Rough evolution equations

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Prologue.
(or, what not to expect)

You will hear something about rough paths, however...

- There will be very few rough paths (at least in the form you could expect)
- I will touch very little on the probabilistic side of the problem. (we describe the "bones" and leave apart the "flesh" of the theory).
- The aim is to give a flavour of the approach. Concrete results and detailed report are being worked over (still. joint work with S. Tindel [Nancy]).

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Outline

- To start

1. Review of the “classical” theory
   - Abstract integration
   - Exercise of deconstruction
   - Rough paths

2. Rough evolution equations
   - Convolution integrals
   - Young theory
   - More irregular noises
   - Fully non-linear case
   - Summary of the approach

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Problem of the day

Our concern today are path-wise solutions for equations of the form

$$y_t = S(t)y_0 + \int_0^t S(t-s)dx_s f(y_s)$$

where S is an analytic semigroup on a Banach space $\mathcal{B}$, $y_0 \in \mathcal{B}$ some initial condition, f some function on $\mathcal{B}$ and dx some irregular noise.

Main example

$\mathcal{B} = L^2([0,1])$, S heat semigroup, x is a gaussian noise with covariance

$$\mathbb{E}[x_u(\xi)x_v(\eta)] = c_{H,\nu}|u-v|^{2H}|\xi-\eta|^{-\nu}, \qquad u,v \in [0,T], \xi,\eta \in [0,1]$$

f some function acting as $f(y)(\xi) = f(y(\xi)), \xi \in [0,1]$.

$H = 1/2$ Brownian motion in time, $\nu = 1$ white noise in space

Act on functions $\varphi \in \mathcal{B}$ as $(x_u\varphi)(\xi) = x_u(\xi)\varphi(\xi), \qquad \xi \in [0,1]$

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Rough paths

- Rough paths are structures on paths for which a meaningful integration theory can be constructed in a general setting.
- We can define and solve differential equations driven by rough paths, the solution has nice continuity property with respect to the data.
- Brownian motion can be used to build a simple non-trivial example of a rough path. (Historically this is the main motivation for the development of rough path theory)
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k-Increments

[M. G. , “Controlling rough paths”, JFA (2004)]

Definition

A k-increment is a continuous function $g : [0, T]^{k+1} \to V$ such that $g_{t_0 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$. Denote them $\mathcal{C}_k(V)$.

Example

- $g \in \mathcal{C}_0$ is a function on $[0, T]$
- Given $f \in \mathcal{C}_0$, set $g_{ts} = f_t - f_s$, then $g \in \mathcal{C}_1$.

Basic fact

$g \in \mathcal{C}_1$ is given by $g_{ts} = f_t - f_s$ for some $f \in \mathcal{C}_0$ iff it satisfy

$$g_{ts} - g_{su} - g_{us} = 0$$

A cocycle property.

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A cochain complex

- Increments forms a cochain complex $(\mathcal{C}_*, \delta)$ with coboundary map

$$\delta : \mathcal{C}_k \to \mathcal{C}_{k+1} \qquad (\delta g)_{t_1 \cdots t_k} = \sum_{i=1}^{k} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_k}$$

- $$\mathcal{C}_0 \xrightarrow{\delta} \mathcal{C}_1 \xrightarrow{\delta} \mathcal{C}_2 \xrightarrow{\delta} \mathcal{C}_3 \xrightarrow{\delta} \cdots$$

  $\delta\delta = 0$ and $\mathrm{Ker}\delta|_{\mathcal{C}_{k+1}} = \mathrm{Im}\delta|_{\mathcal{C}_k}$ so the complex is acyclic.

- In particular, $g \in \mathcal{C}_1$ is a 1-cocycle (or closed 1-increment) if

$$\delta g_{tus} = -g_{us} + g_{ts} - g_{tu} = 0.$$

  Then there exists $f \in \mathcal{C}_0$ such that $g = \delta f$: closed 1-increments are exact.

- (cfr. de-Rham cohomology of $\mathbb{R}^n$: closed differential forms are exact)

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Some (useful) notation...

Definition

For $a \in \mathcal{C}_k$ and $b \in \mathcal{C}_m$ we define the product $ab \in \mathcal{C}_{k+m}$ as

$$(ab)_{t_1 \cdots t_{k+m+1}} = a_{t_1 \cdots t_{k+1}} b_{t_{k+1} \cdots t_{k+m+1}}$$

Notation

When $x, f_1, f_2 \in \mathcal{C}_0$ and smooth, we will mean

$$\left(\int \varphi(x) dx \right)_{ts} = \int_s^t \varphi(x_r) dx_r$$

and

$$\left(\int df_1 df_2 \right)_{ts} = \int_s^t \left(\int_s^u d_r f_{1,r} \right) d_u f_{2,u}$$

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- Easy to check:

$$\delta \int df_1 = 0 \qquad \delta \int df_1 df_2 = \int df_1 \int df_2 = \delta f_1 \delta f_2$$

for any smooth $f_1, f_2 \in \mathcal{C}_0$.

- And more generally

$$\delta \int df_1 \cdots df_n = \sum_{k=1}^{n-1} \int df_1 \cdots df_k \int df_{k+1} \cdots df_n$$

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Norms on Increments

Definition

For $g \in \mathcal{C}_1, h \in \mathcal{C}_2$ let

$$\|g\|_\mu = \sup_{t,s\in[0,T]} \frac{|g_{ts}|}{|t-s|^\mu} \qquad \|h\|_{\rho,\sigma} = \sup_{t,s,u\in[0,T]^3} \frac{|h_{tus}|}{|t-u|^\rho |u-s|^\sigma}$$

and

$$\|h\|_\mu = \inf\left\{\sum_i \|h_i\|_{\rho_i,\mu-\rho_i} : h = \sum_i h_i, 0 < \rho_i < \mu\right\}$$

Denote $\mathcal{C}_k^\mu$ the subset of $\mathcal{C}_k$ with finite $\|\cdot\|_\mu$ norm $(k = 1,2)$.
Let $\mathcal{C}_k^{1+} = \cup_{\mu>1}\mathcal{C}_k^\mu$ – the small increments.

The Λ map

Fact

We have $\mathcal{BC}_1^{1+} = \mathcal{C}_1^{1+} \cap \mathrm{Im}\delta = \{0\}$: no nontrivial small 1-coboundaries.

Theorem

There exists a unique bounded linear map $\Lambda : \mathcal{BC}_2^{1+} \to \mathcal{C}_1^{1+}$ such that

$$\delta\Lambda g = g.$$

($\mathcal{BC}_2^{1+} = \mathcal{C}_2^{1+} \cap \mathrm{Im}\delta$)

If $g \in \mathcal{C}_1$ and $\delta g \in \mathcal{BC}_2^{1+}$, then

$$g = \Lambda\delta g + \delta f$$

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What an integral is made of?

Taylor formula

$$\int_s^t \varphi(x_r)dx_r = \varphi(x_s)\int_s^t dx_r + \int_s^t \left(\int_s^u \varphi'(x_r)dx_r\right) dx_u$$

with our “brand new” notation reads

$$\int \varphi(x)dx = \varphi(x)\int dx + \int \varphi'(x)dxdx$$

as elements of $\mathcal{C}_1$.
We look in more detail to the iterated integral by **dissecting** it:

$$\delta \int \varphi'(x)dxdx = \int \varphi'(x)dx \int dx = \delta\varphi(x)\delta x \in \mathcal{C}_3^2$$

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Young integration

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$$\int \varphi(x)dx = \varphi(x)\delta x + \Lambda(\delta\varphi(x)\delta x)$$

- The integral on the l.h.s is equal to an expression which do not need x to be differentiable.
- Essentially x must be γ-Hölder with $\gamma > 1/2$ – Young integration

Go on…

Again

$$\int \varphi(x)dx = \varphi(x)\int dx + \int \varphi'(x)dxdx$$

But now continue Taylor expansion one step further:

$$\int \varphi(x)dx = \varphi(x)\int dx + \varphi'(x)\int dxdx + \int \varphi''(x)dxdxdx$$

The remainder is now a three-fold integral:

$$\delta \int \varphi''(x)dxdxdx = \underbrace{\int \varphi''(x)dx}_{\delta\varphi'(x)} \int dxdx + \underbrace{\int \varphi''(x)dxdx}_{\delta\varphi(x)-\varphi'(x)\delta x} \int dx$$

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1. Review of the “classical” theory
   - Abstract integration
   - Exercise of deconstruction
   - Rough paths

2. Rough evolution equations
   - Convolution integrals
   - Young theory
   - More irregular noises
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   - Summary of the approach

Rough paths

Putting things together

$$\int \varphi(x)dx = (1-\Lambda\delta)\left[\varphi(x)\delta x + \varphi'(x)\int dxdx\right]$$

(if the argument of Λ is small enough).

- To make sense of the r.h.s we need a small $\int dxdx$ such that

$$\delta \int dxdx = \delta x \delta x$$

  (which is a highly nontrivial non-linear relation).
- $\int dxdx$ is the "Levy area" of the rough path theory.

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- This procedure can be iterated to recover the hierarchy of (Lyons') rough paths which are given by a sequence of iterated integrals of the form

$$\int dx, \quad \int dxdx, \quad \int dxdxdx, \ldots$$

- **Watch out**: to prove smallness of some terms we need geometric rough paths, i.e. which satisfy relations like

$$[(\delta x)_{ts}]^2 = 2\left(\int dxdx\right)_{ts}.$$

(smooth integrals OK, Stratonovich OK, Itô NO! – but we do not need it).

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Difference equations

A remark

Given a rough path $(\int dx, \int dxdx)$ the solutions y of the diff. eqn.

$$dy = \varphi(y)dx$$

is the unique path which satisfy the difference equation

$$\delta y = \varphi(y) \int dx + \varphi'(y)\varphi(y) \int dxdx + r, \qquad r \in \mathcal{C}_1^{1+}$$

The integral equation

This remainder is uniquely determined and we must have

$$\delta y = (1 - \Lambda\delta)[\varphi(y) \int dx + \varphi'(y)\varphi(y) \int dxdx]$$

which can be solved by fixed point method.

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Extended Garcia-Rodemich-Rumsey inequality

To prove path-wise regularity of 1-increments defined by stochastic integrals we have the following useful lemma:

Lemma

For any $\gamma > 0$ and $p \geq 1$ there exists a constant C such that for any $g \in \mathcal{C}_1$

$$\|g\|_\gamma \leq C(U_{\gamma+2/p,p}(g) + \|\delta g\|_\gamma).$$

where

$$U_{\gamma,p}(g) = \left[\int_{[0,T]^2} \left(\frac{|g_{ts}|}{|t-s|^\gamma}\right)^p dtds\right]^{1/p}.$$

This reduces to the well known GRR inequality when $g_{ts} = f_t - f_s$.

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Increments of convolutions

Let $S(t)$, $t \geq 0$ be a semigroup.
Look at

$$g_t = \int_0^t S(t-u) dx_u f(x_u)$$

Then

$$(\delta g)_{ts} = a_{ts} g_s + \int_s^t S(t-u) dx_u f(x_u)$$

with $a_{ts} = S(t-s) - 1$

Remark

$$(\delta a)_{tus} = a_{tu} a_{us}, \qquad t \geq u \geq s \geq 0$$

due to the semigroup property of S.

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A connection

Idea

Introduce the "perturbed" coboundary $\hat{\delta} = (\delta - a)$

$\hat{\delta}$ is a coboundary

$$\hat{\delta}\hat{\delta}f = (\delta - a)(\delta f - af) = \delta^2 f - \delta(af) - a\delta f + aaf$$
$$= -(\delta a - aa)f = 0$$

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$$(\hat{\delta}g)_{ts} = \int_s^t S(t-u)dx_u f(x_u) = \left(\int \hat{d}x f(x)\right)_{ts}$$

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Ingredients

- a scale of Banach spaces $\{\mathcal{B}_\alpha\}_{\alpha\in\mathbb{R}}$ associated with S ($|\varphi|_\alpha = |(-A)^\alpha \varphi|_{\mathcal{B}}$)
- $\Delta_n = \{(t_0, t_1, \ldots, t_n) : T \geq t_0 \geq t_1 \geq \cdots \geq t_n \geq 0\}$.
- n-increments $\hat{\mathcal{C}}_n(V) = C(\Delta_n; V)$ vanishing on diagonals.
- Norms for $g \in \hat{\mathcal{C}}_1$ and $h \in \hat{\mathcal{C}}_2$:

$$\|g\|_{\mu,\alpha} \equiv \sup_{(t,s)\in\Delta_1} \frac{|g_{ts}|_{\mathcal{B}_\alpha}}{|t-s|^\mu}, \quad \text{and} \quad \|h\|_{\gamma,\rho,\alpha} = \sup_{(t,u,s)\in\Delta_2} \frac{|h_{tus}|_{\mathcal{B}_\alpha}}{|t-u|^\gamma |u-s|^\rho}$$

$$\|h\|_{\mu,\alpha} \equiv \inf\left\{\sum_i \|h_i\|_{\rho_i,\mu-\rho_i,\alpha};\ h = \sum_i h_i,\ 0 < \rho_i < \mu\right\},$$

  and corresponding spaces $\hat{\mathcal{C}}_k^{\mu,\alpha}$, $k = 1, 2$.
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The perturbed complex

- We have another acyclic cochain complex $(\hat{\mathcal{C}}_*, \hat{\delta})$

$$\mathcal{B} \to \hat{\mathcal{C}}_0 \xrightarrow{\hat{\delta}} \hat{\mathcal{C}}_1 \xrightarrow{\hat{\delta}} \hat{\mathcal{C}}_2 \xrightarrow{\hat{\delta}} \hat{\mathcal{C}}_3 \xrightarrow{\hat{\delta}} \cdots$$

- 0-cocycles: $f \in \hat{\mathcal{C}}_0, \hat{\delta} f = 0 \Rightarrow f_t = S(t) f_0$
- For $\mu > 1$, there exists a unique bounded operator $\hat{\Lambda} : \mathcal{B}\hat{\mathcal{C}}_2^{\mu,\alpha} \to \mathcal{E}_1^{\mu,\alpha}$ such that

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Young theory

The simplest expansion gives

$$(\hat{\delta}z)_{ts} = \int_s^t \hat{d}x_u f(y_u) = \int_s^t \hat{d}x_u f(y_s) + \int_s^t \hat{d}x_u [f(y_u) - f(y_s)]$$

or in compact notation

$$\hat{\delta}z = \mathcal{J}[\hat{d}xf(y)] = \mathcal{J}(\hat{d}x)f(y) + \mathcal{J}[\hat{d}x\delta f(y)]$$

Assume

$$\hat{\delta}\mathcal{J}[\hat{d}x\delta f(y)] = \mathcal{J}(\hat{d}x)\delta f(y) \in \hat{\mathcal{C}}_2^{1+}$$

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Young evolution equations

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  we find solutions by solving the equation

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  by fixed points methods.
- SPDEs driven by FBM (H > 1/2), joint work with A. Lejay and S. Tindel.
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Young evolution equations

- Given $\mathcal{J}(\hat{d}x) \in \hat{\mathcal{C}}_1(\mathcal{L}(\mathcal{B};\mathcal{B}))$ and the integral problem

  $$y_t = S(t)y_0 + \mathcal{J}_{0t}(\hat{d}x f(y))$$

  we find solutions by solving the equation

  $$\hat{\delta} y = \mathcal{J}(\hat{d}x f(y)) = \mathcal{J}(\hat{d}x) f(y) + \hat{\Lambda}[\mathcal{J}(\hat{d}x) \delta f(y)]$$

  by fixed points methods.
- SPDEs driven by FBM (H > 1/2), joint work with A. Lejay and S. Tindel.
- When $\nu = 1$ (white noise in space) we are limited to $H > 3/4$. In any case this approach is limited to $H > 1/2$.

Outline

- To start

1. Review of the “classical” theory
   - Abstract integration
   - Exercise of deconstruction
   - Rough paths

2. Rough evolution equations
   - Convolution integrals
   - Young theory
   - More irregular noises
   - Fully non-linear case
   - Summary of the approach

A (bi-)linear equation

Let us play with the solution y of the (bi-)linear integral equation

$$y_t = S(t-s)y_s + \int_s^t S(t-u)dx_u y_u.$$

Expand the r.h.s. in a truncated series of iterated integrals:

$$y_t = S(t-s)y_s + \int_s^t S(t-u)dx_u S(u-s)y_s + \int_s^t S(t-u)dx_u \int_s^u S(u-v)dx_v y_v$$

In our notation this reads:

$$\hat{\delta} y = \mathcal{J}(\hat{d}xS)\,y + \mathcal{J}(\hat{d}x\hat{d}x\,y) = \mathcal{J}(\hat{d}xS)\,y + \mathcal{J}(\hat{d}x\hat{d}xS)\,y + \underbrace{\mathcal{J}(\hat{d}x\hat{d}x\hat{d}x\,y)}_{remainder}$$

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Convolution rough paths

- Working a bit we get to

$$\hat{\delta} y = (1 - \hat{\Lambda}\hat{\delta}) \left[\mathcal{J}(\hat{d}xS)y + \mathcal{J}(\hat{d}x\hat{d}xS)y \right]$$

  where we used the fact that $\hat{\delta}\mathcal{J}(\hat{d}x\hat{d}xS) = \mathcal{J}(\hat{d}xS)\,\mathcal{J}(\hat{d}xS)$

- This express the solution y as a function of the couple

$$\mathcal{J}(\hat{d}xS) \qquad \mathcal{J}(\hat{d}x\hat{d}xS)$$

  suitable notion of rough path for this linear convolution equation.

- The solution can be expressed as a series

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A general class of integrands

Controlled paths

A controlled path y is such that exists $y^x \in \hat{\mathcal{C}}_0$ and $y^\sharp \in \hat{\mathcal{C}}_1$

$$\hat{\delta} y = \mathcal{J}(\hat{d}x) y^x + y^\sharp$$

Then

$$\hat{\delta} z = \mathcal{J}(\hat{d}xy) = \mathcal{J}(\hat{d}xS)y + \mathcal{J}(\hat{d}x\hat{\delta}y)$$

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Integration of controlled paths

Controlled paths can be integrated against $\hat{d}x$:

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Non-commutativity

Problem

Controlled paths are not stable under natural maps. $f(y)$ is not controlled in general (even if f is linear).

$$\hat{\delta} z = \mathcal{J}(\hat{d}x f(y))$$

[Non-commutativity of the semigroup with multiplication.]

Idea

Expand $\delta f(y)$ in a Taylor-like series

$$\delta f(y) = B(\delta y \otimes f'(y)) + r$$

r some small remainder

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Non-linear integrals

- $\hat{\delta} z = \mathcal{J}(\hat{d}x f(y)) = \mathcal{J}(\hat{d}x) f(y) + \mathcal{J}(\hat{d}x \delta f(y))$
- Assume y is controlled, and expand $f(y)$:

$$\hat{\delta} z = \mathcal{J}(\hat{d}x) f(y) + \mathcal{J}(\hat{d}x B(\delta y \otimes \mathrm{Id})) f'(y) + \mathcal{J}(\hat{d}x r)$$

- The dissection of the last term gives

$$\hat{\delta} \mathcal{J}(\hat{d}x r) = \mathcal{J}(\hat{d}x) r + \mathcal{J}(\hat{d}x B(\delta y \otimes \mathrm{Id})) \delta f'(y)$$

- The term $\mathcal{J}(\hat{d}x B(\delta y \otimes \mathrm{Id}))$ can be defined using the “linear” theory for the controlled path y

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The linear part

- First, reduce it to having $\hat{\delta} y$ instead of δy,

$$\mathcal{J}(\hat{d}xB(\delta y \otimes \mathrm{Id})) = \mathcal{J}(\hat{d}xB(\hat{\delta} y \otimes \mathrm{Id})) + \underbrace{\mathcal{J}(\hat{d}xB(a \otimes \mathrm{Id}))}_{Q}(y \otimes \mathrm{Id})$$

- then

$$\mathcal{J}(\hat{d}xB(\hat{\delta} y \otimes \mathrm{Id})) = \underbrace{\mathcal{J}(\hat{d}xB(\hat{d}x \otimes \mathrm{Id}))}_{M}(y^x \otimes \mathrm{Id}) + \mathcal{J}(\hat{d}xB(y^\sharp \otimes \mathrm{Id}))$$

- At last

$$\mathcal{J}(\hat{d}xf(y)) = \mathcal{J}(\hat{d}x)f(y) + Q(y \otimes f'(y)) + M(y^x \otimes f'(y)) + \hat{\Lambda}[\cdots]$$

- Note

$$Q : \hat{\mathcal{C}}_1^{\theta_1}(\mathcal{L}(\mathcal{B}_\delta \otimes \mathcal{B}_\delta; \mathcal{B}_\rho)), \quad M : \hat{\mathcal{C}}_1^{\theta_2}(\mathcal{L}(\mathcal{B}_\delta \otimes \mathcal{B}_\delta; \mathcal{B}_\rho))$$

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The full rough path

(Quasi)Theorem

If

$$\hat{\delta} y = n y^x + y^\sharp, \qquad \delta f(y) = B(\delta y \otimes f'(y)) + r$$

then

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Plain talk

- The quadruplet (n, N, Q, M) is the new rough path.

$$n_{ts} : \mathcal{B} \to \mathcal{B}; \qquad N_{ts}, Q_{ts}, M_{ts} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}.$$

- Some expressions

$$\begin{aligned} Q_{ts}(\varphi \otimes \psi) &= [\mathcal{J}(\hat{d}xB(a \otimes \mathrm{Id}))]_{ts}(\varphi \otimes \psi) \\ &= \int_s^t S(t-u)dx_u B[(a_{us}\varphi) \otimes \psi] \end{aligned}$$

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Rough evolution equations

1. Given x prove the existence of path-wise regular versions of n, N, Q, M with right algebraic properties and in the right operator spaces [Estimates on stochastic integrals, Kolmogorov-like criterion, HS norms].
2. Given n, N, Q, M define controlled paths and integrate them against x
3. Find fixed points of the map $y \mapsto S(\cdot)y_0 + \mathcal{J}_{0\cdot}[\hat{d}xf(y)]$ in the space of controlled paths.

Our preferred 1d example

Within this approach we can handle $H > 1/3$ for sufficiently small ν.
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Still not enough to handle space-time white noise.

Rough evolution equations

1. Given x prove the existence of path-wise regular versions of n, N, Q, M with right algebraic properties and in the right operator spaces [Estimates on stochastic integrals, Kolmogorov-like criterion, HS norms].
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3. Find fixed points of the map $y \mapsto S(\cdot)y_0 + \mathcal{J}_{0\cdot}[\hat{d}xf(y)]$ in the space of controlled paths.

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- Lyons' rough paths are particular cases of more general structure (rough increments).
- We tried to emphasize an algorithm for the "creation" of useful notions of rough increments.
- In particular we have shown how to implement rough increment which "support" pathwise solutions of evolution equations.
- To handle space-time white noise (even in the 1d situation) we need to devise an expansion to higher order (analogy with the situation for fBm with $H < 1/3$). Conjecture: We need at least 4th order expansion.
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