let

$$\tilde{\mathcal{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \tilde{S}_{k,\mathbb{A}_k}(\xi_2 - \xi_1, \xi_3 - \xi_2, \ldots, \xi_k - \xi_{k-1})$$

and if $\xi_1, \ldots, \xi_k$ are general then we let

$$\tilde{\mathcal{S}}_{k,\mathbb{A}_k}(\xi_1, \ldots, \xi_k) = \tilde{\mathcal{S}}_{k,\mathbb{A}_k}(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)})$$

where $\sigma \in S_n$ is the permutation such that $\xi_{\sigma(1)} < \cdots < \xi_{\sigma(k)}$. Note that to a family $\tilde{\mathcal{S}}$ invariant under translation and permutation of the time variables it associated a unique family $\tilde{S}$ and viceversa.

We choose now $\mathscr{A} = C_b^0(\mathbb{R})$ and let

$$\tilde{\mathcal{S}}_{k,\mathbb{A}_k}(\xi_1,\ldots,\xi_k) = \mathbb{E}[a_1(X_{\xi_1})\cdots a_k(X_{\xi_k})]$$

when $\xi_1 \leqslant \cdots \leqslant \xi_k$ and the extended via permuations as above. This is a symmetric function which is invariant under translation of the time variables, so we can identify the functions $\tilde{S}$ and have

$$\tilde{S}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1}) = \mathbb{E}[a_1(X_0)a_2(X_{t_1})a_3(X_{t_1+t_2})\cdots a_k(X_{t_1+\cdots+t_{k-1}})]$$

$$=\mathbb{E}[a_1(X_t)a_2(X_{t+t_1})a_3(X_{t+t_1+t_2})\cdots a_k(X_{t+t_1+\cdots+t_{k-1}})]$$

By construction S0,S1 are true and depend only on the linearity of expectations. For S1 we observe that, for example,

$$S_{2,(a_1,a_2)}(0) = \mathbb{E}[a_1(X_0)a_2(X_0)] = \mathbb{E}[(a_1a_2)(X_0)] = S_{1,(a_1a_2)}$$

and similarly for all the other conditions of S1. S2' is true, as easily seen from the definition thanks to convergence in law to prove continuity observing that

$$(X_{\xi_1},\ldots,X_{\xi_{i-1}},X_{\xi},X_{\xi_{i+1}},\ldots,X_{\xi_k}) \xrightarrow[\text{law}]{} (X_{\xi_1},\ldots,X_{\xi_{i-1}},X_{\xi_i},X_{\xi_{i+1}},\ldots,X_{\xi_k})$$

if $\xi \longrightarrow \xi_i$ since the covariance function is continuous in each variable and the characteristic functions converge (by Lévy's theorem this implies convergence in law), and using

$$|\tilde{S}_{k,\mathbb{A}_k}(t_1,\ldots,t_{k-1})| \leqslant \|a_1\|\cdots\|a_k\|,$$

for the boundedness.

Consider now:

$$S_{k+h-1,\theta(\mathbb{A}_h')\mathbb{A}_k}(\hat{\theta}(T_{h-1}'),T_{k-1})$$

$$=\mathbb{E}[a_h'^*(X_{-t_{h-1}'-t_{h-2}'-\cdots-t_1'})\cdots a_2'^*(X_{-t_1})a_1'^*(X_0)a_1(X_0)a_2(X_{t_1})\cdots a_k(X_{t_1+\cdots+t_{k-1}})]$$

Consider also the transformation $\mathbb{R}$ of the process X defined as $\mathbb{R}(X)_t = X_{-t}$. Then

$$S_{k+h-1,\theta(\mathbb{A}_h')\mathbb{A}_k}(\hat{\theta}(T_{h-1}'),T_{k-1}) = \mathbb{E}[a_h'^*(\mathbb{R}(X)_{t_1'+\cdots+t_{h-1}'})\cdots a_1'^*(\mathbb{R}(X)_0)a_1(X_0)a_2(X_{t_1})\cdots a_k(X_{t_1+\cdots+t_{k-1}})]$$

We denote $F \in \mathscr{C}^0_{\text{cyl}}(\mathbb{R}^{\mathbb{R}_+},\mathbb{C})$ if F is a cylindric continuous function, i.e. if there exists k and $\xi_1,\ldots,\xi_k \in \mathbb{R}_+$ such that there exists unique continuous $\tilde{F}:\mathbb{R}^k \longrightarrow \mathbb{C}$ such that $F(X) = \tilde{F}(X_{\xi_1},\ldots,X_{\xi_k})$.

Theorem 73. *$(\tilde{S}_k)_k$ satisfy Axiom S3 iff for any $F \in \mathscr{C}^0_{\text{cyl}}(\mathbb{R}^{\mathbb{R}_+},\mathbb{C})$ we have that*

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}X)}] \geqslant 0.$$

Proof. Only a sketch. The implication $\Longleftarrow$ is the most important for us. Consider

$$F(X) = \sum_{n=1}^{k}\sum_{h=1}^{j_n} \lambda_{n,j}a_{1,n,j}(X_0)a_{2,n,j}(X_{t_1})\ldots a_{n,n,h}(X_{t_{1,n,h}+\cdots+t_{n,n,h}}).$$

Note now that

$$\sum_{n_1,n_2=1}^{k}\sum_{h_1,h_2=1}^{j_{n_1},j_{n_2}}\lambda_{n_1,h_1}\bar{\lambda}_{n_2,h_2}S_{n_1+n_2-1,\theta(\mathbb{A}_{n_2,h_2})\mathbb{A}_{n_1,h_1}}(\hat{\theta}(t_{1,n_1,h1}\ldots t_{1,n_1,h1})\cdots)=\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}]\geqslant 0.$$

by hypothesis. The converse is also true because the functions of the form $a_1(x_1)\cdots a_k(x_k)$ are dense in $C_\infty^0(\mathbb{R}^k)$ and this last algebra is dense in $C_b^0(\mathbb{R}^k)$ wrt. the pointwise convergence with uniform bounds. □

Definition 74. *A process $\tilde{X}$ such that for all $F\in C_{\text{cyl}}^0(\mathbb{R}^{\mathbb{R}_+},\mathbb{C})$ we have $\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}]\geqslant 0$ it is called a reflection positive process.*

Lemma 75. *Consider (Y_1,Y_2) taking values in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ which are Gaussian random variables with covariance*

$$\operatorname{Cov}(Y)=\begin{pmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}$$

with $B_{i,j}=\operatorname{Cov}(Y_i,Y_j)$. Then Y_1 given Y_2 is a Gaussian random variable and the conditional covariance is given by

$$\operatorname{Cov}(\mathbb{E}(Y_1|Y_2))=B_{11}-B_{12}B_{22}^{-1}B_{21}$$

we are assuming that B_{22} is non-singular.

Proof. Exercise. □

Lemma 76. *If $\eta_1,\ldots,\eta_h\geqslant 0$ and $\xi_1,\ldots,\xi_k\geqslant 0$ then $Y_1=(X_{-\eta_1},\ldots,X_{-\eta_h})$ is conditionally independent of $Y_2=(X_{\xi_1},\ldots,X_{\xi_k})$ given X_0, where X is the OU process above.*

Proof. We have by simple inspection

$$\operatorname{Cov}((Y_1,Y_2,X_0))=\begin{pmatrix} C_1 & B_1^T & D_1^T\\ B_1 & C_2 & D_2^T\\ D_1 & D_2 & 1/2\theta\end{pmatrix}$$

with

$$D_1=\left(\frac{e^{-\theta\eta_1}}{2\theta},\ldots,\frac{e^{-\theta\eta_h}}{2\theta}\right),\quad D_2=\left(\frac{e^{-\theta\xi_1}}{2\theta},\ldots,\frac{e^{-\theta\xi_k}}{2\theta}\right)$$

and

$$(B_1)_{i,j}=\frac{e^{-\theta(\eta_i+\xi_j)}}{2\theta}.$$

So

$$\operatorname{Cov}((Y_1,Y_2)|X_0)=\begin{pmatrix} C_1 & B_1^T\\ B_1 & C_2\end{pmatrix}-(2\theta)(D_1,D_2)^T(D_1,D_2)$$

with

$$(D_1, D_2)^T (D_1, D_2) = \begin{pmatrix} \tilde{D}_1 & B_1^T \\ B_1 & \tilde{D}_2 \end{pmatrix}$$

so finally one has

$$\mathrm{Cov}((Y_1, Y_2)|X_0) = \begin{pmatrix} \tilde{C}_1 & 0 \\ 0 & \tilde{C}_2 \end{pmatrix}$$

for some matrices $\tilde{C}_1, \tilde{C}_2$. The important observation is that the antidiagonal is zero. (check as exercise). From this form of the covariance this implies that Y_1, Y_2 are independent given X_0. □

We are going now to prove

Theorem 77. *The OU process X is reflection positive.*

Proof. Take $F \in C^0_{\mathrm{cyl}}(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$, so $F = \tilde{F}(X_{\xi_1}, \ldots, X_{\xi_k})$ with $\xi_1, \ldots, \xi_k \geqslant 0$ as above. By the conditional independence (annd the complex-linearity of the expectation) we have

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}] = \mathbb{E}[\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}|X_0]]$$

$$= \mathbb{E}[\mathbb{E}[F(X)|X_0]\overline{\mathbb{E}[F(\mathbb{R}(X))|X_0]}]$$

Now we observe that X is invariant wrt. reflections so

$$\mathbb{E}[F(\mathbb{R}(X))|X_0] = \mathbb{E}[F(\mathbb{R}(X))|\mathbb{R}(X_0)] = \mathbb{E}[F(X)|X_0]$$

and we obtain

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}] = \mathbb{E}[|\mathbb{E}[F(X)|X_0]|^2] \geqslant 0.$$ □

As a consequence we obtain that $(\tilde{\mathcal{S}}_k)_k$ satisfy axioms S0,S1,S2,S3 and by the reconstruction theorem there exists $(\mathcal{H}, Q_0, (K(t))_{t \geqslant 0}, h_0)$ such that $(\tilde{\mathcal{S}}_k)_k$ are the associated extended Schwinger functions.

Now we are interested in explicitly describing these objects in this particular situation.

In this case we can prove that the free algebra $\mathcal{F}$ introduced in the reconstruction is isomorphic to the algebra $\mathcal{F}_X \subseteq C^0_c(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$ by identifying

$$\tilde{Q}_0(a_0)\tilde{K}(t_1)\tilde{Q}_0(a_1)\cdots\tilde{K}(t_{k-1})\tilde{Q}_0(a_k)$$

with

$$a_0(X_0)a_1(X_{t_1})\cdots a_k(X_{t_1+\cdots+t_{k-1}})$$

and extending this map by linearity. We leave as an exercise to prove the isomorphism (as algebras). Under this ispomorphis if $F, G \in \mathcal{F}_X$ then we also have that the Hermitian form $\langle \rangle_{\mathcal{F}}$ can be represented probabilistically as

$$\langle F, G \rangle_{\mathcal{F}_X} = \mathbb{E}[F(X)\overline{G(\mathbb{R}(X))}]$$

which we know to be non-negative and Hermitian. Let $\mathcal{N}_X := \{F \in \mathcal{F}_X | \langle F, F \rangle_{\mathcal{F}_X} = 0\} \subseteq \mathcal{F}_X$

Remark 78. If $F \in \mathcal{F}_X$ then there exists a version $\mathbb{E}[F|X_0]$ which belongs to $\mathcal{F}_X$, indeed the conditional expectation can be written as $\mathbb{E}[F|X_0] = F(L_F X_0)$ for soem linear map L_F depending on F

Lemma 79. *We have*

$$F - \mathbb{E}[F|X_0] \in \mathcal{N}_X$$

Proof. Observe that

$$\mathbb{E}[(F - \mathbb{E}[F|X_0])\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}]$$

$$= \mathbb{E}[\mathbb{E}[(F - \mathbb{E}[F|X_0])\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}|X_0]]$$

$$= \mathbb{E}[\mathbb{E}[(F - \mathbb{E}[F|X_0])|X_0][\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F|X_0])}|X_0]] = 0$$

since clearly $\mathbb{E}[(F - \mathbb{E}[F|X_0])|X_0] = 0$. □

So from an algebraic point of view we have that $\hat{\mathcal{H}} = \mathcal{F}_X \backslash \mathcal{N}_X$ is just $C_b^0(\mathbb{R}, \mathbb{C})$ where the map $\mathcal{F}_X \longrightarrow \hat{\mathcal{H}}$ is just the conditional expectation $F \mapsto \mathbb{E}[F|X_0]$. That $\hat{\mathcal{H}} = C_b^0(\mathbb{R}, \mathbb{C})$ is clear since $\mathbb{E}[a_0(X_0)|X_0] = a_0(X_0)$ so it is a surjective mapping. Moreover the scalar product can be written

$$\langle f, g \rangle_{\hat{\mathcal{H}}} = \mathbb{E}[f(X_0)\overline{g(X_0)}] = \int_{\mathbb{R}} f(z)\overline{g(z)} \underbrace{\frac{e^{-\theta z^2/2}}{(2\pi/\theta)^{1/2}} \mathrm{d}z}_{\mu_\theta(\mathrm{d}z)}$$

and as a consequence $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}, \mu_\theta)$ moreover $(Q_0(a)f)(z) = a(z)f(z)$. Recall now that $\mathbb{K}(t)F = \tilde{K}(t)A$ which under our isomorphism it is send to a translation of the time variable:

$$\mathbb{K}(t)F(X) = F(X_{t+\cdot}).$$

In particular $\mathbb{K}(t)f(X_0) = f(X_t)$ and we have

$$(K(t)f)(X_0) = \mathbb{E}[\mathbb{K}(t)f(X_0)|X_0] = \mathbb{E}[f(X_t)|X_0]$$

This conditional expectation can be written explicitly since $\mathrm{Cov}(X_t, X_0) = (2\theta)e^{-\theta t}$ and so

$$X_t = e^{-\theta t} X_0 + (1 - e^{-2\theta t})^{1/2} N_\theta$$

where $N_\theta \sim \mathcal{N}(0, 1/2\theta)$ and it is independent of X_0, then

$$K(t)f(z) = \mathbb{E}[f(X_t)|X_0 = z] = \mathbb{E}[f(e^{-\theta t}z + (1 - e^{-2\theta t})^{1/2}N_\theta)] = \int_{\mathbb{R}} f(e^{-\theta t}z + (1 - e^{-2\theta t})^{1/2}y)\mu_\theta(\mathrm{d}y).$$

Obviously $h_0 = 1 \in L^2(\mathbb{R}, \mathbb{C}, \mu_\theta)$. From the explicit expression of $(K(t))_{t \geqslant 0}$ one can check again that it is a stronly continuous contraction semigroup. This is called the Ornstein–Uhlenbeck semigroup.

This is not what is done usually in quantum mechanics since the usual space there is taken to be $L^2(\mathbb{R},\lambda)$ where λ is the Lebesgue measure, not μ_θ. The map connecting the two representations is

$$f\in\hat{\mathscr{H}}\longrightarrow\tilde{f}(z)=f(z)\,\frac{e^{-\theta z^2/4}}{(2\pi/\theta)^{1/4}}\in\tilde{\mathscr{H}}=L^2(\mathbb{R},\lambda)$$

Let's compute the generator H of $K(t)$:

$$-Hf(z)=\lim_{t\longrightarrow 0}\frac{K(t)f(z)-f(z)}{t}=\lim_{t\longrightarrow 0}\int_{\mathbb{R}}\frac{f(e^{-\theta t}z+(1-e^{-2\theta t})^{1/2}y)-f(z)}{t}\mu_\theta(\mathrm{d}y)$$

By Taylor expansion:

$$=\lim_{t\longrightarrow 0}\int_{\mathbb{R}}\frac{f'(z)((e^{-\theta t}-1)z+(1-e^{-2\theta t})^{1/2}y)+\frac{1}{2}f''(z)((e^{-\theta t}-1)z+(1-e^{-2\theta t})^{1/2}y)^2+O(t^{3/2})}{t}\mu_\theta(\mathrm{d}y)$$

and since μ_θ has zero first moment we have

$$=\lim_{t\longrightarrow 0}\int_{\mathbb{R}}\frac{f'(z)(e^{-\theta t}-1)z+\frac{1}{2}f''(z)((1-e^{-2\theta t})^{1/2}y)^2+O(t^{3/2})}{t}\mu_\theta(\mathrm{d}y)$$

$$=\lim_{t\longrightarrow 0}\frac{f'(z)(-\theta t)z+\frac{1}{2}f''(z)(1-e^{-2\theta t})(1/2\theta)+O(t^{3/2})}{t}=-\theta f'(z)+\frac{1}{4}f''(z)$$

so on $\hat{\mathscr{H}}$ we have

$$Hf(z)=\theta f'(z)-\frac{1}{4}f''(z)$$

and the same operator on $\tilde{\mathscr{H}}$ has the form

$$\tilde{H}f(z)=-\theta z^2\tilde{f}(z)-\frac{1}{4}\Delta\tilde{f}(z)$$

and this is usually called the Schrödinger representation of the harmonic oscillator, indeed note that

$$\tilde{H}=\frac{1}{4}P^2+Q^2\frac{\theta^2}{2}$$

which if interpreted classically is the Hamiltonian of the harmonic oscillator.

Therefore we have proven that the quantum mechanical harmonic oscillator is related via the reconstruction theorem with the Ornstein–Uhlenbeck process.

10 Euclidean processes

In this section we take a state space M and a stochastic process $(X_t)_{t\in\mathbb{R}}$ taking values in M and take $\mathscr{A}$ a subset of the continuous functions on M large enough (so that $\mathscr{A}$ characterise the measures on M) and we define the Schwinger functions as before, i.e. as

$$\mathcal{S}_{k,\mathbb{A}_k}(\xi_1,\ldots,\xi_k)=\mathbb{E}[a_1(X_{\xi_1})\cdots a_k(X_{\xi_k})]$$

and we will show that the properties

S0. Linearity in $a \in \mathscr{A}$

S1. Compatibility conditions

S2. Laplace transform of a positively supported distribution

S2'. Boundedness and continuity in t

S3. Reflection positivity.

become suitable probabilistic properties of $(X_t)_{t\in\mathbb{R}}$. We are then going to characterise some classes of processes which have these properties (and therefore which give rise to quantum mechanical dynamics).

Why it is simpler to use this strategy (to construct QM models)? Essentially because probabilistic tools are usually easier to use/more powerful than functional analityc tools in Hilbert spaces. So the probabilistic model should be considered a special and versatile representation of a quantum system.

We consider now $\mathscr{A} \subset C_b^0(M)$ where M is topological space.

We introduce now Axiom N (Nelson positivity).

Definition 80. *A family $(S_k)_k$ is Nelson positive is for all $t_1, \ldots, t_{k-1} \in \mathbb{R}_+$ there exists $\mu_{t_1,\ldots,t_{k-1}}$ a Radon probability measure on M^k such that*

$$S_{k,(a_1,\ldots,a_k)}(t_1,\ldots,t_{k-1}) = \int_{M^k} a_1(x_1)\cdots a_k(x_k)\mu_{t_1,\ldots,t_{k-1}}(\mathrm{d}x_1\cdots\mathrm{d}x_k)$$

Remark 81. In particular, if $a_1,\ldots,a_k \geqslant 0$ in $\mathscr{A}$ i.e. $a_i = b_i b_i^*$ then

$$S_{k,(a_1,\ldots,a_k)}(t_1,\ldots,t_{k-1}) = \int_{M^k} |b_1(x_1)\cdots b_k(x_k)|^2 \mu_{t_1,\ldots,t_{k-1}}(\mathrm{d}x_1\cdots\mathrm{d}x_k) \geqslant 0.$$

This justify the name of positivity.

On M we need to assume also that

(*). $\mathscr{A}^{\otimes k}$ (the linear combination of functions of the form $a_1(x_1)\cdots a_k(x_k)$) generates $C_b^0(M^k;\mathbb{C})$ with respect to the topology of pointwise convergence with uniform bounds.

For example, this holds, if $M = \mathbb{R}^m$ and $\mathscr{A}$ is the space of continuous functions vanishing at ∞ on M,

Theorem 82. *$(S_k)_k$ satisfy Axioms (N, S1, S2, S3) is equivalent to the existence of a stochastic process $X: \Omega \times \mathbb{R} \longrightarrow M$ such that*

1. $$\mathcal{S}_{k,\mathbb{A}_k}(\xi_1,\ldots,\xi_k) = \mathbb{E}[a_1(X_{\xi_1})\cdots a_k(X_{\xi_k})]$$

2. *$(X_{\xi_1},\ldots,X_{\xi_i},\ldots,X_{\xi_k}) \longrightarrow (X_{\xi_1},\ldots,X_{\xi},\ldots,X_{\xi_k})$ in law as $\xi_i \longrightarrow \xi \in \mathbb{R}$.*

3. *For any $s \in \mathbb{R}$ we have that $(X_{s+t})_{t\in\mathbb{R}}$ has the same law of X, i.e. the law of X is invariant under translation*

4. Recall that $\mathbb{R}(X)_t = X_{-t}$ and that $F \in C^0_{\text{cyl}}(\mathbb{R}^{\mathbb{R}_+}; \mathbb{C})$ with $F(X) = \tilde{F}(X_{\xi_1}, \ldots, X_{\xi_k})$, then we have that

$$\mathbb{E}[F(X) F(\mathbb{R}(X))] \geqslant 0,$$

i.e. the process X is reflection positive.

Proof. The direction $\Leftarrow$ is the same in the case $M = \mathbb{R}$ and X the OU process, we did in the last lectures. The reverse direction $\Rightarrow$ goes as follows. If there exists a process satisfying condition 1 using the technical hypothesis $(*)$ we can prove 2,3,4. Indeed if $\mathcal{S}$ satisfies Axiom S0,S1,S2,S3 the process X satisfies 4 for $F = \sum_m \lambda_m a_{1,m}(x_1) \cdots a_{k,m}(x_k)$ but by $(*)$ the functions of this form are dense in $C^0_{\text{cyl}}(M^{\mathbb{R}_+}, \mathbb{C})$ with respect to the pointwise convergence with uniform bounds so 4 follows from dominated convergence theorem. For 2 we do the the case involving only one function:

$$\mathcal{S}_{1,(a_1)}(\xi_1) = \mathbb{E}[a_1(X_{\xi_1})]$$

but $S2'$ implies $\lim_{\xi_1 \to \xi} \mathcal{S}_{1,(a_1)}(\xi_1) = \mathcal{S}_{1,(a_1)}(\xi) = \mathbb{E}[a_1(X_\xi)]$ but they are dense in $C^0_b(M, \mathbb{C})$ and one can argument the convergence in law. For 3 one uses the fact that the function are invariant under translations and $(*)$. It remains now to prove 1, i.e. the existence of such a process. By N we have that

$$S_{k,(a_1,\ldots,a_k)}(t_1, \ldots, t_{k-1}) = \int_{M^k} a_1(y_1) \cdots a_k(y_k) \mu_{t_1,\ldots,t_{k-1}}(\mathrm{d}y_1 \cdots \mathrm{d}y_k)$$

for some Radon probability measure $\mu_{t_1,\ldots,t_{k-1}}$. We consider the process $(X_\xi)_\xi$ with marginals given by $\mu_{t_1,\ldots,t_{k-1}}$. The law of X is unique (if exists) because of $(*)$. By Axiom $S1$ (compatibility conditions), in particular the fact that $\mathcal{S}_{k,(a_1,\ldots,a_{i-1},1,a_{i+1},\ldots a_k)}(\xi_1, \ldots, \xi_k) = \mathcal{S}_{k-1,(a_1,\ldots,a_{i-1},a_{i+1},\ldots a_k)}(\xi_1, \ldots, \not{\xi}_i, \ldots, \xi_k)$ and this implies that $(\mu_{T_k})_{T_k}$ are a compatible family of finite dimensional marginals, and by Kolmogorov's extension theorem there exists a probability measure $\mathbb{P}$ on $\Omega = M^{\mathbb{R}}$ witth the product σ-algebra and with marginals given by $\mu_{\xi_1,\ldots,\xi_k}$. So we can take on Ω the process X: $M^{\mathbb{R}} \times \mathbb{R} \longrightarrow M$ given by $X(\omega)(t) = \omega(t)$. □

The most difficult of the conditions is the reflection positivity. There is no "easy" way to check for it, however is a quite robust property which pass easily to the limit. In this second property it lies its usefulness.

Situations in which one can check easily for reflection positivity are two. The first is when dealing with Gaussian processes, then second in when dealing with Markov processes.

We focus today on the Gaussian case. Let $M = \mathbb{R}^m$ and $\mathcal{A} = C^0_b(M; \mathbb{C})$ and X_t a Gaussian process taking values in $\mathbb{R}^m$ with mean zero. For $\alpha \in \mathbb{R}^m$ we can define $\alpha \cdot X_t = \sum \alpha_i X^i_t$. A Gaussian process is uniquely characterised by its covariance function

$$r^{ij}(t,s) = \mathbb{E}[X^i_t X^j_s].$$

If X satisfies condition 3 then we have that $r^{ij}(t,s)$ is only a function of $t-s$, i.e. $r^{ij}(t,s) = r^{ij}(t-s)$. The continuity in distribution is equivalent to require that $t \mapsto r^{i,j}(t)$ is continuous. This can be verified using the characteristic function (exercise). What about reflection positivity?

Theorem 83. *If X is a reflection positive process then for all $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ and $\xi_1, \ldots, \xi_k \in \mathbb{R}$ we have*

$$\sum_{i,j=1}^{k} \langle \alpha_i, r(\xi_i + \xi_j) \bar{\alpha}_j \rangle_{\mathbb{R}^m} \geqslant 0. \tag{11}$$

Proof. We prove in the scalar case $m=1$ and $M=\mathbb{R}$, the general case follows similarly. We consider $f_n \in \mathscr{A} \longrightarrow x$ in $\mathbb{R}$ and such that $|f(x)| \leqslant |x|$, e.g. $f_n(x) = (-n) \vee (x \wedge n)$. Let $F_n(x) = \sum_i \alpha_i f_n(X_{\xi_i})$, then

$$0 \leqslant \mathbb{E}[F_n(X)\overline{F_n(\mathbb{R}(X))}] = \sum_{i,j=1}^{k} \alpha_i \bar{\alpha}_j \mathbb{E}[f_n(X_{\xi_i}) f_n(X_{-\xi_j})] \longrightarrow \sum_{i,j=1}^{k} \langle \alpha_i, r(\xi_i + \xi_j) \bar{\alpha}_j \rangle_{\mathbb{R}^m}$$

by Lebesgue dominated convergence theorem. □

Theorem 84. *(Wick's theorem) Let $(Y_1, \ldots, Y_k)$ be a centred Gaussian vector, then for r even and $i_1, \ldots, i_r$ chosen among $\{1, \ldots, k\}$ we have*

$$\mathbb{E}[Y_{i_1} \cdots Y_{i_r}] = \sum_{\{(i,j)\}} \prod_{(i,j) \in \{(i,j)\}} \mathbb{E}[Y_i Y_j]$$

where $\{(i,j)\}$ run over the perfect matches of $\{i_1, \ldots, i_r\}$. If r is odd then the expectation is zero.

Proof. Let $\Sigma_{i,j} = \mathbb{E}[Y_i, Y_j]$ and we have that the moment generating function is given by

$$\mathbb{E}[e^{\alpha \cdot Y}] = e^{\frac{1}{2}\langle \alpha, \Sigma \alpha \rangle}$$

then

$$\mathbb{E}[Y_{i_1} \cdots Y_{i_r}] = \frac{\partial^r}{\partial \alpha_{i_1} \cdots \partial \alpha_{i_r}}\bigg|_{\alpha=0} \mathbb{E}[e^{\alpha \cdot Y}] = \cdots = \sum_{\{(i,j)\}} \prod_{(i,j) \in \{(i,j)\}} \mathbb{E}[Y_i Y_j].$$

□

Lemma 85. *Let $(Y_1, \ldots, Y_k)$ be Gaussian with mean zero, then there are polynomials $p_N(x) \in C^0(\mathbb{R}^k, \mathbb{R})$ indexed by $N = \{i_1, \ldots, i_r\}$ with r even or odd of the form*

$$p_N(x) = x_{i_1} \cdots x_{i_r} - \sum_{M : M < N} c_M p_M(x)$$

where M is of degree less then N. These polynomials are orthogonal wrt. the Gaussian measure, i.e.

$$\mathbb{E}[p_N(Y) p_M(Y)] = 0$$

for $\deg(N) \neq \deg(M)$.

Proof. If the covariance matrix Σ is non-singular we apply a form Gram–Schmidt orthogonalisation. For Σ general we can find a subset of the Gaussians whose covariance is non-singular and express the rest of the random variables by linear combinations of this subset and use the previous method. □

The lemma on orthogonal polinomials holds actually for any random variable (for which polinomials are integrable). In the Gaussian case we can prove that the polynomial depends only on the variables we are considering. Let us give here the version of the lemma that we are going to actually use.

Lemma 86. *Let $(Y_1,\ldots,Y_k)$ in $\mathbb{R}^k$ be Gaussian random variables. Use $N=\{i_1,\ldots,i_r\}$ for multiindices. There exists polynomials $p_N(y_{i_1},\ldots,y_{i_r})$ such that*

$$p_N(y_{i_1},\ldots,y_{i_r}) = y_{i_1}\cdots y_{i_r} + \textit{lower order polynomial.}$$

and

$$\mathbb{E}[p_N(Y_{i_1},\ldots,Y_{i_r})p_{N'}(Y_{i'_1},\ldots,Y_{i'_{r'}})] = 0$$

if $r\neq r'$. Moreover introducing the notion of Wick product we have: the Wick product is $:Y_{i_1}\cdots Y_{i_r}: = p_N(Y_{i_1},\ldots,Y_{i_r})$ which is characterised by the properties

$$\frac{\partial}{\partial Y_{i_j}} :Y_{i_1}\cdots Y_{i_r}: = :Y_{i_1}\cdots \not{Y}_{i_j}\cdots Y_{i_r}:, \qquad \mathbb{E}[:Y_{i_1}\cdots Y_{i_r}:] = 0.$$

Note that $:Y_i: = Y_i$.

Proof. The proof is based on Wick's theorem. If Q_1, Q_2 are two polynomials $Q_1(Y_{i_1},\ldots,Y_{i_r})$ and $Q_2(Y_{j_1},\ldots,Y_{j_\ell})$ then

$$\mathbb{E}[Q_1(Y_{i_1},\ldots,Y_{i_r})Q_2(Y_{j_1},\ldots,Y_{j_\ell})] = \sum_{p,q} \mathbb{E}[Y_{i_p}Y_{j_q}]\mathbb{E}\left[\left(\frac{\partial}{\partial Y_{i_p}}Q_1(Y_{i_1},\ldots,Y_{i_r})\right)\left(\frac{\partial}{\partial Y_{j_q}}Q_2(Y_{j_1},\ldots,Y_{j_\ell})\right)\right]$$

which can be proven by integration by parts on monomials and then extended by linearity. We want to prove now that

$$\mathbb{E}[:Y_{i_1}\cdots Y_{i_r}::Y_{j_1}\cdots Y_{j_\ell}:] = 0$$

for $r\neq\ell$. The proof is by induction on $r+\ell$, when $r+\ell=1$ we have $\mathbb{E}[:Y_i:] = \mathbb{E}[Y_i] = 0$. Otherwise we use the above formula to have

$$\mathbb{E}[:Y_{i_1}\cdots Y_{i_r}::Y_{j_1}\cdots Y_{j_\ell}:] = \sum_{p,q} \mathbb{E}[Y_{i_p}Y_{j_q}]\mathbb{E}\left[:Y_{i_1}\cdots \not{Y}_{i_p}\cdots Y_{i_r}::Y_{j_1}\cdots \not{Y}_{j_q}\cdots Y_{j_\ell}:\right] = 0$$

using the induction hypothesis. □

Theorem 87. *Assume that the covariance r satisfies*

$$\sum_{i,j=1}^{k} \langle \alpha_i, r(t_i+t_j)\bar{\alpha}_j\rangle_{\mathbb{R}^m} \geq 0. \tag{12}$$

for all $\alpha_i\in\mathbb{C}$ and $t_i\in\mathbb{R}$. Then X is a reflection positive process.

Proof. The first step is to prove that reflection positivity holds for polynomials and then extended to arbitrary functions. Take a cylindrical polynomial $Q(X) = \tilde{Q}(X_{\xi_1},\ldots,X_{\xi_k})$ for some $k\geq 1$ and $\xi_1,\ldots,\xi_k\in\mathbb{R}$. This polynomial can be expanded in Wick products (since they span the space of all polynomials). We consier the scalar case, the vector case just involve heavier notation. We have

$$\tilde{Q}(X_{\xi_1},\ldots,X_{\xi_k}) = \sum \lambda_{i_1,\ldots,i_r} :X_{\xi_{i_1}}\cdots X_{\xi_{i_r}}:$$

with $\lambda_{i_1,\ldots,i_r} \in \mathbb{C}$. Note that if we let $:X_{\xi_{i_1}} \cdots X_{\xi_{i_r}}: = f(X)$ then $:X_{-\xi_{i_1}} \cdots X_{-\xi_{i_r}}: = f(\mathbb{R}(X))$ since the covariance is invariant under reflections. Then

$$\mathbb{E}[Q(X)\overline{Q(\mathbb{R}(X))}] = \sum \lambda_{i_1,\ldots,i_r} \overline{\lambda_{j_1,\ldots,j_\ell}} \mathbb{E}[:X_{\xi_{i_1}} \cdots X_{\xi_{i_r}}::X_{-\xi_{j_1}} \cdots X_{-\xi_{j_\ell}}:]$$

$$= \sum \lambda_{i_1,\ldots,i_r} \overline{\lambda_{j_1,\ldots,j_\ell}} \sum_{\text{pairings } (i_q,j_p)} \prod r(\xi_{i_q} + \xi_{j_p})$$

where we use that if $r = \ell$ we have

$$\mathbb{E}[:X_{\xi_{i_1}} \cdots X_{\xi_{i_r}}::X_{\xi_{j_1}} \cdots X_{\xi_{j_\ell}}:] = \sum_{q,p} \mathbb{E}[X_{\xi_q} X_{-\xi_p}] \mathbb{E}\left[:X_{\xi_{i_1}} \cdots \not{X}_{\xi_q} \cdots X_{\xi_{i_r}}::X_{\xi_{j_1}} \cdots \not{X}_{\xi_p} \cdots X_{\xi_{j_\ell}}:\right]$$

and proceeding with this we obtain the equality above. We have now to show that the above expression is positive, we know that the matrix $(r(\xi_{i_q} + \xi_{j_p}))_{p,q}$ is positive definite and so the above expression can be written as $\langle v_1, v_2 \rangle_{\oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}}$ where on the vector space $\oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}$ we consider the scalar products where on $\mathbb{R}^k$ we consider the product

$$\sum_i \alpha_i \bar{\alpha}_j r(\xi_i + \xi_j)$$

while on $(\mathbb{R}^k)^{\otimes \ell}$ we use the tesorization of this scalar product, i.e. for $p_1 \otimes \cdots \otimes p_\ell \in (\mathbb{R}^k)^{\otimes \ell}$ we let

$$\langle p_1 \otimes \cdots \otimes p_\ell, p_1 \otimes \cdots \otimes p_\ell \rangle = \sum_{\text{pairings } (i_q,j_p)} \prod \langle p_{i_q}, p_{j_p} \rangle$$

and finally we identify

$$v_1 = (\lambda_1, \ldots, \lambda_k) \oplus (\lambda_{1,2}, \lambda_{1,3} \ldots) \oplus \cdots \in \oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}.$$

Then we deduce that $\langle v_1, v_1 \rangle_{\oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}} \geq 0$ since it is a positive definite scalar product on $\oplus_{i=1}^{\deg(Q)} (\mathbb{R}^k)^{\otimes \ell}$. We conclude that $\mathbb{E}[Q(X)\overline{Q(\mathbb{R}(X))}] \geq 0$.

Now we approximate $\exp(i\alpha X_{\xi_j})$ by polynomials and then we can extend the positivity to convex linear combinations of complex exponentials on $\mathbb{R}^k$. But these are dense in $C_b^0(\mathbb{R}^k, \mathbb{C})$ and therefore we can extend the reflection positivity to all functions in $C_{\text{cyl}}^0(\mathbb{R}^{\mathbb{R}_+}, \mathbb{C})$. □

Theorem 88. *A gaussian process X satisfies conditions* $(1, 2, 3, 4)$ *iff r is continuous, translation invariant and such that eq. (12) holds. In the scalar case this holds iff r is completely monotone and bounded and translation invariant.*

Recall that complete monotonocity is exaclty the condition eq. (12) in the scalar case and this implies that there exists a positive and bounded measure μ on $\mathbb{R}_+$ such that

$$r(t) = \int_0^\infty e^{-|t|s} \mu(\mathrm{d}s).$$

Recall that

$$r(t) = \frac{1}{2\theta} e^{-\theta|t|}$$

is the covariance of the Ornstein–Uhlenbeck process. So the theorem says that the reflection positive Gaussian processes are positive combinations of OU processes.

For example, if μ is a sum of Dirac deltas in $(\theta_k)_k$ then one can obtain a Gaussian process with covariance r taking the sum of independent OU processes with parameter θ_k.

Let us now give a look at reflection positivity for Markovian processes.

Definition 89. *A process $(X_t)_{t\in\mathbb{R}}$ is Markovian if $F\in C^0_{\text{cyl}}(M^{[t,+\infty]},\mathbb{C})$ then for all $\xi_1,\ldots,\xi_k\leq t$*

$$\mathbb{E}[F(X)|X_t,X_{\xi_1},\ldots,X_{\xi_k}]=\mathbb{E}[F(X)|X_t]$$

almost surely.

Definition 90. *The process X is said to be symmetric with respect to time reflections if $\mathbb{R}(X)$ has the same law as X.*

Lemma 91. *If X is Markovian then $F\in C_{\text{cyl}}(M^{[t,+\infty)},\mathbb{C})$ and $G\in C_{\text{cyl}}(M^{(-\infty,t]},\mathbb{C})$ then $F(X)$ and $G(X)$ are conditionally independent given X_t.*

Proof. Assuming that $G(X)=\tilde{G}(X_{\xi_1},\ldots,X_{\xi_k})$ with $\xi_1,\ldots,\xi_k\leq t$ we have

$$\mathbb{E}[e^{i\alpha F(X)}e^{i\beta G(X)}|X_t]=\mathbb{E}[\mathbb{E}[e^{i\alpha F(X)}|X_t,X_{\xi_1},\ldots,X_{\xi_k}]e^{i\beta G(X)}|X_t]$$

$$=\mathbb{E}[\mathbb{E}[e^{i\alpha F(X)}|X_t]e^{i\beta G(X)}|X_t]=\mathbb{E}[e^{i\alpha F(X)}|X_t]\mathbb{E}[e^{i\beta G(X)}|X_t]$$

so this proves conditional independence. □

Theorem 92. *Let X be a Markovian process symmetric with respect to time reflections, then it is reflection positive.*

Proof. Take $F\in C_{\text{cyl}}(M^{\mathbb{R}_+},\mathbb{C})$ then by the above lemma

$$\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}]=\mathbb{E}[\mathbb{E}[F(X)\overline{F(\mathbb{R}(X))}|X_0]]=\mathbb{E}[\mathbb{E}[\overline{F(\mathbb{R}(X))}|X_0]\mathbb{E}[F(X)X_0]]$$

$$=\mathbb{E}[\overline{\mathbb{E}[F(\mathbb{R}(X))|X_0]}\mathbb{E}[F(X)X_0]]$$

$$=\mathbb{E}[\overline{\mathbb{E}[F(\mathbb{R}(X))|\mathbb{R}X_0]}\,\mathbb{E}[F(X)X_0]]=\mathbb{E}[\,|\mathbb{E}[F(X)X_0]|^2]\geq 0,$$

where we used that the law is invariant under time reflection. □

The converse implication of the above lemma is also true. Note that the OU process has exactly this property and therefore it means that the OU process is Markovian and since it is symmetric wrt. time reflections we have another proof that that OU process is reflection positive.

If we want to prove the other properties required for the reconstruction theorem we need that X is continuous in distribution and that it is invariant (in law) under translations. These properties can be obtained analysing the transition kernel of the Markov process.

Let us remark that X_t as an M-valued random variable has a law $\nu = \mathrm{Law}(X_t)$ which is independent of $t \in \mathbb{R}$. Then we can build $\mathscr{H} = L^2(\nu)$, with $h_0 = 1$ and $K(t) f \in L^2(\nu)$ is given explictly by

$$\mathbb{E}[f(X_t)|X_0] = (K(t)f)(X_0).$$

The proof is essentially the same we gave for the OU process. The key observation is that if $F \in C_{\mathrm{cyl}}(M^{\mathbb{R}_+}; \mathbb{C})$ we have that

$$\mathbb{E}[(F(X) - \mathbb{E}[F(X)|X_0])\overline{(F(\mathbb{R}(X)) - \mathbb{E}[F(\mathbb{R}(X))|X_0])}] = 0$$

by Markov property and symmetry under reflections. This allows to identify $\mathscr{H} = L^2(\nu)$ and Q_0 is given by multiplication : $Q_0(a) f = a(x) f(x)$.

Consider a Gaussian process with $r(0) = \mathbb{I}$, then X is Markovian iff $r(t+s) = r(t) r(s)$ (as matrices) for $t, s \geqslant 0$. More generally $r(t, s) = r(t, u) r(u, s)$ for all $s \leqslant u \leqslant t$. So in particular, in the scalar case the process is reflection positive iff it is an OU process.

To construct reflection positive processes which are Markovian but not Gaussian we can take the solution $(X_t)_t$ of a stochastic differential equation of the form

$$\mathrm{d}X_t = \frac{\nabla \rho(X_t)}{2\rho(X_t)} \mathrm{d}t + \mathrm{d}W_t,$$

where $\rho \in C^2(\mathbb{R}^m, \mathbb{R}_{>0})$ and $\int \rho(x) \mathrm{d}x = 1$. And take $\mathrm{Law}(X_{\xi_1}, \ldots, X_{\xi_k})$ to be given by the solution of the SDE starting at X_{ξ_1} with law $\rho \mathrm{d}x$. One can check that this is a consistent assignment of finite dimensional distributions giving a continuous, stationary (i.e. invariant in law under translation), Markov process which is moreover invariant under time reflection. Therefore it defines a reflection positive process to which the reconstruction theorem can be applied. In the case where ρ is Gaussian, then X is the OU process. However if ρ is not Gaussian this procedure gives a large class of reflection positive processes and therefore a large class of quantum dynamics where the Hamiltonian operator H has the form

$$H = -\Delta + V(x)$$

for some function V.

The course ends here.