Applications of controlled paths

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Controlled paths

Controlled paths are paths which "looks like" a *given* path which often is random (but not necessarily).

This proximity allows a great deal of computations to be carried on explicitly on the base path and extends also to all controlled paths.

Successful approach which mixes functional analysis and probability.

Basic analogies

- Itô processes

$$dX_t = f_t dM_t + g_t dt$$

- Amplitude modulation

$$f(t) = g(t)\sin(\omega t)$$

with $|\operatorname{supp}\hat{g}| \ll \omega$.

[Joint work with H. Bessaih, R. Catellier, K. Chouk, A. Deya, P. Imkeller, N. Perkowski, F. Russo, S. Tindel]

Some interesting problems (I)

- Compute the value of a 1-form φ over an irregular curve γ

$$\int_\gamma \varphi = \int_0^1 \langle \varphi(\gamma_\sigma), \mathrm{d}\gamma_\sigma \rangle$$

- Solve driven ODEs (and SDEs) [Lyons]

$$\partial_t y(t) = f(y(t))\partial_t x(t), \qquad y(0) = y_0$$

  Study the Itô map $x(\cdot) \mapsto y(T)$.

- Study models of vortex filaments $\gamma : [0, T] \times [0, 1] \to \mathbb{R}^3$

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_\sigma(t) = v^{\gamma(t)}(\gamma_\sigma(t)), \qquad v^{\gamma(t)}(x) = \int_0^1 A(x - \gamma_\sigma(t))\mathrm{d}_\sigma\gamma_\sigma(t)$$

  when the initial condition $\gamma(0)$ is not regular, e.g. a Brownian loop.

Some interesting problems (II)

Define and solve the following kind of stochastic partial differential equations.

- Burgers equations: $u \in [0,T] \times \mathbb{T} \to \mathbb{R}^n$

$$\partial_t u = \Delta u + f(u)Du + \xi$$

  with $\xi : [0,T] \times \mathbb{T} \to \mathbb{R}^n$ space-time white noise.
- Parabolic Anderson model: $u \in [0,T] \times \mathbb{T}^2 \to \mathbb{R}$

$$\partial_t u = \Delta u + f(u)\xi$$

  with $\xi : \mathbb{T}^2 \to \mathbb{R}$ space white noise.
- Kardar-Parisi-Zhang equation [Hairer]

$$\partial_t h = \Delta h + "(Du)^2 - \infty" + \xi$$

  with $\xi : [0,T] \times \mathbb{T} \to \mathbb{R}^n$ space-time white noise.

Some interesting problems (III)

Define and study regularisation by noise phenomena in ODE/PDEs.

- Averaging along irregular curves $w : [0, 1] \to \mathbb{R}^d$

$$f \mapsto T^w_{s,t} f(x) = \int_s^t f(x + w_s) \mathrm{d}s$$

- Differential equations with distributional vector fields and additive perturbations

$$x_t = x_0 + \int_0^t b(x_s) \mathrm{d}s + w_t$$

- Non-linear dispersive equations with irregular modulation

$$\partial_t u(t, \xi) = \partial^3_\xi u(t, \xi) \xi(t) + \partial_\xi u(t, \xi)^2$$

with ξ a time distribution (e.g. $\xi = \partial_t w$).

Here w is (for example) a sample path of a fractional Brownian motion.

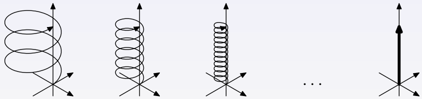
What can go wrong?

Consider the sequence of functions $x^n : \mathbb{R} \to \mathbb{R}^2$

$$x(t) = \frac{1}{n}(\cos(2\pi n^2 t), \sin(2\pi n^2 t))$$

then $x^n(\cdot) \to 0$ in $C^\gamma([0,T];\mathbb{R}^2)$ for any $\gamma < 1/2$. But

$$I(x^{n,1}, x^{n,2})(t) = \int_0^t x^{n,1}(s)\partial_t x^{n,2}(s)\mathrm{d}s \to \frac{t}{2}$$



$$I(x^{n,1}, x^{n,2})(t) \not\to I(0,0)(t) = 0$$

The definite integral $I(\cdot,\cdot)(t)$ is not a continuous map $C^\gamma \times C^\gamma \to \mathbb{R}$ for $\gamma < 1/2$.

(Cyclic microscopic processes can produce macroscopic results. Resonances.)

Functional analysis is not enough

Consider the random functions $(X^n, Y^n) : \mathbb{R} \to \mathbb{R}^2$

$$X^N(t) = \sum_{1 \leqslant n \leqslant N} \frac{g_n}{n} \cos(2\pi n t) + \frac{g_n'}{n} \sin(2\pi n t)$$

$$Y^N(t) = \sum_{1 \leqslant n \leqslant N} \frac{g_n}{n} \sin(2\pi n t) - \frac{g_n'}{n} \cos(2\pi n t)$$

where $(g_n, g_n')_{n \geqslant 1}$ are iid normal variables. Then

$$I(X^N, Y^N)(1) = \int_0^1 X^N(s) \partial_s Y^N(s) \mathrm{d}s = 2\pi \sum_{1 \leqslant n \leqslant N} \frac{g_n^2 + (g_n')^2}{n} \to +\infty$$

almost surely as $N \to \infty$.

No continuous map on a space of paths can represent the integral I and allow Brownian motion at the same time.

Stochastic calculus is not enough

Itô theory has been very successful in handling integrals on Brownian motion (and similar objects) are related differential equations. Key requirements:

- A "temporal" structure (filtration, adapted processes).
- A probability space.
- Martingales.

However sometimes:

- No (natural) temporal structure (no past/future, multidimensional problems, Brownian sheets)
- Results independent of the probabilistic structure (many probabilities) or of exceptional sets (continuity of Itô map with respect to the data).
- No (convenient) martingales around (SDEs driven by fractional Brownian motion).

Young integral

Let f, g two smooth functions, consider the bilinear form

$$I(f,g)_t = \int_0^t f_t \mathrm{d}g_r$$

Then Young proved that

$$I : C^\rho \times C^\gamma \to C^\gamma$$

provided $\gamma + \rho > 1$. Moreover $h = I(f,g)$ is the unique function which satisfy

$$h_t - h_s = f_s(g_t - g_s) + O(|t-s|^{\gamma+\rho}) \quad \text{or} \quad h_t - h_s = \lim_{|\Pi_{s,t}| \to 0} \sum_{t_i \in \Pi_{s,t}} f_{t_i}(g_{t_{i+1}} - g_{t_i})$$

This result does not (and cannot) cover Brownian motion, for example it cannot handle

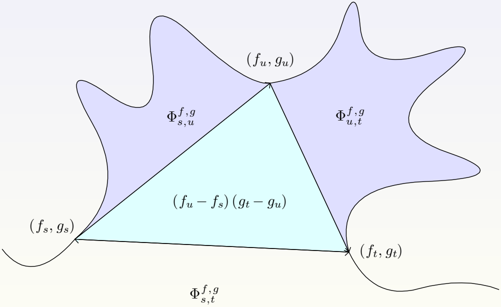
$$\int_0^t f(B_s) \mathrm{d}B_s.$$

The area

Let $f \in C^\rho$ and $g \in C^\gamma$ with $\gamma + \rho > 1$. If $\gamma + \rho > 1$ we can define $\Phi^{f,g} : [0,T]^2 \to \mathbb{R}$ as

$$\Phi^{f,g}_{s,t} = \int_s^t (f_u - f_s) \mathrm{d} g_u$$

$$\Phi^{f,g}_{s,t} - \Phi^{f,g}_{s,u} - \Phi^{f,g}_{u,t} = (f_s - f_u)(g_u - g_t) \quad \text{and} \quad |\Phi^{f,g}_{s,t}| \lesssim |t-s|^{\rho+\gamma}$$



Beyond Young: Controlled paths

When $\gamma + \rho \leqslant 1$ **assume** that exists $\Phi^{f,g}$ such that

$$\Phi^{f,g}_{s,t} - \Phi^{f,g}_{s,u} - \Phi^{f,g}_{u,t} = (f_s - f_u)(g_u - g_t) \quad \text{and} \quad |\Phi^{f,g}_{s,t}| \lesssim |t-s|^{\rho+\gamma}$$

Then for any h such that

$$h_t - h_s = h'_s(f_t - f_s) + O(|t-s|^{\rho+\theta})$$

with $h' \in C^\theta$, if $\rho + \gamma + \theta > 1$ there exists a unique function z such that

$$z_t - z_s = h_s(g_t - g_s) + h'_s \Phi^{f,g}_{s,t} + O(|t-s|^{\gamma+\rho+\theta})$$

and

$$z_t - z_s = \lim_{|\Pi_{s,t}| \to 0} \sum_{t_i \in \Pi_{s,t}} h_{t_i}(g_{t_{i+1}} - g_{t_i}) + h'_{t_i} \Phi^{f,g}_{t_i,t_{i+1}} = \oint_s^t h_r \mathrm{d} g_r$$

Observe that

$$\oint_s^t f_r \mathrm{d} g_r = f_s(g_t - g_s) + \Phi^{f,g}_{s,t}$$

(Controlled) Rough differential equations

Rough path ($3\gamma > 1$): $x \in C^\gamma([0,T],\mathbb{R}^d)$, $\mathbb{X} \in \mathcal{C}_2^{2\gamma}([0,T]^2;\mathbb{R}^d \otimes \mathbb{R}^d)$ with

$$\mathbb{X}^{i,j}_{s,t} = \mathbb{X}^{i,j}_{s,u} + \mathbb{X}^{i,j}_{u,t} + (x^i_u - x^i_s)(x^j_t - x^j_u), \qquad |\mathbb{X}_{s,t}| \lesssim |t-s|^{2\gamma}$$

Controlled path: $y \in \mathcal{D}_x$: $y^x \in C^\gamma$ and $y^\sharp \in \mathcal{C}_2^{2\gamma}$

$$y_t - y_s = y^x_s(x_t - x_s) + y^\sharp_{s,t}$$

Stability upon **non-linear maps**: $z = f(y) \in \mathcal{D}_x$

$$z_t - z_s = f'(y_s)y^x_s(x_t - x_s) + z^\sharp_{s,t}$$

Stability upon **integration**: $h = \oint z\mathrm{d}x$

$$h_t - h_s = f(y_s)(x_t - x_s) + f'(y_s)y^x_s\mathbb{X}_{s,t} + O(|t-s|^{3\gamma})$$

Result: good theory for **differential equations** driven by rough paths

$$y_t = \xi + \oint_0^t f(y_s)\mathrm{d}x_s \qquad \text{(integral form)}$$

$$y_t - y_s = f(y_s)(x_t - x_s) + f'(y_s)f(y_s)\mathbb{X}_{s,t} + O(|t-s|^{3\gamma}) \qquad \text{(diff form)}$$

Young integral as multiplication of distributions

We would like to extend the controlled approach to more general problem. The real difficulty has been to give a meaning to the product

$$f_t \partial_t g_t$$

as a distribution. Young theory works if $f \in C^\rho$, $g \in C^\gamma$ with $\gamma + \rho > 1$.

In general the point-wise product $\mu(F, G) = FG$ of two distributions F, G is a continuous map

$$\mu : B^s_{\infty,\infty} \times B^r_{\infty,\infty} \to B^{\min(s,r)}_{\infty,\infty}$$

for $s, r \in \mathbb{R}$ and $s + r > 0$.

$B^s_{\infty,\infty}$ is a Besov space. It coincides with C^γ if $\gamma \in (0, 1)$. We will abuse the language and set $C^s = B^s_{\infty,\infty}$. If $g \in C^\gamma$ then $\dot{g} = \partial_t g \in C^{\gamma-1}$ and Young integral says that $f\dot{g}$ is meaningful if $\gamma - 1 + \rho > 0$ and $f\dot{g} \in C^{\gamma-1}$.

Advantage of this point of view: no reference to the dimension of the parameter space. The non-trivial thing is the **multiplication**, not the integral.

Littlewood-Paley blocks and Hölder-Besov spaces

We will measure regularity in Hölder-Besov spaces $C^\gamma = B^\gamma_{\infty,\infty}$.

$f \in C^\gamma, \gamma \in \mathbb{R}$ iff

$$\|\Delta_i f\|_{L^\infty} \lesssim 2^{-i\gamma}, \qquad i \geqslant 0$$

$$\mathcal{F}(\Delta_i f)(\xi) = \rho(2^{-i}|\xi|)\hat{f}(\xi)$$

where $\rho : \mathbb{R} \to \mathbb{R}_+$ is a smooth function with support in $[1/2, 5/2]$ and such that $\rho(x) = 1$ if $x \in [1,2]$ and there exists $\theta : \mathbb{R} \to \mathbb{R}_+$ smooth and with support $[0,1]$ such that $\theta(|x|) + \sum_{i\geqslant 0} \rho(2^{-i}|x|) = 1$ for all $x \in \mathbb{R}$.

$$\mathcal{F}(\Delta_{-1} f)(\xi) = \theta(|\xi|)\hat{f}(\xi).$$

$$f = \sum_{i\geqslant -1} \Delta_i f$$

Paraproducts

Deconstruction of a product: $f \in C^\rho, g \in C^\gamma$

$$fg = \sum_{i,j \geqslant -1} \Delta_i f \Delta_j g = \pi_<(f,g) + \pi_\circ(f,g) + \pi_>(f,g)$$

$$\pi_<(f,g) = \pi_>(g,f) = \sum_{i<j-1} \Delta_i f \Delta_j g \qquad \pi_\circ(f,g) = \sum_{|i-j|\leqslant 1} \Delta_i f \Delta_j g$$

Paraproduct (Bony, Meyer et al.)

$$\pi_<(f,g) \in C^{\min(\gamma+\rho,\gamma)}$$

$$\pi_\circ(f,g) \in C^{\gamma+\rho} \qquad \text{if } \gamma + \rho > 0$$

Young integral: $\gamma, \rho \in (0,1)$

$$fDg = \underbrace{\pi_<(f,Dg)}_{C^{\gamma-1}} + \underbrace{\pi_\circ(f,Dg) + \pi_>(f,Dg)}_{C^{\gamma+\rho-1}}$$

Recall

$$\int_s^t f_u \mathrm{d}g_u = f_s(g_t - g_s) + O(|t-s|^{\gamma+\rho})$$

(Para)controlled structure

Idea

Use the paraproduct to *define* a controlled structure. We say $y \in \mathcal{D}_x^{\gamma,\rho}$ if $x \in C^\gamma$

$$y = \pi_<(y^x, x) + y^\sharp$$

with $y^x \in C^\rho$ and $y^\sharp \in C^{\gamma+\rho}$.

Paralinearization. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function and $x \in C^\gamma, \gamma > 0$. Then

$$\varphi(x) = \pi_<(\varphi'(x), x) + C^{2\gamma}$$

[Compare with: $\varphi(x_t) - \varphi(x_s) = \varphi'(x_s)(x_t - x_s) + O(|t-s|^\gamma)$]

▷ A first commutator: $f, g \in C^\rho$, $x \in C^\gamma$

$$\pi_<(f, \pi_<(g,h)) = \pi_<(fg, h) + C^{\gamma+\rho}$$

Stability. $(\rho \geqslant \gamma)$

$$\varphi(y) = \pi_<(\varphi'(y)y^x, x) + C^{\gamma+\rho}$$

A key commutator

All the difficulty is concentrated in the "resonating" term

$$\pi_\circ(f,g) = \sum_{|i-j|\leqslant 1} \Delta_i f \Delta_j g$$

which however is smoother than $\pi_<(f,g)$.

Paraproducts decouple the problem from the source of the problem.

Commutator

The linear form $R(f,g,h) = \pi_\circ(\pi_<(f,g),h) - f\pi_\circ(g,h)$ satisfies

$$\|R(f,g,h)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha \|g\|_\beta \|h\|_\gamma$$

with $\alpha \in (0,1)$, $\beta+\gamma < 0$, $\alpha+\beta+\gamma > 0$.

Paradifferential calculus allow algebraic computations to simplify the form of the resonating terms ($\pi_\circ$).

The Besov area

Concrete example. Let B be a d-dimensional Brownian motion (or a regularisation B^ε) and φ a smooth function. Then $B \in C^\gamma$ for $\gamma < 1/2$.

$$\varphi(B)DB = \pi_<(\varphi(B), DB) + \pi_\circ(\varphi(B), DB) + \underbrace{\pi_>(\varphi(B), DB)}_{C^{2\gamma-1}}$$

and

$$\varphi(B) = \pi_<(\varphi'(B), B) + C^{2\gamma}$$

Then

$$\pi_\circ(\varphi(B), DB) = \pi_\circ(\pi_<(\varphi'(B), B), DB) + \underbrace{\pi_\circ(C^{2\gamma}, DB)}_{\text{OK}}$$

$$= \pi_<(\varphi'(B), \pi_\circ(B, DB)) + C^{3\gamma-1}$$

Finally

$$\varphi(B)DB = \pi_<(\varphi(B), DB) + \pi_<(\varphi'(B), \underbrace{\pi_\circ(B, DB)}_{\text{"Besov area"}}) + \pi_>(\varphi(B), DB) + C^{3\gamma-1}$$

The Besov area (II)

The Besov area $\pi_\circ(B, DB)$ can be defined and studied efficiently using Gaussian arguments:

$$\pi_\circ(B^\varepsilon, DB^\varepsilon) \to \pi_\circ(B, DB)$$

almost surely in $C^{2\gamma-1}$ as $\varepsilon \to 0$.

Remark. If $d = 1$

$$\pi_\circ(B, DB) = \frac{1}{2}(\pi_\circ(B, DB) + \pi_\circ(DB, B)) = \frac{1}{2}D\pi_\circ(B, B)$$

which is well defined.

Tools: Besov embeddings $L^p(\Omega; C^\theta) \to L^p(\Omega; B^{\theta'}_{p,p}) \simeq B^{\theta'}_{p,p}(L^p(\Omega))$, Gaussian hypercontractivity $L^p(\Omega) \to L^2(\Omega)$, explicit L^2 computations.

Au delá des paraproduits

Assume $x \in C^\gamma$ and $\pi_\circ(x, Dx) \in C^{2\gamma-1}$ with $3\gamma - 1 > 0$.

For any controlled $z = \pi_<(z^x, x) + z^\sharp \in \mathcal{D}_x^{\gamma,\gamma}$ define

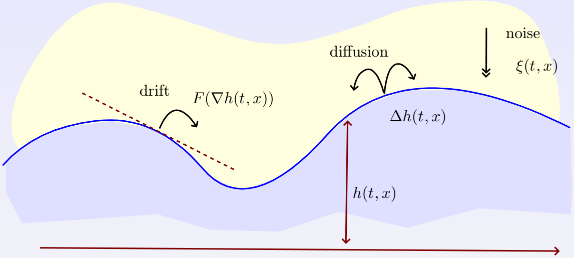
$$zDx = \pi_<(z, Dx) + z^x \pi_\circ(x, Dx) + \pi_>(z, Dx) + \underbrace{Q(z^x, z^\sharp, x, Dx)}_{C^{3\gamma-1}} \in \mathcal{D}_{Dx}^{\gamma-1,\gamma}$$

Solutions to DE. A solution is given by $y \in \mathcal{D}_x^{\gamma,\gamma}$ such that

$$Dy = \varphi(y) Dx$$

distributionally, where the r.h.s. is defined in the sense above. Existence and uniqueness in $\mathcal{D}_x^{\gamma,\gamma}$.

The Kardar–Parisi–Zhang equation



Large scale dynamics of the height $h : [0,T] \times \mathbb{T} \to \mathbb{R}$ of an interface

$$\partial_t h \simeq \Delta h + F(Dh) + \xi$$

The universal limit should coincide with the large scale fluctuations of the KPZ equation

$$\partial_t h = \Delta h + [(Dh)^2 - \infty] + \xi$$

with $\xi : [0,T] \times \mathbb{T} \to \mathbb{R}$ space-time white noise

$$\mathbb{E}[\xi(x,t)\xi(x',t')] = \delta(x-x')\delta(t-t').$$

Stochastic Burgers equation

Take $u = Dh$

$$\partial_t u = \Delta u + Du^2 + D\xi$$

Mild formuation

$$u_t = P_t u_0 + \int_0^t P_{t-s} D\xi_s \mathrm{d}s + \int_0^t P_{t-s} Du_s^2 \mathrm{d}s = X_t + B_t(u, u)$$

Driving term

For all $\gamma < -1/2$, almost surely

$$X = P_\cdot u_0 + \int_0^\cdot P_{\cdot - s} D\xi_s \in C([0, T], C^\gamma(\mathbb{T})) = C^\gamma$$

is an OU process with invariant measure the white noise on $\mathbb{T}$.

Bilinear map

For $f \in C^\alpha, g \in C^\beta$ with $\alpha + \beta > 0$

$$B(f, g) = \int_0^\cdot DP_{\cdot - s}(f_s g_s)\mathrm{d}s \in C^{\min(\alpha,\beta)+1-}$$

Structure of the solution

$$u = X + B(u,u) = X + B(X,X) + 2B(X,B(u,u)) + B(B(u,u),B(u,u)) = \cdots$$

$$X^{\vee} = B(X,X), \quad X^{\curlyvee} = B(X,X^{\vee}), \quad X^{\Psi} = B(X,X^{\curlyvee}), \quad X^{\mathrm{W}} = B(X^{\vee},X^{\vee})$$

Theorem

$$X \in C^{-1/2-}, X^{\vee} \in C^{0-}, \quad X^{\curlyvee}, X^{\Psi} \in C^{1/2-}, \quad X^{\mathrm{W}} \in C^{1-}.$$

$$u = X + X^{\vee} + u^{\geqslant 2}$$

$$u^{\geqslant 2} = 2X^{\curlyvee} + X^{\mathrm{W}} + 2B(X,u^{\geqslant 2}) + 2B(X^{\vee},u^{\geqslant 2}) + B(u^{\geqslant 2},u^{\geqslant 2})$$

Since we expect $u^{\geqslant 2} \simeq X^{\curlyvee} \in \mathcal{C}^{1/2-}$ the term $B(X,u^{\geqslant 2})$ is problematic.

(Para)Controlled structure

$$B(X, u^{\geqslant 2}) = \underbrace{B_<(u^{\geqslant 2}, X)}_{C^{1/2-}} + \underbrace{B_\circ(u^{\geqslant 2}, X)}_{C^{1-}} + \underbrace{B_>(u^{\geqslant 2}, X)}_{C^{1-}}$$

We say that u is controlled if

$$u = X + X^{\vee} + 2X^{\curlyvee} + B_<(u', X) + u^\sharp$$

with $u' \in C^{-1/2-}$ and $u^\sharp \in C^{1-}$.

For a controlled distribution we can decompose $B(u,u)$ as

$$B(u,u) = X^{\vee} + 2X^{\curlyvee} + 4X^{\curlyvee\!\curlyvee} + X^{\vee\!\vee} + 4B(X^{\curlyvee}, X^{\curlyvee})$$

$$+2B(X, B_<(u', X)) + 2B(X, u^\sharp) + 2B(X^{\vee} + 2X^{\curlyvee}, B_<(u', X) + u^\sharp)$$

$$+B(B_<(u', X) + u^\sharp, B_<(u', X) + u^\sharp)$$

with

$$B(X, B_<(u', X)) = B_<(X, B_<(u', X)) + B_\circ(X, B_<(u', X)) + B_>(X, B_<(u', X))$$

(Para)Controlled structure (II)

$$\begin{aligned} B_\circ(X, B_<(u', X))(t) &= \int_0^t \int_0^s DP_{t-s}\pi_\circ(X_s, DP_{s-r}\pi_<(u'_r, X_r))\mathrm{d}r\mathrm{d}s \\ &= \int_0^t \int_0^s DP_{t-s}\pi_\circ(X_s, \pi_<(u'_r, DP_{s-r}X_r))\mathrm{d}r\mathrm{d}s + C^{1-} \\ &= \int_0^t \int_r^t DP_{t-s}[u'_r\pi_\circ(X_s, DP_{s-r}X_r)\mathrm{d}s]\mathrm{d}r + C^{1-} \end{aligned}$$

And by probabilistic estimates we can get

$$\|\pi_\circ(X_s, DP_{s-r}X_r)\|_{0-} \lesssim |s-r|^{-1+}$$

which implies

$$X^\Diamond(u')_t = \int_0^t \int_r^t DP_{t-s}[u'_r\pi_\circ(X_s, DP_{s-r}X_r)\mathrm{d}s]\mathrm{d}r \in C^{1-}$$

Paracontrolled fixed point

It is now easy to show that in the space of controlled distribution of form

$$u = X + X^{\vee} + 2X^{\curlyvee} + B_<(u', X) + u^\sharp$$

with $u' \in C^{-1/2-}$ and $u^\sharp \in C^{1-}$ the fixed point equation $u = \Gamma(u)$ with

$$\Gamma(u) = v = X + X^{\vee} + 2X^{\curlyvee} + 2B_<(\Gamma'(u), X) + X^{\diamond}(u') + \Gamma^\sharp(u)$$

$$\Gamma'(u) = 2B_<(u', X) + 4X^{\curlyvee} + 2u^\sharp$$

$$\Gamma^\sharp(u) = X^{\vee\vee} + 4B_{\circ>}(X^{\curlyvee}, X) + 4B(X^{\curlyvee}, X^{\curlyvee}) + 2X^{\diamond}(u') + 2B_>(B_<(u', X), X) + B_{\circ>}(u^\sharp, X)$$
$$+ 2B(X^{\vee} + 2X^{\curlyvee}, B_<(u', X) + u^\sharp) + B(B_<(u', X) + u^\sharp, B_<(u', X) + u^\sharp)$$

admits a unique solution which is an continuous function of the data of the problem:

$$u_0, X, X^{\vee}, X^{\curlyvee}, X^{\veebar}, X^{\vee\vee}, X^{\diamond}.$$

Merci