Excursions on rough paths

(Or, thoughts about the integration of irregular functions)

Massimiliano Gubinelli

Dipartimento di Matematica Applicata “U. Dini”
Università di Pisa

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Prologue.

(or, what not to expect)

- You will hear something about rough paths, however…
- There will be very little rough paths (at least in the form you could expect)
- There will be no reference to probability (but stochastic analysis is the main application)
- Enjoy the open landscape

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Outline

1. Increments
   - Abstract integration
   - Exercise of deconstruction
   - Rough paths
2. Variations
   - Convolution integrals
   - Multiparameter integrals

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k-Increments

Definition

A k-increment is a continuous function $g : [0, T]^{k+1} \to V$ such that $g_{t_0 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$. Denote them $\mathcal{C}_k(V)$.

Example

- $g \in \mathcal{C}_0$ is a function on $[0, T]$
- Given $f \in \mathcal{C}_1$, set $g_{ts} = f_t - f_s$, then $g \in \mathcal{C}_1$.

Basic fact

$g \in \mathcal{C}_1$ is given by $g_{ts} = f_t - f_s$ for some $f \in \mathcal{C}_0$ iff it satisfy

$$g_{ts} - g_{su} - g_{us} = 0$$

A cocycle property.

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A cochain complex

- Increments forms a cochain complex $(\mathcal{C}_*, \delta)$ with coboundary map

$$\delta : \mathcal{C}_k \to \mathcal{C}_{k+1} \qquad (\delta g)_{t_1 \cdots t_k} = \sum_{i=1}^{k} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_k}$$

- $$\mathcal{C}_0 \xrightarrow{\delta} \mathcal{C}_1 \xrightarrow{\delta} \mathcal{C}_2 \xrightarrow{\delta} \mathcal{C}_3 \xrightarrow{\delta} \cdots$$

  $\delta\delta = 0$ and $\mathrm{Ker}\delta|_{\mathcal{C}_{k+1}} = \mathrm{Im}\delta|_{\mathcal{C}_k}$ so the complex is acyclic.

- In particular, $g \in \mathcal{C}_1$ is a 1-cocycle (or closed 1-increment) if

$$\delta g_{tus} = -g_{us} + g_{ts} - g_{tu} = 0.$$

  Then there exists $f \in \mathcal{C}_0$ such that $g = \delta f$: closed 1-increments are exact.

- (cfr. de-Rham cohomology of $\mathbb{R}^n$: closed differential forms are exact)

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$$(ab)_{t_1 \cdots t_{k+m+1}} = a_{t_1 \cdots t_{k+1}} b_{t_{k+1} \cdots t_{k+m+1}}$$

Notation

When $x, f_1, f_2 \in \mathcal{C}_0$ and smooth, we will mean

$$\left(\int \varphi(x) dx \right)_{ts} = \int_s^t \varphi(x_r) dx_r$$

and

$$\left(\int df_1 df_2 \right)_{ts} = \int_s^t \left(\int_s^u d_r f_{1,r} \right) d_u f_{2,u}$$

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…and interesting relations

- Easy to check:

$$\delta \int df_1 = 0 \qquad \delta \int df_1 df_2 = \int df_1 \int df_2 = \delta f_1 \delta f_2$$

for any smooth $f_1, f_2 \in \mathcal{C}_0$.

- And more generally

$$\delta \int df_1 \cdots df_n = \sum_{k=1}^{n-1} \int df_1 \cdots df_k \int df_{k+1} \cdots df_n$$

- Moral: δ splits interated integral into "simpler" objects (and Λ put them together again...)

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Norms on Increments

Definition

For $g \in \mathcal{C}_1, h \in \mathcal{C}_2$ let

$$\|g\|_\mu = \sup_{t,s\in[0,T]} \frac{|g_{ts}|}{|t-s|^\mu} \qquad \|h\|_{\rho,\sigma} = \inf \sup_{t,s,u\in[0,T]^3} \frac{|h_{tus}|}{|t-u|^\rho |u-s|^\sigma}$$

and

$$\|h\|_\mu = \inf \left\{ \sum_i \|h_i\|_{\rho_i,\mu-\rho_i} : h = \sum_i h_i, 0 < \rho_i < \mu \right\}$$

Denote $\mathcal{C}_k^\mu$ the subset of $\mathcal{C}_k$ with finite $\|\cdot\|_\mu$ norm $(k = 1, 2)$.
Let $\mathcal{C}_k^{1+} = \cup_{\mu>1} \mathcal{C}_k^\mu$ – the small increments.

The Λ map

Fact

We have $\mathcal{BC}_1^{1+} = \mathcal{C}_1^{1+} \cap \text{Im}\delta = \{0\}$: nontrivial small 1-increments cannot be exact.

Theorem

There exists a unique bounded linear map $\Lambda : \mathcal{BC}_2^{1+} \to \mathcal{C}_1^{1+}$ such that

$$\delta\Lambda g = g.$$

($\mathcal{BC}_2^{1+} = \mathcal{C}_2^{1+} \cap \text{Im}\delta$)

If $g \in \mathcal{C}_1$ and $\delta g \in \mathcal{C}_2^{1+}$, then

$$g = \Lambda\delta g + \delta f$$

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Outline

What an integral is made of?

Taylor formula

$$\int_s^t \varphi(x_r)dx_r = \varphi(x_s)\int_s^t dx_r + \int_s^t \left(\int_s^u \varphi'(x_r)dx_r\right) dx_u$$

with our “brand new” notation reads

$$\int \varphi(x)dx = \varphi(x)\int dx + \int \varphi'(x)dxdx$$

as elements of $\mathcal{C}_1$.
We look in more detail to the iterated integral by **dissecting** it:

$$\delta \int \varphi'(x)dxdx = \int \varphi'(x)dx \int dx = \delta\varphi(x)\delta x \in \mathcal{C}_3^2$$

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Young integration

- Then
$$\int \varphi(x)dx = \varphi(x)\delta x + \Lambda\left(\delta\varphi(x)\delta x\right)$$
- The integral on the l.h.s is equal to an expression which do not need x to be differentiable.
- Essentially x must be γ-Hölder with $\gamma > 1/2$ – Young integration

Go on. . .

Again

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But now continue Taylor expansion one step further:

$$\int \varphi(x)dx = \varphi(x)\int dx + \varphi'(x)\int dxdx + \int \varphi''(x)dxdxdx$$

The remainder is now a three-fold integral:

$$\delta\int \varphi''(x)dxdxdx = \underbrace{\int \varphi''(x)dx}_{\delta\varphi'(x)}\int dxdx + \underbrace{\int \varphi''(x)dxdx}_{\delta\varphi(x)-\varphi'(x)\delta x}\int dx$$

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Rough paths

Putting things together

$$\int \varphi(x)dx = (1 - \Lambda\delta)\left[\varphi(x)\delta x + \varphi'(x)\int dxdx\right]$$

(if the argument of Λ is small enough).

- To make sense of the r.h.s we need a small $\int dxdx$ such that

$$\delta \int dxdx = \delta x \delta x$$

  (which is a highly nontrivial non-linear relation).
- $\int dxdx$ is the “Levy area” of the rough path theory.

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- This procedure can be iterated to recover the hierarchy of (Lyons') rough paths which are given by a sequence of iterated integrals of the form

$$\int dx, \quad \int dxdx, \quad \int dxdxdx, \dots$$

- **Watch out**: to prove smallness of some terms we need geometric rough paths, i.e. which satisfy relations like

$$[(\delta x)_{st}]^2 = 2\left(\int dxdx\right)_{ts}.$$

(smooth integrals OK, Stratonovich OK, Itô NO! – but we do not need it).

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Outline

Increments of convolutions

Let $S(t) = e^{-\lambda t}$, $t \geq 0$ a (semi-)group of contractions on the real line. Look at

$$g_t = \int_0^t S(t-u)\varphi(x_u)dx_u$$

Then

$$(\delta g)_{ts} = a_{ts}g_s + \int_s^t S(t-u)\varphi(x_u)dx_u$$

with $a_{ts} = S(t-s) - 1$

The perturbed complex

Idea

Introduce the "perturbed" coboundary $\hat{\delta} = (\delta - a)$

- $\hat{\delta}\hat{\delta} = 0$ using $\delta a = aa$
- We have another acyclic cochain complex $(\mathcal{C}_*, \hat{\delta})$

$$\mathcal{C}_0 \xrightarrow{\hat{\delta}} \mathcal{C}_1 \xrightarrow{\hat{\delta}} \mathcal{C}_2 \xrightarrow{\hat{\delta}} \mathcal{C}_3 \xrightarrow{\hat{\delta}} \cdots$$

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$$\hat{\delta}\hat{\Lambda}g = g.$$

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SPDEs driven by FBM (H > 1/2), joint work with A. Lejay and S. Tindel).

The perturbed complex

Idea

Introduce the "perturbed" coboundary $\hat{\delta} = (\delta - a)$

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A (bi-)linear equation

Let us play with the solution y of the (bi-)linear integral equation

$$y_t = S(t-s)y_s + \int_s^t S(t-u)dx_u y_u.$$

Expand the r.h.s. in a truncated series of iterated integrals:

$$y_t = S(t-s)y_s + \int_s^t S(t-u)dx_u S(u-s)y_s + \int_s^t S(t-u)dx_u \int_s^u S(u-v)dx_v y_v$$

In our notation this reads:

$$\hat{\delta} y = y \int \tilde{d}x + \int \hat{d}x(\hat{d}x\, y) = y \int \tilde{d}x + y \int \hat{d}x\tilde{d}x + \int \hat{d}x\hat{d}x(\hat{d}x\, y)$$

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Convolution rough paths

- Working a bit we get to

$$\hat{\delta} y = (1 - \hat{\Lambda}\hat{\delta}) \left[y \int \tilde{d}x + y \int \hat{d}x\tilde{d}x \right]$$

  where we used the fact that $\hat{\delta} \int \hat{d}x\tilde{d}x = \int \tilde{d}x \int \tilde{d}x$
- This express the solution y as a function of the couple

$$\int \tilde{d}x \qquad \int \hat{d}x\tilde{d}x$$

  suitable notion of rough path for convolution equations.
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Outline

2d Integrals

- With regular $f, g : \mathbb{R}^2 \to \mathbb{R}$ define 2d integrals as

$$\iint_{(x_1,y_1)}^{(x_2,y_2)} f dg := \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy f(x,y) \partial_1 \partial_2 g(x,y)$$

- ∂_1 and ∂_2 are the partial derivatives wrt. the first and the second coordinate, respectively.
- Another possibility, for a triple f, g, h

$$\iint_{(x_1,y_1)}^{(x_2,y_2)} f d_1 g d_2 h := \iint_{(x_1,y_1)}^{(x_2,y_2)} f(x,y) \partial_1 g(x,y) \partial_2 h(x,y)\, dxdy$$

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2d exact increments

- Then

$$\iint_{(x_1,y_1)}^{(x_2,y_2)} dg = g(x_2,y_2) - g(x_1,y_2) - g(x_2,y_1) + g(x_1,y_1)$$
$$=: (\delta g)(x_1,y_1,x_2,y_2)$$

which identify the natural “two-dimensional” increment δg of a function g.

The complex of 2d increments

In complete analogy with the 1-d case we have:

- 2-d cochains $\mathcal{C}_{k,l}$ which are k-increments in the first direction and l increments in the second direction.
- 2-d coboundary map $\delta = \delta_1 \delta_2$ which is given by the successive application of 1-d coboundaries in the two directions.
- the complex

$$\mathcal{C}_{0,0} \xrightarrow{\delta} \mathcal{C}_{1,1} \xrightarrow{\delta} \mathcal{C}_{2,2} \xrightarrow{\delta} \mathcal{C}_{3,3} \xrightarrow{\delta} \cdots$$

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Expansion of 2d integrals

For a two-dimensional quantity we can write down the following expansion

$$\iint_{(x_1,y_1)}^{(x_2,y_2)} f(x,y)dg(x,y) = -f(x_1,y_1)(\delta g)(x_1,y_1,x_2,y_2)$$
$$+\int_{y_1}^{y_2} f(x_1,y)d_2[g(x_2,y)-g(x_1,y)] + \int_{x_1}^{x_2} f(x,y_1)d_1[g(x,y_2)-g(x,y_1)]$$
$$+\iint_{(x_1,y_1)}^{(x_2,y_2)} \left[\iint_{(x_1,y_1)}^{(x,y)} df(u,v)\right] dg(x,y)$$

We can set up a convenient notation in which this equation reads

$$\iint fdg = \underbrace{f\iint dg}_{f\delta g} + \underbrace{\int_1 f\int_2 dg + \int_2 f\int_1 dg}_{\text{boundary integrals}} + \underbrace{\iint dfdg}_{\text{remainder}}$$

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2d dissection

Our preferred exercise:

$$\iint \varphi(x)dx = -\varphi(x)\int dx + \int_1 \varphi(x)\int_2 dx + \int_2 \varphi(x)\int_1 dx + \iint d\varphi(x)dx$$

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Strategy to control the expansion

$$\iint \varphi'(x)dxdx = -\varphi'(x)\int dxdx + \int_1 \varphi'(x)\int_2 dxdx + \int_2 \varphi'(x)\int_1 dxdx + \iint d\varphi'(x)dxdx$$

This expression seems complicated, however it shows that, in order to control the l.h.s. we need two ingredients:

1. Being able to define essentially one-dimensional integrals like

$$\int_1 \varphi(x)\int_2 dx, \int_1 \varphi'(x)\int_2 dxdx, \int_1 \varphi''(x)\int_2 d_1xd_2dx, \ldots$$

2. Control the remainders given by the three-fold iterated integrals

$$\mathcal{R} := \iint d\varphi'(x)dxdx \qquad \tilde{\mathcal{R}} := \iint d\varphi''(x)d_1xd_2xdx.$$

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The boundary integrals (example)

$$\int_1 \varphi'(x) \int_2 dxdx = \varphi'(x) \iint dxdx - \Lambda_1 \left[\delta_1 \varphi'(x) \iint dxdx + \mathcal{C}_1 \right]$$

with

$$\mathcal{C}_1 := \varphi''(x) \int_1 d_1 x \int_2 dx \int_1 dx - (\Lambda_1 \otimes_1 1)(\mathcal{A}_1 + \mathcal{B}_1)$$

$$\mathcal{A}_1 := \delta_1 \varphi''(x) \int_1 d_1 x \int_2 dx \int_1 dx$$

and

$$\mathcal{B}_1 := \left(\delta_1 \varphi''(x) - \varphi''(x) \delta_1 x \right) \int_1 \int_2 dx \int_1 dx$$

The 2d rough sheet

To define integrals over a Brownian-like sheet x we need at least the following data:

$$\iint dx, \quad \iint d_1 x d_2 x, \quad \iint dx dx, \quad \iint d_1 x d_2 x dx,$$

$$\iint (d_1 x d_2 x)(d_1 x d_2 x), \quad \int_1 d_1 x \int_2 dx \int_1 dx,$$

and some others…

Summary

- Lyons' rough paths seems the tip of an iceberg.
- Flexibility of the approach (many kinds of "rough paths" but same structure)
- *Sometimes* algebra is useful (and interesting).

- Outlook
  - Effective approach to path-wise SPDEs (more in S. Tindel talk).
  - A lot of work to do to fully understand rough sheets.
  - Any use of the higher degrees $C_{k\geq 2}$ in the cochain complex?
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