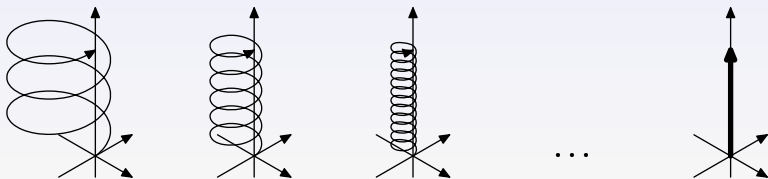# Rough paths in Spain

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Motion in the third dimension

$$dz = ydx - xdy, \qquad z_t = \int_0^t xdy - ydx = \int dxdy - dydx$$

$$(x^n, y^n) \to (0,0), \qquad z^n \to t \neq 0$$

z encode "microscopic" informations on the trajectory (x, y).

Series solution to differential equations

The solution y of the differential equation

$$dy = f(y)dt, \qquad y_0 = \eta$$

has the B-series representation

$$y_t = \eta + \sum_{\tau \in \mathcal{T}} \psi^f(\tau)(\eta) \frac{t^{|\tau|}}{\sigma(\tau)\tau!}$$

Elementary differentials ψ^f defined as

$$\psi^f(\bullet)(\xi) = f(\xi), \qquad \psi^f([\tau^1 \cdots \tau^k]) = f_{\bar{b}}(\eta)\psi^f(\tau^1)(\xi)^{b_1} \cdots \psi^f(\tau^k)(\xi)^{b_k}$$

where $f_\emptyset(\xi) = f(\xi)$ and $f_{\bar{b}}(\xi) = \prod_{i=1}^{|\bar{b}|} \partial_{\xi_{b_i}} f(\xi)$ derivatives of the vectorfields.

Trees

$\mathcal{L}$ finite set. Trees labeled by $\mathcal{L}$, $\mathcal{T}_\mathcal{L}$

$(\tau_1, \cdots, \tau_k) \overset{B^a_+}{\mapsto} \tau = [\tau_1, \cdots, \tau_k]_a$

$$[\bullet] = \quad \quad [\bullet, [\bullet]] = \quad, \quad \quad \text{etc...}$$

- Size of the tree $|\tau|$: $|\bullet| = 1$, $\quad |[\tau_1 \cdots \tau_n]| = 1 + |\tau_1| + \cdots + |\tau_n|$
- Tree factorial $\tau!$: $\bullet! = 1$, $\quad [\tau_1 \cdots \tau_n]! = |[\tau_1 \cdots \tau_n]| \tau_1! \cdots \tau_n!$

Driven differential equations

Given a collection of paths $\{x^a \in C^1([0,T], \mathbb{R})\}_{a \in \mathcal{L}}$, $\eta \in \mathbb{R}^n$
Analytic vectorfields $\{f_a : \mathbb{R}^n \to \mathbb{R}^n\}_{a \in \mathcal{L}}$

The differential equation

$$dy_t = f_a(y_t)dx_t^a, \qquad y_0 = \eta$$

admit locally the series solution

$$y_t = y_s + \sum_{\tau \in \mathcal{T}_{\mathcal{L}}} \frac{1}{\sigma(\tau)} \phi^f(\tau)(y_s) X_{ts}^\tau, \qquad y_0 = \eta$$

where $\phi^f(\bullet_a)(\xi) = f_a(\xi)$,
$\phi^f([\tau^1 \cdots \tau^k]_a)(\xi) = f_{a;b_1\ldots b_k}(\xi) \prod_{i=1}^k [\phi^f(\tau^i)(\xi)]^{b_i}$.

Smooth iterated integrals

Let $X : \mathfrak{T}_{\mathcal{L}} \to \mathcal{C}_2 \subset C([0,T]^2; \mathbb{R})$

$$X_{ts}^{\bullet_a} = \int_s^t dx_u^a, \qquad X_{ts}^{[\tau^1 \cdots \tau^k]_a} = \int_s^t \prod_{i=1}^k X_{us}^{\tau^i} dx_u^a. \tag{1}$$

Extend X to $\mathcal{A}\mathfrak{T}_{\mathcal{L}}$ considering $\mathcal{C}_2$ as an algebra with (commutative) product $(a \circ b)_{ts} = a_{ts} b_{ts}$ for $a, b \in \mathcal{C}_2$. We let $X^1 = 1$.

$$X_{ts}^{\tau_1 \cdots \tau_n} = X_{ts}^{\tau_1} X_{ts}^{\tau_2} \cdots X_{ts}^{\tau_n}, \qquad X^{[\tau_1 \cdots \tau_n]_a} = \int X^{\tau_1 \cdots \tau_n} dx^a$$

Bounds

$$|X_{ts}^{\tau}| \leqslant \frac{(A|t-s|)^{|\tau|}}{\tau!}$$

Example

$$T_{ts}^{\bullet} = t - s, \qquad T_{ts}^{[\tau_1 \cdots \tau_n]} = \int_s^t T_{us}^{\tau_1} \cdots T_{us}^{\tau_n} du$$

By induction: $T_{ts}^{\tau} = (t-s)^{|\tau|}(\tau!)^{-1}$

Lemma (Tree Binomial)

For every $\tau \in \mathfrak{T}$ and $a, b \geqslant 0$ we have

$$(a+b)^{|\tau|} = \sum_i \frac{\tau!}{\tau_i^{(1)}! \tau_i^{(2)}!} a^{|\tau_i^{(1)}|} b^{|\tau_i^{(2)}|} \tag{2}$$

Theorem (tree multiplicative property)

$$X_{ts}^{\tau} = \sum X_{tu}^{\tau^{(1)}} X_{us}^{\tau^{(2)}} = X_{tus}^{\Delta\tau} = X_{tu}^{\tau} + X_{us}^{\tau} + X_{tus}^{\Delta'\tau}$$

Connes-Kreimer coproduct:

$$\Delta : \mathcal{AT} \to \mathcal{AT} \otimes \mathcal{AT}$$

algebra homomorphism defined recursively by

$$\Delta(\tau) = 1 \otimes \tau + \sum_{a \in \mathcal{L}} (B_+^a \otimes \mathrm{id})[\Delta(B_-^a(\tau))]$$

$$B_-^a(B_+^b(\tau_1 \cdots \tau_n)) = \begin{cases} \tau_1 \cdots \tau_n & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

Example

$$\Delta \, \mathbf{Y} = 1 \otimes \mathbf{Y} + \mathbf{Y} \otimes 1 + \bullet \otimes \mathbf{V} + \mathbf{I} \otimes \bullet \bullet + 2 \mathbf{I} \otimes \bullet$$

Notation: $\Delta\tau = \sum \tau^{(1)} \otimes \tau^{(2)}$, reduced coproduct

$$\Delta'\tau = \Delta\tau - 1 \otimes \tau - \tau \otimes 1$$

Some examples

Forests with $|\tau| \leqslant 3$

$$\bullet, \, \mathbf{\mathord{\mathbf{\textbf{\i}}}}, \, \bullet\bullet, \, \mathbf{\mathord{\mathbf{\textbf{\i}}}}, \, \bullet\mathbf{\mathord{\mathbf{\textbf{\i}}}}, \, \bullet\bullet\bullet, \, \mathbf{Y}$$

$$\Delta'\,\mathbf{\mathord{\textbf{\i}}} = \bullet \otimes \bullet$$

$$\Delta'(\bullet\bullet) = 2\bullet \otimes \bullet$$

$$\Delta'\,\mathbf{\mathord{\textbf{\i}}} = \mathbf{\mathord{\textbf{\i}}} \otimes \bullet + \bullet \otimes \mathbf{\mathord{\textbf{\i}}}$$

$$\Delta'(\bullet\mathbf{\mathord{\textbf{\i}}}) = \bullet\otimes\bullet\bullet + \bullet\bullet\otimes\bullet + \mathbf{\mathord{\textbf{\i}}}\otimes\bullet + \bullet\otimes\mathbf{\mathord{\textbf{\i}}}$$

$$\Delta'(\bullet^3) = 3\bullet^2\otimes\bullet + 3\bullet\otimes\bullet^2$$

$$\Delta'\,\mathbf{Y} = \bullet\otimes\bullet\bullet + 2\mathbf{\mathord{\textbf{\i}}}\otimes\bullet$$

$$\delta X_{tus}^{\mathbf{\mathord{\textbf{\i}}}} = X_{tu}^{\bullet} X_{us}^{\bullet}$$

$$\delta X_{tus}^{\bullet\bullet} = 2 X_{tu}^{\bullet} X_{us}^{\bullet}$$

$$\delta X_{tus}^{\mathbf{\mathord{\textbf{\i}}}} = X_{tu}^{\mathbf{\mathord{\textbf{\i}}}} X_{us}^{\bullet} + X_{tu}^{\bullet} X_{us}^{\mathbf{\mathord{\textbf{\i}}}}$$

$$\delta X_{tus}^{\bullet\mathbf{\mathord{\textbf{\i}}}} = X_{tu}^{\bullet} X_{us}^{\bullet\bullet} + X_{tu}^{\bullet\bullet} X_{us}^{\bullet} + X_{tu}^{\mathbf{\mathord{\textbf{\i}}}} X_{us}^{\bullet} + X_{tu}^{\bullet} X_{us}^{\mathbf{\mathord{\textbf{\i}}}}$$

$$\delta X_{tus}^{\bullet^3} = 3 X_{tu}^{\bullet^2} X_{us}^{\bullet} + 3 X_{tu}^{\bullet} X_{us}^{\bullet^2}$$

$$\delta X_{tus}^{\mathbf{Y}} = X_{tu}^{\bullet} X_{us}^{\bullet\bullet} + 2 X_{tu}^{\mathbf{\mathord{\textbf{\i}}}} X_{us}^{\bullet}$$

Structure of solution to DDEs

Write $y_s^\tau = \phi^f(\tau)(y_s)/\sigma(\tau)$ so that

$$y_t - y_s = \sum_{\tau \in \mathcal{T}_\mathcal{L}} X_{ts}^\tau y_s^\tau$$

For any $\tau \in \mathcal{T}_\mathcal{L} \cup \{\emptyset\}$ we have

$$y_t^\tau - y_s^\tau = \sum_{\sigma \in \mathcal{T}_\mathcal{L}, \rho \in \mathcal{F}_\mathcal{L}} c'(\sigma, \tau, \rho) X_{ts}^\rho y_s^\sigma$$

c' counting function of reduced coproduct:

$$\Delta'\sigma = \sum_{\tau, \rho} c'(\sigma, \tau, \rho)\tau \otimes \rho$$

Series truncation

Take the simplest truncation

$$y_t - y_s = f(y_s)X_{ts}^\bullet + r_{ts}$$

Is r_{ts} really negligible?

Take a partition $\{\tau_i\}_i$ of $[s,t]$ and try to recover the path from accumulation of small increments:

$$y_t - y_s = \sum_i (y_{\tau_{i+1}} - y_{\tau_i}) \simeq \sum_i f(y_{\tau_i})X_{\tau_{i+1}\tau_i}^\bullet$$

Adding a point c between the points a, b of a partition gives a contribution of

$$f(y_c)X_{bc}^\bullet + f(y_a)X_{ab}^\bullet - f(y_a)X_{ac}^\bullet = (f(y_c) - f(y_a))X_{bc}^\bullet \sim O(|X_{bc}^\bullet||X_{ac}^\bullet|)$$

where we used the relation $X_{ac}^\bullet = X_{ab}^\bullet + X_{bc}^\bullet$

The sewing map

Assume a_{ts} given and solve

$$\delta f_{ts} = f_t - f_s = a_{ts} + r_{ts}$$

with $r_{ts} = o(|t - s|)$. If a solution (f, r) exists it is unique.

Solution exists if $\delta a_{tus} = a_{ts} - a_{tu} - a_{us}$ is small ($\delta a_{tus} = o(|t - s|)$).

Also $\delta \delta g = 0$ which gives $\delta a = -\delta r$.

The map $\Lambda : \delta a \mapsto r$ is called the sewing map :

$$\delta f = a - \Lambda \delta a = (1 - \Lambda \delta)a$$

$$\delta(1 - \Lambda \delta)a = \delta a - \delta \Lambda \delta a = \delta a - \delta a = 0 \Rightarrow a = \delta f$$

Examples

- Convergence of sums:

$$S_{t0} = \sum_i a_{t_{i+1} t_i} = \sum_i (\delta f)_{t_{i+1} t_i} + \sum_i (r)_{t_{i+1} t_i}$$

$$= (\delta f)_{t0} + \sum_i o(|t_{i+1} - t_i|) \to f_t - f_0$$

- Young integrals: $x \in C^\gamma$, $\gamma > 1/2$. Take $a_{ts} = \varphi(x_s)\delta x_{ts}$

$$\delta a_{tus} = \delta\varphi(x)_{tu}\delta x_{us} = o(|t-s|^{2\gamma})$$

$$\Rightarrow \delta f = (1 - \Lambda\delta)a =: \int \varphi(x)dx$$

(Step-2) Rough paths

Rough integrals ($\gamma > 1/3$):

$$X_{ts}^{\bullet} = \delta x_{ts}$$

$$\delta X_{tus}^{[\bullet]} = X_{tu}^{\bullet} X_{us}^{\bullet}$$

$$X^{\bullet} \in \mathcal{C}_2^{\gamma}, X^{[\bullet]} \in \mathcal{C}_2^{2\gamma}$$

Take

$$a_{ts} = \varphi(x_s)X_{ts}^{\bullet} + \varphi'(x_s)X_{ts}^{[\bullet]}$$

$$\delta a_{tus} = (-\delta\varphi(x)_{tu} + \varphi'(x_u)X_{tu}^{\bullet})X_{us}^{\bullet} - \delta\varphi'(x)_{tu}X_{us}^{[\bullet]} = o(|t-s|^{3\gamma})$$

$$\oint \varphi(x)dx = (1 - \Lambda\delta)(a)$$

- Continuous map:

$$(\varphi, X^{\bullet}, X^{[\bullet]}) \mapsto \oint \varphi(x)dx$$

- Renormalized sums:

$$\sum_i (\varphi(x_{t_i})X^{\bullet}_{t_{i+1}t_i} + \varphi'(x_{t_i})X^{[\bullet]}_{t_{i+1}t_i}) \to \oint \varphi(x)dx$$

- $\oint \oint dxdx = X^{[\bullet]}$
- A finite number of iterated integrals determines all the other integrals.

$$\delta X^{[[\bullet]]]} = X^{[[\bullet]]}X^{\bullet} + X^{[\bullet]}X^{[[\bullet]]}$$

$$\Rightarrow X^{[[[\bullet]]]} = \Lambda(X^{[[\bullet]]}X^{\bullet} + X^{[\bullet]}X^{[[\bullet]]})$$

and $X^{[[[\bullet]]]}_{ts} = O(|t-s|^{3\gamma})$. And so on. . .

Geometric rough paths

For smooth paths we have the shuffle relation :

$$\int dx^{a_1} \cdots dx^{a_n} \circ \int dx^{b_1} \cdots dx^{b_m} = \sum_{\overline{c} \in \mathrm{Sh}(\overline{a}, \overline{b})} \int dx^{c_1} \cdots dx^{c_{n+m}}$$

This relation reduces X^τ for $\tau \in \mathcal{T}_\mathcal{L}$ to a linear combination of $\{X^\sigma\}_{\sigma \in \mathcal{T}_\mathcal{L}^{\mathrm{Chen}}}$.

Geometric rough-paths are the closure of smooth rough paths.

Growing a branched rough path

Fix $\gamma \in (0,1]$, consider $q_\gamma : \mathcal{F} \to \mathbb{R}_+$ on forests as $q_\gamma(\tau) = 1$ for $|\tau| \leqslant 1/\gamma$ and

$$q_\gamma(\tau) = 1, \text{ if } |\tau| \leqslant 1/\gamma \qquad q_\gamma(\tau) = \frac{1}{2^{\gamma|\tau|}} \sum q_\gamma(\tau^{(1)}) q_\gamma(\tau^{(2)}) \text{ otherwise}$$

$$q_\gamma(\tau_1 \cdots \tau_n) = q_\gamma(\tau_1) \cdots q_\gamma(\tau_n).$$

Given a partial homomorphism $X : \mathcal{A}_n \mathcal{T}_\mathcal{L} \to \mathcal{C}_2$ satisfying the multiplicative property

$$|X_{ts}^\tau| \leqslant B A^{|\tau|} q_\gamma(\tau) |t-s|^{\gamma|\tau|}, \qquad \tau \in \mathcal{T}_\mathcal{L}^n$$

with $\gamma(n+1) > 1$, then $\exists!\ X : \mathcal{A}\mathcal{T}_\mathcal{L} \to \mathcal{C}_2$ with same bounds $\forall \tau \in \mathcal{T}_\mathcal{L}$.

Construction via the equation: $X^\tau = \Lambda(X^{\Delta'\tau})$.

Speed of growth?

Conjecture

$$q_\gamma(\tau) \asymp C(\tau!)^{-\gamma}$$

True for linear Chen trees $\mathcal{T}^{\mathrm{Chen}}$:

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(k!)^\gamma (n!)^\gamma} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(n!)^\gamma}, \qquad \gamma \in (0,1], \ a,b \geqslant 0$$

Variant of Lyons' neo-classical inequality

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(\gamma k)! [\gamma(n-k)n]!} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(\gamma n)!}$$

"neo-classical tree inequality"?

$$\sum \frac{a^{\gamma|\tau^{(1)}|} b^{\gamma|\tau^{(2)}|}}{(\tau^{(1)}!)^\gamma (\tau^{(2)}!)^\gamma} \leqslant c_\gamma \frac{(a+b)^{\gamma|\tau|}}{(\tau!)^\gamma}$$

OK for $\gamma = 1$: tree binomial formula.

Controlled paths

Definition

Let n the largest integer such that $n\gamma \leqslant 1$. For any $\kappa \in (1/(n+1), \gamma]$ a path y is a κ-*weakly controlled* by X if

$$\delta y = \sum_{\tau \in \mathcal{F}_{\mathcal{L}}^{n-1}} X^\tau y^\tau + y^\sharp, \qquad \delta y^\tau = \sum_{\sigma \in \mathcal{F}_{\mathcal{L}}^{n-1}} \sum_\rho c'(\sigma, \tau, \rho) X^\rho y^\sigma + y^{\tau,\sharp}, \qquad \tau \in \mathcal{F}_{\mathcal{L}}^{n-1}$$

with $y^\tau \in \mathcal{C}_2^{|\tau|\kappa}, y^{\sharp,\tau} \in \mathcal{C}_2^{(n-|\tau|)\kappa}$. Then we write $y \in \mathcal{Q}_\kappa(X; V)$.

Lemma (Stability)

Let $\varphi \in C_b^n(\mathbb{R}^k, \mathbb{R})$ and $y \in \mathcal{Q}_\kappa(X; \mathbb{R}^k)$, then $z_t = \varphi(y_t)$ is a weakly controlled path, $z \in \mathcal{Q}_\kappa(X; \mathbb{R})$ where its coefficients are given by

$$z^\tau = \sum_{m=1}^{n-1} \sum_{|\bar{b}|=m} \frac{\varphi_{\bar{b}}(y)}{m!} \sum_{\substack{\tau_1,\ldots,\tau_m \in \mathcal{F}_{\mathcal{L}}^{n-1} \\ \tau_1 \cdots \tau_m = \bar{\tau}}} y^{\tau_1,b_1} \cdots y^{\tau_m,b_m}, \qquad \tau \in \mathcal{F}_{\mathcal{L}}^{n-1}$$

Integration of controlled paths

Theorem

The integral maps $\{I^a\}_{a \in \mathcal{L}}$ can be extended to maps $I^a : \mathcal{Q}_\kappa(X) \to \delta\mathcal{Q}_\kappa(X)$

$$y \in \mathcal{Q}_\kappa(X) \mapsto \delta z = I^a(y) = X^{\bullet_a} z^{\bullet_a} + \sum_{\tau \in \mathcal{T}^n_{\mathcal{L}}} X^\tau z^\tau + z^\flat, \tag{3}$$

where $z^\flat \in \mathcal{C}_2^{\kappa(n+1)}$, $z^{\bullet_a} = y$, $z^{[\tau]_a} = y^\tau$ and zero otherwise.

Remark

If $y \in \mathcal{Q}_\kappa(X; \mathbb{R}^n \otimes \mathbb{R}^d)$ then $\{J^b(\cdot) = \sum_{a \in \mathcal{L}} I^a(y^{ab} \cdot)\}_{b \in \mathcal{L}_1}$ defines a family of integrals with an associated branched rough path Y indexed by $\mathcal{T}_{\mathcal{L}_1}$. An explicit recursion is

$$Y^{\bullet_b} = \sum_{a \in \mathcal{L}} I^a(y^{ab}), \qquad Y^{[\tau^1 \cdots \tau^k]_b} = \sum_{a \in \mathcal{L}} I^a(y^{ab} Y^{\tau^1} \circ \cdots \circ Y^{\tau^k}), \qquad b \in \mathcal{L}_1$$

Example

$$\delta y = X^\bullet y^\bullet + X^{\text{\ding{89}}} y^{\text{\ding{89}}} + X^{\bullet\bullet} y^{\bullet\bullet} + X^{\curlyvee} y^{\curlyvee} + X^{\text{\ding{89}}\bullet} y^{\text{\ding{89}}\bullet} + X^{\curlyvee} y^{\curlyvee} + X^{\bullet\bullet\bullet} y^{\bullet\bullet\bullet} + X^{\text{\ding{89}}} y$$

$$\delta y^\bullet = X^\bullet (y^{\text{\ding{89}}} + 2y^{\bullet\bullet}) + X^{\text{\ding{89}}} (y^{\text{\ding{89}}} + y^{\text{\ding{89}}\bullet}) + X^{\bullet\bullet}(y^{\text{\ding{89}}\bullet} + y^{\curlyvee} + 3y^{\bullet\bullet\bullet}) + y^{\bullet,\sharp}$$

$$\delta y^{\text{\ding{89}}} = X^\bullet (y^{\text{\ding{89}}\bullet} + 2y^{\curlyvee} + y^{\text{\ding{89}}}) + y^{\text{\ding{89}},\sharp}$$

$$\delta y^{\bullet\bullet} = X^\bullet (y^{\text{\ding{89}}\bullet} + y^{\bullet\bullet\bullet}) + y^{\bullet\bullet,\sharp}$$

$$\delta y^{\curlyvee} = y^{\curlyvee,\sharp} \qquad \delta y^{\text{\ding{89}}\bullet} = y^{\text{\ding{89}}\bullet,\sharp} \qquad \delta y^{\bullet\bullet\bullet} = y^{\bullet\bullet\bullet,\sharp} \qquad \delta y^{\text{\ding{89}}} = y^{\text{\ding{89}},\sharp}$$

$$\delta z = \delta I(y) = X^\bullet y + X^{\text{\ding{89}}} y^\bullet + X^{\text{\ding{89}}} y^{\text{\ding{89}}} + X^{\curlyvee} y^{\bullet\bullet} + X^{\curlyvee} y^{\text{\ding{89}}\bullet} + X^{\text{\ding{89}}} y^{\curlyvee} + X^{\curlyvee} y^{\bullet\bullet\bullet} + $$
$$= X^\bullet z^\bullet + X^{\text{\ding{89}}} z^{\text{\ding{89}}} + X^{\text{\ding{89}}} z^{\text{\ding{89}}} + X^{\curlyvee} z^{\curlyvee} + z^\sharp$$

with

$$z^\flat = \Lambda \left[X^\bullet y^\sharp + X^{\text{\ding{89}}} y^{\bullet,\sharp} + X^{\text{\ding{89}}} y^{\text{\ding{89}},\sharp} + X^{\curlyvee} y^{\bullet\bullet,\sharp} + X^{\text{\ding{89}}} y^{\curlyvee,\sharp} + X^{\curlyvee} y^{\text{\ding{89}}\bullet,\sharp} + X^{\curlyvee} y^{\bullet\bullet\bullet,\sharp} \right.$$

Rough differential equations

Take vectorfields $\{f_a \in C_b^n(\mathbb{R}^k; \mathbb{R}^k)\}_{a \in \mathcal{L}}$ and integral maps $\{I^a\}_{a \in \mathcal{L}}$ and consider the *rough differential equation*

$$\delta y = I^a(f_a(y)), \qquad y_0 = \eta \in \mathbb{R}^k$$

or equivalently

$$y_t = y_s + \sum_{\tau:|\tau| \leqslant n} f_\tau(y_s) X_{ts}^\tau + o(|t-s|)$$

in the time interval $[0, T]$.

This rough differential equation has a global solution $y \in \mathcal{Q}_\gamma(X; \mathbb{R}^k)$ for any initial condition $\eta \in \mathbb{R}^k$. If the vectorfields are C_b^{n+1} the solution is unique and has Lipshitz dependence on data.

Euler methods

An equivalent construction of the solution can be obtained via discretization:

$$y^n_{(k+1)/n} = y^n_{k/n} + \sum_{\tau:\gamma|\tau|\leqslant 1} f_\tau(y^n_{k/n})X^\tau_{(k+1)/n,k/n} \qquad y^n_0 = \eta$$

for $0 \leqslant k/n \leqslant T$. Then

$$y^n \to_{C^\gamma} y \qquad \text{as } n \to \infty$$

PDE examples

- The KdV equation as a rough path
- The Navier-Stokes equation
- Heat equation with multiplicative noise
- Nonlinear parabolic evolution equations

The KdV equation

Our toy equation: the 1d periodic KdV equation

$$\begin{cases} \partial_t u(t, \xi) + \partial_\xi^3 u(t, \xi) + \frac{1}{2}\partial_\xi u(t, \xi)^2 = 0 \\ u(0, \xi) = u_0(\xi) \end{cases} \qquad (t, \xi) \in \mathbb{R} \times \mathbb{T}$$

with initial condition $u_0 \in H^\alpha(\mathbb{T})$, $\mathbb{T} = [-\pi, \pi]$.

- Low regularity theory [Bourgain, Kenig, Ponce, Vega, Colliander, Keel, Staffilani, Takaoka, Tao, ...]. Global solutions for initial conditions in $H^{-1/2}$.
- No uniformly continuous dependence on initial conditions for $\alpha < -1/2$.
- Using complete integrability the solution map can be extended by continuity up to $\alpha = -1$ [Kappeler, Topalov].

Mild form (Duhamel's formula)

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\partial_\xi u(s)^2 \, ds$$

Linear part: Airy group $\mathcal{F}(U(t)\varphi)(k) = e^{-ik^3 t}\hat{\varphi}(k)$, $k \in \mathbb{Z}$.

Abstract formulation

Taking advantage of the fact that U is a group (not just semi-group) operate the change of variables $v(t) = U(-t)u(t)$:

$$v(t) = u_0 + \int_0^t U(-s)\partial_\xi [U(s)v(s)]^2 \, ds$$

For all $s < t$

$$v_t = v_s + \int_s^t \dot{X}_\sigma(v_\sigma, v_\sigma) d\sigma$$

with $v_0 = u_0$.

Bilinear operator

$$\mathcal{F}\dot{X}_\sigma(\varphi_1, \varphi_2)(k) = ik \sum_{k_1+k_2=k} e^{-(k^3-k_1^3-k_2^3)\sigma} \hat{\varphi}_1(k_1)\hat{\varphi}_2(k_2)$$

Expansion

$$v_t - v_s = X_{ts}^{\bullet}(v_s^{\times 2}) + X_{ts}^{\mathbf{I}}(v_s^{\times 3}) + X_{ts}^{\mathbf{I}}(v_s^{\times 4}) + X_{ts}^{\mathbf{Y}}(v_s^{\times 4}) + r_{ts}$$

where r_{ts} is a remainder term.
Tree-indexed multi-linear operators appear:

$$X_{ts}^{\bullet}(\varphi_1, \varphi_2) = \int_s^t \dot{X}_\sigma(\varphi_1, \varphi_2) d\sigma$$

$$X_{ts}^{\mathbf{I}}(\varphi_1, \varphi_2, \varphi_3) = \int_s^t \dot{X}_\sigma(X_{\sigma s}^{\bullet}(\varphi_1, \varphi_2), \varphi_3) d\sigma$$

$$X_{ts}^{\mathbf{Y}}(\varphi_1, \ldots, \varphi_4) = \int_s^t \dot{X}_\sigma(X_{\sigma s}^{\bullet}(\varphi_1, \varphi_2), X_{\sigma s}^{\bullet}(\varphi_3, \varphi_4)) d\sigma$$

etc...

$$X_{ts}^{\mathbf{I}}(\varphi_1, \varphi_2, \varphi_3) = X_{tu}^{\mathbf{I}}(\varphi_1, \varphi_2, \varphi_3) + X_{us}^{\mathbf{I}}(\varphi_1, \varphi_2, \varphi_3) + X_{tu}^{\bullet}(X_{tu}^{\bullet}(\varphi_1, \varphi_2), \varphi_3)$$

Regularity

Main observation: cancellations of high-frequency oscillations

$$\mathcal{F}X_{ts}^{\bullet}(\varphi_1, \varphi_2)(k) = \sum_{k_1+k_2=k} \left[ik \int_s^t e^{-(k^3-k_1^3-k_2^3)\sigma} d\sigma \right] \hat{\varphi}_1(k_1)\hat{\varphi}_2(k_2)$$

$$= \sum_{k_1+k_2=k} \left[\frac{e^{i(3kk_1k_2)t} - e^{i(3kk_1k_2)s}}{3k_1k_2} \right] \hat{\varphi}_1(k_1)\hat{\varphi}_2(k_2)$$

Easy analysis gives

$$\|X_{ts}^{\bullet}\|_{\mathcal{L}(H^\alpha)} \lesssim |t-s|^\gamma$$

with $\gamma \leqslant 1/2$, $\gamma < 1 + \alpha$, $\gamma < \alpha/3 + 1/2$.

Compensations of space and time regularity.
In any case always beyond Young integration $\Rightarrow$ a real rough path !

Rough path regularity

Theorem

$$\|X_{ts}^{\bullet}\|_{\mathcal{L}(H^{\alpha})} + \|X_{ts}^{\bullet\bullet}\|_{\mathcal{L}(H^{\alpha})}^{1/2} \lesssim |t-s|^{\gamma}$$

for $\gamma \leqslant 1/2$, $\gamma < 1 + \alpha$, $\gamma < \alpha/3 + 1/2$, $\alpha \geqslant -1/2$.

Taking $\gamma = (1/3)+$ we get $\alpha = 1/2+$, almost as good as standard theory.

Impossible to go beyond the $\alpha = -1/2$ boundary?
What about expansions to third order?

Fixed point

KdV is a rough equation which can be solved with fixed-point method and rough path estimates.

- Existence: show that there exists a converging sequence $(y^n)_n$ of paths in H^α satisfying

$$y_t^{n+1} = y_s^{n+1} + X^\bullet(y_s^n, y_s^n) + X^{\smash{\tiny\bullet\!\bullet}}(y_s^n, y_s^n, y_s^{n-1}) + O(|t-s|^{3\gamma})$$

where $3\gamma > 1$.

- Uniqueness: prove that there exists a unique path y satisfying

$$y_t = y_s + X^\bullet(y_s, y_s) + X^{\smash{\tiny\bullet\!\bullet}}(y_s, y_s, y_s) + O(|t-s|^{1+})$$

Existence and uniqueness of (local) rough path solution for any initial condition in H^α with $\alpha > -1/2$ (since $\gamma > 1/3$)

Series solution

Explicit series representation of the solution:

$$v_t = v_s + \sum_\tau X_{st}^\tau (v_s^{\times d(\tau)})$$

Assuming $3\gamma > 1$ it is possible to obtain estimates for higher order operators (e.g. $X^{\curlyvee}$ and $X^{\vcenter{\hbox{$\vdots$}}}$) using the estimates on $X^\bullet$ and $X^{\vcenter{\hbox{$\vdots$}}}$.

$$\|X_{ts}^\tau\|_{\mathcal{L}H^\alpha} \lesssim C^{|\tau|} q_\gamma(\tau)|t-s|^{\gamma|\tau|}$$

where $(q_\gamma(\tau))_\tau$ is a universal sequence of numbers depending only on γ.

Problem: I do not know the asymptotics as $|\tau| \to \infty$ of $q_\gamma(\tau)$.

L^2 conservation law

Original operator satisfies a basic symmetry wrt L^2 scalar product:

$$\langle \varphi_1, \dot{X}_s(\varphi_2, \varphi_3) \rangle + \langle \varphi_2, \dot{X}_s(\varphi_1, \varphi_3) \rangle + \langle \varphi_3, \dot{X}_s(\varphi_2, \varphi_1) \rangle = 0$$

Lemma

$$\langle \varphi, X_{ts}(\varphi, \varphi) \rangle = 0 \qquad 2\langle \varphi, X_{ts}^2(\varphi, \varphi, \varphi) \rangle + \langle X_{ts}(\varphi, \varphi), X_{ts}(\varphi, \varphi) \rangle = 0$$

This implies some cancellations for the rough solution to KdV:

$$\begin{aligned}
\langle v_t, v_t \rangle - \langle v_s, v_s \rangle &= 2\langle X_{ts}(v_s, v_s) + X_{ts}^2(v_s, v_s, v_s), v_s \rangle \\
&\quad + \langle X_{ts}(v_s, v_s), X_{ts}(v_s, v_s) \rangle + O(|t-s|^{3\gamma}) \\
&= O(|t-s|^{3\gamma})
\end{aligned}$$

Theorem (Integral conservation law)

If v is a rough solution of KdV then $|v_t|_{L^2} = |v_0|_{L^2}$ for any t.

Galerkin approximations

Consider projection P_N onto modes with $|k| < N$ and the approximate evolution

$$\partial_t u^{(N)} + \partial_\xi^3 u^{(N)} + \frac{1}{2} P_N \partial_\xi (u^{(N)})^2 = 0, \qquad u^{(N)}(0) = P_N u_0$$

Then $v_t^{(N)} = U(-t) u_t^{(N)}$ are rough solution to the equation

$$v_t^{(N)} = v_s^{(N)} + X_{ts}^{\bullet,(N)}(v_s^{(N)}) + X_{ts}^{\mathbf{\cdot},(N)}(v_s^{(N)}) + O(|t-s|^{3\gamma})$$

where $X^{\bullet,(N)} = P_N X^\bullet (P_N \times P_N)$ and where the trilinear operator $X^{\mathbf{\cdot},(N)}$ is defined as

$$X_{ts}^{\mathbf{\cdot},(N)}(\varphi_1, \varphi_2, \varphi_3) = 2 \int_s^t d\sigma \int_s^\sigma d\sigma_1 P_N \dot{X}_\sigma(P_N \varphi_1, P_N \dot{X}_{\sigma_1}(P_N \varphi_2, P_N \varphi_3))$$

We (almost) have

$$X^{\bullet,(N)} \to X^{\bullet}, \qquad X^{\mathbf{\cdot},(N)} \to X^{\mathbf{\cdot}}$$

and the continuity of rough path estimates would imply that $v^{(N)} \to v$: convergence of the Galerkin approximations in Holder norm.

Back to reality: unfortunately $X^{\mathbf{\cdot},(N)} \not\to X^{\mathbf{\cdot}}$ but we can modify the Galerkin approximation to overcome this difficulty and complete the picture.

It is well known that naive Galerkin does not work due to symplectic non-squeezing property of the KdV flow (which is Hamiltonian).

Euler scheme

For any $n > 0$ let $v_0^n = u_0$ and

$$v_i^n = X_{i/n,j/n}^{\bullet}(v_{i-1}^n) + X_{i/n,j/n}^{\text{\textbullet}}(v_{i-1}^n)$$

for $i \geqslant 1$.

Theorem

Let $\Delta_i^n = v_i^n - v_{i/n}$ then

$$\sup_{0 \leqslant i < j \leqslant nT} \frac{|\Delta_i^n - \Delta_j^n|_{H^\alpha}}{|i - j|^\gamma} = O(n^{1-3\gamma}).$$

The combination of this scheme with the Galerkin approximation discussed before provide an implementable numerical approximation scheme for the solutions of KdV with low regularity initial conditions with explicit rates of convergence.

The NS equation as a rough path

The *d*-dimensional NS equation has the abstract form

$$u_t = S_t u_0 + \int_0^t S_{t-s} B(u_s, u_s)\, ds. \tag{4}$$

S bounded semi-group on $\mathcal{B}$, B symmetric bilinear operator.
Define

$$X_{ts}^{\bullet}(\varphi^{\times 2}) = \int_s^t S_{t-u} B(S_{u-s}\varphi, S_{u-s}\varphi)\, du$$

$$X_{ts}^{[\tau^1]}(\varphi^{\times(d(\tau^1)+1)}) = \int_s^t S_{t-u} B(X_{us}^{\tau^1}(\varphi^{\times d(\tau^1)}), S_{u-s}\varphi)\, du$$

and

$$X_{ts}^{[\tau^1 \tau^2]}(\varphi^{\times(d(\tau^1)+d(\tau^2))}) = \int_s^t S_{t-u} B(X_{us}^{\tau^1}(\varphi^{\times d(\tau^1)}), X_{us}^{\tau^2}(\varphi^{\times d(\tau^2)}))\, du$$

where $d(\tau)$ is a degree function.

Bounds on the operators and regularity

For suitable Banach space $\mathcal{B}$

$$|X_{ts}^{\tau}(\varphi^{\times d(\tau)})|_{\mathcal{B}} \leqslant C \frac{|t-s|^{\varepsilon|\tau|}}{(\tau!)^{\varepsilon}} |\varphi|_{\mathcal{B}}^{d(\tau)}$$

where $\varepsilon \geqslant 0$ is a constant depending on $\mathcal{B}$.

Norm convergent series representation

$$u_t = S_{t-s} u_s + \sum_{\tau \in \mathcal{T}_B} X_{ts}^{\tau}(u_s^{\times d(\tau)})$$

gives local solution, global for small data (for $\varepsilon = 0$).

Regularity: $|\hat{u}(k)| \leqslant C e^{-|k|\sqrt{t}}$

[Le Jan & Sznitman, Cannone & Planchon, Sinai, Gallavotti]

Linear evolution equations

Consider the SPDE

$$\frac{\partial}{\partial t}y(t,\xi) = \Delta_\xi y(t,\xi) + y(t,\xi)\frac{\partial^2}{\partial t\partial\xi}x(t,\xi)$$

on the 1d torus $\mathbb{T}$ with initial condition $\overline{y} \in L^2(\mathbb{T})$.
Mild formulation

$$y(t,\xi) = \int_{\mathbb{T}} G_t(\xi - \xi')\overline{y}(\xi')d\xi' + \int_0^t\int_{\mathbb{T}} G_{t-s}(\xi - \xi')y_s(\xi')x(ds,d\xi')$$

▶ Distributional Gaussian noise

$$\mathbb{E}x(dt,d\xi)x(dt',d\xi') \sim \delta(t-t')Q(\xi - \xi')$$

with $Q(\xi) = \sum_{n\in\mathbb{Z}} \lambda_n^{-\nu}e_n(\xi)$, $\Delta e_n = -\lambda_n e_n$.

▶ G_t kernel of the analytic semigroup $S_t = e^{t\Delta}$

Compact form

$$y_t = S_t \overline{y} + \int_0^t S_{t-s} dx_s y_s$$

Iterative solution methods make appear *operator-valued* increments:

$$X_{ts}^1(\varphi) = \int_s^t S_{t-u} dx_u S_{u-s} \varphi \quad X_{ts}^2(\varphi) = \int_s^t \int_s^u S_{t-u} dx_u S_{u-v} dx_v S_{v-s} \varphi \cdots$$

with kernels given by

$$X_{ts}^1(\varphi)(\xi) = \int_{\mathbb{T}} \left[\int_s^t \int_{\mathbb{T}} G_{t-u}(\xi - \xi') x(du, d\xi') G_{u-s}(\xi' - \xi'') \right] \varphi(\xi'') d\xi''$$

and similar formulas for X^n.

Regularity

Theorem

If X^n, $n = 1, 2, 3$ are defined by Itô integrals then

$$X^n \in \mathcal{C}_2^{\gamma + (n-1)\kappa}(\mathcal{L}_{HS}(H^\eta; H^{-\rho})) \cap \mathcal{C}_2^{n\kappa}(\mathcal{L}_{HS}(H^\eta; H^\eta))$$

for any $\eta > 1/4$, $\gamma > \kappa$ satisfying

$$\kappa < 1/4 - \eta + \overline{\nu}/2 \qquad and \qquad \gamma < 1/2$$

where $\rho + \eta = \gamma - \kappa$ and $\overline{\nu} = \min(\nu, 1/2)$.

Proof: extended Garsia-Rodemich-Rumsey inequality for convolutional increments + Gaussian estimates

Convolutional multiplicative proprerty: for $0 \leqslant s \leqslant u \leqslant t$,

$$X_{ts}^n(\varphi) = X_{tu}^n(S_{u-s}\varphi) + S_{t-s}X_{us}^n(\varphi) + \sum_{k=1}^{n-1} X_{tu}^{n-k}(X_{us}^k(\varphi))$$

Assume that $\gamma + 3\kappa > 1$, then

$$H = X^1X^3 + X^2X^2 + X^3X^1 \in \mathcal{C}_3^{\gamma+3\kappa}(\mathcal{L}_{HS}(H^\eta; H^{-\rho}))$$

so we can define X^4, and so on : $X^n \in \mathcal{C}_2^{n\kappa}(\mathcal{L}_{HS}(H^\eta; H^\eta))$

Convergent series expansion

$$y_t = S_{t-s}y_s + \sum_{n=1}^{\infty} X_{ts}^n y_s.$$

Non-linear equations

Consider now:

$$y(t, \xi) = \int_{\mathbb{T}} G_t(\xi - \xi')\overline{y}(\xi')d\xi' + \int_0^t \int_{\mathbb{T}} G_{t-s}(\xi - \xi')[y_s(\xi')]^2 x(ds, d\xi')$$

with a bilinear non-linearity. Abstractly:

$$y_t = S_t\overline{y} + \int_0^t S_{t-s}dx_s B(y_s, y_s).$$

Again, tree-indexed incremental operators:

$$X_{ts}^{\bullet}(\varphi^1, \varphi^2) = \int_s^t S_{t-u}dx_u B(S_{u-s}\varphi^1, S_{u-s}\varphi^2)$$

$$X_{ts}^{\bullet\!\bullet}(\varphi^1, \varphi^2, \varphi^3) = \int_s^t S_{t-u}dx_u B\left(S_{u-s}\varphi^1, X_{us}^{\bullet}(\varphi^2, \varphi^3)\right)$$

$$X_{ts}^{\mathbf{Y}}(\varphi^1, \varphi^2, \varphi^3, \varphi^4) = \int_s^t S_{t-u}dx_u B\left(X_{us}^{\bullet}(\varphi^1, \varphi^2), X_{us}^{\bullet}(\varphi^3, \varphi^4)\right)$$

Thanks

Resonances

A natural boundary appears at $\alpha = -1/2$ due to a *resonance* in $X_{ts}^{\bullet\bullet}$. (non-linear interaction + linear propagation)

Decomposition

$$X_{ts}^{\bullet\bullet} = \hat{X}_{ts}^{\bullet\bullet} + (t-s)\Phi$$

where $\|\hat{X}_{ts}^{\bullet\bullet}\|_{\mathcal{L}(H^\alpha)} \lesssim |t-s|^{2\gamma}$ for all $\gamma \leqslant 1/2, \gamma < 1 + \alpha, \gamma < \alpha/3 + 1/2$.

But $\|\Phi\|_{\mathcal{L}(H^\alpha)} \lesssim 1$ if and only if $\alpha \geqslant -1/2$.

Same kind of difficulty appears for fBM when $H < 3/4$ (resonances in Fourier variable computations).

See recent works [Unterberger, Tindel-Nualart] about extension of rough paths beyond the $H = 1/4$ boundary.

Still open problem (heavy computations to handle 3rd order terms).

Additive stochastic forcing

Noisy KdV

$$\partial_t u + \partial_\xi^3 u + \partial_\xi u^2 = \Phi \partial_t \partial_\xi B$$

where $\partial_t \partial_\xi B$ a white noise on $\mathbb{R} \times \mathbb{T}$ and where Φ is a linear operator such that $\Phi e_k = \lambda_k e_k$ where $\{e_k\}_{k \in \mathbb{Z}}$ is the trigonometric basis and where $\lambda_0 = 0$.

Rewrite as

$$v_t = v_s + w_t - w_s + \int_s^t \dot{X}_\sigma(v_\sigma, v_\sigma) d\sigma$$

where $w_t = U(-t)\Phi \partial_\xi B(t, \cdot)$ and expand the solution for small $t - s$

$$v_t - v_s = w_t - w_s + \int_s^t \dot{X}_\sigma(v_s, v_s) d\sigma$$

$$+2 \underbrace{\int_s^t \dot{X}_\sigma(v_s, w_\sigma - w_s) d\sigma}_{X_{ts}^w(v_s)} + 2\int_s^t \dot{X}_\sigma\left(v_s, \int_s^\sigma \dot{X}_{\sigma'}(v_s, v_s) d\sigma'\right) d\sigma + \text{remainder}$$

Rough equation

$$v_t = v_s + X_{ts}^{\bullet}(v_s) + X_{ts}^{\bullet}(v_s) + w_t - w_s + X_{ts}^{w}(v_s) + O(|t-s|^{1+})$$

where it appears the (random) cross iterated integral:

$$X_{ts}^{w}(\varphi) = \int_s^t d\sigma \dot{X}_\sigma(\varphi, w_\sigma - w_s).$$

Under natural assumptions on Φ, almost surely

$$\|w_t - w_s\|_{H^\alpha} + \|X_{ts}^{w}\|_{\mathcal{L}H^\alpha}^{1/2} \lesssim |t-s|^\gamma.$$

Theorem

Existence and uniqueness of rough solutions to the noisy KdV

This cover the results of [De Bouard-Debussche-Tsutsumi].

On the meaning of the rough solutions

Theorem

Let v be the unique solution of the (transformed) rough KdV equation and let $u(t) = U(t)v(t)$. Let $\mathcal{N}(\varphi)(t,\xi) = \partial_\xi(\varphi(t,\xi)^2)/2$ for smooth functions φ. Then in the sense of distributions we have

$$\mathcal{N}(P_N u) \to \mathcal{N}(u)$$

and

$$\partial_t u + \partial_\xi^3 u + \mathcal{N}(u) = 0$$

is satisfied in distributional sense.

The nonlinear term is not always defined, but it is defined on the KdV solution.

Uniqueness of weak solutions

Using rough path theory we can prove that the nonlinear term is defined of every controlled path:

Any path y in H^α such that

$$y_t = y_s + X_{ts}^\bullet(z_s) + O(|t-s|^{2\gamma})$$

for some other path z_s regular enough enjoy the property that

$$\mathcal{N}(P_N y) \to \mathcal{N}(y)$$

distributionally: the non-linear term is well-defined.

In the space of controlled paths weak solutions to KdV are well-defined and unique.

Power series solutions to dispersive equations

Power series solutions to dispersive equations have been recently explored

- [Christ] (modified) non-linear Schödinger equation

$$\partial_t u + i\partial_\xi^2 u + (|u|^2 - \int |u|^2)u = 0$$

- [Nguyen] (modified) modified-KdV

$$\partial_t u + \partial_\xi^3 u + (u^2 - \int u^2)\partial_\xi u = 0$$

In both cases the existence result can be interpreted as the existence of a rough solution. Rough path theory gives also a way to enforce uniqueness of these weak solutions.