Some infinite dimensionals rough-paths

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Rough paths

- T. Lyons (Oxford): an integration theory for irregular signals.
- Nonlinear systems y_t driven by a (non-differentiable) noise x_t

$$dy = f(y)dx$$

- The output y is a nice function of the iterated integrals of x:

$$\left(x, \int dxdx, \cdots, \int dx^{\otimes n} \right) \xrightarrow{\Phi} y$$

We can consider only a finite number of them. No need of formal series.

Algebraically

The increment $\delta y_{ts} = y_t - y_s$ of the solution of the integral equation $y = \int f(y) dx$ has a natural expansion

$$\delta y_{ts} = \int_s^t f(y_u) dx_u = f(y_s) \int_s^t dx_u + f(y_s) f'(y_s) \int_s^t \int_s^u dx_w dx_u + \text{h.o.t.}$$

If we neglect the h.o.t. it belongs to the linear span of the iterated integrals

$$X^1_{ts} = x_t - x_s \quad \ldots \quad X^n_{ts} = \int_s^t X^{n-1}_{us} dx_u$$

and we write it as $[\ (fX)_{ts} = f_s X_{ts}\ ]$

$$\delta y = y^1 X^1 + y^2 X^2 + \cdots$$

where $y^1_t = f(y_t)$, $y^2_t = f(y_t) f'(y_t)$, and so forth.

The solution of the ODE is identified with the fixed point of

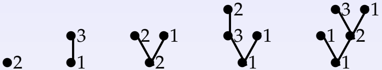
$$y \xrightarrow{\substack{\text{nonlinear}\\ \text{map}}} f(y) \xrightarrow{\text{integration}} I(f(y))$$

for ys whose increments can be (partially) expanded on the $\{X^n\}_n$

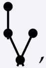
Phenomenology of rough-paths

Trees

$\mathcal{L}$ finite set. Trees labeled by $\mathcal{L}$, $\mathcal{T}_{\mathcal{L}}$



$$(\tau_1, \cdots, \tau_k) \overset{B_+^a}{\mapsto} \tau = [\tau_1, \cdots, \tau_k]_a$$

$[\bullet] =$  $\quad [\bullet, [\bullet]] =$ , $\quad$ etc...

$$\Delta(\tau) = 1 \otimes \tau + \sum_{a \in \mathcal{L}} (B_+^a \otimes \mathrm{id})[\Delta(B_-^a(\tau))]$$

$$B_-^a(B_+^b(\tau_1 \cdots \tau_n)) = \begin{cases} \tau_1 \cdots \tau_n & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

Differential equations (*à la Butcher*)

The solution y of the differential equation

$$dy = f(y)dt, \qquad y_0 = \eta$$

has the B-series representation

$$y_t = \eta + \sum_{\tau \in \mathcal{T}} \psi^f(\tau)(\eta) \frac{t^{|\tau|}}{\sigma(\tau)\tau!}$$

Elementary differentials ψ^f defined as

$$\psi^f(\bullet)(\xi) = f(\xi), \qquad \psi^f([\tau^1 \cdots \tau^k]) = f_{\bar{b}}(\eta)\psi^f(\tau^1)(\xi)^{b_1} \cdots \psi^f(\tau^k)(\xi)^{b_k}$$

where $f_\emptyset(\xi) = f(\xi)$ and $f_{\bar{b}}(\xi) = \prod_{i=1}^{|\bar{b}|} \partial_{\xi_{b_i}} f(\xi)$ derivatives of the vectorfields.

Driven differential equations

Given a collection of paths $\{x^a \in C^1([0,T],\mathbb{R})\}_{a\in\mathcal{L}}$, $\eta \in \mathbb{R}^n$
Analytic vectorfields $\{f_a : \mathbb{R}^n \to \mathbb{R}^n\}_{a\in\mathcal{L}}$

Theorem

The differential equation

$$dy_t = f_a(y_t)dx_t^a, \qquad y_0 = \eta$$

admit locally the series solution

$$y_t = y_s + \sum_{\tau\in\mathcal{T}_{\mathcal{L}}} \frac{1}{\sigma(\tau)} \phi^f(\tau)(y_s) X_{ts}^{\tau}, \qquad y_0 = \eta$$

where $\phi^f(\bullet_a)(\xi) = f_a(\xi)$, $\phi^f([\tau^1 \cdots \tau^k]_a)(\xi) = f_{a;b_1\ldots b_k}(\xi) \prod_{i=1}^k [\phi^f(\tau^i)(\xi)]^{b_i}$.

Smooth iterated integrals

Let $X : \mathcal{T}_{\mathcal{L}} \to \mathcal{C}_2 \subset C([0,T]^2; \mathbb{R})$

$$X^{\bullet_a}_{ts} = \int_s^t dx^a_u, \qquad X^{[\tau^1 \cdots \tau^k]_a}_{ts} = \int_s^t \prod_{i=1}^k X^{\tau^i}_{us} dx^a_u. \tag{1}$$

Extend X to $\mathcal{A}\mathcal{T}_{\mathcal{L}}$ considering $\mathcal{C}_2$ as an algebra with (commutative) product $(a \circ b)_{ts} = a_{ts} b_{ts}$ for $a, b \in \mathcal{C}_2$. We let $X^1 = 1$.

$$X^{\tau_1 \cdots \tau_n}_{ts} = X^{\tau_1}_{ts} X^{\tau_2}_{ts} \cdots X^{\tau_n}_{ts}, \qquad X^{[\tau_1 \cdots \tau_n]_a} = \int X^{\tau_1 \cdots \tau_n} dx^a$$

Bounds

$$|X^{\tau}_{ts}| \leqslant \frac{(A|t-s|)^{|\tau|}}{\tau!}$$

Theorem (Tree multiplicative property)

$$X^{\tau}_{ts} = \sum X^{\tau^{(1)}}_{tu} X^{\tau^{(2)}}_{us} = X^{\Delta\tau}_{tus}$$

Example

$$T^{\bullet}_{ts} = t - s, \qquad T^{[\tau_1 \cdots \tau_n]}_{ts} = \int_s^t T^{\tau_1}_{us} \cdots T^{\tau_n}_{us} du$$

By induction: $T^{\tau}_{ts} = (t-s)^{|\tau|}(\tau!)^{-1}$

Lemma (Tree Binomial)

For every $\tau \in \mathcal{T}$ *and* $a, b \geqslant 0$ *we have*

$$(a+b)^{|\tau|} = \sum_i \frac{\tau!}{\tau_i^{(1)}!\tau_i^{(2)}!} a^{|\tau_i^{(1)}|} b^{|\tau_i^{(2)}|} \tag{2}$$

Structure of solution to DDEs

Write $y_s^\tau = \phi^f(\tau)(y_s)/\sigma(\tau)$ so that

$$y_t - y_s = \sum_{\tau \in \mathcal{T}_\mathcal{L}} X_{ts}^\tau y_s^\tau$$

Lemma

For any $\tau \in \mathcal{T}_\mathcal{L} \cup \{\emptyset\}$ *we have*

$$y_t^\tau - y_s^\tau = \sum_{\sigma \in \mathcal{T}_\mathcal{L}, \rho \in \mathcal{F}_\mathcal{L}} c'(\sigma, \tau, \rho) X_{ts}^\rho y_s^\sigma$$

c' counting function of reduced coproduct: $\Delta'\sigma = \sum_{\tau,\rho} c'(\sigma, \tau, \rho)\tau \otimes \rho$.

Integration of increments

- Q: Given $a \in \mathcal{C}_2$ can we find $f \in \mathcal{C}_1$ such that $f_t - f_s = a_{ts}$?
  $a_{ts} - a_{tu} - a_{us} = (f_t - f_s) - (f_t - f_u) - (f_u - f_s) = 0$ (obstruction)
- Increments: $\mathcal{C}_n \subset C([0,1]^n, V)$, $g \in \mathcal{C}_n$ iff $g_{t_1 \cdots t_n} = 0$ when $t_i = t_{i+1}$
  Coboundary: $\delta f_{ts} = f_t - f_s$, $\delta g_{tus} = g_{ts} - g_{tu} - g_{us}, \ldots, \delta^2 = 0$

  $$0 \to \mathbb{R} \to \mathcal{C}_1 \xrightarrow{\delta} \mathcal{C}_2 \xrightarrow{\delta} \mathcal{C}_3 \xrightarrow{\delta} \cdots$$

  Then $\delta f = a \Leftrightarrow \delta a = 0$.
- Small 2-increments cannot be exact: $a_{ts} = o(|t-s|) \Rightarrow a \neq \delta f$
- Unique decomposition: $a = \delta f + o(|t-s|)$?
  Yes, if obstruction δa is small:

Theorem

If $\delta a_{tus} = o(|t-s|)$, then $\exists!\, f \in \mathcal{C}_1, r \in \mathcal{C}_2^{1+}$ such that

$$a = \delta f + r, \qquad \delta f = (1 - \Lambda\delta)a$$

Examples

- Convergence of sums:

$$S_{t0} = \sum_i a_{t_{i+1}t_i} = \sum_i (\delta f)_{t_{i+1}t_i} + \sum_i (r)_{t_{i+1}t_i} = (\delta f)_{t0} + \sum_i o(|t_{i+1} - t_i|) \to f_t - f_0$$

- Young integrals: $x \in C^\gamma, \gamma > 1/2$ $a_{ts} = \varphi(x_s)\delta x_{ts}$

$$\delta a_{tus} = \delta\varphi(x)_{tu}\delta x_{us} = o(|t-s|^{2\gamma}) \Rightarrow \delta f = (1 - \Lambda\delta)a =: \int \varphi(x)dx$$

- NCG & Λ map. $L^2(\mathbb{R})$, $dg = [F,g]$, $F^2 = c$,

$$(df)_{ts} = \frac{f_t - f_s}{t-s} \qquad \int f dg = \mathrm{Tr}_\omega(fdg) = \frac{1}{2c}\mathrm{Tr}_\omega(Fdfdg)$$

so

$$\Lambda(\delta f \delta g)_{-\infty,\infty} = -\frac{1}{2c}\mathrm{Tr}_\omega(Fdfdg)$$

(Step-2) Rough paths

Rough integrals: $X^\bullet = \delta x$, $\delta X^{[\bullet]} = X^\bullet X^\bullet$, $X^\bullet \in \mathcal{C}_2^\gamma$, $X^{[\bullet]} \in \mathcal{C}_2^{2\gamma}$ $(\gamma > 1/3)$

$$\oint \varphi(x)dx = (1 - \Lambda\delta)(\underbrace{\varphi(x)X^\bullet + \varphi'(x)X^{[\bullet]}}_{\delta(\cdot)\in\mathcal{C}_3^{3\gamma>1}}),$$

$$\delta(\varphi(x)X^\bullet + \varphi'(x)X^{[\bullet]}) = (-\delta\varphi(x) + \varphi'(x)X^\bullet)X^\bullet - \delta\varphi'(x)X^{[\bullet]}$$

- Continuous map: $(\varphi, X^\bullet, X^{[\bullet]}) \mapsto \oint \varphi(x)dx$
- Renormalized sums: $\sum_i(\varphi(x_{t_i})X^\bullet_{t_{i+1}t_i} + \varphi'(x_{t_i})X^{[\bullet]}_{t_{i+1}t_i}) \to \oint \varphi(x)dx$
- $\oint dxdx = X^{[\bullet]}$
- A finite number of iterated integrals determines all the other integrals.

Branched rough paths

The only data we need to build the family $\{X^\tau\}_{\tau \in \mathcal{T}_\mathcal{L}}$ is a family of maps $\{I^a\}_{a \in \mathcal{L}}$ from $\mathcal{C}_2$ to $\mathcal{C}_2$ satisfying certain properties.

Definition

An *integral* is a linear map $I : \mathcal{D}_I \to \mathcal{D}_I$ on an unital sub-algebra $\mathcal{D}_I \subset \mathcal{C}_2^+$ for which $I(hf) = I(h)f, \forall h \in \mathcal{D}_I, f \in \mathcal{C}_1$ and

$$\delta I(h) = I(e)h + \sum_i I(h^{1,i})h^{2,i} \qquad \text{when } h \in \mathcal{D}_I,\ \delta h = \sum_i h^{1,i}h^{2,i} \text{ and } h^{1,i} \in \mathcal{D}_I$$

Then $X^{\bullet_a} = I^a(e)$, $\quad X^{[\tau^1 \cdots \tau^k]_a} = I^a(X^{\tau^1 \cdots \tau^k})$, $\quad X^{\tau^1 \cdots \tau^k} = X^{\tau^1} \circ \cdots \circ X^{\tau^k}$.

Tree multiplicative property still holds: $\delta X^\tau = X^{\Delta' \tau}$.

Geometric rough paths

Integrals are not necessarily Rota-Baxter maps: e.g. ItÃŹ stochastic integral

$$\int_0^t f_s dx_s \int_0^t g_s dx_s = \int_0^t f_s \int_0^s g_u dx_u dx_s + \int_0^t g_s \int_0^s f_u dx_u dx_s + \int_0^t f_s g_s ds$$

$$I(f)I(g) = I(I(f)g) + I(fI(g)) + J(fg), \qquad J(f)I(g) = J(fI(g)) + I(gJ(f))$$

Solution to $dy = ydx$, $y_0 = 1$: $y_t = \exp(x_t - t/2)$.
When they are Rota-Baxter we have shuffle relations:

$$I^{a_1}(\cdots I^{a_n}(1)) \circ I^{b_1}(\cdots I^{b_m}(1)) = \sum_{\overline{c} \in \mathrm{Sh}(\overline{a}, \overline{b})} I^{c_1}(\cdots I^{c_{n+m}}(1)) \tag{3}$$

This relation reduces X^τ for $\tau \in \mathcal{T}_\mathcal{L}$ to a linear combination of $\{X^\sigma\}_{\sigma \in \mathcal{T}_\mathcal{L}^{\text{Chen}}}$.
These are *geometric rough-paths*: the closure of smooth rough paths.

Growing a branched rough path

Fix $\gamma \in (0,1]$, consider $q_\gamma : \mathcal{F} \to \mathbb{R}_+$ on forests as $q_\gamma(\tau) = 1$ for $|\tau| \leqslant 1/\gamma$ and

$$q_\gamma(\tau) = 1, \text{ if } |\tau| \leqslant 1/\gamma \qquad q_\gamma(\tau) = \frac{1}{2^{\gamma|\tau|}} \sum q_\gamma(\tau^{(1)}) q_\gamma(\tau^{(2)}) \text{ otherwise}$$

$q_\gamma(\tau_1 \cdots \tau_n) = q_\gamma(\tau_1) \cdots q_\gamma(\tau_n)$.

Theorem

Given a partial homomorphism $X : \mathcal{A}_n \mathcal{T}_{\mathcal{L}} \to \mathcal{C}_2$ satisfying the multiplicative property

$$|X^\tau_{ts}| \leqslant B A^{|\tau|} q_\gamma(\tau) |t-s|^{\gamma|\tau|}, \qquad \tau \in \mathcal{T}^n_{\mathcal{L}} \tag{4}$$

with $\gamma(n+1) > 1$, then $\exists!\, X : \mathcal{A}\mathcal{T}_{\mathcal{L}} \to \mathcal{C}_2$ such that eq. (4) holds $\forall \tau \in \mathcal{T}_{\mathcal{L}}$.

Construction via the equation: $X^\tau = \Lambda(X^{\Delta' \tau})$.

Speed of growth

Conjecture

$$q_\gamma(\tau) \asymp C(\tau!)^{-\gamma}$$

True for linear Chen trees $\mathcal{T}^{\text{Chen}}$:

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(k!)^\gamma (n!)^\gamma} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(n!)^\gamma}, \qquad \gamma \in (0,1], \ \ a,b \geqslant 0$$

Variant of Lyons' neo-classical inequality

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(\gamma k)![\gamma(n-k)n]!} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(\gamma n)!}$$

"neo-classical tree inequality"?

$$\sum \frac{a^{\gamma|\tau^{(1)}|} b^{\gamma|\tau^{(2)}|}}{(\tau^{(1)}!)^\gamma (\tau^{(2)}!)^\gamma} \leqslant c_\gamma \frac{(a+b)^{\gamma|\tau|}}{(\tau!)^\gamma}$$

OK for $\gamma = 1$: tree binomial formula.

Controlled paths

Definition

Let n the largest integer such that $n\gamma \leqslant 1$. For any $\kappa \in (1/(n+1), \gamma]$ a path y is a κ-*weakly controlled* by X if

$$\delta y = \sum_{\tau \in \mathcal{F}_{\mathcal{L}}^{n-1}} X^{\tau} y^{\tau} + y^{\sharp}, \qquad \delta y^{\tau} = \sum_{\sigma \in \mathcal{F}_{\mathcal{L}}^{n-1}} \sum_{\rho} c'(\sigma,\tau,\rho) X^{\rho} y^{\sigma} + y^{\tau,\sharp}, \qquad \tau \in \mathcal{F}_{\mathcal{L}}^{n-1}$$

with $y^{\tau} \in \mathcal{C}_2^{|\tau|\kappa}, y^{\sharp,\tau} \in \mathcal{C}_2^{(n-|\tau|)\kappa}$. Then we write $y \in \mathcal{Q}_{\kappa}(X;V)$.

Lemma (Stability)

Let $\varphi \in C_b^n(\mathbb{R}^k, \mathbb{R})$ *and* $y \in \mathcal{Q}_{\kappa}(X;\mathbb{R}^k)$, *then* $z_t = \varphi(y_t)$ *is a weakly controlled path,* $z \in \mathcal{Q}_{\kappa}(X;\mathbb{R})$ *where its coefficients are given by*

$$z^{\tau} = \sum_{m=1}^{n-1} \sum_{|\bar{b}|=m} \frac{\varphi_{\bar{b}}(y)}{m!} \sum_{\substack{\tau_1,\dots,\tau_m \in \mathcal{F}_{\mathcal{L}}^{n-1} \\ \tau_1 \cdots \tau_m = \tau}} y^{\tau_1,b_1} \cdots y^{\tau_m,b_m}, \qquad \tau \in \mathcal{F}_{\mathcal{L}}^{n-1}$$

Integration of controlled paths

Theorem

The integral maps $\{I^a\}_{a\in\mathcal{L}}$ *can be extended to maps* $I^a : \mathcal{Q}_\kappa(X) \to \delta\mathcal{Q}_\kappa(X)$

$$y \in \mathcal{Q}_\kappa(X) \mapsto \delta z = I^a(y) = X^{\bullet_a} z^{\bullet_a} + \sum_{\tau\in\mathcal{T}^n_{\mathcal{L}}} X^\tau z^\tau + z^\flat, \tag{5}$$

where $z^\flat \in \mathcal{C}_2^{\kappa(n+1)}$, $z^{\bullet_a} = y$, $z^{[\tau]_a} = y^\tau$ *and zero otherwise.*

Remark

If $y \in \mathcal{Q}_\kappa(X; \mathbb{R}^n \otimes \mathbb{R}^d)$ then $\{J^b(\cdot) = \sum_{a\in\mathcal{L}} I^a(y^{ab}\cdot)\}_{b\in\mathcal{L}_1}$ defines a family of integrals with an associated branched rough path Y indexed by $\mathcal{T}_{\mathcal{L}_1}$. An explicit recursion is

$$Y^{\bullet_b} = \sum_{a\in\mathcal{L}} I^a(y^{ab}), \qquad Y^{[\tau^1\cdots\tau^k]_b} = \sum_{a\in\mathcal{L}} I^a(y^{ab} Y^{\tau^1} \circ \cdots \circ Y^{\tau_k}), \qquad b \in \mathcal{L}_1$$

Example

$$\delta y = X^{\bullet} y^{\bullet} + X^{[\bullet]} y^{[\bullet]} + X^{\bullet\bullet} y^{\bullet\bullet} + X^{[\bullet\bullet]} y^{[\bullet\bullet]} + X^{[\bullet]\bullet} y^{[\bullet]\bullet} + X^{[\bullet\bullet]} y^{[\bullet\bullet]} + X^{\bullet\bullet\bullet} y^{\bullet\bullet\bullet} + X^{[[\bullet]]} y^{[[\bullet]]} + y^{\sharp}$$

$$\delta y^{\bullet} = X^{\bullet}(y^{[\bullet]} + 2y^{\bullet\bullet}) + X^{[\bullet]}(y^{[[\bullet]]} + y^{[\bullet]\bullet}) + X^{\bullet\bullet}(y^{[\bullet]\bullet} + y^{[\bullet\bullet]} + 3y^{\bullet\bullet\bullet}) + y^{\bullet,\sharp}$$

$$\delta y^{[\bullet]} = X^{\bullet}(y^{[\bullet]\bullet} + 2y^{[\bullet\bullet]} + y^{[[\bullet]]}) + y^{[\bullet],\sharp}$$

$$\delta y^{\bullet\bullet} = X^{\bullet}(y^{[\bullet]\bullet} + y^{\bullet\bullet\bullet}) + y^{\bullet\bullet,\sharp}$$

$$\delta y^{[\bullet\bullet]} = y^{[\bullet\bullet],\sharp} \qquad \delta y^{[\bullet]\bullet} = y^{[\bullet]\bullet,\sharp} \qquad \delta y^{\bullet\bullet\bullet} = y^{\bullet\bullet\bullet,\sharp} \qquad \delta y^{[[\bullet]]} = y^{[[\bullet]],\sharp}$$

$$\begin{aligned}\delta z = \delta I(y) &= X^{\bullet} y + X^{[\bullet]} y^{\bullet} + X^{[[\bullet]]} y^{[\bullet]} + X^{[\bullet\bullet]} y^{\bullet\bullet} + X^{[\bullet[\bullet]]} y^{[\bullet]\bullet} + X^{[[\bullet\bullet]]} y^{[\bullet\bullet]} + X^{[\bullet\bullet\bullet]} y^{\bullet\bullet\bullet} + X^{[[[\bullet]]]} y^{[[\bullet]]} + z^{\flat} \\ &= X^{\bullet} z^{\bullet} + X^{[\bullet]} z^{[\bullet]} + X^{[[\bullet]]} z^{[[\bullet]]} + X^{[\bullet\bullet]} z^{[\bullet\bullet]} + z^{\sharp}\end{aligned}$$

with

$$z^{\flat} = \Lambda\left[X^{\bullet} y^{\sharp} + X^{[\bullet]} y^{\bullet,\sharp} + X^{[[\bullet]]} y^{[\bullet],\sharp} + X^{[\bullet\bullet]} y^{\bullet\bullet,\sharp} + X^{[[\bullet\bullet]]} y^{[\bullet\bullet],\sharp} + X^{[\bullet[\bullet]]} y^{[\bullet]\bullet,\sharp} + X^{[\bullet\bullet\bullet]} y^{\bullet\bullet\bullet,\sharp}\right].$$

Rough differential equations

Take vectorfields $\{f_a \in C_b^n(\mathbb{R}^k;\mathbb{R}^k)\}_{a\in\mathcal{L}}$ and integral maps $\{I^a\}_{a\in\mathcal{L}}$ and consider the *rough differential equation*

$$\delta y = I^a(f_a(y)), \qquad y_0 = \eta \in \mathbb{R}^k \tag{6}$$

in the time interval $[0, T]$.

Theorem

The rough differential equation (6) has a global solution $y \in \mathcal{Q}_\gamma(X;\mathbb{R}^k)$ for any initial condition $\eta \in \mathbb{R}^k$. If the vectorfields are C_b^{n+1} the solution is unique and has Lipshitz dependence on data.

The KdV equation

1d periodic KdV equation:

$$\partial_t u(t,\xi) + \partial_\xi^3 u(t,\xi) + \frac{1}{2}\partial_\xi u(t,\xi)^2 = 0, \quad u(0,\xi) = u_0(\xi), \qquad (t,\xi) \in \mathbb{R} \times \mathbb{T}$$

where initial condition $u_0 \in H^\alpha(\mathbb{T})$, $\mathbb{T} = [-\pi, \pi]$. Linear part: Airy group $U(t)$ (isometries on H^α). Go to Fourier variables and let $v_t = U(-t)u_t$:

$$v_t(k) = v_0(k) + \frac{ik}{2} \sum_{k_1}' \int_0^t e^{-i3kk_1k_2 s} v_s(k_1) v_s(k_2)\, ds, \quad t \in [0,T], k \in \mathbb{Z}_*$$

where $k_2 = k - k_1$ and $v_0(k) = u_0(k)$. Restrict to $v_0(0) = 0$. It has the form

$$v_t = v_s + \int_s^t \dot{X}_\sigma(v_\sigma, v_\sigma) d\sigma, \qquad t,s \in [0,T].$$

where $\dot{X}_\sigma(\varphi,\varphi) = \frac{ik}{2} \sum_{k_1}' e^{-i3kk_1k_2\sigma} \varphi(k_1)\varphi(k_2)$.

The KdV equation

Expansion

$$\delta v_{ts} = X^{\bullet}(v^{\times 2}) + X^{[\bullet]}(v^{\times 3}) + X^{[[\bullet]]}(v^{\times 4}) + X^{[\bullet\bullet]}(v^{\times 4}) + r \tag{7}$$

with multi-linear operators X^{τ}:

$$X^{\bullet}_{ts}(\varphi_1, \varphi_2) = \int_s^t \dot{X}_\sigma(\varphi_1, \varphi_2) d\sigma;$$

$$X^{[\tau^1]}_{ts}(\varphi_1, \dots, \varphi_{m+1}) = \int_s^t \dot{X}_\sigma(X^{\tau^1}_{\sigma s}(\varphi_1, \dots, \varphi_m), \varphi_{m+1}) d\sigma$$

and

$$X^{[\tau^1 \tau^2]}_{ts}(\varphi_1, \dots, \varphi_{m+n}) = \int_s^t \dot{X}_\sigma(X^{\tau^1}_{\sigma s}(\varphi_1, \dots, \varphi_m), X^{\tau^2}_{\sigma s}(\varphi_{m+1}, \dots, \varphi_{m+n})) d\sigma.$$

Eq.7 is a rough equation which can be solved with fixed-point:

$$\delta v = (1 - \Lambda\delta)[X^{\bullet}(v^{\times 2}) + X^{[\bullet]}(v^{\times 3})]$$

Shadows of the conservation law

Lemma

$$\langle \varphi_1, \dot{X}_s(\varphi_2, \varphi_3)\rangle + \langle \varphi_2, \dot{X}_s(\varphi_1, \varphi_3)\rangle + \langle \varphi_3, \dot{X}_s(\varphi_2, \varphi_1)\rangle = 0, \qquad s \in [0, T]$$

$$\langle \varphi, X_{ts}(\varphi, \varphi)\rangle = 0 \qquad 2\langle \varphi, X^2_{ts}(\varphi, \varphi, \varphi)\rangle + \langle X_{ts}(\varphi, \varphi), X_{ts}(\varphi, \varphi)\rangle = 0$$

$$\begin{aligned}[\delta\langle v, v\rangle]_{ts} &= 2\langle X_{ts}(v_s, v_s) + X^2_{ts}(v_s, v_s, v_s) + v^\flat_{ts}, v_s\rangle \\ &\quad + \langle X_{ts}(v_s, v_s), X_{ts}(v_s, v_s)\rangle + 2\langle X_{ts}(v_s, v_s), v^\sharp_{ts}\rangle + \langle v^\sharp_{ts}, v^\sharp_{ts}\rangle \\ &= 2\langle v^\flat_{ts}, v_s\rangle + 2\langle X_{ts}(v_s, v_s), v^\sharp_{ts}\rangle + \langle v^\sharp_{ts}, v^\sharp_{ts}\rangle = O(|t-s|^{3\gamma})\end{aligned}$$

Theorem (Integral conservation law)

If v is a solution of KdV then $|v_t|_0^2 = |v_0|_0^2$ for any t.

The NS equation

The d-dimensional NS equation (or the Burgers' equation) have the abstract form

$$u_t = S_t u_0 + \int_0^t S_{t-s} B(u_s, u_s)\, ds. \tag{8}$$

S bounded semi-group on $\mathcal{B}$, B symmetric bilinear operator.
Define $d(\tau)$-multilinear operator by

$$X_{ts}^{\bullet}(\varphi^{\times 2}) = \int_s^t S_{t-u} B(S_{u-s}\varphi, S_{u-s}\varphi) du$$

$$X_{ts}^{[\tau^1]}(\varphi^{\times (d(\tau^1)+1)}) = \int_s^t S_{t-u} B(X_{us}^{\tau^1}(\varphi^{\times d(\tau^1)}), \varphi) du$$

and

$$X_{ts}^{[\tau^1 \tau^2]}(\varphi^{\times (d(\tau^1)+d(\tau^2))}) = \int_s^t S_{t-u} B(X_{us}^{\tau^1}(\varphi^{\times d(\tau^1)}), X_{us}^{\tau^2}(\varphi^{\times d(\tau^2)})) du$$

where $d(\tau)$ is an appropriate degree function.

Bounds on the operators and regularity

The X operators allow bounds in $\mathcal{B}$ of the form

$$|X^{\tau}(\varphi^{\times d(\tau)})|_{\mathcal{B}} \leqslant C\frac{|t-s|^{\varepsilon|\tau|}}{(\tau!)^{\varepsilon}}|\varphi|_{\mathcal{B}}^{d(\tau)}$$

where $\varepsilon \geqslant 0$ is a constant depending on the particular Banach space $\mathcal{B}$ we choose.
We have the (norm convergent) series representation

$$u_t = S_t u_0 + \sum_{\tau \in \mathcal{T}_B} X_{t0}^{\tau}(u_0^{\times d(\tau)}) \tag{9}$$

which gives local solutions to NS.
Regularity: $|u(k)| \leqslant Ce^{-|k|\sqrt{t}}$ by controlling growth of the terms in the series.

Convolution integrals

- A cochain complex $(\hat{C}_*, \hat{\delta})$ adapted to the study of convolution integrals.
- Coboundary $\tilde{\delta}h = \delta h - ah - ha$ with $a_{ts} = S_{t-s} - \mathrm{Id}$ the 2-increment associated to the semi-group (parallel transport).
- Associated integration theory ($\tilde{\Lambda}$-map as inverse to $\tilde{\delta}$).
- Algebraic relations , e.g.:

$$\tilde{\delta} X^{\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}}(\varphi^{\times 3}) = X^{\bullet}(X^{\bullet}(\varphi^{\times 2}), \varphi)$$

- Applications to stochastic partial differential equations (SPDEs):

$$u_t = S_t u_0 + \int_0^t S_{t-s} dw_s f(u_s)$$

Perspectives & open problems

- Rough integrals as renormalized integrals
- Growth of X and generalized B-series:

$$\sum_{\tau} c_\tau \frac{a^{|\tau|}}{(\tau!)^{\varepsilon}}$$

- Birkhoff decomposition for PDEs (cf. ERGE)
- Scaling in PDEs (RG):
  - Blowup of solutions via series methods (cf. Sinai for cNS)
  - Long-time asymptotics
- Nonperturbative solutions of DSE
- Hochschild cohomology for $(\mathcal{C}, \delta)$