Trees, rough integration and differential equations

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Rough paths

- T. Lyons (Oxford): an integration theory for irregular signals.
- Nonlinear systems y_t driven by a (non-differentiable) noise x_t

$$dy = f(y)\,dx$$

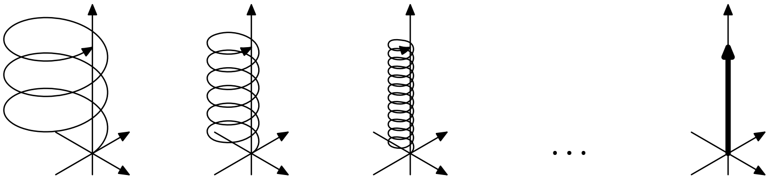
- The output y is a nice function of the iterated integrals of x:

$$\left(x, \int dx dx, \cdots, \int dx^{\otimes n}\right) \xrightarrow{\Phi} y$$

  Can consider only a finite number of them. No formal series.
- Applications: stochastic analysis, sound compression algorithms.

Motion in a third direction

$$dz = ydx - xdy, \qquad z_t = \int_0^t xdy - ydx = \int dxdy - dydx$$

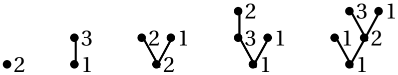


$$(x^n, y^n) \to (0,0), \qquad z^n \to t \neq 0$$

z encode “microscopic” informations on the trajectory (x, y).

Trees

$\mathscr{L}$ finite set. Trees labeled by $\mathscr{L}$, $\mathscr{T}_{\mathscr{L}}$



$$(\tau_1, \cdots, \tau_k) \overset{B_+^a}{\longmapsto} \tau = [\tau_1, \cdots, \tau_k]_a$$

$[\bullet] =$  $\quad [\bullet, [\bullet]] =$ , etc...

$$\Delta(\tau) = 1 \otimes \tau + \sum_{a \in \mathscr{L}} (B_+^a \otimes \mathrm{id})[\Delta(B_-^a(\tau))]$$

$$B_-^a(B_+^b(\tau_1 \cdots \tau_n)) = \begin{cases} \tau_1 \cdots \tau_n & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

Differential equations (*à la Butcher*)

The solution y of the differential equation

$$dy = f(y)\,dt, \qquad y_0 = \eta$$

has the B-series representation

$$y_t = \eta + \sum_{\tau \in \mathscr{T}} \psi^f(\tau)(\eta) \frac{t^{|\tau|}}{\sigma(\tau)\tau!}$$

Elementary differentials ψ^f defined as

$$\psi^f(\bullet)(\xi) = f(\xi), \qquad \psi^f([\tau^1 \cdots \tau^k]) = f_{\bar{b}}(\eta)\psi^f(\tau^1)(\xi)^{b_1} \cdots \psi^f(\tau^k)(\xi)^{b_k}$$

where $f_\emptyset(\xi) = f(\xi)$ and $f_{\bar{b}}(\xi) = \prod_{i=1}^{|\bar{b}|} \partial_{\xi_{b_i}} f(\xi)$ derivatives of the vectorfields.

Driven differential equations

Given a collection of paths $\{x^a \in C^1([0,T],\mathbb{R})\}_{a\in\mathcal{L}}$, $\eta \in \mathbb{R}^n$
Analytic vectorfields $\{f_a : \mathbb{R}^n \to \mathbb{R}^n\}_{a\in\mathcal{L}}$

Theorem

The differential equation

$$dy_t = f_a(y_t)\,dx^a_t, \qquad y_0 = \eta$$

admit locally the series solution

$$y_t = y_s + \sum_{\tau\in\mathcal{T}_{\mathcal{L}}} \frac{1}{\sigma(\tau)} \phi^f(\tau)(y_s) X^\tau_{ts}, \qquad y_0 = \eta$$

where $\phi^f(\bullet_a)(\xi) = f_a(\xi)$, $\phi^f([\tau^1\cdots\tau^k]_a)(\xi) = f_{a;b_1\dots b_k}(\xi)\prod_{i=1}^k[\phi^f(\tau^i)(\xi)]^{b_i}$.

Smooth iterated integrals

Let $X : \mathscr{T}_{\mathscr{L}} \to \mathscr{C}_2 \subset C([0,T]^2;\mathbb{R})$

$$X^{\bullet_a}_{ts} = \int_s^t dx^a_u, \qquad X^{[\tau^1 \cdots \tau^k]_a}_{ts} = \int_s^t \prod_{i=1}^k X^{\tau^i}_{us} dx^a_u. \tag{1}$$

Extend X to $\mathscr{A}\mathscr{T}_{\mathscr{L}}$ considering $\mathscr{C}_2$ as an algebra with (commutative) product $(a \circ b)_{ts} = a_{ts} b_{ts}$ for $a, b \in \mathscr{C}_2$. We let $X^1 = 1$.

$$X^{\tau_1 \cdots \tau_n}_{ts} = X^{\tau_1}_{ts} X^{\tau_2}_{ts} \cdots X^{\tau_n}_{ts}, \qquad X^{[\tau_1 \cdots \tau_n]_a} = \int X^{\tau_1 \cdots \tau_n} dx^a$$

Bounds

$$|X^\tau_{ts}| \le \frac{(A|t-s|)^{|\tau|}}{\tau!}$$

Theorem (Tree multiplicative property)

$$X^\tau_{ts} = \sum X^{\tau^{(1)}}_{tu} X^{\tau^{(2)}}_{us} = X^{\Delta\tau}_{tus}$$

Example

$$T^{\bullet}_{ts} = t - s, \qquad T^{[\tau_1 \cdots \tau_n]}_{ts} = \int_s^t T^{\tau_1}_{us} \cdots T^{\tau_n}_{us} \, du$$

By induction: $T^{\tau}_{ts} = (t-s)^{|\tau|}(\tau!)^{-1}$

Lemma (Tree Binomial)

For every $\tau \in \mathscr{T}$ and $a, b \geq 0$ we have

$$(a+b)^{|\tau|} = \sum_i \frac{\tau!}{\tau_i^{(1)}!\tau_i^{(2)}!} a^{|\tau_i^{(1)}|} b^{|\tau_i^{(2)}|} \tag{2}$$

Structure of solution to DDEs

Write $y_s^\tau = \phi^f(\tau)(y_s)/\sigma(\tau)$ so that

$$y_t - y_s = \sum_{\tau \in \mathcal{T}_{\mathcal{L}}} X_{ts}^\tau y_s^\tau$$

Lemma

For any $\tau \in \mathcal{T}_{\mathcal{L}} \cup \{\emptyset\}$ *we have*

$$y_t^\tau - y_s^\tau = \sum_{\sigma \in \mathcal{T}_{\mathcal{L}}, \rho \in \mathcal{F}_{\mathcal{L}}} c'(\sigma, \tau, \rho) X_{ts}^\rho y_s^\sigma$$

c' counting function of reduced coproduct: $\Delta' \sigma = \sum_{\tau,\rho} c'(\sigma,\tau,\rho) \tau \otimes \rho$.

Integration of increments

- Q: Given $a \in \mathscr{C}_2$ can we find $f \in \mathscr{C}_1$ such that $f_t - f_s = a_{ts}$?
  $a_{ts} - a_{tu} - a_{us} = (f_t - f_s) - (f_t - f_u) - (f_u - f_s) = 0$ (obstruction)
- Increments: $\mathscr{C}_n \subset C([0,1]^n, V)$, $g \in \mathscr{C}_n$ iff $g_{t_1 \cdots t_n} = 0$ when $t_i = t_{i+1}$
  Coboundary: $\delta f_{ts} = f_t - f_s$, $\delta g_{tus} = g_{ts} - g_{tu} - g_{us}$, $\ldots$, $\delta^2 = 0$

$$0 \to \mathbb{R} \to \mathscr{C}_1 \xrightarrow{\delta} \mathscr{C}_2 \xrightarrow{\delta} \mathscr{C}_3 \xrightarrow{\delta} \cdots$$

  Then $\delta f = a \Leftrightarrow \delta a = 0$.
- Small 2-increments cannot be exact: $a_{ts} = o(|t-s|) \Rightarrow a \neq \delta f$
- Unique decomposition: $a = \delta f + o(|t-s|)$?
  Yes, if obstruction δa is small:

Theorem

If $\delta a_{tus} = o(|t-s|)$, then $\exists!\, f \in \mathscr{C}_1, r \in \mathscr{C}_2^{1+}$ such that

$$a = \delta f + r, \qquad \delta f = (1 - \Lambda\delta) a$$

Examples

- Convergence of sums:

$$S_{t0} = \sum_i a_{t_{i+1}t_i} = \sum_i (\delta f)_{t_{i+1}t_i} + \sum_i (r)_{t_{i+1}t_i} = (\delta f)_{t0} + \sum_i o(|t_{i+1} - t_i|) \to f_t - f_0$$

- Young integrals: $x \in C^\gamma, \gamma > 1/2$ $a_{ts} = \varphi(x_s)\delta x_{ts}$

$$\delta a_{tus} = \delta\varphi(x)_{tu}\delta x_{us} = o(|t-s|^{2\gamma}) \Rightarrow \delta f = (1 - \Lambda\delta)a =: \int \varphi(x)dx$$

- NCG & Λ map. $L^2(\mathbb{R})$, $dg = [F, g]$, $F^2 = c$,

$$(df)_{ts} = \frac{f_t - f_s}{t - s} \qquad \int f dg = \mathrm{Tr}_\omega(fdg) = \frac{1}{2c}\mathrm{Tr}_\omega(Fdfdg)$$

so

$$\Lambda(\delta f \delta g)_{-\infty,\infty} = -\frac{1}{2c}\mathrm{Tr}_\omega(Fdfdg)$$

(Step-2) Rough paths

Rough integrals: $X^\bullet = \delta x$, $\delta X^{[\bullet]} = X^\bullet X^\bullet$, $X^\bullet \in \mathscr{C}_2^\gamma$, $X^{[\bullet]} \in \mathscr{C}_2^{2\gamma}$ $(\gamma > 1/3)$

$$\oint \varphi(x)dx = (1 - \Lambda\delta)\underbrace{(\varphi(x)X^\bullet + \varphi'(x)X^{[\bullet]})}_{\delta(\cdot)\in\mathscr{C}_3^{3\gamma>1}},$$

$$\delta(\varphi(x)X^\bullet + \varphi'(x)X^{[\bullet]}) = (-\delta\varphi(x) + \varphi'(x)X^\bullet)X^\bullet - \delta\varphi'(x)X^{[\bullet]}$$

- Continuous map: $(\varphi, X^\bullet, X^{[\bullet]}) \longmapsto \oint \varphi(x)dx$
- Renormalized sums: $\sum_i(\varphi(x_{t_i})X^\bullet_{t_{i+1}t_i} + \varphi'(x_{t_i})X^{[\bullet]}_{t_{i+1}t_i}) \to \oint \varphi(x)dx$
- $\oint dxdx = X^{[\bullet]}$
- A finite number of iterated integrals determines all the other integrals.

Branched rough paths

The only data we need to build the family $\{X^\tau\}_{\tau\in\mathcal{T}_{\mathcal{L}}}$ is a family of maps $\{I^a\}_{a\in\mathcal{L}}$ from $\mathscr{C}_2$ to $\mathscr{C}_2$ satisfying certain properties.

Definition

An *integral* is a linear map $I:\mathscr{D}_I\to\mathscr{D}_I$ on an unital sub-algebra $\mathscr{D}_I\subset\mathscr{C}_2^+$ for which $I(hf)=I(h)f, \forall h\in\mathscr{D}_I, f\in\mathscr{C}_1$ and

$$\delta I(h)=I(e)h+\sum_i I(h^{1,i})h^{2,i} \qquad \text{when } h\in\mathscr{D}_I,\, \delta h=\sum_i h^{1,i}h^{2,i} \text{ and } h^{1,i}\in\mathscr{D}_I$$

Then $X^{\bullet_a}=I^a(e)$, $\qquad X^{[\tau^1\cdots\tau^k]_a}=I^a(X^{\tau^1\cdots\tau^k})$, $\qquad X^{\tau^1\cdots\tau^k}=X^{\tau^1}\circ\cdots\circ X^{\tau^k}$.

Tree multiplicative property still holds: $\delta X^\tau=X^{\Delta'\tau}$.

Geometric rough paths

Integrals are not necessarily Rota-Baxter maps: e.g. Itô stochastic integral

$$\int_0^t f_s dx_s \int_0^t g_s dx_s = \int_0^t f_s \int_0^s g_u dx_u dx_s + \int_0^t g_s \int_0^s f_u dx_u dx_s + \int_0^t f_s g_s ds$$

$$I(f)I(g) = I(I(f)g) + I(fI(g)) + J(fg), \qquad J(f)I(g) = J(fI(g)) + I(gJ(f))$$

Solution to $dy = ydx$, $y_0 = 1$: $y_t = \exp(x_t - t/2)$.
When they are Rota-Baxter we have shuffle relations:

$$I^{a_1}(\cdots I^{a_n}(1)) \circ I^{b_1}(\cdots I^{b_m}(1)) = \sum_{\overline{c} \in \mathrm{Sh}(\overline{a}, \overline{b})} I^{c_1}(\cdots I^{c_{n+m}}(1)) \tag{3}$$

This relation reduces X^τ for $\tau \in \mathcal{T}_{\mathcal{L}}$ to a linear combination of $\{X^\sigma\}_{\sigma \in \mathcal{T}_{\mathcal{L}}^{\mathrm{Chen}}}$.
These are *geometric rough-paths*: the closure of smooth rough paths.

Growing a branched rough path

Fix $\gamma \in (0,1]$, consider $q_\gamma : \mathscr{F} \to \mathbb{R}_+$ on forests as $q_\gamma(\tau) = 1$ for $|\tau| \le 1/\gamma$ and

$$q_\gamma(\tau) = 1, \text{ if } |\tau| \le 1/\gamma \qquad q_\gamma(\tau) = \frac{1}{2^{\gamma|\tau|}} \sum q_\gamma(\tau^{(1)}) q_\gamma(\tau^{(2)}) \text{ otherwise}$$

$q_\gamma(\tau_1 \cdots \tau_n) = q_\gamma(\tau_1) \cdots q_\gamma(\tau_n)$.

Theorem

Given a partial homomorphism $X : \mathscr{A}_n \mathscr{T}_{\mathscr{L}} \to \mathscr{C}_2$ *satisfying the multiplicative property*

$$|X^\tau_{ts}| \le B A^{|\tau|} q_\gamma(\tau) |t-s|^{\gamma|\tau|}, \qquad \tau \in \mathscr{T}^n_{\mathscr{L}} \tag{4}$$

with $\gamma(n+1) > 1$, *then* $\exists!\, X : \mathscr{A}\mathscr{T}_{\mathscr{L}} \to \mathscr{C}_2$ *such that eq. (4) holds* $\forall \tau \in \mathscr{T}_{\mathscr{L}}$.

Construction via the equation: $X^\tau = \Lambda(X^{\Delta'\tau})$.

Speed of growth

Conjecture

$$q_\gamma(\tau) \asymp C(\tau!)^{-\gamma}$$

True for linear Chen trees $\mathscr{T}^{\text{Chen}}$:

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(k!)^\gamma (n!)^\gamma} \le c_\gamma \frac{(a+b)^{\gamma n}}{(n!)^\gamma}, \qquad \gamma \in (0,1], \ a,b \ge 0$$

Variant of Lyons' neo-classical inequality

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(\gamma k)![\gamma(n-k)n]!} \le c_\gamma \frac{(a+b)^{\gamma n}}{(\gamma n)!}$$

"neo-classical tree inequality"?

$$\sum \frac{a^{\gamma|\tau^{(1)}|} b^{\gamma|\tau^{(2)}|}}{(\tau^{(1)}!)^\gamma (\tau^{(2)}!)^\gamma} \le c_\gamma \frac{(a+b)^{\gamma|\tau|}}{(\tau!)^\gamma}$$

OK for $\gamma = 1$: tree binomial formula.

Controlled paths

Definition

Let n the largest integer such that $n\gamma \le 1$. For any $\kappa \in (1/(n+1), \gamma]$ a path y is a κ-*weakly controlled* by X if

$$\delta y = \sum_{\tau \in \mathscr{F}_{\mathscr{L}}^{n-1}} X^\tau y^\tau + y^\sharp, \qquad \delta y^\tau = \sum_{\sigma \in \mathscr{F}_{\mathscr{L}}^{n-1}} \sum_\rho c'(\sigma,\tau,\rho) X^\rho y^\sigma + y^{\tau,\sharp}, \qquad \tau \in \mathscr{F}_{\mathscr{L}}^{n-1}$$

with $y^\tau \in \mathscr{C}_2^{|\tau|\kappa}, y^{\sharp,\tau} \in \mathscr{C}_2^{(n-|\tau|)\kappa}$. Then we write $y \in \mathscr{Q}_\kappa(X;V)$.

Lemma (Stability)

Let $\varphi \in C_b^n(\mathbb{R}^k,\mathbb{R})$ and $y \in \mathscr{Q}_\kappa(X;\mathbb{R}^k)$, then $z_t = \varphi(y_t)$ is a weakly controlled path, $z \in \mathscr{Q}_\kappa(X;\mathbb{R})$ where its coefficients are given by

$$z^\tau = \sum_{m=1}^{n-1} \sum_{|\bar b|=m} \frac{\varphi_{\bar b}(y)}{m!} \sum_{\substack{\tau_1,\dots,\tau_m \in \mathscr{F}_{\mathscr{L}}^{n-1} \\ \tau_1\cdots\tau_m=\tau}} y^{\tau_1,b_1}\cdots y^{\tau_m,b_m}, \qquad \tau \in \mathscr{F}_{\mathscr{L}}^{n-1}$$

Integration of controlled paths

Theorem

The integral maps $\{I^a\}_{a\in\mathcal{L}}$ *can be extended to maps* $I^a : \mathcal{Q}_\kappa(X) \to \delta\mathcal{Q}_\kappa(X)$

$$y \in \mathcal{Q}_\kappa(X) \mapsto \delta z = I^a(y) = X^{\bullet_a} z^{\bullet_a} + \sum_{\tau\in\mathcal{T}^n_{\mathcal{L}}} X^\tau z^\tau + z^\flat, \tag{5}$$

where $z^\flat \in \mathcal{C}_2^{\kappa(n+1)}$, $z^{\bullet_a} = y$, $z^{[\tau]_a} = y^\tau$ *and zero otherwise.*

Remark

If $y \in \mathcal{Q}_\kappa(X; \mathbb{R}^n \otimes \mathbb{R}^d)$ then $\{J^b(\cdot) = \sum_{a\in\mathcal{L}} I^a(y^{ab}\cdot)\}_{b\in\mathcal{L}_1}$ defines a family of integrals with an associated branched rough path Y indexed by $\mathcal{T}_{\mathcal{L}_1}$. An explicit recursion is

$$Y^{\bullet_b} = \sum_{a\in\mathcal{L}} I^a(y^{ab}), \qquad Y^{[\tau^1\cdots\tau^k]_b} = \sum_{a\in\mathcal{L}} I^a(y^{ab} Y^{\tau^1} \circ \cdots \circ Y^{\tau_k}), \qquad b \in \mathcal{L}_1$$

Example

$$\delta y = X^{\bullet} y^{\bullet} + X^{\mathbf{I}} y^{\mathbf{I}} + X^{\bullet\bullet} y^{\bullet\bullet} + X^{\mathbf{V}} y^{\mathbf{V}} + X^{\mathbf{I}\bullet} y^{\mathbf{I}\bullet} + X^{\mathbf{V}} y^{\mathbf{V}} + X^{\bullet\bullet\bullet} y^{\bullet\bullet\bullet} + X^{\mathbf{I}_3} y^{\mathbf{I}_3} + y^{\sharp}$$

$$\delta y^{\bullet} = X^{\bullet}(y^{\mathbf{I}} + 2y^{\bullet\bullet}) + X^{\mathbf{I}}(y^{\mathbf{I}_3} + y^{\mathbf{I}\bullet}) + X^{\bullet\bullet}(y^{\mathbf{I}\bullet} + y^{\mathbf{V}} + 3y^{\bullet\bullet\bullet}) + y^{\bullet,\sharp}$$

$$\delta y^{\mathbf{I}} = X^{\bullet}(y^{\mathbf{I}\bullet} + 2y^{\mathbf{V}} + y^{\mathbf{I}_3}) + y^{\mathbf{I},\sharp}$$

$$\delta y^{\bullet\bullet} = X^{\bullet}(y^{\mathbf{I}\bullet} + y^{\bullet\bullet\bullet}) + y^{\bullet\bullet,\sharp}$$

$$\delta y^{\mathbf{V}} = y^{\mathbf{V},\sharp} \qquad \delta y^{\mathbf{I}\bullet} = y^{\mathbf{I}\bullet,\sharp} \qquad \delta y^{\bullet\bullet\bullet} = y^{\bullet\bullet\bullet,\sharp} \qquad \delta y^{\mathbf{I}_3} = y^{\mathbf{I}_3,\sharp}$$

$$\begin{aligned} \delta z = \delta I(y) &= X^{\bullet} y + X^{\mathbf{I}} y^{\bullet} + X^{\mathbf{I}_3} y^{\mathbf{I}} + X^{\mathbf{V}} y^{\bullet\bullet} + X^{\mathbf{V}'} y^{\mathbf{I}\bullet} + X^{\mathbf{Y}} y^{\mathbf{V}} + X^{\mathbf{W}} y^{\bullet\bullet\bullet} + X^{\mathbf{I}_4} y^{\mathbf{I}_3} + z^{\flat} \\ &= X^{\bullet} z^{\bullet} + X^{\mathbf{I}} z^{\mathbf{I}} + X^{\mathbf{I}_3} z^{\mathbf{I}_3} + X^{\mathbf{V}} z^{\mathbf{V}} + z^{\sharp} \end{aligned}$$

with

$$z^{\flat} = \Lambda\left[X^{\bullet} y^{\sharp} + X^{\mathbf{I}} y^{\bullet,\sharp} + X^{\mathbf{I}_3} y^{\mathbf{I},\sharp} + X^{\mathbf{V}} y^{\bullet\bullet,\sharp} + X^{\mathbf{Y}} y^{\mathbf{V},\sharp} + X^{\mathbf{V}'} y^{\mathbf{I}\bullet,\sharp} + X^{\mathbf{W}} y^{\bullet\bullet\bullet,\sharp}\right].$$

Rough differential equations

Take vectorfields $\{f_a \in C_b^n(\mathbb{R}^k;\mathbb{R}^k)\}_{a\in\mathscr{L}}$ and integral maps $\{I^a\}_{a\in\mathscr{L}}$ and consider the *rough differential equation*

$$\delta y = I^a(f_a(y)), \qquad y_0 = \eta \in \mathbb{R}^k \tag{6}$$

in the time interval $[0, T]$.

Theorem

The rough differential equation (6) has a global solution $y \in \mathscr{Q}_\gamma(X;\mathbb{R}^k)$ for any initial condition $\eta \in \mathbb{R}^k$. If the vectorfields are C_b^{n+1} the solution is unique and has Lipshitz dependence on data.

The KdV equation

1d periodic KdV equation:

$$\partial_t u(t,\xi) + \partial_\xi^3 u(t,\xi) + \frac{1}{2}\partial_\xi u(t,\xi)^2 = 0, \quad u(0,\xi) = u_0(\xi), \qquad (t,\xi) \in \mathbb{R}\times\mathbb{T}$$

where initial condition $u_0 \in H^\alpha(\mathbb{T})$, $\mathbb{T} = [-\pi,\pi]$. Linear part: Airy group $U(t)$ (isometries on H^α). Go to Fourier variables and let $v_t = U(-t)u_t$:

$$v_t(k) = v_0(k) + \frac{ik}{2}\sum_{k_1}' \int_0^t e^{-i3kk_1k_2 s} v_s(k_1)v_s(k_2)\,ds, \quad t\in[0,T], k\in\mathbb{Z}_*$$

where $k_2 = k - k_1$ and $v_0(k) = u_0(k)$. Restrict to $v_0(0) = 0$. It has the form

$$v_t = v_s + \int_s^t \dot{X}_\sigma(v_\sigma, v_\sigma)\,d\sigma, \qquad t,s\in[0,T].$$

where $\dot{X}_\sigma(\varphi,\varphi) = \frac{ik}{2}\sum_{k_1}' e^{-i3kk_1k_2\sigma}\varphi(k_1)\varphi(k_2)$.

The KdV equation

Expansion

$$\delta v_{ts} = X^{\bullet}(v^{\times 2}) + X^{[\bullet]}(v^{\times 3}) + X^{[[\bullet]]}(v^{\times 4}) + X^{[\bullet\bullet]}(v^{\times 4}) + r \tag{7}$$

with multi-linear operators X^{τ}:

$$X^{\bullet}_{ts}(\varphi_1,\varphi_2) = \int_s^t \dot{X}_\sigma(\varphi_1,\varphi_2)\,d\sigma;$$

$$X^{[\tau^1]}_{ts}(\varphi_1,\dots,\varphi_{m+1}) = \int_s^t \dot{X}_\sigma(X^{\tau^1}_{\sigma s}(\varphi_1,\dots,\varphi_m),\varphi_{m+1})\,d\sigma$$

and

$$X^{[\tau^1\tau^2]}_{ts}(\varphi_1,\dots,\varphi_{m+n}) = \int_s^t \dot{X}_\sigma(X^{\tau^1}_{\sigma s}(\varphi_1,\dots,\varphi_m),X^{\tau^2}_{\sigma s}(\varphi_{m+1},\dots,\varphi_{m+n}))\,d\sigma.$$

Eq.7 is a rough equation which can be solved with fixed-point:

$$\delta v = (1-\Lambda\delta)[X^{\bullet}(v^{\times 2}) + X^{[\bullet]}(v^{\times 3})]$$

Shadows of the conservation law

Lemma

$$\langle \varphi_1, \dot{X}_s(\varphi_2,\varphi_3)\rangle + \langle \varphi_2, \dot{X}_s(\varphi_1,\varphi_3)\rangle + \langle \varphi_3, \dot{X}_s(\varphi_2,\varphi_1)\rangle = 0, \qquad s \in [0,T]$$

$$\langle \varphi, X_{ts}(\varphi,\varphi)\rangle = 0 \qquad 2\langle \varphi, X^2_{ts}(\varphi,\varphi,\varphi)\rangle + \langle X_{ts}(\varphi,\varphi), X_{ts}(\varphi,\varphi)\rangle = 0$$

$$\begin{aligned}[\delta\langle v,v\rangle]_{ts} &= 2\langle X_{ts}(v_s,v_s) + X^2_{ts}(v_s,v_s,v_s) + v^\flat_{ts}, v_s\rangle \\ &\quad + \langle X_{ts}(v_s,v_s), X_{ts}(v_s,v_s)\rangle + 2\langle X_{ts}(v_s,v_s), v^\sharp_{ts}\rangle + \langle v^\sharp_{ts}, v^\sharp_{ts}\rangle \\ &= 2\langle v^\flat_{ts}, v_s\rangle + 2\langle X_{ts}(v_s,v_s), v^\sharp_{ts}\rangle + \langle v^\sharp_{ts}, v^\sharp_{ts}\rangle = O(|t-s|^{3\gamma})\end{aligned}$$

Theorem (Integral conservation law)

If v is a solution of KdV then $|v_t|_0^2 = |v_0|_0^2$ for any t.

The NS equation

The d-dimensional NS equation (or the Burgers' equation) have the abstract form

$$u_t = S_t u_0 + \int_0^t S_{t-s} B(u_s, u_s)\, ds. \tag{8}$$

S bounded semi-group on $\mathscr{B}$, B symmetric bilinear operator.
Define $d(\tau)$-multilinear operator by

$$X^{\bullet}_{ts}(\varphi^{\times 2}) = \int_s^t S_{t-u} B(S_{u-s}\varphi, S_{u-s}\varphi)\, du$$

$$X^{[\tau^1]}_{ts}(\varphi^{\times(d(\tau^1)+1)}) = \int_s^t S_{t-u} B(X^{\tau^1}_{us}(\varphi^{\times d(\tau^1)}), \varphi)\, du$$

and

$$X^{[\tau^1\tau^2]}_{ts}(\varphi^{\times(d(\tau^1)+d(\tau^2))}) = \int_s^t S_{t-u} B(X^{\tau^1}_{us}(\varphi^{\times d(\tau^1)}), X^{\tau^2}_{us}(\varphi^{\times d(\tau^2)}))\, du$$

where $d(\tau)$ is an appropriate degree function.

Bounds on the operators and regularity

The X operators allow bounds in $\mathscr{B}$ of the form

$$|X^\tau(\varphi^{\times d(\tau)})|_{\mathscr{B}} \leq C\frac{|t-s|^{\varepsilon|\tau|}}{(\tau!)^{\varepsilon}}|\varphi|_{\mathscr{B}}^{d(\tau)}$$

where $\varepsilon \geq 0$ is a constant depending on the particular Banach space $\mathscr{B}$ we choose.

We have the (norm convergent) series representation

$$u_t = S_t u_0 + \sum_{\tau \in \mathscr{T}_B} X^\tau_{t0}(u_0^{\times d(\tau)}) \tag{9}$$

which gives local solutions to NS.

Regularity: $|u(k)| \leq Ce^{-|k|\sqrt{t}}$ by controlling growth of the terms in the series.

Convolution integrals

- A cochain complex $(\hat{C}_*, \hat{\delta})$ adapted to the study of convolution integrals.
- Coboundary $\tilde{\delta} h = \delta h - ah - ha$ with $a_{ts} = S_{t-s} - \text{Id}$ the 2-increment associated to the semi-group (parallel transport).
- Associated integration theory ($\tilde{\Lambda}$-map as inverse to $\tilde{\delta}$).
- Algebraic relations , e.g.:

$$\tilde{\delta} X^{\substack{\bullet \\ \vert \\ \bullet}}(\varphi^{\times 3}) = X^{\bullet}(X^{\bullet}(\varphi^{\times 2}), \varphi)$$

- Applications to stochastic partial differential equations (SPDEs):

$$u_t = S_t u_0 + \int_0^t S_{t-s} dw_s f(u_s)$$

Perspectives & open problems

- Rough integrals as renormalized integrals
- Growth of X and generalized B-series:

$$\sum_\tau c_\tau \frac{a^{|\tau|}}{(\tau!)^\varepsilon}$$

- Birkhoff decomposition for PDEs (cf. ERGE)
- Scaling in PDEs (RG):
  - Blowup of solutions via series methods (cf. Sinai for cNS)
  - Long-time asymptotics
- Nonperturbative solutions of DSE
- Hochschild cohomology for $(\mathscr{C}, \delta)$