Controlled paths and regularization in (S)PDEs

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Outline

I would like to show a zoo of regularization phenomena (for ODE/PDE/SPDEs) which share similar structural properties.

- Davie's theorem for SDEs with bounded drift
- Korteweg–de Vries equation with distributional initial condition
- Schrödinger equation with random dispersion
- Stochastic Burgers equation with derivative white noise perturbation

Go fast!

Consider this integral equation in $\mathbb{R}$:

$$x_t = x_0 + \int_0^t b(x_s)\mathrm{d}s + \lambda t, \qquad 0 \leqslant t \leqslant 1$$

where b is a continuous and **bounded** vectorfield.

Let $x_t = \lambda t + u_t$ and $G'(x) = b(x)/(\lambda + b(x))$:

$$u_t = x_0 + \int_0^t b(\lambda s + u_s)\mathrm{d}s = x_0 + \int_0^t (\lambda + b(\lambda s + u_s))G'(\lambda s + u_s)\mathrm{d}s$$

$$= x_0 + G(\lambda t + u_t) - G(x_0)$$

If λ is large this reformulation of the equation implies uniqueness.

Define $\sigma_t(x) = \int_0^t b(\lambda s + x)\mathrm{d}s$ and note that ($B' = b/\lambda$):

$$|\sigma_t(x) - \sigma_t(y)| = |[B(\lambda t + x) - B(\lambda t + y)] + [B(x) - B(y)]| \leqslant C|x - y|$$

In the same spirit: global solution of 2d Euler equation under strong rotation.

Davie's phenomenon

Consider this integral equation in $\mathbb{R}^d$:

$$x_t = x_0 + \int_0^t b(s, x_s)\mathrm{d}s + w_t, \qquad 0 \leqslant t \leqslant 1$$

where $w \in C([0,1]; \mathbb{R}^d)$ is a path picked according Wiener measure and b is a generic **bounded** vectorfield.

A. M. Davie showed that there exist a full measure set $\Gamma \subset C([0,1]; \mathbb{R}^d)$ such that every $w \in \Gamma$ admits only one solution $x \in C([0,1]; \mathbb{R}^d)$ to the integral equation.

Related work: [Veretennikov, Krylov-Röckner, Flandoli-Priola-G.]

Smoothing effect of typical brownian paths

Let $x_t = w_t + u_t$, then $u_t = u_0 + \int_0^t b(s, w_s + u_s)\mathrm{d}s, \qquad 0 \leqslant t \leqslant 1.$

Interpret the equation as a Young integral equation:

$$u_t = u_0 + \int_0^t \sigma_{\mathrm{d}s}(u_s), \qquad 0 \leqslant t \leqslant 1$$

where $\sigma_{t,s}(x) = \int_s^t b(r, w_r + x)\mathrm{d}r$ and $\int_0^t \sigma_{\mathrm{d}s}(u_s) = \lim_{\Delta t \to 0} \sum_i \sigma_{t_{i+1},t_i}(u_{t_i})$.

$$\mathbb{E}\left[|\sigma_{t,s}(x) - \sigma_{t,s}(y)|^p\right] \leqslant C_p |x-y|^p |t-s|^{p/2}$$

$\Rightarrow$ The random field $x \mapsto \sigma_{t,s}(x)$ is almost surely **log-Lipshitz**.

$\Rightarrow$ The log-Lipshitz regularity of σ is enough for the uniqueness of the ODE.

Extension to fBM (with R. Catellier)

1d periodic KdV equation

$$\begin{cases} \partial_t u(t,\xi) + \partial_\xi^3 u(t,\xi) + \frac{1}{2}\partial_\xi u(t,\xi)^2 = 0 \\ u(0,\xi) = u_0(\xi) \end{cases} \qquad (t,\xi) \in \mathbb{R} \times \mathbb{T}$$

with initial condition $u_0 \in H^\alpha(\mathbb{T})$, $\mathbb{T} = [-\pi, \pi]$.

We look for solutions for any $\alpha > -1/2$.

Airy group

$$\mathcal{F}(U(t)\varphi)(k) = e^{-ik^3 t}\hat{\varphi}(k), \qquad k \in \mathbb{Z}.$$

Mild form

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\partial_\xi u(s)^2 \, ds$$

Abstract formulation

After the change of variables $v(t) = U(-t)u(t)$:

$$v(t) = v_0 + \int_0^t \underbrace{U(-s)\partial_\xi [U(s)v(s)]^2}_{\dot{X}_s(v_s, v_s)} ds$$

$$= v_0 + \int_0^t \dot{X}_s(v_s, v_s) ds = v_0 + \int_0^t X^\bullet_{\mathrm{ds}}(v_s, v_s)$$

with $X^\bullet_{ts}(\varphi_1, \varphi_2) = \int_s^t \dot{X}_\sigma(\varphi_1, \varphi_2) d\sigma$.

Now

$$\|X^\bullet_{ts}\|_{\mathcal{L}(H^\alpha)} \lesssim |t-s|^\gamma$$

for $\gamma < 1/2$, $\gamma < 1 + \alpha$, $\gamma < \alpha/3 + 1/2$, $\alpha \geqslant -1/2$.

The time regularity of $X^\bullet$ is not enough to use Young integrals.

Rough integral

Assume that v is *controlled* by $X^{\bullet}$:

$$v_t = v_s + X^{\bullet}_{ts}(w_s, w_s) + O(|t-s|^{2\gamma})$$

and define

$$\int_0^t X_{\mathrm{ds}}(v_s, v_s) = \lim_{\Delta t \to 0} \sum_i X^{\bullet}_{t_{i+1},t_i}(v_{t_i}, v_{t_i}) + X^{\substack{\bullet \\ | \\ \bullet}}_{t_{i+1},t_i}(v_{t_i}, w_{t_i}, w_{t_i})$$

with

$$X^{\substack{\bullet \\ | \\ \bullet}}_{ts}(\varphi_1, \varphi_2, \varphi_3) = \int_s^t \dot{X}_\sigma(\varphi_1, X^{\bullet}_{\sigma s}(\varphi_2, \varphi_3)) d\sigma$$

$$\| X^{\substack{\bullet \\ | \\ \bullet}}_{ts} \|_{\mathcal{L}(H^\alpha)} \lesssim |t-s|^{2\gamma}$$

Then this rough integral is well defined and the equation

$$v_t = v_0 + \int_0^t X_{\mathrm{ds}}(v_s, v_s)$$

has a unique (local) solution in the space of controlled paths with values in H^α.

Uniqueness of weak solutions

Using rough path theory we can prove that the nonlinear term is defined of every controlled path:

Let $\mathcal{N}(\varphi)(t,\xi) = \partial_\xi(\varphi(t,\xi)^2)/2$ for smooth functions φ. Any path y in H^α such that

$$y_t = y_s + X^{\bullet}_{ts}(z_s) + O(|t-s|^{2\gamma})$$

for some other path z_s regular enough enjoy the property that

$$\mathcal{N}(P_N y) \to \mathcal{N}(y)$$

as space-time distribution. The non-linear term is well-defined.

The solution we found satisfy

$$\partial_t u + \partial^3_\xi u + \mathcal{N}(u) = 0$$

as space-time distribution.
In the space of controlled paths these solutions are unique.

Additive stochastic forcing

Noisy KdV

$$\partial_t u + \partial_\xi^3 u + \partial_\xi u^2 = \Phi \partial_t \partial_\xi B$$

where $\partial_t \partial_\xi B$ a white noise on $\mathbb{R} \times \mathbb{T}$ and where Φ is a linear operator such that $\Phi e_k = \lambda_k e_k$ where $\{e_k\}_{k \in \mathbb{Z}}$ is the trigonometric basis and where $\lambda_0 = 0$.

Rewrite as

$$v_t = v_s + w_t - w_s + \int_s^t \dot{X}_\sigma(v_\sigma, v_\sigma) d\sigma$$

where $w_t = U(-t)\Phi \partial_\xi B(t, \cdot)$.

Rough equation

For any path controlled in the sense that

$$v_t = v_s + w_t - w_s + X^{\bullet}_{ts}(z_s) + O(|t-s|^{2\gamma})$$

define

$$\int_0^t X_{\mathrm{ds}}(v_s, v_s) = \lim_{\Delta t \to 0} \sum_i X^{\bullet}_{t_{i+1},t_i}(v_{t_i}, v_{t_i}) + X^{\substack{\bullet\\|\\\bullet}}_{t_{i+1},t_i}(v_{t_i}, z_{t_i}, z_{t_i}) + X^{w}_{t_{i+1},t_i}(v_{t_i})$$

where it appears the (random) cross iterated integral:

$$X^{w}_{ts}(\varphi) = \int_s^t d\sigma \dot{X}_\sigma(\varphi, w_\sigma - w_s).$$

Under natural assumptions on Φ: $\|w_t - w_s\|_{H^\alpha} + \|X^{w}_{ts}\|^{1/2}_{\mathcal{L}H^\alpha} \lesssim |t-s|^\gamma$ and it is possible to solve

$$v_t = v_0 + \int_0^t X_{\mathrm{d}\sigma}(v_\sigma, v_\sigma) + w_t$$

obtaning existence and uniqueness of rough solutions to the noisy KdV.

This cover the results of [De Bouard-Debussche-Tsutsumi].

Power series solutions to dispersive equations

Power series solutions to dispersive equations have been recently explored

- [Christ] (modified) non-linear Schödinger equation

$$\partial_t u + i\partial_\xi^2 u + (|u|^2 - \int |u|^2)u = 0$$

- [Nguyen] (modified) modified-KdV

$$\partial_t u + \partial_\xi^3 u + (u^2 - \int u^2)\partial_\xi u = 0$$

In both cases the existence result can be interpreted as the existence of a rough solution. Rough path theory gives also a way to enforce uniqueness of these weak solutions.

Schrödinger equation with random dispersion

Consider the (Stratonovich-) stochastic Schrödinger equation

$$d\phi_t = i\Delta\phi_t \circ \mathrm{d}B_t + |\phi_t|^2\phi_t \mathrm{d}t$$

for $\phi : [0, T] \times \mathbb{T} \to \mathbb{C}$.

[Debussche–De Bouard]

Let

$$U_t = e^{i\Delta B_t}$$

so that distributionally

$$dU_t = i\Delta U_t \circ \mathrm{d}B_t$$

and set $\psi_t = U_t^{-1}\phi_t$. Then

$$\psi_t = \psi_0 + \int_0^t U_s^{-1}(|U_s\psi_s|^2 U_s\psi_s)\mathrm{d}s.$$

Regularization

Define

$$X_t(\theta) = \int_0^t U_s^{-1}(|U_s\theta|^2 U_s\theta)\mathrm{d}s$$

It turns out that this random map has the following pathwise regularity

$$\|X_t(\theta) - X_s(\theta)\|_{L^2(\mathbb{T})} \lesssim |t-s|^{\gamma} \|\theta\|^3_{L^2(\mathbb{T})}$$

for some $\gamma > 1/2$.

Then take any $\theta \in C^{\gamma}([0,T], L^2(\mathbb{T}))$ and consider

$$\lim_{\Delta t \to 0} \sum_i [X_{t_{i+1}}(\theta_{t_i}) - X_{t_i}(\theta_{t_i})] = \int_0^t X_{\mathrm{d}s}(\theta_s).$$

By Young theory we have the estimate

$$\left\| \int_0^{\cdot} X_{\mathrm{d}s}(\theta_s) \right\|_{C^{\gamma}([0,T],L^2(\mathbb{T}))} \lesssim_X \|\theta\|^3_{C^{\gamma}([0,T],L^2(\mathbb{T}))}.$$

Existence and uniqueness of global solutions

The Schrödinger equation can be reformulated as a Young equation

$$\psi_t = \psi_0 + \int_0^t X_{\mathrm{d}s}(\psi_s)$$

giving rise to local solutions

$$\psi \in C^\gamma([0, T_*], L^2(\mathbb{T})).$$

The L^2 conservation law allows to obtain global solutions.

Standard arguments for Young equations allows to prove convergence of the Euler scheme

$$\psi_{t_{i+1}} = \psi_{t_i} + [X_{t_{i+1}}(\psi_{t_i}) - X_{t_i}(\psi_{t_i})]$$

Stochastic Burgers equation

[joint work with M. Jara]

Here the stochastic Burgers equation on $\mathbb{T} = [-\pi, \pi]$

$$\mathrm{d}u_t = \frac{1}{2}\partial_\xi^2 u_t(\xi)\mathrm{d}t + \frac{1}{2}\partial_\xi (u_t(\xi))^2\mathrm{d}t + \partial_\xi \mathrm{d}W_t$$

where $\mathrm{d}W_t$ is space-time white noise.

The solution u would like to be the derivative of the solution of the Kardar–Parisi–Zhang equation

$$\mathrm{d}h_t = \frac{1}{2}\partial_\xi^2 h_t(\xi)\mathrm{d}t + \frac{1}{2}(\partial_\xi h_t(\xi))^2\mathrm{d}t + \mathrm{d}W_t. \tag{1}$$

which is believed to capture the macroscopic behavior of a large class of surface growth phenomena.

Problems with the weak formulation

For sufficiently smooth test functions $\varphi : \mathbb{T} \to \mathbb{R}$ look for solutions of

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial_\xi^2 \varphi) \mathrm{d}s + \int_0^t \langle \partial_\xi \varphi, B(u_s) \rangle \mathrm{d}s + W_t(\partial_\xi \varphi)$$

where $B(u_s)(\xi) = (u_s(\xi))^2$.

- We would like to start the equation from initial condition u_0 which is space white noise, this is expected to be an invariant measure.
- The linearized equation

$$X_t(\varphi) = u_0(\varphi) + \int_0^t X_s(\partial_\xi^2 \varphi) \mathrm{d}s + W_t(\partial_\xi \varphi)$$

  has trajectories which look like white noise in space.

$\Rightarrow$ The nonlinear term $B(u_s)$ is not defined.

Lazy smoothing estimation

Call "good" a process y such that

$$y_t(\varphi) = y_0(\varphi) + \int_0^t v_s(\partial_\xi^2 \varphi)\mathrm{d}s + \mathcal{A}_t(\varphi) + W_t(\partial_\xi \varphi)$$

where

- $\mathcal{A}_t(\varphi)$ is a zero-quadratic variation process
- y_t is space-time white noise at all times
- The reversed process $\hat{y}_t = y_{T-t}$ has the same properties with drift $\widehat{\mathcal{A}} = -\mathcal{A}$.

Forward/backward Itô trick

Adding Itô formula for the finite quadratic variation process y

$$h(y_t) = h(y_0) + \int_0^t L^0 h(y_s)\mathrm{d}s + \int_0^t Dh(y_s)\mathrm{d}\mathcal{A}_s + M_t^+$$

(here L^0 is the OU generator) with Itô formula for the backward process

$$h(y_{T-t}) = h(y_T) + \int_T^{T-t} L^0 h(y_{T-s})\mathrm{d}s - \int_T^{T-t} Dh(y_{T-s})\mathrm{d}\mathcal{A}_{T-s} + M_t^-$$

gives

$$M_t^+ - M_{T-t}^- + M_T^- = \int_0^t 2L^0 h(y_s)\mathrm{d}s$$

Easy to find an H such that $2L^0 H = \partial_\xi B$ which allows to replace the Burgers drift

$$\int_0^t \partial_\xi B(y_s) \mathrm{d}s$$

with a sum of forward and backward martingales such that

$$\langle M^\pm(\varphi)\rangle_T = \int_0^T \mathcal{E}(\langle \varphi, H(\cdot)\rangle)(y_s) \mathrm{d}s$$

where

$$\mathcal{E}(h)(x) = \sum_{q \in \mathbb{Z}_0} q^2 (D_q h)(x)^2.$$

The function $\mathcal{E}(\langle \varphi, H(\cdot)\rangle)(y_s)$ is now well defined for y_s sampled according white noise and we can estimate it.

Formulation of the equation

Let $B_\varepsilon(x) = B(\rho_\varepsilon * x)$ a regularization of the non-linearity.

By previous arguments we have that for good processes y this limit exists

$$\lim_{\varepsilon \to 0} \int_0^t \langle \varphi, \partial_\xi B_\varepsilon(y_s) \rangle \mathrm{d}s = \mathcal{B}_t(\varphi)$$

and we can use it to define the drift in the Burgers equation.

A solution u of the Burgers equation is a good process such that

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial_\xi^2 \varphi) \mathrm{d}s + \mathcal{B}_t(\varphi) + W_t(\partial_\xi \varphi)$$

The Itô trick provides compactness estimates for Galerkin approximation. Uniqueness is open (in this approach).

The process $\mathcal{B}_t(\varphi)$ is only $3/2-$ Hölder in time.

Thanks