Some infinite dimensional rough paths

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Oxford, February 22nd, 2010

Outline

KdV equation

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- Tree-indexed rough paths
- Approximation schemes
- An "orthodox" meaning for the rough solution

Other examples

- The Navier-Stokes equation
- Heat equation with multiplicative noise
- Nonlinear parabolic evolution equations

The KdV equation

Our toy equation: the 1d periodic KdV equation

$$\begin{cases} \partial_t u(t,\xi) + \partial_\xi^3 u(t,\xi) + \frac{1}{2}\partial_\xi u(t,\xi)^2 = 0 \\ u(0,\xi) = u_0(\xi) \end{cases} \qquad (t,\xi) \in \mathbb{R} \times \mathbb{T}$$

with initial condition $u_0 \in H^\alpha(\mathbb{T})$, $\mathbb{T} = [-\pi, \pi]$.

- Low regularity theory [Bourgain, Kenig, Ponce , Vega, Colliander, Keel, Staffilani, Takaoka, Tao, . . .]. Global solutions for initial conditions in $H^{-1/2}$.
- No uniformly continuous dependence on initial conditions for $\alpha < -1/2$.
- Using complete integrability the solution map can be extended by continuity up to $\alpha = -1$ [Kappeler, Topalov].

Mild form (Duhamel's formula)

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\partial_\xi u(s)^2 \, ds$$

Linear part: Airy group $\mathcal{F}(U(t)\varphi)(k) = e^{-ik^3 t}\hat{\varphi}(k)$, $k \in \mathbb{Z}$.

Abstract formulation

Taking advantage of the fact that U is a group (not just semi-group) operate the change of variables $v(t) = U(-t)u(t)$:

$$v(t) = u_0 + \int_0^t U(-s)\partial_\xi [U(s)v(s)]^2 \, ds$$

For all $s < t$

$$v_t = v_s + \int_s^t \dot{X}_\sigma(v_\sigma, v_\sigma) d\sigma$$

with $v_0 = u_0$.
Bilinear operator

$$\mathcal{F}\dot{X}_\sigma(\varphi_1, \varphi_2)(k) = ik \sum_{k_1+k_2=k} e^{-(k^3-k_1^3-k_2^3)\sigma} \hat{\varphi}_1(k_1)\hat{\varphi}_2(k_2)$$

Expansion

$$v_t - v_s = X^{\bullet}_{ts}(v_s^{\times 2}) + X^{[\bullet]}_{ts}(v_s^{\times 3}) + X^{[[\bullet]]}_{ts}(v_s^{\times 4}) + X^{[\bullet\bullet]}_{ts}(v_s^{\times 4}) + r_{ts}$$

where r_{ts} is a remainder term.
Tree-indexed multi-linear operators appear:

$$X^{\bullet}_{ts}(\varphi_1, \varphi_2) = \int_s^t \dot{X}_\sigma(\varphi_1, \varphi_2) d\sigma$$

$$X^{[\bullet]}_{ts}(\varphi_1, \varphi_2, \varphi_3) = \int_s^t \dot{X}_\sigma(X^{\bullet}_{\sigma s}(\varphi_1, \varphi_2), \varphi_3) d\sigma$$

$$X^{[\bullet\bullet]}_{ts}(\varphi_1, \dots, \varphi_4) = \int_s^t \dot{X}_\sigma(X^{\bullet}_{\sigma s}(\varphi_1, \varphi_2), X^{\bullet}_{\sigma s}(\varphi_3, \varphi_4)) d\sigma$$

etc...

Regularity

Main observation: cancellations of high-frequency oscillations

$$\mathcal{F}X^{\bullet}_{ts}(\varphi_1,\varphi_2)(k) = \sum_{k_1+k_2=k}\left[ik\int_s^t e^{-(k^3-k_1^3-k_2^3)\sigma}d\sigma\right]\hat{\varphi}_1(k_1)\hat{\varphi}_2(k_2)$$

$$= \sum_{k_1+k_2=k}\left[\frac{e^{i(3kk_1k_2)t}-e^{i(3kk_1k_2)s}}{3k_1k_2}\right]\hat{\varphi}_1(k_1)\hat{\varphi}_2(k_2)$$

Easy analysis gives

$$\|X^{\bullet}_{ts}\|_{\mathcal{L}(H^\alpha)} \lesssim |t-s|^\gamma$$

with $\gamma \leqslant 1/2, \gamma < 1+\alpha, \gamma < \alpha/3 + 1/2$.

Compensations of space and time regularity.
In any case always beyond Young integration $\Rightarrow$ a real rough path !

Rough path regularity

Theorem

$$\|X^{\bullet}_{ts}\|_{\mathcal{L}(H^{\alpha})} + \|X^{\bullet\!-\!\bullet}_{ts}\|^{1/2}_{\mathcal{L}(H^{\alpha})} \lesssim |t-s|^{\gamma}$$

for $\gamma \leqslant 1/2$, $\gamma < 1+\alpha$, $\gamma < \alpha/3 + 1/2$, $\alpha \geqslant -1/2$.

Taking $\gamma = (1/3)+$ we get $\alpha = 1/2+$, almost as good as standard theory.

Impossible to go beyond the $\alpha = -1/2$ boundary?
What about expansions to third order?

Resonances

A natural boundary appears at $\alpha = -1/2$ due to a *resonance* in $X^{\substack{\bullet\\ \bullet}}_{ts}$. (non-linear interaction + linear propagation)

Decomposition

$$X^{\substack{\bullet\\ \bullet}}_{ts} = \hat{X}^{\substack{\bullet\\ \bullet}}_{ts} + (t-s)\Phi$$

where $\|\hat{X}^{\substack{\bullet\\ \bullet}}_{ts}\|_{\mathcal{L}(H^\alpha)} \lesssim |t-s|^{2\gamma}$ for all $\gamma \leqslant 1/2, \gamma < 1+\alpha, \gamma < \alpha/3 + 1/2$.

But $\|\Phi\|_{\mathcal{L}(H^\alpha)} \lesssim 1$ if and only if $\alpha \geqslant -1/2$.

Same kind of difficulty appears for fBM when $H < 3/4$ (resonances in Fourier variable computations).

See recent works [Unterberger, Tindel-Nualart] about extension of rough paths beyond the $H = 1/4$ boundary.

Still open problem (heavy computations to handle 3rd order terms).

Fixed point

KdV is a rough equation which can be solved with fixed-point method and rough path estimates.

- Existence: show that there exists a converging sequence $(y^n)_n$ of paths in H^α satisfying

$$y_t^{n+1} = y_s^{n+1} + X^{\bullet}(y_s^n, y_s^n) + X^{\substack{\bullet \\ | \\ \bullet}}(y_s^n, y_s^n, y_s^{n-1}) + O(|t-s|^{3\gamma})$$

  where $3\gamma > 1$.
- Uniqueness: prove that there exists a unique path y satisfying

$$y_t = y_s + X^{\bullet}(y_s, y_s) + X^{\substack{\bullet \\ | \\ \bullet}}(y_s, y_s, y_s) + O(|t-s|^{1+})$$

Theorem

Existence and uniqueness of (local) rough path solution for any initial condition in H^α with $\alpha > -1/2$ (since $\gamma > 1/3$)

Chen on trees

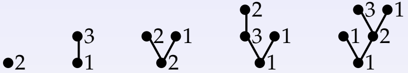
The main ingredient of the proof is the multiplicative properties of the operator $X^{\substack{\bullet \\ | \\ \bullet}}$:

$$X^{\substack{\bullet \\ | \\ \bullet}}_{ts}(\varphi_1, \varphi_2, \varphi_3) = X^{\substack{\bullet \\ | \\ \bullet}}_{tu}(\varphi_1, \varphi_2, \varphi_3) + X^{\substack{\bullet \\ | \\ \bullet}}_{us}(\varphi_1, \varphi_2, \varphi_3) + X^{\bullet}_{tu}(X^{\bullet}_{tu}(\varphi_1, \varphi_2), \varphi_3)$$

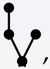
At higher order this kind of relations involves non-commutative generalization of Chen's relation: rough path indexed by trees and the Hopf algebra of Connes-Kreimer.

Trees

$\mathcal{L}$ finite set. Trees labeled by $\mathcal{L}$, $\mathcal{T}_{\mathcal{L}}$



$$(\tau_1, \cdots, \tau_k) \overset{B_+^a}{\mapsto} \tau = [\tau_1, \cdots, \tau_k]_a$$

$[\bullet] =$  $\quad [\bullet, [\bullet]] =$ , $\quad$ etc...

- Size of the tree $|\tau|$: $|\bullet| = 1, \qquad |[\tau_1 \cdots \tau_n]| = 1 + |\tau_1| + \cdots + |\tau_n|$
- Tree factorial $\tau!$: $\bullet! = 1, \qquad [\tau_1 \cdots \tau_n]! = |[\tau_1 \cdots \tau_n]|\tau_1! \cdots \tau_n!$

Theorem (tree multiplicative property)

$$X^{\tau}_{ts} = \sum X^{\tau^{(1)}}_{tu} X^{\tau^{(2)}}_{us} = X^{\Delta\tau}_{tus} = X^{\tau}_{tu} + X^{\tau}_{us} + X^{\Delta'\tau}_{tus}$$

Connes-Kreimer coproduct:

$$\Delta : \mathcal{A}\mathcal{T} \to \mathcal{A}\mathcal{T} \otimes \mathcal{A}\mathcal{T}$$

algebra homomorphism defined recursively by

$$\Delta(\tau) = 1 \otimes \tau + \sum_{a \in \mathcal{L}} (B^a_+ \otimes \mathrm{id})[\Delta(B^a_-(\tau))]$$

$$B^a_-(B^b_+(\tau_1 \cdots \tau_n)) = \begin{cases} \tau_1 \cdots \tau_n & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

Example

Notation: $\Delta\tau = \sum \tau^{(1)} \otimes \tau^{(2)}$, reduced coproduct

$$\Delta'\tau = \Delta\tau - 1 \otimes \tau - \tau \otimes 1$$

Some examples

Forests with $|\tau| \leqslant 3$

$$\bullet,\ [\bullet],\ \bullet\bullet,\ [[\bullet]],\ \bullet[\bullet],\ \bullet\bullet\bullet,\ [\bullet\bullet]$$

$$\Delta'[\bullet] = \bullet \otimes \bullet$$

$$\Delta'(\bullet\bullet) = 2\bullet \otimes \bullet$$

$$\Delta'[[\bullet]] = [\bullet] \otimes \bullet + \bullet \otimes [\bullet]$$

$$\Delta'(\bullet[\bullet]) = \bullet \otimes \bullet\bullet + \bullet\bullet \otimes \bullet + [\bullet] \otimes \bullet + \bullet \otimes [\bullet]$$

$$\Delta'(\bullet^3) = 3\bullet^2 \otimes \bullet + 3\bullet \otimes \bullet^2$$

$$\Delta'[\bullet\bullet] = \bullet \otimes \bullet\bullet + 2[\bullet] \otimes \bullet$$

$$\delta X^{[\bullet]}_{tus} = X^{\bullet}_{tu} X^{\bullet}_{us}$$

$$\delta X^{\bullet\bullet}_{tus} = 2X^{\bullet}_{tu} X^{\bullet}_{us}$$

$$\delta X^{[[\bullet]]}_{tus} = X^{[\bullet]}_{tu} X^{\bullet}_{us} + X^{\bullet}_{tu} X^{[\bullet]}_{us}$$

$$\delta X^{\bullet[\bullet]}_{tus} = X^{\bullet}_{tu} X^{\bullet\bullet}_{us} + X^{\bullet\bullet}_{tu} X^{\bullet}_{us} + X^{[\bullet]}_{tu} X^{\bullet}_{us} + X^{\bullet}_{tu} X^{[\bullet]}_{us}$$

$$\delta X^{\bullet^3}_{tus} = 3X^{\bullet^2}_{tu} X^{\bullet}_{us} + 3X^{\bullet}_{tu} X^{\bullet^2}_{us}$$

$$\delta X^{[\bullet\bullet]}_{tus} = X^{\bullet}_{tu} X^{\bullet\bullet}_{us} + 2X^{[\bullet]}_{tu} X^{\bullet}_{us}$$

Series solution

Explicit series representation of the solution:

$$v_t = v_s + \sum_{\tau} X^{\tau}_{st}(v_s^{\times d(\tau)})$$

Assuming $3\gamma > 1$ it is possible to obtain estimates for higher order operators (e.g. $X^{\vee}$ and $X^{\vdots}$) using the estimates on $X^{\bullet}$ and $X^{:}$.

$$\|X^{\tau}_{ts}\|_{\mathcal{L}H^{\alpha}} \lesssim C^{|\tau|} q_{\gamma}(\tau) |t-s|^{\gamma|\tau|}$$

where $(q_{\gamma}(\tau))_{\tau}$ is a universal sequence of numbers depending only on γ.

Problem: I do not know the asymptotics as $|\tau| \to \infty$ of $q_{\gamma}(\tau)$.

Speed of growth

Conjecture

$$q_\gamma(\tau) \asymp C(\tau!)^{-\gamma}$$

True for linear Chen trees $\mathcal{T}^{\text{Chen}}$:

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(k!)^\gamma((n-k)!)^\gamma} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(n!)^\gamma}, \qquad \gamma \in (0,1],\ \ a,b \geqslant 0$$

Variant of Lyons' neo-classical inequality

$$\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma(n-k)}}{(\gamma k)![\gamma(n-k)n]!} \leqslant c_\gamma \frac{(a+b)^{\gamma n}}{(\gamma n)!}$$

"neo-classical tree inequality" ?

$$\sum \frac{a^{\gamma|\tau^{(1)}|} b^{\gamma|\tau^{(2)}|}}{(\tau^{(1)}!)^\gamma(\tau^{(2)}!)^\gamma} \overset{???}{\leqslant} c_\gamma \frac{(a+b)^{\gamma|\tau|}}{(\tau!)^\gamma}$$

OK for $\gamma = 1$: tree binomial formula.

L^2 conservation law

Original operator satisfies a basic symmetry wrt L^2 scalar product:

$$\langle \varphi_1, \dot{X}_s(\varphi_2, \varphi_3) \rangle + \langle \varphi_2, \dot{X}_s(\varphi_1, \varphi_3) \rangle + \langle \varphi_3, \dot{X}_s(\varphi_2, \varphi_1) \rangle = 0$$

Lemma

$$\langle \varphi, X_{ts}(\varphi, \varphi) \rangle = 0 \qquad 2\langle \varphi, X^2_{ts}(\varphi, \varphi, \varphi) \rangle + \langle X_{ts}(\varphi, \varphi), X_{ts}(\varphi, \varphi) \rangle = 0$$

This implies some cancellations for the rough solution to KdV:

$$\begin{aligned} \langle v_t, v_t \rangle - \langle v_s, v_s \rangle &= 2\langle X_{ts}(v_s, v_s) + X^2_{ts}(v_s, v_s, v_s), v_s \rangle \\ &\quad + \langle X_{ts}(v_s, v_s), X_{ts}(v_s, v_s) \rangle + O(|t-s|^{3\gamma}) \\ &= O(|t-s|^{3\gamma}) \end{aligned}$$

Theorem (Integral conservation law)

If v is a rough solution of KdV then $|v_t|_{L^2} = |v_0|_{L^2}$ for any t.

Galerkin approximations

Consider projection P_N onto modes with $|k| < N$ and the approximate evolution

$$\partial_t u^{(N)} + \partial_\xi^3 u^{(N)} + \frac{1}{2} P_N \partial_\xi (u^{(N)})^2 = 0, \qquad u^{(N)}(0) = P_N u_0$$

Then $v_t^{(N)} = U(-t) u_t^{(N)}$ are rough solution to the equation

$$v_t^{(N)} = v_s^{(N)} + X_{ts}^{\bullet,(N)}(v_s^{(N)}) + X_{ts}^{\overset{\bullet}{\bullet},(N)}(v_s^{(N)}) + O(|t-s|^{3\gamma})$$

where $X^{\bullet,(N)} = P_N X^{\bullet}(P_N \times P_N)$ and where the trilinear operator $X^{\overset{\bullet}{\bullet},(N)}$ is defined as

$$X_{ts}^{\overset{\bullet}{\bullet},(N)}(\varphi_1, \varphi_2, \varphi_3) = 2 \int_s^t d\sigma \int_s^\sigma d\sigma_1 P_N \dot{X}_\sigma (P_N \varphi_1, P_N \dot{X}_{\sigma_1}(P_N \varphi_2, P_N \varphi_3))$$

We (almost) have

$$X^{\bullet,(N)} \to X^{\bullet}, \qquad X^{\substack{\bullet\\|\\\bullet},(N)} \to X^{\substack{\bullet\\|\\\bullet}}$$

and the continuity of rough path estimates would imply that $v^{(N)} \to v$: convergence of the Galerkin approximations in Holder norm.

Back to reality: unfortunately $X^{\substack{\bullet\\|\\\bullet},(N)} \not\to X^{\substack{\bullet\\|\\\bullet}}$ but we can modify the Galerkin approximation to overcome this difficulty and complete the picture.

It is well known that naive Galerkin does not work due to symplectic non-squeezing property of the KdV flow (which is Hamiltonian).

Euler scheme

For any $n > 0$ let $v_0^n = u_0$ and

$$v_i^n = X^{\bullet}_{i/n,j/n}(v_{i-1}^n) + X^{\substack{\bullet\\|\\\bullet}}_{i/n,j/n}(v_{i-1}^n)$$

for $i \geqslant 1$.

Theorem

Let $\Delta_i^n = v_i^n - v_{i/n}$ *then*

$$\sup_{0 \leqslant i < j \leqslant nT} \frac{|\Delta_i^n - \Delta_j^n|_{H^\alpha}}{|i-j|^\gamma} = O(n^{1-3\gamma}).$$

The combination of this scheme with the Galerkin approximation discussed before provide an implementable numerical approximation scheme for the solutions of KdV with low regularity initial conditions with explicit rates of convergence.

Additive stochastic forcing

Noisy KdV

$$\partial_t u + \partial_\xi^3 u + \partial_\xi u^2 = \Phi \partial_t \partial_\xi B$$

where $\partial_t \partial_\xi B$ a white noise on $\mathbb{R} \times \mathbb{T}$ and where Φ is a linear operator such that $\Phi e_k = \lambda_k e_k$ where $\{e_k\}_{k \in \mathbb{Z}}$ is the trigonometric basis and where $\lambda_0 = 0$.

Rewrite as

$$v_t = v_s + w_t - w_s + \int_s^t \dot{X}_\sigma(v_\sigma, v_\sigma) d\sigma$$

where $w_t = U(-t)\Phi \partial_\xi B(t, \cdot)$ and expand the solution for small $t - s$

$$v_t - v_s = w_t - w_s + \int_s^t \dot{X}_\sigma(v_s, v_s) d\sigma$$

$$+2 \underbrace{\int_s^t \dot{X}_\sigma(v_s, w_\sigma - w_s) d\sigma}_{X^w_{ts}(v_s)} + 2 \int_s^t \dot{X}_\sigma\Big(v_s, \int_s^\sigma \dot{X}_{\sigma'}(v_s, v_s) d\sigma'\Big) d\sigma + \text{remainder}$$

Rough equation

$$v_t = v_s + X^{\bullet}_{ts}(v_s) + X^{\substack{\bullet \\ | \\ \bullet}}_{ts}(v_s) + w_t - w_s + X^{w}_{ts}(v_s) + O(|t-s|^{1+})$$

where it appears the (random) cross iterated integral:

$$X^{w}_{ts}(\varphi) = \int_s^t d\sigma \dot{X}_{\sigma}(\varphi, w_{\sigma} - w_s).$$

Under natural assumptions on Φ, almost surely

$$\|w_t - w_s\|_{H^{\alpha}} + \|X^{w}_{ts}\|^{1/2}_{\mathcal{L}H^{\alpha}} \lesssim |t-s|^{\gamma}.$$

Theorem

Existence and uniqueness of rough solutions to the noisy KdV

This cover the results of [De Bouard-Debussche-Tsutsumi].

On the meaning of the rough solutions

Theorem

Let v be the unique solution of the (transformed) rough KdV equation and let $u(t) = U(t)v(t)$. Let $\mathcal{N}(\varphi)(t,\xi) = \partial_\xi(\varphi(t,\xi)^2)/2$ for smooth functions φ. Then in the sense of distributions we have

$$\mathcal{N}(P_N u) \to \mathcal{N}(u)$$

and

$$\partial_t u + \partial_\xi^3 u + \mathcal{N}(u) = 0$$

is satisfied in distributional sense.

The nonlinear term is not always defined, but it is defined on the KdV solution.

Uniqueness of weak solutions

Using rough path theory we can prove that the nonlinear term is defined of every controlled path:

Any path y in H^α such that

$$y_t = y_s + X^{\bullet}_{ts}(z_s) + O(|t-s|^{2\gamma})$$

for some other path z_s regular enough enjoy the property that

$$\mathcal{N}(P_N y) \to \mathcal{N}(y)$$

distributionally: the non-linear term is well-defined.

In the space of controlled paths weak solutions to KdV are well-defined and unique.

Power series solutions to dispersive equations

Power series solutions to dispersive equations have been recently explored

- [Christ] (modified) non-linear Schödinger equation

$$\partial_t u + i\partial_\xi^2 u + (|u|^2 - \int |u|^2)u = 0$$

- [Nguyen] (modified) modified-KdV

$$\partial_t u + \partial_\xi^3 u + (u^2 - \int u^2)\partial_\xi u = 0$$

In both cases the existence result can be interpreted as the existence of a rough solution. Rough path theory gives also a way to enforce uniqueness of these weak solutions.

The NS equation as a rough path

The d-dimensional NS equation has the abstract form

$$u_t = S_t u_0 + \int_0^t S_{t-s} B(u_s, u_s)\, ds. \tag{1}$$

S bounded semi-group on $\mathcal{B}$, B symmetric bilinear operator.
Define

$$X^{\bullet}_{ts}(\varphi^{\times 2}) = \int_s^t S_{t-u} B(S_{u-s}\varphi, S_{u-s}\varphi) du$$

$$X^{[\tau^1]}_{ts}(\varphi^{\times (d(\tau^1)+1)}) = \int_s^t S_{t-u} B(X^{\tau^1}_{us}(\varphi^{\times d(\tau^1)}), S_{u-s}\varphi) du$$

and

$$X^{[\tau^1\tau^2]}_{ts}(\varphi^{\times (d(\tau^1)+d(\tau^2))}) = \int_s^t S_{t-u} B(X^{\tau^1}_{us}(\varphi^{\times d(\tau^1)}), X^{\tau^2}_{us}(\varphi^{\times d(\tau^2)})) du$$

where $d(\tau)$ is a degree function.

Bounds on the operators and regularity

For suitable Banach space $\mathcal{B}$

$$|X^{\tau}_{ts}(\varphi^{\times d(\tau)})|_{\mathcal{B}} \leqslant C\frac{|t-s|^{\varepsilon|\tau|}}{(\tau!)^{\varepsilon}}|\varphi|^{d(\tau)}_{\mathcal{B}}$$

where $\varepsilon \geqslant 0$ is a constant depending on $\mathcal{B}$.

Norm convergent series representation

$$u_t = S_{t-s}u_s + \sum_{\tau\in\mathcal{T}_B} X^{\tau}_{ts}(u_s^{\times d(\tau)})$$

gives local solution, global for small data (for $\varepsilon = 0$).

Regularity: $|\hat{u}(k)| \leqslant Ce^{-|k|\sqrt{t}}$

[Le Jan & Sznitman, Cannone & Planchon, Sinai, Gallavotti]

Linear evolution equations

[joint work with S. Tindel]

Consider the SPDE

$$\frac{\partial}{\partial t} y(t,\xi) = \Delta_\xi y(t,\xi) + y(t,\xi) \frac{\partial^2}{\partial t \partial \xi} x(t,\xi)$$

on the 1d torus $\mathbb{T}$ with initial condition $\overline{y} \in L^2(\mathbb{T})$.
Mild formulation

$$y(t,\xi) = \int_{\mathbb{T}} G_t(\xi-\xi')\overline{y}(\xi')d\xi' + \int_0^t \int_{\mathbb{T}} G_{t-s}(\xi-\xi') y_s(\xi') x(ds, d\xi')$$

- Distributional Gaussian noise

$$\mathbb{E} x(dt, d\xi) x(dt', d\xi') \sim \delta(t-t') Q(\xi-\xi')$$

  with $Q(\xi) = \sum_{n\in\mathbb{Z}} \lambda_n^{-\gamma} e_n(\xi)$, $\Delta e_n = -\lambda_n e_n$.
- G_t kernel of the analytic semigroup $S_t = e^{t\Delta}$

Compact form

$$y_t = S_t\overline{y} + \int_0^t S_{t-s}dx_s y_s$$

Iterative solution methods make appear *operator-valued* increments:

$$X^1_{ts}(\varphi) = \int_s^t S_{t-u}dx_u S_{u-s}\varphi \quad X^2_{ts}(\varphi) = \int_s^t\int_s^u S_{t-u}dx_u S_{u-v}dx_v S_{v-s}\varphi \cdots$$

with kernels given by

$$X^1_{ts}(\varphi)(\xi) = \int_{\mathbb{T}}\left[\int_s^t\int_{\mathbb{T}} G_{t-u}(\xi-\xi')x(du,d\xi')G_{u-s}(\xi'-\xi'')\right]\varphi(\xi'')d\xi''$$

and similar formulas for X^n.

Regularity

Theorem

If X^n, $n = 1, 2, 3$ *are defined by Itô integrals then*

$$X^n \in \mathcal{C}_2^{\gamma+(n-1)\kappa}(\mathcal{L}_{HS}(H^\eta; H^{-\rho})) \cap \mathcal{C}_2^{n\kappa}(\mathcal{L}_{HS}(H^\eta; H^\eta))$$

for any $\eta > 1/4, \gamma > \kappa$ *satisfying*

$$\kappa < 1/4 - \eta + \overline{\nu}/2 \qquad \textit{and} \qquad \gamma < 1/2$$

where $\rho + \eta = \gamma - \kappa$ *and* $\overline{\nu} = \min(\nu, 1/2)$.

Proof: extended Garsia-Rodemich-Rumsey inequality for convolutional increments + Gaussian estimates

Convolutional multiplicative proprerty: for $0 \leqslant s \leqslant u \leqslant t$,

$$X^n_{ts}(\varphi) = X^n_{tu}(S_{u-s}\varphi) + S_{t-s}X^n_{us}(\varphi) + \sum_{k=1}^{n-1} X^{n-k}_{tu}(X^k_{us}(\varphi))$$

Assume that $\gamma + 3\kappa > 1$, then

$$H = X^1X^3 + X^2X^2 + X^3X^1 \in \mathcal{C}_3^{\gamma+3\kappa}(\mathcal{L}_{\mathrm{HS}}(H^\eta; H^{-\rho}))$$

so we can define X^4, and so on : $X^n \in \mathcal{C}_2^{n\kappa}(\mathcal{L}_{\mathrm{HS}}(H^\eta; H^\eta))$

Convergent series expansion

$$y_t = S_{t-s}y_s + \sum_{n=1}^{\infty} X^n_{ts}y_s.$$

Non-linear equations

Consider now:

$$y(t,\xi) = \int_{\mathbb{T}} G_t(\xi - \xi')\overline{y}(\xi')d\xi' + \int_0^t \int_{\mathbb{T}} G_{t-s}(\xi - \xi')[y_s(\xi')]^2 x(ds, d\xi')$$

with a bilinear non-linearity. Abstractly:

$$y_t = S_t\overline{y} + \int_0^t S_{t-s} dx_s B(y_s, y_s).$$

Again, tree-indexed incremental operators:

$$X^{\bullet}_{ts}(\varphi^1, \varphi^2) = \int_s^t S_{t-u} dx_u B(S_{u-s}\varphi^1, S_{u-s}\varphi^2)$$

$$X^{\mathrm{I}}_{ts}(\varphi^1, \varphi^2, \varphi^3) = \int_s^t S_{t-u} dx_u B\left(S_{u-s}\varphi^1, X^{\bullet}_{us}(\varphi^2, \varphi^3)\right)$$

$$X^{\mathrm{Y}}_{ts}(\varphi^1, \varphi^2, \varphi^3, \varphi^4) = \int_s^t S_{t-u} dx_u B\left(X^{\bullet}_{us}(\varphi^1, \varphi^2), X^{\bullet}_{us}(\varphi^3, \varphi^4)\right)$$

Conclusions

- Rough paths appear naturally in (infinite-dimensional) deterministic problems
- They are operators which encode "microscropic information" on the dynamics
- Rough paths are indexed by trees
- Examples of non-linear distributions

Thanks