Regularization by oscillations

Massimiliano Gubinelli

CEREMADE
Université Paris Dauphine

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Outline

I would like to discuss three different regularization phenomena due to the presence of noise which however share similar structural properties.

- Davie's theorem for SDEs
- Stochastic Burgers equation
- Schrödinger equation with random dispersion

Davie's phenomenon

Consider this integral equation in $\mathbb{R}^d$:

$$x_t = \int_0^t b(s, x_s)\mathrm{d}s + w_t, \qquad 0 \leqslant t \leqslant 1$$

where $w \in C([0,1];\mathbb{R}^d)$ is a path picked according Wiener measure and b is a generic **bounded** vectorfield.

A. M. Davie showed that there exist a full measure set $\Gamma \subset C([0,1];\mathbb{R}^d)$ such that every $w \in \Gamma$ admits only one solution $x \in C([0,1];\mathbb{R}^d)$ to the integral equation.

Easy consequence: any discrete scheme will converge to this solution (in particular non-adapted schemes).

Related work: [Veretennikov, Krylov-Röckner, Flandoli-Priola-G.]

Smoothing effect of typical brownian paths

Let $x_t = u_t + w_t$:

$$u_t = \int_0^t b(s, w_s + u_s)\mathrm{d}s, \qquad 0 \leqslant t \leqslant 1$$

Key fact

$$\mathbb{E}\left[\left(\int_s^t (b(r, W_r + x) - b(r, W_r + y))\mathrm{d}r\right)^p\right] \leqslant C_p |x - y|^p |t - s|^{p/2}$$

$\Rightarrow$ The random field $(t, s, x) \mapsto \int_s^t b(r, W_r + x)\mathrm{d}r$ is almost surely almost **Lipshitz** in the space variable.

$\Rightarrow$ Via an approximation argument this results in uniqueness for the ODE.

A simple argument, or the Itô trick

Consider the Fourier transform of local time of a Brownian motion. By Itô formula

$$Y(t,\xi) = \int_0^t e^{i\xi\cdot W_s}\mathrm{d}s = 2\frac{e^{i\xi\cdot W_t} - e^{i\xi\cdot W_0}}{|\xi|^2} - \frac{2i\xi}{|\xi|^2}\cdot\int_0^t e^{i\xi\cdot W_s}\mathrm{d}W_s$$

And for large $\xi \in \mathbb{R}^d$

$$\mathbb{E}\left[|Y(t,\xi)|^p\right] \leqslant C_p|\xi|^{-p}|t|^{p/2}$$

so there is a gain of one power of ξ with respect to a trivial estimate.

Alternative approach using Gaussian computations

$$\mathbb{E}\left[|Y(t,\xi)|^2\right] = \int_0^t\int_0^t \mathbb{E}[e^{i\xi\cdot(B_r - B_s)}]\mathrm{d}s\mathrm{d}r$$

$$= 2\int_0^t\int_0^r e^{-|\xi|^2(r-s)/2}\mathrm{d}s\mathrm{d}r \lesssim \frac{t}{|\xi|}$$

$\Rightarrow$ Extensions to fBM [ongoing work with R. Catellier].

Stochastic Burgers equation

[joint work with M. Jara]

Here the stochastic Burgers equation on $\mathbb{T} = [-\pi, \pi]$

$$\mathrm{d}u_t = \frac{1}{2}\partial_\xi^2 u_t(\xi)\mathrm{d}t + \frac{1}{2}\partial_\xi (u_t(\xi))^2 \mathrm{d}t + \partial_\xi \mathrm{d}W_t$$

where $W_t(\xi) = \sum_{k\in\mathbb{Z}_0} e_k(\xi)\beta_t^k$ with $\mathbb{Z}_0 = \mathbb{Z}\backslash\{0\}$, $e_k(\xi) = e^{ik\xi}/\sqrt{2\pi}$ and $(\beta_t^k)_{t\geqslant 0, k\in\mathbb{Z}_0}$ are complex Brownian motions with $(\beta_t^k)^* = \beta_t^{-k}$ and covariance $\mathbb{E}[\beta_t^k \beta_t^q] = \delta_{q+k=0} t$.

The solution u would like to be the derivative of the solution of the Kardar–Parisi–Zhang equation

$$\mathrm{d}h_t = \frac{1}{2}\partial_\xi^2 h_t(\xi)\mathrm{d}t + \frac{1}{2}(\partial_\xi h_t(\xi))^2 \mathrm{d}t + \mathrm{d}W_t. \qquad (1)$$

which is believed to capture the macroscopic behavior of a large class of surface growth phenomena.

Problems with the weak formulation

For sufficiently smooth test functions $\varphi : \mathbb{T} \to \mathbb{R}$ look for solutions of

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial_\xi^2 \varphi)\mathrm{d}s + \int_0^t \langle \partial_\xi \varphi, B(u_s) \rangle \mathrm{d}s + W_t(\partial_\xi \varphi)$$

where $B(u_s)(\xi) = (u_s(\xi))^2$.

- We would like to start the equation from initial condition u_0 which is space white noise, this is expected to be an invariant measure.
- The linearized equation

  $$X_t(\varphi) = u_0(\varphi) + \int_0^t X_s(\partial_\xi^2 \varphi)\mathrm{d}s + W_t(\partial_\xi \varphi)$$

  has trajectories which look like white noise in space.

$\Rightarrow$ The nonlinear term $B(u_s)$ is not defined.

Smoothing

Write $u_t = X_t + v_t$, then

$$v_t(\varphi) = \int_0^t v_s(\partial_\xi^2 \varphi)\mathrm{d}s + \int_0^t \langle \partial_\xi \varphi, B(X_s + v_s)\rangle \mathrm{d}s$$

The covariance of the OU process is

$$\mathbb{E}[X_t(e_k)X_s(e_m)] = \delta_{m+k=0} e^{-m^2|t-s|/2}$$

Key fact

The quantity

$$\int_0^t \langle \partial_\xi \varphi, B(X_s)\rangle \mathrm{d}s$$

is well defined due to the rapid space-time decorrelation of the OU process.

Lazy smoothing estimation

Call "good" a process y such that

$$y_t(\varphi) = y_0(\varphi) + \int_0^t v_s(\partial_\xi^2 \varphi)\mathrm{d}s + \mathcal{A}_t(\varphi) + W_t(\partial_\xi \varphi)$$

where

- $\mathcal{A}_t(\varphi)$ is a zero-quadratic variation process
- y_t is space-time white noise at all times
- The reversed process $\hat{y}_t = y_{T-t}$ has the same properties with drift $\widehat{\mathcal{A}} = -\mathcal{A}$.

Forward/backward Itô trick

Adding Itô formula for the finite quadratic variation process y

$$h(y_t) = h(y_0) + \int_0^t L^0 h(y_s)\mathrm{d}s + \int_0^t Dh(y_s)\mathrm{d}\mathcal{A}_s + M_t^+$$

(here L^0 is the OU generator) with Itô formula for the backward process

$$h(y_{T-t}) = h(y_T) + \int_T^{T-t} L^0 h(y_{T-s})\mathrm{d}s - \int_T^{T-t} Dh(y_{T-s})\mathrm{d}\mathcal{A}_{T-s} + M_t^-$$

gives

$$M_t^+ - M_{T-t}^- + M_T^- = \int_0^t 2L^0 h(y_s)\mathrm{d}s$$

Easy to find an H such that $2L^0 H = \partial_\xi B$ which allows to replace the Burgers drift

$$\int_0^t \partial_\xi B(y_s)\mathrm{d}s$$

with a sum of forward and backward martingales such that

$$\langle M^\pm(\varphi)\rangle_T = \int_0^T \mathcal{E}(\langle\varphi, H(\cdot)\rangle)(y_s)\mathrm{d}s$$

where

$$\mathcal{E}(h)(x) = \sum_{q\in\mathbb{Z}_0} q^2 (D_q h)(x)^2.$$

The function $\mathcal{E}(\langle\varphi, H(\cdot)\rangle)(y_s)$ is now well defined for y_s sampled according white noise and we can estimate it.

Formulation of the equation

Let $B_\varepsilon(x) = B(\rho_\varepsilon * x)$ a regularization of the non-linearity.

By previous arguments we have that for good processes y this limit exists

$$\lim_{\varepsilon\to 0}\int_0^t \langle \varphi, \partial_\xi B_\varepsilon(y_s)\rangle \mathrm{d}s = \mathcal{B}_t(\varphi)$$

and we can use it to define the drift in the Burgers equation.

A solution u of the Burgers equation is a good process such that

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial_\xi^2 \varphi)\mathrm{d}s + \mathcal{B}_t(\varphi) + W_t(\partial_\xi \varphi)$$

The Itô trick provides compactness estimates for Galerkin approximation.
Uniqueness is open (in this approach).

The process $\mathcal{B}_t(\varphi)$ is only $3/2-$ Hölder in time.

Schrödinger equation with random dispersion

Consider the (Stratonovich-) stochastic Schrödinger equation

$$d\phi_t = i\Delta\phi_t \circ \mathrm{d}B_t + |\phi_t|^2\phi_t \mathrm{d}t$$

for $\phi : [0, T] \times \mathbb{T} \to \mathbb{C}$.

[Debussche–De Bouard]

Let

$$U_t = e^{i\Delta B_t}$$

so that distributionally

$$dU_t = i\Delta U_t \circ \mathrm{d}B_t$$

and set $\psi_t = U_t^{-1}\phi_t$. Then

$$\psi_t = \psi_0 + \int_0^t U_s^{-1}(|U_s\psi_s|^2 U_s\psi_s)\mathrm{d}s.$$

Regularization

Define

$$X_t(\theta) = \int_0^t U_s^{-1}(|U_s\theta|^2 U_s\theta)\mathrm{d}s$$

It turns out that this random map has the following pathwise regularity

$$\|X_t(\theta) - X_s(\theta)\|_{L^2(\mathbb{T})} \lesssim |t-s|^\gamma \|\theta\|^3_{L^2(\mathbb{T})}$$

for some $\gamma > 1/2$.

Then take any $\theta \in C^\gamma([0,T], L^2(\mathbb{T}))$ and consider

$$\lim_{\Delta t \to 0} \sum_i [X_{t_{i+1}}(\theta_{t_i}) - X_{t_i}(\theta_{t_i})] = \int_0^t X_{\mathrm{d}s}(\theta_s).$$

By Young theory we have the estimate

$$\left\| \int_0^\cdot X_{\mathrm{d}s}(\theta_s) \right\|_{C^\gamma([0,T],L^2(\mathbb{T}))} \lesssim_X \|\theta\|^3_{C^\gamma([0,T],L^2(\mathbb{T}))}.$$

Existence and uniqueness of global solutions

The Schrödinger equation can be reformulated as a Young equation

$$\psi_t = \psi_0 + \int_0^t X_{ds}(\psi_s)$$

giving rise to local solutions

$$\psi \in C^\gamma([0, T_*], L^2(\mathbb{T})).$$

The L^2 conservation law allows to obtain global solutions.

Standard arguments for Young equations allows to prove convergence of the Euler scheme

$$\psi_{t_{i+1}} = \psi_{t_i} + [X_{t_{i+1}}(\psi_{t_i}) - X_{t_i}(\psi_{t_i})]$$

Similar phenomena for the deterministic KdV equation

$$\partial_t u_t = \partial^3_\xi u_t + \partial_\xi u_t^2.$$

with initial conditions in $H^\alpha(\mathbb{T})$ for $\alpha > -1/2$. [see paper on CPAA]

Thanks