Stochastic transport equation and non-Lipshitz SDEs

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The linear transport equation (classically)

Given $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ smooth vectorfield, $\overline{u}$ smooth. Consider the Cauchy problem in $\mathbb{R}_+ \times \mathbb{R}^d$

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) = 0 \\ u(0,x) = \overline{u}(x) \end{cases} \tag{1}$$

and the flow generated by b :

$$\begin{cases} \partial_t \Phi_{s,t}(x) = b(t, \Phi_{s,t}(x)) \\ \Phi_{s,s}(x) = x \end{cases}$$

Solutions to (1) are constant on the trajectories of b :

$$\frac{d}{dt} u(t, \Phi_{0,t}(x)) = \partial_t u(t, \Phi_{0,t}(x)) + \partial_t \Phi_{0,t}(x) \cdot \nabla u(t, \Phi_{0,t}(x)) = 0$$

Method of characteristics

The unique solution to (1) is $u(t,x) = \overline{u}(\Phi_{0,t}^{-1}(x))$.

Non-smooth vectorfields

Weak formulation

$$\begin{cases} \partial_t u + \operatorname{div}(bu) - (\operatorname{div} b)u = 0 \\ u(0,x) = \overline{u}(x) \end{cases}$$

Testing with smooth θ

$$\int \theta(x) u(t,x)dx = \int \theta(x)\overline{u}(x)dx$$
$$+ \int_0^t ds \int (u(s,x)b(s,x) \cdot \nabla\theta(x) + u(s,x)\theta(x)\operatorname{div} b(s,x))dx$$

- Existence of L^∞ weak solutions when $b \in L^p$, $\operatorname{div} b \in L^1_{\text{loc}}$ and $\overline{u} \in L^\infty$
- **[DiPerna-Lions]** Renormalized solutions: uniqueness and stability of L^∞ weak solutions when $b \in L^1(W^{1,p}) \cap L^\infty$ and $\operatorname{div} b \in L^\infty$
- **[Ambrosio]** Renormalized solutions for BV vectorfields
- Use the transport equation to select a flow Φ defined *almost everywhere*

SDEs with non-smooth coefficients

Idea:

Perturb the equation of characteristics by an additive Brownian noise acting on all components.

Why?

Consider the SDE in $\mathbb{R}^d$

$$dX_t = b(t, X_t)dt + dW_t, \qquad X_0 = x_0$$

- Strong solutions for b Lipshitz (+ linear growth) by fixed point method
- **[Veretennikov]** b bounded $\Rightarrow$ uniqueness of strong solutions
- **[Krylov-Röckner]** Strong uniqueness for b in Sobolev spaces
- **[Davie]** b bounded $\Rightarrow$ unique solution for a.e. Brownian path

$\Rightarrow$ *The noise regularizes the flow of the vectorfield b* $\Leftarrow$

Stochastic flow

To implement the method of characteristics we need information on *dependence on initial conditions*.

Definition

A *stochastic flow* is a family of maps $\{\Phi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d\}_{0 \leqslant s \leqslant t \leqslant T}$ such that

- $\Phi_{s,t}(x)$ is $\sigma(\{W_r - W_q\}_{s \leqslant q \leqslant r \leqslant t})$ measurable for any $x \in \mathbb{R}^d$, $0 \leqslant s \leqslant t \leqslant T$;
- $\lim_{t \to s+} \Phi_{s,t}(x) = x$, a.s. for any x, s, t;
- $\Phi_{u,t}(\Phi_{s,u}(x)) = \Phi_{s,t}(x)$

Theorem (Kunita)

If $b \in C^{1,\alpha}$ then there exists a $C^{1,\alpha'}$-stochastic flow $\Phi_{s,t}$ for any $\alpha' < \alpha$ solving the SDE

$$\Phi_{s,t}(x) = x + \int_s^t b(u, \Phi_{s,u}(x))du + W_t - W_s$$

for any $x \in \mathbb{R}^d$.

The Itô trick (I)

The regularization effect can be understood easily in the case $b(t,x) = b(x)$. Consider

$$X_t = x + \int_0^t b(X_s)ds + W_t$$

Try the *Itô trick*: interpret the integral over time as a correction in an Itô formula:

$$G(X_t) = G(x) + \int_0^t \nabla G(X_s)dW_s + \int_0^t LG(X_s)ds$$

with $L = \Delta/2 + b \cdot \nabla$. Assume that we can solve the elliptic problem

$$\lambda G - LG = b$$

for some $\lambda > 0$ (maybe very large), then

$$X_t + G(X_t) = x + G(x) + W_t + \int_0^t \nabla G(X_s)dW_s - \int_0^t \lambda G(X_s)ds$$

where G "has two derivatives more" than b. Setting $\psi(x) = x + G(x)$ we get

$$\psi(X_t) = \psi(x) + \int_0^t \nabla\psi(X_s)dW_s - \int_0^t \lambda G(X_s)ds$$

The Itô trick (II)

Theorem (Elliptic estimates)

For any $\epsilon > 0$, $\epsilon' < \epsilon$, $b \in C^{\epsilon}$, the elliptic equation $\lambda G - LG = b$ has a solution $G \in C^{2,\epsilon}$ for which $\|G\|_{2,\epsilon'} \to 0$ as $\lambda \to \infty$.

For λ large enough $\nabla\psi = 1 + \nabla G$ is invertible and ψ has inverse ψ^{-1}.
Let $Y_t = \psi(X_t)$, $y = \psi(x)$:

$$Y_t = y + \int_0^t \tilde{\sigma}(Y_s)dW_s + \int_0^t \tilde{b}(Y_s)ds$$

where $\tilde{\sigma}(y) = \nabla\psi \circ \psi^{-1}(y)$ and $\tilde{b}(y) = \lambda G \circ \psi^{-1}(y)$.

We have $\tilde{\sigma} \in C^{1,\epsilon'}$, $\tilde{b} \in C^{2,\epsilon'}$ and there exists a $C^{1,\epsilon'}$-stochastic flow φ solving

$$\varphi_{s,t}(y) = y + \int_s^t \tilde{\sigma}(\varphi_{s,u}(y))dW_u + \int_0^t \tilde{b}(\varphi_{s,u}(y))du$$

Stochastic flow for C^ϵ vectorfields

By letting $\phi_{s,t} = \psi^{-1} \circ \varphi_{s,t} \circ \psi$ we obtain a $C^{1,\epsilon'}$ stochastic flow satisfying

$$\phi_{s,t}(x) = x + \int_s^t b(\phi_{s,u}(x))du + W_t - W_s$$

- this flow is the unique strong solution to the SDE
- it does not depend on the choice of λ.
- we have an equation for $\nabla\phi_{s,t}(x)$:

$$\nabla\psi(\phi_{s,t}(x))\nabla\phi_{s,t}(x) = \nabla\psi(x) + \int_s^t \lambda\nabla G(\phi_{s,u}(x))\nabla\phi_{s,u}(x)du$$
$$+ \int_s^t \nabla^2\psi(\phi_{s,u}(x))\nabla\phi_{s,u}(x)dW_u$$

- by a stopping procedure we can assume b locally in C^ϵ (+ linear growth)

Push-forward

For smooth b we have

$$\int \theta(\phi_{s,t}(x))dx = \int \theta(x)\frac{dx}{J_{s,t}(x)}$$

where $J_{s,t}(x) = |\det \nabla\phi_{s,t}(x)|$ (Jacobian determinant) satisfy the differential equation

$$\frac{d}{dt}J_{s,t}(x) = \operatorname{div} b(\phi_{s,t}(x))\, J_{s,t}(x), \qquad J_{s,s}(x) = 1.$$

(the stochastic perturbation is solenoidal). Then

$$J_{s,t}(x) = \exp\left(\int_s^t \operatorname{div} b(\phi_{s,u}(x))du\right)$$

For $b \in C^\epsilon$ by an approximation procedure and another Itô trick we get

$$J_{s,t}(x) = \exp\left(\Gamma(\phi_{s,t}(x)) - \Gamma(x) + \int_s^t \nabla\Gamma(\phi_{s,u}(x))dW_u + \int_s^t \lambda\Gamma(\phi_{s,u}(x))du\right)$$

where $\Gamma \in C^{1,\epsilon'}$ solve $\lambda\Gamma - L\Gamma = \operatorname{div} b$ in the sense of distributions.

Stochastic transport equation

The simplest stochastic perturbation which is compatible with the method of characteristics leads to the Stratonovich SPDE

$$\begin{cases} d_t u_t + b_t \cdot \nabla u_t \, dt + \sum_{i=1}^{d} \nabla_i u_t \circ dW_t^i = 0 \\ u_0(x) = \overline{u}(x) \end{cases}$$

and to the related SDE for the flow of characteristics:

$$\begin{cases} d_t \Phi_{s,t}(x) = b(t, \Phi_{s,t}(x)) dt + dW_t \\ \Phi_{s,s}(x) = x \end{cases}$$

Euristically we must have again $u_t(x) = \overline{u}(\Phi_{0,t}^{-1}(x))$.

Assume that b is locally bounded and $\operatorname{div} b \in L^q_{\mathrm{loc}}$.

Definition

Given $\overline{u} \in L^p_{\mathrm{loc}}$, for some $p \geqslant 1$ a solution of the stochastic transport equation (STE) in L^p_{loc} is a measurable function $(u(t,x,\omega), t \geqslant 0, x \in \mathbb{R}^d, \omega \in \Omega)$ such that

(i) for P-a.e. $\omega \in \Omega$, $x \in \mathbb{R}^d$, $R > 0$, $\sup_{t\in[0,T]} \int_{B(x,R)} |u(t,x,\omega)|^p \, dx < \infty$

(ii) for any test function $\theta \in C^0_0(\mathbb{R}^d)$, the process $t \mapsto \int_{\mathbb{R}^d} u(t,x)\theta(x)dx$ is continuous and $\mathcal{F}_t$-adapted;

(iii) for any test function $\theta \in C^\infty_0(\mathbb{R}^d)$, the process $t \mapsto \int_{\mathbb{R}^d} u(t,x)\theta(x)dx$ is an $\mathcal{F}_t$-semimartingale satisfying

$$\int_{\mathbb{R}^d} u(t,x)\theta(x)dx = \int_{\mathbb{R}^d} \overline{u}(x)\theta(x)dx + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x) D_i\theta(x)dx \right) \circ dW^i_s$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} u(s,x)[b(x) \cdot \nabla\theta(x) + \operatorname{div} b(x)\theta(x)]dx$$

Main result

Theorem

Assume $b \in C^\epsilon$ and $\operatorname{div} b \in L^q$ and $\epsilon > d/q$. The STE has a unique solution u for any $\overline{u} \in L^p_{\text{loc}}$ and $u(t,x) = \overline{u}(\phi_{0,t}^{-1}(x))$.

Note that by the pushforward formula

$$\int_{\mathbb{R}^d} f(x) g \circ \phi_{s,t}(x) J_{s,t}(x) dx = \int_{\mathbb{R}^d} f \circ \phi_{s,t}^{-1}(x) g(x) dx$$

with $J_{s,t}(x) \leqslant C$ locally. So if $f \in L^p_{\text{loc}}, g \in L^q_{\text{loc}}$ we have $f \circ \phi_{s,t}^{-1} \in L^p_{\text{loc}}$ and

$$\int_A |f \circ \phi_{s,t}^{-1}(x)|^p dx = \int_{\phi_{s,t}^{-1}(A)} |f(x)|^p J_{s,t}(x) dx < \infty.$$

Existence

First we need to prove that $\int u(t,x)\theta(x)dx$ is a semimartingale.
Let $\phi_t = \phi_{0,t}$. Take a smooth test function θ, by Itô formula

$$\theta(\phi_t(y)) = \theta(y) + \int_0^t L^b\theta(\phi_s(y))ds + \int_0^t \nabla\theta(\phi_s(y))\cdot dW_s.$$

Let $J_t^\varepsilon(y)$ the Jacobian determinant of the flow ϕ_t^ε for the regularized vectorfield b^ε. Since b^ε is smooth: $dJ_t^\varepsilon(y) = \operatorname{div} b^\varepsilon(\phi_t(y))J_t^\varepsilon(y)dt$.
Then

$$\begin{aligned}\int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J_t^\varepsilon(y)dy &= \int_{\mathbb{R}^d} u_0(y)\theta(y)dy + \int_0^t ds\int_{\mathbb{R}^d} u_0(y)L^b\theta(\phi_s(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t ds\int_{\mathbb{R}^d} u_0(y)\theta(\phi_s(y))\operatorname{div} b^\varepsilon(\phi_s(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t dW_s\cdot\int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s(y))J_s^\varepsilon(y)dy\end{aligned}$$

In the limit $\varepsilon \to 0$ each term converges so

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J_t^\varepsilon(y)dy = \int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J_t(y)dy = \int_{\mathbb{R}^d} u(t,y)\theta(y)dy$$

is a semi-martingale.

Next we need to prove that the semimartingale $\int u(t,x)\theta(x)dx$ satisfy the stochastic transport equation.
By the Stratonovic-Itô formula

$$\theta(\phi_t(y)) = \theta(y) + \int_0^t b \cdot \nabla\theta(\phi_s(y))ds + \int_0^t \nabla\theta(\phi_s(y)) \circ dW_s.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^d} u_0(y)\theta(\phi_t(y))J_t^\varepsilon(y)dy &= \int_{\mathbb{R}^d} u_0(y)\theta(y)dy + \int_0^t ds \int_{\mathbb{R}^d} u_0(y)b \cdot \nabla\theta(\phi_s(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t ds \int_{\mathbb{R}^d} u_0(y)\theta(\phi_s(y))\operatorname{div} b^\varepsilon(\phi_s(y))J_s^\varepsilon(y)dy \\ &+ \int_0^t dW_s \circ \int_{\mathbb{R}^d} u_0(y)\nabla\theta(\phi_s(y))J_s^\varepsilon(y)dy\end{aligned}$$

and take the limit $\varepsilon \to 0$ to conclude.

Uniqueness

Goal

Prove that, if $u(t,x)$ solve the STE then we must have $u(t,x) = \overline{u}(\phi_t^{-1}(x))$.

We start by smoothing u. Define

$$u_\varepsilon(t,y) = \int_{\mathbb{R}^d} u(t,x)\vartheta_\varepsilon\,(y-x)\,dx, \quad u_{0,\varepsilon}(y) = \int_{\mathbb{R}^d} u_0(x)\vartheta_\varepsilon\,(y-x)\,dx.$$

Since u is a solution to STE we get

$$\begin{aligned} u_\varepsilon(t,y) = u_{0,\varepsilon}(y) &+ \int_0^t \left[\int_{\mathbb{R}^d} u(s,x)b(x)\cdot\nabla_x\vartheta_\varepsilon(y-x)dx\right] ds \\ &+ \int_0^t ds \int_{\mathbb{R}^d} u(s,x)\operatorname{div} b(x)\vartheta_\varepsilon(y-x)dx \\ &+ \sum_{i=1}^d \int_0^t \left[\int_{\mathbb{R}^d} u(s,x)D_{x_i}\vartheta_\varepsilon\,(y-x)\,dx\right] \circ dW_s^i \end{aligned}$$

Let $b^\delta = \vartheta_\delta * b$ and let ϕ^δ the associated flow.
By Stratonovich version of Itô-Wentzel calculus

$$\frac{d}{dt} u_\varepsilon(t, \phi_t^\delta(x)) = \left\{ \int u(t,x') \left[(b(x') - b^\delta(y)) \cdot \nabla_{x'} \vartheta_\varepsilon (y - x') + \operatorname{div} b(x') \vartheta_\varepsilon (y - x') \right] dx' \right\}$$

Test against $\rho \in C_0^\infty(\mathbb{R}^d)$ and perform a change of variables

$$\frac{d}{dt} \int_{\mathbb{R}^d} u_\varepsilon(t, \phi_t^\delta x) \rho(x) dx$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(t,x') \left[\left[b(x') - b^\delta(y) \right] \cdot \nabla_{x'} \vartheta_\varepsilon \left(y - x' \right) + \operatorname{div} b(x') \vartheta_\varepsilon (y - x') \right]_{y = \phi_t^\delta(x)} dx' \rho(x) dx$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(t,x') \left[\left[b(x') - b^\delta(y) \right] \cdot \nabla_{x'} \vartheta_\varepsilon \left(y - x' \right) + \operatorname{div} b(x') \vartheta_\varepsilon (y - x') \right] dx' \rho \left((\phi_t^\delta)^{-1}(y) \right)$$

By an integration by parts this is equal to

$$= \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \vartheta_\varepsilon \left(y - x' \right) \left[b(x') - b^\delta(y) \right] \cdot \nabla_y \left[\rho \left((\phi_t^\delta)^{-1}(y) \right) J_t^\delta(y) \right] dy \right] u(t,x') dx'$$
$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\operatorname{div} b(x') - \operatorname{div} b^\delta(y) \right] \vartheta_\varepsilon(y - x') \rho((\phi_t^\delta)^{-1}(y)) J_t^\delta(y) dy\, u(t,x') dx'$$

We want to show that both contributions go to zero as $\varepsilon \to 0$ and $\delta \to 0$

First term

$$A^{\delta} = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \vartheta_{\varepsilon}(y - x') \left[b(x') - b^{\delta}(y) \right] \cdot \nabla_y \left[\rho \left((\phi_t^{\delta})^{-1} y \right) J_t^{\delta}(y) \right] dy$$
$$= \left[b(x') - b^{\delta}(x') \right] \cdot \nabla_{x'} \left[\rho \left((\phi_t^{\delta})^{-1}(x') \right) J_t^{\delta}(x') \right]$$

We can prove that

$$|\nabla \left[\rho \left((\phi_t^{\delta})^{-1}(\cdot) \right) J_t^{\delta}(\cdot) \right] | \lesssim \delta^{\beta}$$

locally as $\delta \to 0$ for any $\beta < -d/q$. Moreover

$$|b - b^{\delta}| \lesssim \delta^{\epsilon}$$

so $|A_{\delta}| \lesssim \delta^{\epsilon+\beta} \to 0$ as soon as $\epsilon + \beta > 0$.

Second term

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\operatorname{div} b(x') - \operatorname{div} b^\delta(y)\right]\vartheta_\varepsilon(y-x')\rho((\phi_t^\delta)^{-1}(y))J_t^\delta(y)dy\,u(t,x')dx'$$

$$= \int_{\mathbb{R}^d}\operatorname{div} b(x')\left(\int_{\mathbb{R}^d}\vartheta_\varepsilon(y-x')\rho((\phi_t^\delta)^{-1}(y))J_t^\delta(y)dy\right)u(t,x')dx'$$

$$- \int_{\mathbb{R}^d}\operatorname{div} b^\delta(y)\rho((\phi_t^\delta)^{-1}(y))J_t^\delta(y)u_\varepsilon(t,y)dy$$

and both terms converge, as $\varepsilon \to 0$ followed by $\delta \to 0$ to

$$\int_{\mathbb{R}^d}\operatorname{div} b(y)\rho(\phi_t^{-1}(y))J_t(y)u(t,y)dy$$

so their difference converge to zero.

We obtained

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left[\int_{\mathbb{R}^d} u_\varepsilon(t,\phi_t^\delta x)\rho(x)\,dx - \int_{\mathbb{R}^d} u_\varepsilon(0,x)\rho(x)\,dx\right] = 0.$$

Now

$$\int_{\mathbb{R}^d} u_\varepsilon(t,\phi_t^\delta x)\rho(x)\,dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_\varepsilon(t,y)\vartheta_\varepsilon(\phi_t^\delta(x)-y)\rho(x)\,dxdy$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u_\varepsilon(t,y)\vartheta_\varepsilon(z-y)\rho\left((\phi_t^\delta)^{-1}(z)\right) J_t^\delta((\phi_t^\delta)^{-1}(z))^{-1}dzdy$$

$$\to \int_{\mathbb{R}^d} u(t,z)\rho\left(\phi_t^{-1}(z)\right) J_t(\phi_t^{-1}(z))^{-1}dz$$

This yields

$$\int_{\mathbb{R}^d} u(t,z)\rho\left(\phi_t^{-1}(z)\right) J_t(\phi_t^{-1}(z))^{-1}dz = \int_{\mathbb{R}^d} u(0,x)\rho(x)\,dx$$

for every $\rho(x) \in C_0^\infty(\mathbb{R}^d)$. Choosing ρ appropriately we get

$$\int_{\mathbb{R}^d} u(t,z)\rho(z)dz = \int_{\mathbb{R}^d} u(0,x)\rho(\phi_t(x))J_t(x)dx = \int_{\mathbb{R}^d} u(0,\phi_t^{-1}(y))\rho(y)dy.$$

Counterexamples to certain extensions

Example (Random vectorfields)

Take $b(t,x) = \sqrt{|x - W_t|}$, then

$$dX_t = b(t, X_t)dt + dW_t = \sqrt{|X_t - W_t|}dt + dW_t.$$

By the change of variables $Y_t = X_t - W_t$ we obtain

$$dY_t = \sqrt{|Y_t|}dt$$

so path-wise uniqueness is impossible in general.

Not so artificial...

Consider a 2d stochastic Euler equation in vorticity variables

$$\partial_t \xi (t,x) + (u (t,x) \cdot \nabla \xi (t,x))\, dt + \nabla \xi (t,x) \circ dW (t) = 0$$

where $\xi = \partial_2 u_1 - \partial_1 u_2$.
Formally equivalent to the "system" of stochastic ordinary equations

$$dX_t^a = \left[\int_{\mathbb{R}^2} K(X_t^a - X_t^{a'}) \xi_0(X_t^{a'}) da' \right] dt + dW_t, \qquad a \in \mathbb{R}^2$$

for a suitable kernel K, ξ_0 being the initial condition of the vorticity equation.
By the change of variable $Y_t^a = X_t^a - W_t$ we obtain

$$dY_t^a = \left[\int_{\mathbb{R}^2} K(Y_t^a - Y_t^{a'}) \xi_0(X_t^{a'}) da' \right] dt$$

The equation for (Y_t^a) corresponds to the classical vorticity equation

$$\frac{\partial_t \xi' (t,x)}{\partial t} + (u' (t,x) \cdot \nabla \xi' (t,x))\, dt = 0 \qquad \xi' = \partial_2 u_1' - \partial_1 u_2'$$

with initial condition ξ_0.

Possible way out

Consider a more complex (infinite-dimensional) noise:

$$dX_t^a = \left[\int_{\mathbb{R}^2} K(X_t^a - X_t^{a'})\xi_0(X_t^{a'})da'\right] dt + \sum_{k=1}^{\infty} \sigma_k(X_t^a)dW_t^k, \qquad a \in \mathbb{R}^2$$

where each point X_a is moved "almost" independently of the others.
Seems useful to require

$$\sum_{k=1}^{\infty} \sigma_k(x)\sigma_k(y) = a(|x-y|)$$

with $a(r) \simeq r^\alpha$ as $r \to 0$, $\alpha > 0$. This in order to hope some regularizing effect of the noise over the deterministic (and singular) drift.
Connection with the theory of stochastic flows of Le Jan-Raimond.