A Constructive Introduction to Quantum Field Theory

Lecture 1: Relativistic quantum fields and imaginary time

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Abstract

We recall the Wightman axioms for a relativistic scalar field, explain why quantum fields must be operator-valued distributions, and discuss the Källén–Lehmann representation of the two-point function. Using a lattice regularisation of the quantum wave equation on a spatial circle we show how positivity of the energy allows the analytic continuation to imaginary time, leading to the Feynman–Kac (path integral) representation and to the identification of Euclidean field theories with continuum limits of short-range lattice spin systems at criticality. We conclude with the Osterwalder–Schrader axioms. Exercises are collected at the end of each section. These notes follow the slides of Lecture 1.

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Introduction

Constructive quantum field theory aims at constructing, with full mathematical rigour, examples of *interacting* relativistic quantum fields, i.e. models satisfying a small set of axioms which encode quantum mechanics, special relativity and causality. In spacetime dimension $d = 2$ and $d = 3$ this programme has been carried out for several models ($P(\varphi)_2$, Yukawa$_2$, Φ^4_3, sine-Gordon, ...), while in $d = 4$ no interacting model is known, and the scaling limits of critical lattice φ^4_4 models are Gaussian (trivial) [ADC21].

The main lesson of the late 1960s and 1970s [Sym69, Nel73a, Nel73b, OS73, OS75, GRS75] is that it is much more efficient to work in *imaginary time*. The relativistic theory is then encoded in a probability measure on distributions on $\mathbb{R}^d$, which looks like the continuum limit of a classical lattice spin system at a critical point. All modern approaches (cluster expansions, renormalisation group, stochastic quantisation, variational methods) work at this Euclidean level.

Plan. Section 1 recalls the Minkowski picture: Wightman axioms, the need for distributions, the Källén–Lehmann representation, the quantum wave equation. Section 2 explains the passage to imaginary time on a lattice regularisation of the field on a circle, the Feynman–Kac formula, the statistical mechanics interpretation and the Osterwalder–Schrader axioms. Lecture 2 (Section 3 of its notes) studies the first example, the Gaussian free field. Section 3 outlines the plan of the course (seven lectures).

Conventions. $\hbar = c = 1$. Spacetime dimension is $d = 1+s$, $s \geqslant 1$ the number of space dimensions. Inner products $\langle f, g\rangle = \int \bar{f} g$ are antilinear in the first slot. For $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f}(k) = \int \mathrm{e}^{-\mathrm{i}k\cdot x} f(x)\,\mathrm{d}x$. Constants C may change from line to line.

1 Relativistic quantum fields in Minkowski space

1.1 Kinematics

Minkowski space is $\mathbb{R}^{1,s} \simeq \mathbb{R}^d$ with points $x = (x^0, \mathbf{x})$ and bilinear form $x\cdot y = x^0 y^0 - \mathbf{x}\cdot\mathbf{y}$, $x^2 = x\cdot x$. Two points are *spacelike separated* if $(x-y)^2 < 0$. The open forward light cone is $V_+ = \{p : p^0 > |\mathbf{p}|\}$, with closure $\bar{V}_+$. The restricted Lorentz group $\mathcal{L}_+^\uparrow$ is the connected component of the identity of the group of linear maps preserving $x\cdot y$; the restricted Poincaré group is $\mathcal{P}_+^\uparrow = \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow$, with $(a,\Lambda)(b,\Lambda') = (a+\Lambda b, \Lambda\Lambda')$, acting on spacetime by $x \mapsto \Lambda x + a$.

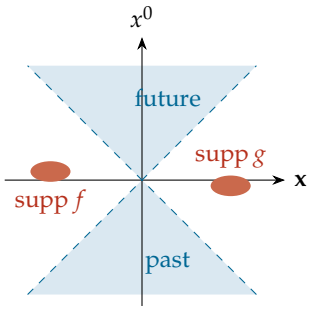


Figure 1.1. Minkowski space for $s = 1$. The shaded regions are timelike to the origin. The supports of f and g are spacelike separated, so $[\varphi(f), \varphi(g)] = 0$ by (W3). In momentum space the same forward cone $\bar{V}_+$ carries the joint spectrum of $(H, \mathbf{P})$.

By Wigner's theorem a relativistic quantum system carries a (projective) unitary representation of $\mathcal{P}_+^\uparrow$ on its Hilbert space of states. For a scalar field we may assume a true representation.

1.2 The Wightman axioms

Definition 1.1. **(Wightman QFT of a hermitian scalar field)** *A Wightman theory is given by the following data, subject to the following conditions.*

(W0) ***Relativistic quantum theory.***
A separable Hilbert space $\mathcal{H}$ and a strongly continuous unitary representation U of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$. Write $U(a,1) = e^{iP\cdot a} = e^{i(P^0 a^0 - \mathbf{P}\cdot\mathbf{a})}$ for commuting self-adjoint generators $P = (P^0, \mathbf{P})$; $H := P^0$ is the Hamiltonian.

Spectrum condition: *the joint spectrum of $(P^0, \dots, P^s)$ is contained in $\bar{V}_+$; in particular $H \geqslant |\mathbf{P}| \geqslant 0$.*

Vacuum: *there is a unit vector Ω, unique up to a phase among the translation invariant vectors, with $U(a,\Lambda)\,\Omega = \Omega$ for all $(a,\Lambda) \in \mathcal{P}_+^\uparrow$.*

(W1) ***Fields.***
There is a dense subspace $\mathcal{D} \subset \mathcal{H}$ and, for every $f \in \mathcal{S}(\mathbb{R}^d)$, an operator $\varphi(f)$ with domain $\mathcal{D}$ such that $\Omega \in \mathcal{D}$, $\varphi(f)\,\mathcal{D} \subset \mathcal{D}$, $U(a,\Lambda)\,\mathcal{D} \subset \mathcal{D}$; for $\chi, \psi \in \mathcal{D}$ the map $f \mapsto \langle \chi, \varphi(f)\,\psi\rangle$ is a tempered distribution; and $\langle \chi, \varphi(f)\,\psi\rangle = \langle \varphi(\bar{f})\,\chi, \psi\rangle$.

(W2) ***Covariance.***
$U(a,\Lambda)\,\varphi(f)\,U(a,\Lambda)^{-1} = \varphi(f_{(a,\Lambda)})$ on $\mathcal{D}$, where $f_{(a,\Lambda)}(x) = f(\Lambda^{-1}(x-a))$.

(W3) ***Locality.***
If the supports of f and g are spacelike separated, then $[\varphi(f), \varphi(g)]\,\psi = 0$ for all $\psi \in \mathcal{D}$.

(W4) ***Cyclicity of the vacuum.***
The span of $\{\varphi(f_1)\cdots\varphi(f_n)\,\Omega \colon n \geqslant 0, f_j \in \mathcal{S}(\mathbb{R}^d)\}$ is dense in $\mathcal{H}$.

Formally $\varphi(f) = \int \varphi(x)\, f(x)\, \mathrm{d}x$, and $\varphi(x^0, \mathbf{x}) = e^{iHx^0}\varphi(0,\mathbf{x})e^{-iHx^0}$ is the Heisenberg-picture field. For $d \geqslant 3$, Lorentz invariance of the vacuum follows from translation invariance and uniqueness: $U(0,\Lambda)\,\Omega$ is translation invariant, hence equal to $\chi(\Lambda)\,\Omega$ for a character χ of $\mathcal{L}_+^\uparrow$, and $\chi \equiv 1$ because $\mathcal{L}_+^\uparrow$ has no nontrivial one-dimensional unitary representations. For $d = 2$ the boosts form the abelian group $\mathbb{R}$; there one may always replace $U(a,\Lambda)$ by $\chi(\Lambda)^{-1}U(a,\Lambda)$, which is why we include Lorentz invariance in (W0). The *Wightman functions*

$$W_n(f_1, \dots, f_n) := \langle \Omega, \varphi(f_1)\cdots\varphi(f_n)\,\Omega\rangle \tag{1.1}$$

extend, by the Schwartz kernel theorem, to tempered distributions $W_n \in \mathcal{S}'(\mathbb{R}^{dn})$. They satisfy: (a) *positivity*: $\sum_{j,k} W_{j+k}(g_j^* \otimes g_k) \geqslant 0$ for every finite sequence $g_j \in \mathcal{S}(\mathbb{R}^{dj})$, with $g^*(x_1,\dots,x_j) = \overline{g(x_j,\dots,x_1)}$; (b) *Poincaré invariance*; (c) *spectral condition*: $W_n(x_1,\dots,x_n) = w_n(x_1 - x_2, \dots, x_{n-1} - x_n)$ with $\hat{w}_n$ supported in $\bar{V}_+^{n-1}$ (for a suitable sign convention); (d) *locality*: W_n is unchanged under exchange of two adjacent spacelike separated arguments; (e) *cluster property*, equivalent to uniqueness of the vacuum (Remark 2.7).

Theorem 1.2. **(Wightman reconstruction [SW64])** *Any sequence $(W_n)_{n\geqslant 0}$ of tempered distributions with $W_0 = 1$ satisfying (a)–(d) is the sequence of Wightman functions of a theory satisfying (W1)–(W4) and (W0) except possibly uniqueness of the vacuum, unique up to unitary equivalence. The vacuum is unique iff (e) holds.*

So constructing a QFT amounts to constructing its Wightman functions. For the axioms, the reconstruction theorem and the algebraic viewpoint see [SW64, BSZ92, Haa96]. The difficulty is that the positivity condition (a) is hard to verify for candidate Wightman functions: it involves all W_n at once, and the objects one can actually construct live in imaginary time, whereas (a) refers to their real-time boundary values. The Euclidean approach of Section 2 replaces it by reflection positivity, which can be checked on regularised (e.g. lattice) approximations and is preserved under the limits one has to take. Note also that the axioms say nothing about dynamics: there is no Lagrangian, and the interaction is hidden in the higher Wightman functions.

Remark 1.3. (Minimal requirements) The axioms (W0)–(W4) are a minimal transcription of what we expect of any relativistic quantum theory with a local quantity. (W0) encodes quantum mechanics (states in a Hilbert space), Poincaré symmetry (Wigner's theorem), and stability: the energy is bounded below, which together with Lorentz covariance gives the spectrum condition, and there is a ground state. (W1) and (W2) require a covariant quantity localised in spacetime regions. (W3) is Einstein causality: measurements in spacelike separated regions do not interfere. (W4) says that the field is complete: all states are produced from the vacuum by local operations. Nothing is assumed about dynamics, Lagrangians, perturbation theory or particles. These either follow as theorems (PCT and the spin–statistics connection [SW64]; particles and scattering states from an isolated mass shell, by Haag–Ruelle theory [Haa96]) or are additional structure. A construction of a relativistic QFT must at least produce a model of these axioms, which is why they are our target. They are often supplemented by the *time-slice axiom* (primitive causality): the fields smeared in an arbitrarily thin time slice already generate all operators. It expresses the existence of a dynamical law and is the abstract counterpart of a well-posed Cauchy problem [Haa96, §II.1].

Remark 1.4. (Local algebras and fields) For a bounded open region $O \subset \mathbb{R}^d$, let $\mathfrak{A}(O)$ be the von Neumann algebra generated by the bounded functions of the smeared fields $\varphi(f)$, f real with $\operatorname{supp} f \subset O$. (This requires the closures of the $\varphi(f)$ to be self-adjoint and to commute strongly at spacelike separation, which holds in the models we construct.) The map $O \mapsto \mathfrak{A}(O)$ is a *Haag–Kastler net* [HK64, Haa96]. It satisfies isotony, $\mathfrak{A}(O_1) \subset \mathfrak{A}(O_2)$ for $O_1 \subset O_2$; Einstein causality, $[\mathfrak{A}(O_1), \mathfrak{A}(O_2)] = 0$ for spacelike separated regions; covariance, $U(a,\Lambda)\,\mathfrak{A}(O)\,U(a,\Lambda)^* = \mathfrak{A}(\Lambda O + a)$; and the spectrum condition, with a vacuum vector. In the algebraic point of view the net *is* the physical content of the theory: it specifies which observables can be measured in each region, and nothing else. A field is a choice of coordinates for the net. Different fields can generate the same net, and fields that are local relative to each other (the Borchers class of φ, which contains for instance all Wick polynomials of a free field) describe the same physics, e.g. the same scattering matrix. Conversely, fields need not be observable: fermion fields anticommute at spacelike separation, and charged fields are not invariant under the gauge group.

We nevertheless need fields, for three reasons. (a) The net refers to regions, not to points. Fields provide the pointlike localised quantities in which dynamics (field equations, Lagrangians), interactions (local products such as $:\varphi^4:$) and the short-distance structure (operator product expansion, scaling dimensions) are expressed. (b) Correlation functions of fields are the concrete objects that constructive methods produce and estimate: the Euclidean measures of these lectures. (c) The two descriptions are close but not equivalent. The Wightman axioms alone do not guarantee a net of local algebras of bounded operators, and a net does not guarantee a Wightman field [Haa96, §III.1]. Under a phase-space condition, Fredenhagen and Hertel recover pointlike localised fields from a net by shrinking the regions to points [FH81]. In this course a QFT is constructed as a Euclidean measure for a field, and its local algebras are those generated by the reconstructed field.

Remark 1.5. (The vacuum, and states without a vacuum) In Minkowski space the vacuum is singled out by symmetry and stability: it is the Poincaré invariant state, and by the spectrum condition it has the lowest energy in every Lorentz frame. In the Wightman framework its existence is an axiom. In the constructive approach it is a result: it arises as the limit $T \to \infty$ of the Euclidean measures (Table 2.1), and its uniqueness is equivalent to clustering (Remark 2.7). On a curved spacetime, for instance an expanding Friedmann–Robertson–Walker universe, there is no Poincaré group. Without a timelike Killing field there is neither a conserved energy nor a preferred vacuum, and the notion of particle becomes observer dependent (particle

creation in expanding universes, and in gravitational collapse, the Hawking effect; already in flat space, the Unruh effect shows that the notion of particle depends on the observer). The algebraic framework still makes sense: nets of local algebras can be defined locally and covariantly [BFV03]. But the physically acceptable states must be selected by a *local* condition, the Hadamard condition: at short distance, the two-point function has the same singularity as the Minkowski vacuum. In $d=4$ it reads, schematically,

$$W_2(x,y)=\frac{1}{4\pi^2}\left(\frac{U(x,y)}{\sigma_\varepsilon(x,y)}+V(x,y)\log\sigma_\varepsilon(x,y)\right)+w(x,y),$$

where $\sigma(x,y)$ is the squared geodesic distance, taken positive at spacelike separation (so that $W_2=1/(4\pi^2\sigma_\varepsilon)$ for the massless field in Minkowski space), and $\sigma_\varepsilon(x,y)=\sigma(x,y)+2\mathrm{i}\varepsilon\,(t(x)-t(y))+\varepsilon^2$, $\varepsilon\downarrow 0$, implements the positive-frequency $\mathrm{i}\varepsilon$ prescription (t a time function), the coefficients U,V are determined by the metric and the field equation, and w is smooth and state dependent. By Radzikowski's theorem this is equivalent to a *microlocal spectrum condition*: the wave front set of W_2 contains only positive frequencies [Rad96]. This is the local substitute for positivity of the energy. Hadamard states are the states in which Wick powers and the expectation of the stress–energy tensor can be renormalised, by subtracting the geometric part of W_2 (point splitting). This is what semiclassical gravity and cosmology need [Wal94]; examples are the Bunch–Davies state of de Sitter space, used for inflationary perturbations, and the states of low energy on Friedmann–Robertson–Walker spacetimes (adiabatic states of finite order are, in general, only approximately Hadamard). In the language of these lectures, Remark 1.10(iii) and its Euclidean version (Remark 3.7) show that in flat space every theory is at least as singular as the free field. The Hadamard condition prescribes exactly the free singularity, and the Wick ordering of Lecture 2 is the flat-space version of the subtraction of the Hadamard parametrix.

1.3 Why operator-valued distributions?

The free field. Let $m>0$, $\omega(\mathbf{p})=(|\mathbf{p}|^2+m^2)^{1/2}$, and let $\mathrm{d}\Omega_m(\mathbf{p})=\mathrm{d}\mathbf{p}/((2\pi)^s 2\,\omega(\mathbf{p}))$ be the Lorentz invariant measure on the mass hyperboloid $H_m^+=\{p\colon p^2=m^2,\ p^0>0\}$, parametrised by $\mathbf{p}$. On the Fock space over $\mathfrak{h}=L^2(\mathbb{R}^s,\mathrm{d}\Omega_m)$ let $a(\mathbf{p}),a^*(\mathbf{p})$ be the annihilation/creation operator-valued distributions, $[a(\mathbf{p}),a^*(\mathbf{q})]=(2\pi)^s 2\omega(\mathbf{p})\,\delta(\mathbf{p}-\mathbf{q})$, $a(\mathbf{p})\,\Omega=0$. The free field is

$$\varphi(x)=\int \mathrm{d}\Omega_m(\mathbf{p})\ [a(\mathbf{p})\mathrm{e}^{-\mathrm{i}p\cdot x}+a^*(\mathbf{p})\mathrm{e}^{\mathrm{i}p\cdot x}]_{p^0=\omega(\mathbf{p})}. \tag{1.2}$$

For $f\in\mathcal{S}(\mathbb{R}^d)$ one gets $\varphi(f)\,\Omega=a^*(\tilde f)\,\Omega$ with $\tilde f(\mathbf{p})=\int f(x)\mathrm{e}^{\mathrm{i}p\cdot x}\mathrm{d}x|_{p^0=\omega(\mathbf{p})}$, and

$$W_2(x-y)=\int \mathrm{d}\Omega_m(\mathbf{p})\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)},\qquad \|\varphi(f)\,\Omega\|^2=\int \mathrm{d}\Omega_m(\mathbf{p})\,|\tilde f(\mathbf{p})|^2<\infty. \tag{1.3}$$

But formally $\|\varphi(x)\,\Omega\|^2=W_2(0)=\int \mathrm{d}\Omega_m=+\infty$ for $s\geqslant 1$: field fluctuations at a point are infinite. The divergence is ultraviolet (large momenta). It is logarithmic for $s=1$ and power-like for $s\geqslant 2$. This is not an artefact of the free field.

Proposition 1.6. (No fields at a point; Wightman [Wig64]) *Assume (W0)–(W2) and suppose there is $\psi\in\mathcal{H}$ such that $\varphi(f)\,\Omega=\int f(x)\,U(x,1)\,\psi\,\mathrm{d}x$ for all $f\in\mathcal{S}(\mathbb{R}^d)$, i.e. "$\varphi(0)\,\Omega=\psi$ exists". Then $\psi=\langle\Omega,\psi\rangle\,\Omega$, hence $\varphi(f)\,\Omega=\langle\Omega,\psi\rangle(\int f)\,\Omega$.*

Lemma 1.7. *Let $d\geqslant 2$ and let μ be a finite positive Borel measure on $\bar V_+$ invariant under $\mathcal{L}_+^\uparrow$. Then $\mu=\mu(\{0\})\,\delta_0$.*

Proof. A boost of rapidity θ in the x^1 direction acts as $p^0 \pm p^1 \mapsto \mathrm{e}^{\pm\theta}(p^0 \pm p^1)$, leaving the other components unchanged. Let $S_j^{\pm} = \{p \in \bar{V}_+ \colon 2^j \leqslant p^0 \pm p^1 < 2^{j+1}\}$, $j \in \mathbb{Z}$. The boost with $\mathrm{e}^{\theta} = 2$ maps S_j^+ onto S_{j+1}^+, so $\mu(S_j^+)$ is independent of j. These sets are disjoint and μ is finite, so $\mu(S_j^+) = 0$. The same argument applies to S_j^-. Finally $\bigcup_j (S_j^+ \cup S_j^-) = \bar{V}_+ \setminus \{0\}$, since $p^0 \pm p^1 \geqslant 0$ on $\bar{V}_+$ and $p^0 + p^1 = p^0 - p^1 = 0$ forces $p = 0$. □

Proof of Proposition 1.6. By covariance, $U(0,\Lambda)\,\varphi(f)\,\Omega = \varphi(f_{(0,\Lambda)})\,\Omega$. Since $U(0,\Lambda)\,U(x,1) = U(\Lambda x,1)\,U(0,\Lambda)$, after the change of variables $x \mapsto \Lambda x$ this reads

$$\int f(x)\; U(\Lambda x,1)\,(U(0,\Lambda)\,\psi - \psi)\,\mathrm{d}x = 0 \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^d).$$

Letting $f \to \delta_0$ gives $U(0,\Lambda)\,\psi = \psi$. Let μ be the spectral measure of ψ for the translations, $\langle \psi, U(a,1)\,\psi\rangle = \int \mathrm{e}^{\mathrm{i}p\cdot a}\mu(\mathrm{d}p)$. It is finite with mass $\|\psi\|^2$, supported in $\bar{V}_+$ by the spectrum condition, and Lorentz invariant because ψ is. By Lemma 1.7, $\langle \psi, U(a,1)\,\psi\rangle = \|\psi\|^2$ for all a, hence $U(a,1)\,\psi = \psi$ (equality in Cauchy–Schwarz), and $\psi \in \mathbb{C}\,\Omega$ by uniqueness of the vacuum. □

Remark 1.8. So the only theories with point-like fields have $\varphi(f)\,\Omega \propto \Omega$; using locality and the Reeh–Schlieder theorem one then shows that φ is a multiple of the identity [SW64, Wig64]. Physically (Bohr–Rosenfeld), measuring a field at a sharp point requires infinite energy. Quantum fields are therefore *operator-valued distributions*, and every nonlinear expression in $\varphi(x)$ (interaction terms, field equations, composite operators) requires a definition. This is the root of renormalisation.

1.4 The two-point function: the Källén–Lehmann representation

The free two-point function (1.3) is the Fourier transform of the invariant measure on the mass shell. In every Wightman theory the two-point function is a positive superposition of such free ones. For $M \geqslant 0$ let $W_2^{(M)}$ be the two-point function (1.3) of the free field of mass M. For $M = 0$ and $d \geqslant 3$ it is defined with the invariant measure $\mathrm{d}\Omega_0 = \mathrm{d}\mathbf{p}/((2\pi)^s 2|\mathbf{p}|)$ on the boundary of the cone; for $d = 2$ the measure $\mathrm{d}p/|p|$ is not locally finite at $p = 0$, and $W_2^{(0)}$ is not defined. By translation invariance of Ω, $\langle \Omega, \varphi(f)\,\Omega\rangle = v \int f$ for some $v \in \mathbb{R}$.

Theorem 1.9. (Källén–Lehmann [Kal52, Leh54]) *Assume (W0)–(W2). There is a positive Borel measure ρ on $[0,\infty)$, the* spectral measure *of the field, such that*

$$W_2(x-y) = v^2 + \int_{[0,\infty)} \rho(\mathrm{d}M^2)\; W_2^{(M)}(x-y) \tag{1.4}$$

in the sense of tempered distributions. The measure ρ grows at most polynomially and is such that the right-hand side is tempered; in particular $\rho(\{0\}) = 0$ if $d = 2$.

Sketch. Let $\psi_f := \varphi(f)\,\Omega - v(\int f)\,\Omega \perp \Omega$. By hermiticity $\langle \psi_{\bar{f}}, \psi_g\rangle = W_2(f \otimes g) - v^2 \int f \int g$, and by covariance $U(a,1)\,\psi_f = \psi_{f_{(a,1)}}$. Hence the truncated function $W_2^T := W_2 - v^2$ is a distribution of positive type in $x - y$. By the Bochner–Schwartz theorem (equivalently, the spectral theorem for the translations restricted to the closed span of the ψ_f) it is the Fourier transform of a positive tempered measure μ on momentum space. The spectrum condition gives $\operatorname{supp}\mu \subset \bar{V}_+$. Since $\psi_f \perp \Omega$ and Ω is the only translation invariant vector, $\mu(\{0\}) = 0$. Covariance and $U(0,\Lambda)\,\Omega = \Omega$ make μ invariant under $\mathcal{L}_+^{\uparrow}$. The $\mathcal{L}_+^{\uparrow}$-orbits in $\bar{V}_+ \setminus \{0\}$ are the hyperboloids H_M^+, $M > 0$, and (for $d \geqslant 3$) the punctured boundary of the cone, $M = 0$. Each carries a unique invariant measure up to normalisation, namely $\mathrm{d}\Omega_M$. (For $d = 2$ the two half-lines $p^0 = \pm p^1 > 0$ of the cone boundary are orbits too, but their invariant measures $\mathrm{d}p^{\pm}/p^{\pm}$ are not locally finite at 0, so a tempered μ cannot charge them; hence $\rho(\{0\}) = 0$.) Disintegrating μ along $M = (p^2)^{1/2}$ gives $\mu = \int \rho(\mathrm{d}M^2)\; \Omega_M$, which is (1.4). □

Remark 1.10. (Reading the spectrum off ρ) (i) *Free fields.* The free field has $\rho = \delta_{m^2}$. Conversely, if $\rho = \delta_{m^2}$ with $m > 0$ then $\varphi - v$ is a free field of mass m (Jost–Schroer theorem [SW64]; it uses the full Wightman axioms, in particular locality (W3)). So an interacting theory must have $\rho \neq \delta_{m^2}$; this is not sufficient, since generalised free fields have arbitrary ρ and no interaction (Exercise 1.6).

(ii) *Particles.* An atom $Z\,\delta_{m^2}$ of ρ isolated from the rest of its support, $Z > 0$, signals a particle of mass m: an isolated eigenvalue m of the mass operator $(H^2 - \mathbf{P}^2)^{1/2}$. The remaining part of ρ describes multi-particle states. In the absence of bound states it starts at $(2m)^2$, or at $(3m)^2$ if the theory is invariant under $\varphi \mapsto -\varphi$. For weakly coupled $P(\varphi)_2$ models the isolated one-particle shell was established in [GJS74].

(iii) *No point fields, again.* Formally $W_2^T(0) = \int \rho(\mathrm{d}M^2) \int \mathrm{d}\Omega_M = \infty$ for any $\rho \neq 0$ when $d \geqslant 2$. The singularity at coincident points of a nontrivial Wightman field is at least as strong as that of the free field. Remark 3.7 gives a precise Euclidean version of this statement.

(iv) *Canonical fields and field-strength renormalisation.* If the sharp-time fields exist and satisfy the CCR (1.6), then $\int \rho(\mathrm{d}M^2) = 1$ (Exercise 1.5). Hence $Z = \rho(\{m^2\}) \leqslant 1$, with equality iff $\rho = \delta_{m^2}$. The time-zero fields of $P(\varphi)_2$ models satisfy the CCR [GJ87]. In $d = 4$, perturbation theory gives $\int \rho = \infty$, i.e. "$Z = 0$": the renormalised field cannot be canonical, and must be rescaled by a divergent factor (wave-function renormalisation).

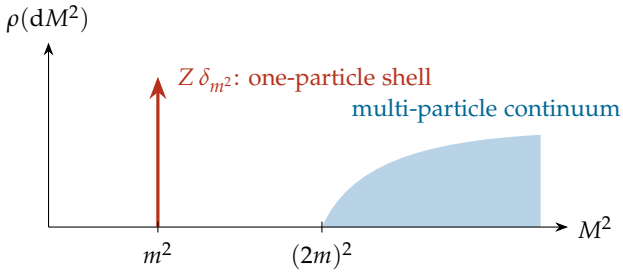


Figure 1.2. Typical spectral measure of an interacting field with mass gap (Theorem 1.9 and Remark 1.10): an isolated atom at the one-particle mass ($0 < Z \leqslant 1$ for a canonical field), and a continuum starting at $(2m)^2$, or at $(3m)^2$ in the presence of the symmetry $\varphi \mapsto -\varphi$. The free field has $\rho = \delta_{m^2}$.

1.5 The quantum wave equation

Since $(-p^2 + m^2)\mathrm{e}^{\pm \mathrm{i}p\cdot x} = 0$ on the mass shell, the free field (1.2) solves the Klein–Gordon equation in the sense of distributions:

$$(\Box + m^2)\,\varphi = 0, \qquad \Box = \partial_t^2 - \Delta_{\mathbf{x}}. \tag{1.5}$$

Moreover, the *sharp-time fields* $\varphi(t, g) = \int \varphi(t, \mathbf{x})\, g(\mathbf{x})\mathrm{d}\mathbf{x}$ and $\pi(t, g)$, with $\pi = \partial_t \varphi$, are well defined for $g \in \mathcal{S}(\mathbb{R}^s)$ (Exercise 1.3), and satisfy the canonical commutation relations (CCR)

$$[\varphi(t,\mathbf{x}), \pi(t,\mathbf{y})] = \mathrm{i}\delta(\mathbf{x}-\mathbf{y}), \qquad [\varphi(t,\mathbf{x}), \varphi(t,\mathbf{y})] = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})] = 0. \tag{1.6}$$

One may think of φ as the solution of a *linear wave equation with quantum initial data*: $(\varphi(0,\cdot), \pi(0,\cdot))$ satisfy the CCR and are "distributed" according to the vacuum.

Free field on a spatial circle. For $s = 1$ and space $\mathbb{T}_L = \mathbb{R}/L\,\mathbb{Z}$, the momenta are $k \in \frac{2\pi}{L}\mathbb{Z}$, $\omega_k = (k^2 + m^2)^{1/2}$, and

$$\varphi(0,x) = \sum_k (2L\omega_k)^{-1/2}(a_k \mathrm{e}^{\mathrm{i}kx} + a_k^* \mathrm{e}^{-\mathrm{i}kx}), \qquad [a_k, a_{k'}^*] = \delta_{kk'}, \qquad H_0 = \sum_k \omega_k a_k^* a_k. \tag{1.7}$$

The free field is an infinite collection of independent harmonic oscillators, one per mode. With an ultraviolet cutoff $|k| \leqslant R$,

$$\langle \Omega, \varphi_R(0,x)^2 \Omega \rangle = \sum_{|k| \leqslant R} \frac{1}{2L\omega_k} = \frac{1}{2\pi} \log \frac{R}{m} + O(1) \xrightarrow{R\to\infty} \infty. \tag{1.8}$$

Interactions. The classical nonlinear Klein–Gordon equation $(\Box + m^2)u + \lambda u^3 = 0$ is Hamiltonian with $H = \int [\frac{1}{2}\pi^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}m^2u^2 + \frac{\lambda}{4}u^4]\mathrm{d}\mathbf{x}$. Its quantum version would be the operator equation

$$(\Box + m^2)\varphi + \lambda \varphi^3 = 0, \tag{1.9}$$

together with the CCR (1.6). This immediately runs into three problems.

i. $\varphi(x)^3$ has no meaning, being the cube of a distribution. It must be *renormalised*. In $d=2$ Wick ordering $:\varphi^3:$ suffices (cf. (1.8) and Section 3.3); in $d=3$ a further mass counterterm is needed in the equation (and a vacuum-energy counterterm in the Hamiltonian); in $d=4$ the theory is expected to be trivial, as is proved for the scaling limits of critical lattice φ^4_4 models [ADC21].

ii. *Haag's theorem* [Haa96]: in infinite volume the interacting field at fixed time cannot be unitarily equivalent to the free one, so there is no interaction picture. The interacting theory lives in a representation of the CCR inequivalent to the Fock one.

iii. Even in finite volume, $H = H_0 + \lambda \int_{\mathbb{T}_L} :\varphi(0,x)^4: \mathrm{d}x$ is the sum of a positive operator and an operator unbounded below. That it is bounded below was Nelson's breakthrough [Nel66], based on hypercontractivity. Its self-adjointness and the construction of the corresponding local Heisenberg dynamics, with finite propagation speed, are due to Glimm–Jaffe ([GJ68] and its sequels) and Segal [Seg70]; see [BSZ92] for a textbook account of this Hamiltonian approach. Removing the volume cutoff for φ^4_2 was achieved by Glimm and Jaffe in a series of difficult papers; the full set of Wightman axioms for weakly coupled $P(\varphi)_2$ models was established in [GJS74], already with Euclidean methods. For Φ^4_3 see [GJ73, FO76].

The Euclidean approach (Symanzik, Nelson, Osterwalder–Schrader, Guerra–Rosen–Simon) simplified this enormously. We now explain how it arises.

Exercises

Exercise 1.1. (The free two-point function) (a) Show that W_2 in (1.3) is a Lorentz invariant tempered distribution solving $(\Box + m^2)W_2 = 0$, whose Fourier transform $\int \mathrm{e}^{\mathrm{i}p\cdot x} W_2(x)\,\mathrm{d}x$ (Minkowski product) is $2\pi\,\theta(p^0)\,\delta(p^2 - m^2)$. (b) For $d=2$ show that $W_2(0,x) = (2\pi)^{-1}K_0(m|x|)$ and deduce the logarithmic singularity at $x=0$. (c) Show that $[\varphi(x), \varphi(0)] = W_2(x) - W_2(-x)$ vanishes for spacelike x, i.e. the free field satisfies (W3). *Hint:* boost x to the plane $x^0 = 0$ and use the symmetry $\mathbf{p} \mapsto -\mathbf{p}$ of $\mathrm{d}\Omega_m$.

Exercise 1.2. (Invariant measures on the hyperboloid) Show that $\int_{\mathbb{R}^d} \theta(p^0)\,\delta(p^2 - m^2)\,g(p)\,\mathrm{d}p = \int g(\omega(\mathbf{p}), \mathbf{p})\,\frac{\mathrm{d}\mathbf{p}}{2\omega(\mathbf{p})}$, that this is the unique $\mathcal{L}^\uparrow_+$-invariant measure on H^+_m up to constants, and give a direct proof that H^+_m carries no finite invariant measure.

Exercise 1.3. (Which restrictions make sense?) For a distribution T on $\mathbb{R}^d$, the free field "$\varphi(T)\,\Omega$" is defined iff $\int \mathrm{d}\Omega_m |\tilde{T}|^2 < \infty$. Decide for which of the following this holds: (a) $T = \delta_x$; (b) $T = \delta(t) \otimes g(\mathbf{x})$ with $g \in \mathcal{S}(\mathbb{R}^s)$ (sharp time); (c) $T = h(t) \otimes \delta(\mathbf{x})$ with $h \in \mathcal{S}(\mathbb{R})$; (d) $T = \delta'(t) \otimes g(\mathbf{x})$, which gives $-\pi(0,g)$. Verify the CCR (1.6).

Exercise 1.4. (Analyticity from the spectrum condition) (a) Let $H \geqslant 0$ be self-adjoint. Show that $z \mapsto \langle \psi, \mathrm{e}^{-\mathrm{i}zH}\chi\rangle$ is bounded and analytic on $\{\mathrm{Im}z<0\}$, continuous up to the boundary, and is determined by its values on the negative imaginary half-line. (b) Using (W0), show that $a \mapsto U(a,1)\,\psi$ extends to a bounded analytic $\mathcal{H}$-valued function on the tube $\mathbb{R}^d + \mathrm{i}V_+$.

Exercise 1.5. (A canonical sum rule) (a) For the free field of mass M, write $[\varphi(x),\varphi(0)] = W_2^{(M)}(x) - W_2^{(M)}(-x)$ and show that $\partial_t[\varphi(t,\mathbf{x}),\varphi(0,0)]|_{t=0} = -\mathrm{i}\delta(\mathbf{x})$, independently of M. (b) Assume a Wightman field has sharp-time fields satisfying (1.6). Using (1.4), show (at least formally) that $\int \rho(\mathrm{d}M^2) = 1$. Deduce $0 \leqslant Z \leqslant 1$ in Remark 1.10(iv).

Exercise 1.6. (Generalised free fields) Let $\varphi_1, \dots, \varphi_n$ be independent free fields of distinct masses $M_1, \dots, M_n > 0$, on the tensor product of their Fock spaces, and $\varphi = \sum_i c_i \varphi_i$ with $c_i \in \mathbb{R}\setminus\{0\}$. (a) Show that φ satisfies (W0)–(W4) and has $\rho = \sum_i c_i^2\, \delta_{M_i^2}$. (b) Show that $\prod_i (\Box + M_i^2)\,\varphi = 0$, but that φ satisfies no Klein–Gordon equation of a single mass if $n \geqslant 2$. (c) Show that all truncated Wightman functions of order $\geqslant 3$ vanish, so there is no scattering. (d) Extend the construction to an arbitrary polynomially bounded ρ (with $\rho(\{0\}) = 0$ if $d = 2$).

Exercise 1.7. (Wick ordering on the circle) Using (1.7) and the time evolution $a_k \mapsto a_k \mathrm{e}^{-\mathrm{i}\omega_k t}$, verify (1.5), (1.6) and (1.8). For $g \in C^\infty(\mathbb{T}_L)$ with $\int g \neq 0$ show that $\int g(x)\,\varphi_R(0,x)^2 \mathrm{d}x\,\Omega$ diverges as $R \to \infty$, while $\int g(x) {:}\varphi_R(0,x)^2{:}\,\mathrm{d}x\,\Omega$ converges in Fock space, where ${:}\varphi_R^2{:} := \varphi_R^2 - \langle \Omega, \varphi_R^2 \Omega\rangle$.

2 From real to imaginary time

We work with a regularised version of (1.9) with finitely many degrees of freedom, where everything is rigorous, and follow how the quantum evolution turns into a classical probability measure.

2.1 A lattice Hamiltonian on the circle

Fix $L > 0$, $N \in \mathbb{N}$, $\varepsilon = L/N$, and the periodic lattice $\Lambda_\varepsilon = \varepsilon\,\mathbb{Z}/L\,\mathbb{Z} \subset \mathbb{T}_L$ (N sites). Configurations are $\varphi = (\varphi_x)_{x\in\Lambda_\varepsilon} \in \mathbb{R}^{\Lambda_\varepsilon}$. In the Schrödinger representation $\mathcal{H}_\varepsilon = L^2(\mathbb{R}^{\Lambda_\varepsilon}, \mathrm{d}\varphi)$, φ_x acts by multiplication and $\pi_x := -\mathrm{i}\varepsilon^{-1}\partial/\partial\varphi_x$, so that $[\varphi_x, \pi_y] = \mathrm{i}\varepsilon^{-1} 1_{x=y}$, the lattice version of $\mathrm{i}\delta(x-y)$. Let V be a polynomial bounded below (e.g. $V(u) = \lambda u^4$) and

$$H_\varepsilon := \sum_{x\in\Lambda_\varepsilon} \varepsilon\left[\tfrac{1}{2}\pi_x^2 + \tfrac{1}{2}(\nabla_\varepsilon\varphi)_x^2 + \tfrac{1}{2}m^2\varphi_x^2 + V(\varphi_x)\right] = -\frac{1}{2\varepsilon}\Delta_{\mathbb{R}^{\Lambda_\varepsilon}} + U_\varepsilon(\varphi), \tag{2.1}$$

where $(\nabla_\varepsilon\varphi)_x = (\varphi_{x+\varepsilon} - \varphi_x)/\varepsilon$ and $U_\varepsilon(\varphi) = \varepsilon\sum_x [\frac{1}{2}(\nabla_\varepsilon\varphi)_x^2 + \frac{1}{2}m^2\varphi_x^2 + V(\varphi_x)]$. A short computation (Exercise 2.1) shows that the Heisenberg fields $\varphi_x(t) = \mathrm{e}^{\mathrm{i}tH_\varepsilon}\varphi_x \mathrm{e}^{-\mathrm{i}tH_\varepsilon}$ satisfy

$$\partial_t^2\varphi_x(t) = (\Delta_\varepsilon\varphi(t))_x - m^2\varphi_x(t) - V'(\varphi_x(t)), \qquad (\Delta_\varepsilon\varphi)_x = \frac{\varphi_{x+\varepsilon} + \varphi_{x-\varepsilon} - 2\varphi_x}{\varepsilon^2}, \tag{2.2}$$

a lattice regularisation of the quantum wave equation (1.9), now with a perfectly rigorous meaning.

H_ε is a Schrödinger operator on $\mathbb{R}^N$ with a confining potential ($U_\varepsilon \geqslant c|\varphi|^2 - C$ since $m > 0$). Hence it is essentially self-adjoint on C_c^∞, bounded below, with compact resolvent, and $\mathrm{e}^{-tH_\varepsilon}$ is trace class for $t > 0$. The integral kernel of $\mathrm{e}^{-tH_\varepsilon}$ is strictly positive, so by the Perron–Frobenius argument the ground state energy $E_0 = \inf \mathrm{spec} H_\varepsilon$ is a simple eigenvalue with a strictly positive eigenfunction Ω_ε, the *lattice vacuum* [Sim79]. We set

$$\bar{H} := H_\varepsilon - E_0 \geqslant 0 \qquad \text{(positivity of the energy)}. \tag{2.3}$$

2.2 Positivity of the energy and analytic continuation

Let $F_1,\dots,F_n\colon \mathbb{R}^{\Lambda_\varepsilon}\to\mathbb{C}$ be bounded measurable (acting as multiplication operators) and $F_j(t)=\mathrm{e}^{\mathrm{i}tH_\varepsilon}F_j\mathrm{e}^{-\mathrm{i}tH_\varepsilon}$. Since $\bar H\,\Omega_\varepsilon=0$,

$$W(t_1,\dots,t_n):=\langle\Omega_\varepsilon,F_1(t_1)\cdots F_n(t_n)\,\Omega_\varepsilon\rangle=\langle\Omega_\varepsilon,F_1\mathrm{e}^{-\mathrm{i}(t_1-t_2)\bar H}F_2\cdots\mathrm{e}^{-\mathrm{i}(t_{n-1}-t_n)\bar H}F_n\,\Omega_\varepsilon\rangle. \tag{2.4}$$

Proposition 2.1. (Wick rotation) *W depends only on $\zeta_j=t_j-t_{j+1}$ and extends to a function $\mathcal{W}(\zeta_1,\dots,\zeta_{n-1})$ which is bounded and continuous on $\bar{\mathbb{C}}_-^{n-1}$ and analytic on $\mathbb{C}_-^{n-1}$, $\mathbb{C}_-=\{\mathrm{Im}\zeta<0\}$. At $\zeta_j=-\mathrm{i}(\tau_j-\tau_{j+1})$, $\tau_1\geqslant\dots\geqslant\tau_n$, it gives the* Schwinger functions

$$S(\tau_1,\dots,\tau_n)=\langle\Omega_\varepsilon,F_1\mathrm{e}^{-(\tau_1-\tau_2)\bar H}F_2\cdots\mathrm{e}^{-(\tau_{n-1}-\tau_n)\bar H}F_n\,\Omega_\varepsilon\rangle, \tag{2.5}$$

and the Schwinger functions determine the Wightman functions W.

Proof. By the spectral theorem, $\mathrm{e}^{-\mathrm{i}\zeta\bar H}=\int\mathrm{e}^{-\mathrm{i}\zeta\lambda}\mathrm{d}E(\lambda)$ with $|\mathrm{e}^{-\mathrm{i}\zeta\lambda}|=\mathrm{e}^{\lambda\mathrm{Im}\zeta}\leqslant 1$ for $\lambda\geqslant 0$, $\mathrm{Im}\zeta\leqslant 0$. It is therefore a contraction, strongly continuous on $\bar{\mathbb{C}}_-$ and analytic in the interior. Products of such operators give separately analytic, bounded, continuous functions, hence jointly analytic ones. If $\mathcal{W}$ vanishes on $(-\mathrm{i}\mathbb{R}_+)^{n-1}$, the identity theorem in ζ_1 (with the other variables on the imaginary axis) shows that it vanishes for all $\zeta_1\in\mathbb{C}_-$. Proceeding variable by variable, and using continuity up to the boundary, gives $W\equiv 0$. □

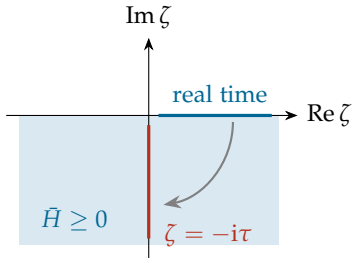


Figure 2.1. Wick rotation (Proposition 2.1). Since $\bar H\geqslant 0$, $\mathrm{e}^{-\mathrm{i}\zeta\bar H}$ is a contraction on the lower half-plane $\mathrm{Im}\zeta\leqslant 0$, analytic in its interior. The Wightman functions on the real axis and the Schwinger functions on the negative imaginary axis are boundary values and restrictions of the same analytic function.

Remark 2.2. This is exactly where positivity of the energy enters: it allows rotating $t\mapsto-\mathrm{i}\tau$ in all time differences simultaneously. It will be used twice more: $\mathrm{e}^{-\tau\bar H}$, $\tau\geqslant 0$, is a contraction semigroup, which can be represented as a path integral (Feynman–Kac); and $\mathrm{e}^{-T\bar H}$ converges to the projection onto $\ker\bar H$ as $T\to\infty$, i.e. onto the vacuum when it is unique (Remark 2.7). (That the Feynman–Kac weights are positive is a separate fact, coming from the kinetic term.) In a relativistic theory, the Wightman functions are boundary values of functions analytic in the forward tube (Exercise 1.4). By Lorentz covariance and locality (the Bargmann–Hall–Wightman theorem and Jost points) these extend to a domain containing all non-coincident Euclidean points, where they define symmetric, Euclidean invariant Schwinger functions [SW64, OS73].

2.3 Discretising imaginary time: the Feynman–Kac formula

The kinetic part of H_ε generates a Gaussian semigroup with kernel

$$\mathrm{e}^{\frac{\delta}{2\varepsilon}\Delta}(\varphi,\varphi')=\left(\frac{2\pi\delta}{\varepsilon}\right)^{-N/2}\exp\left(-\frac{\varepsilon}{2\delta}|\varphi-\varphi'|^2\right).$$

Hence the symmetric Trotter approximant $T_\delta := \mathrm{e}^{-\delta U_\varepsilon/2}\mathrm{e}^{\frac{\delta}{2\varepsilon}\Delta}\mathrm{e}^{-\delta U_\varepsilon/2}$ has the explicit positive kernel

$$T_\delta(\varphi,\varphi') = \Big(\frac{2\pi\delta}{\varepsilon}\Big)^{-N/2}\exp\Bigg(-\frac{\varepsilon}{2\delta}\sum_x(\varphi_x-\varphi'_x)^2-\frac{\delta}{2}\big(U_\varepsilon(\varphi)+U_\varepsilon(\varphi')\big)\Bigg), \tag{2.6}$$

and $T_\delta^{\lfloor\tau/\delta\rfloor}\to\mathrm{e}^{-\tau H_\varepsilon}$ strongly as $\delta\to0$ (Trotter product formula [RS80, Sim79]). Let $T=J\delta$ and consider the space-time lattice

$$\Lambda=\Lambda_{\delta,\varepsilon}^{T,L}:=(\delta\mathbb{Z}/T\mathbb{Z})\times(\varepsilon\mathbb{Z}/L\mathbb{Z}),\qquad \varphi=(\varphi_{\tau,x})_{(\tau,x)\in\Lambda},\quad \varphi_\tau:=(\varphi_{\tau,x})_{x\in\Lambda_\varepsilon}. \tag{2.7}$$

Writing the trace of a product of integral operators as an integral over closed chains, we get $\mathrm{Tr}(T_\delta^J)=(2\pi\delta/\varepsilon)^{-NJ/2}\int_{\mathbb{R}^\Lambda}\mathrm{e}^{-S_{\delta,\varepsilon}(\varphi)}\prod_{u\in\Lambda}\mathrm{d}\varphi_u$ with the *lattice Euclidean action*

$$S_{\delta,\varepsilon}(\varphi)=\sum_{(\tau,x)\in\Lambda}\varepsilon\,\delta\Big[\tfrac12\Big(\frac{\varphi_{\tau+\delta,x}-\varphi_{\tau,x}}{\delta}\Big)^2+\tfrac12\Big(\frac{\varphi_{\tau,x+\varepsilon}-\varphi_{\tau,x}}{\varepsilon}\Big)^2+\tfrac12 m^2\varphi_{\tau,x}^2+V(\varphi_{\tau,x})\Big]. \tag{2.8}$$

This discretises $S(\varphi)=\iint_{\mathbb{T}_T\times\mathbb{T}_L}[\frac12(\partial_\tau\varphi)^2+\frac12(\partial_x\varphi)^2+\frac12 m^2\varphi^2+V(\varphi)]\,\mathrm{d}\tau\,\mathrm{d}x$. Imaginary time and space now play symmetric roles: the Minkowski metric has become Euclidean. Define the probability measure

$$\mu_{\delta,\varepsilon}^{T,L}(\mathrm{d}\varphi):=Z^{-1}\mathrm{e}^{-S_{\delta,\varepsilon}(\varphi)}\prod_{u\in\Lambda}\mathrm{d}\varphi_u. \tag{2.9}$$

Inserting multiplication operators between the factors T_δ gives, for $\tau_1\geqslant\ldots\geqslant\tau_n$ in $\delta\mathbb{Z}$ with $\tau_1-\tau_n<T$,

$$\frac{\mathrm{Tr}(F_1T_\delta^{k_1}F_2T_\delta^{k_2}\cdots F_nT_\delta^{k_n})}{\mathrm{Tr}(T_\delta^J)}=\int F_1(\varphi_{\tau_1})\cdots F_n(\varphi_{\tau_n})\;\mu_{\delta,\varepsilon}^{T,L}(\mathrm{d}\varphi), \tag{2.10}$$

where $k_j=(\tau_j-\tau_{j+1})/\delta$ for $j<n$ and $k_n=J-\sum_{j<n}k_j$.

Proposition 2.3. (Feynman–Kac–Nelson) *For bounded continuous F_j and dyadic $\tau_1\geqslant\ldots\geqslant\tau_n$,*

$$\langle\Omega_\varepsilon,F_1\mathrm{e}^{-(\tau_1-\tau_2)\tilde H}F_2\cdots F_n\Omega_\varepsilon\rangle=\lim_{T\to\infty}\lim_{\delta\to0}\int F_1(\varphi_{\tau_1})\cdots F_n(\varphi_{\tau_n})\;\mu_{\delta,\varepsilon}^{T,L}(\mathrm{d}\varphi). \tag{2.11}$$

Sketch. The Gaussian factor in (2.9) gives precisely the finite-dimensional distributions, at times $\tau\in\delta\mathbb{Z}/T\mathbb{Z}$, of the (unnormalised) Brownian loop measure on $\mathbb{R}^{\Lambda_\varepsilon}$ of length T and generator $\frac{1}{2\varepsilon}\Delta$ (variance t/ε per coordinate). The potential term $\delta\sum_\tau U_\varepsilon(\varphi_\tau)$ is a Riemann sum for $\int_0^T U_\varepsilon(\omega_\tau)\mathrm{d}\tau$. Hence, as $\delta\to0$ (along dyadic δ, say, so that the τ_j stay on the lattice), both sides of (2.10) converge to the Feynman–Kac expression $\mathrm{Tr}(F_1\mathrm{e}^{-(\tau_1-\tau_2)H}\cdots F_n\mathrm{e}^{-(T-\tau_1+\tau_n)H})/\mathrm{Tr}(\mathrm{e}^{-TH})$, written as an integral over Brownian loops weighted by $\exp(-\int_0^T U_\varepsilon)$ [Sim79]. The limit $T\to\infty$ uses that E_0 is simple and isolated (Exercise 2.6). □

Vacuum expectations of the quantum system in imaginary time are thus *moments of a classical probability measure*. As $T\to\infty$ the time-slices $(\varphi_\tau)_{\tau\in\mathbb{R}}$ form a stationary Markov process: the ground-state transform of $\mathrm{e}^{-\tau\tilde H}$, whose one-time marginal is $\Omega_\varepsilon(\varphi)^2\mathrm{d}\varphi$. For $N=1$, $V=0$ this is the Ornstein–Uhlenbeck process (Exercise 2.2 and Figure 2.2).

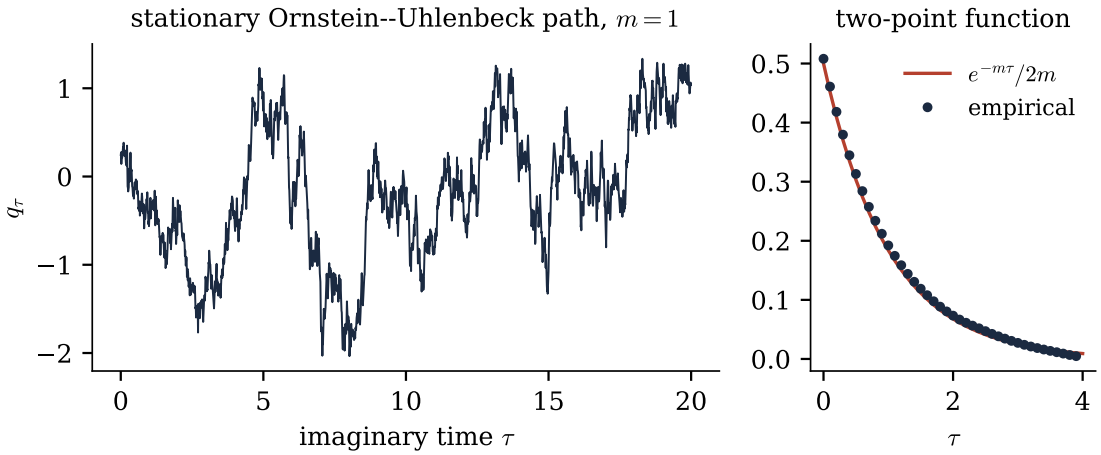


Figure 2.2. Left: a sample of the stationary Ornstein–Uhlenbeck process, the imaginary-time path of one harmonic oscillator, i.e. the GFF in $d=1$ (Exercise 2.2). Right: its empirical two-point function against the Schwinger function $\mathrm{e}^{-m\tau}/2m$.

2.4 Classical statistical mechanics

Take the isotropic lattice $\delta=\varepsilon$. Then (2.8) becomes

$$S_\varepsilon(\varphi)=\sum_{\langle u,v\rangle}\tfrac{1}{2}(\varphi_u-\varphi_v)^2+\varepsilon^2\sum_u\left[\tfrac{1}{2}m^2\varphi_u^2+V(\varphi_u)\right], \tag{2.12}$$

the first sum running over nearest-neighbour pairs of the two-dimensional periodic lattice Λ. So $\mu_\varepsilon\propto\prod_{\langle u,v\rangle}\mathrm{e}^{-\frac{1}{2}(\varphi_u-\varphi_v)^2}\prod_u\nu_\varepsilon(\mathrm{d}\varphi_u)$, with single-site measure $\nu_\varepsilon(\mathrm{d}a)=\mathrm{e}^{-\varepsilon^2(\frac{1}{2}m^2a^2+V(a))}\mathrm{d}a$. This is a *ferromagnetic nearest-neighbour spin system* with continuous, unbounded spins, i.e. a short-range model of classical statistical mechanics. In spacetime dimension d one gets $\varepsilon^{d-2}\sum_{\langle u,v\rangle}$ and $\varepsilon^d\sum_u$ instead (Exercise 2.3). The nearest-neighbour structure makes μ_ε *reflection positive* (Figure 2.3 and Exercise 2.5), the Euclidean shadow of the positivity of $\mathrm{e}^{-\varepsilon H_\varepsilon}$. Conversely, reflection positivity is enough to reconstruct a Hilbert space and a positive Hamiltonian (Exercise 2.7).

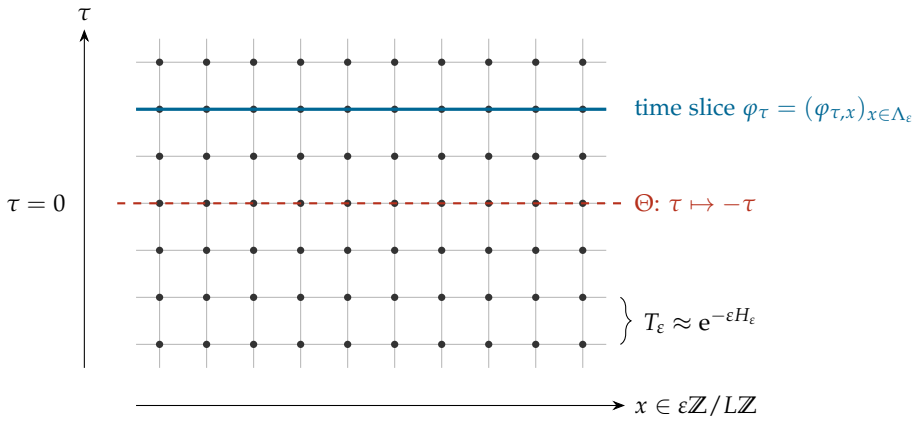


Figure 2.3. The Euclidean space-time lattice. Rows are configurations of the quantum system at fixed imaginary time; the transfer matrix $T_\varepsilon\approx\mathrm{e}^{-\varepsilon H_\varepsilon}$ connects consecutive rows. The nearest-neighbour structure makes the measure reflection positive with respect to Θ.

Quantum theory (real time)	Euclidean field theory	Statistical mechanics
positive Hilbert space, $H \geqslant 0$	reflection positivity	RP of n.n. interactions
$\mathrm{e}^{-\varepsilon H}$	time translation by ε	transfer matrix
vacuum Ω	infinite-volume measure ($T \to \infty$)	infinite-volume Gibbs state
inverse temperature β	length T of the time circle	—
mass gap $E_1 - E_0$	exponential decay of correlations	inverse correlation length $1/\xi$
continuum limit	removal of UV cutoff $\varepsilon \to 0$	critical point, scaling limit
renormalisation	counterterms in the action	tuning parameters to criticality

Table 2.1. A dictionary between the three pictures.

Remark 2.4. (**Continuum limit = critical point**) In lattice units the model (2.12) has mass $\varepsilon m \to 0$, so its correlation length is $\xi \approx (\varepsilon m)^{-1} \to \infty$ lattice spacings (Exercise 2.4). The continuum limit of the regularised QFT is therefore the *scaling limit of a family of short-range spin systems approaching a critical point*. Conversely, any reflection positive, short-range critical model with a nontrivial scaling limit yields a candidate Euclidean QFT. This is Wilson's point of view on renormalisation [WK74]. For instance, the Griffiths–Simon construction realises φ^4 single-spin measures as limits of normalised sums of ferromagnetically coupled Ising spins [GS73], and the critical 4D Ising and φ^4_4 models have Gaussian scaling limits [ADC21]. See [GRS75] for the systematic use of lattice approximations for $P(\varphi)_2$.

Remark 2.5. (**Renormalisation already appears**) For $V(a) = \lambda a^4$, the limit $\varepsilon \to 0$ at fixed (m, λ) does not produce an interacting theory: the Gaussian part alone has $c_\varepsilon := \mathbb{E}[\varphi_u^2] = \frac{1}{2\pi} \log \frac{1}{\varepsilon m} + O(1)$ (compare (1.8)). One must take ε-dependent couplings, e.g. $V_\varepsilon(a) = \lambda\,(a^4 - 6\,c_\varepsilon\,a^2)$ (a divergent mass counterterm), i.e. λ times the Wick power $:a^4:$ of Section 3.3 up to a constant. In $d = 2$ this is all that is needed; in $d = 3$ more counterterms are required.

2.5 Path integrals and the Osterwalder–Schrader axioms

Formally, as $\varepsilon \to 0$ and $L, T \to \infty$, (2.9) becomes the Euclidean path integral

$$\mu(\mathrm{d}\varphi) \text{ ``}=\text{''} Z^{-1} \exp\Big(-\int_{\mathbb{R}^d} \Big[\tfrac{1}{2}|\nabla\varphi|^2 + \tfrac{1}{2}m^2\varphi^2 + V(\varphi)\Big]\mathrm{d}x\Big)\,\mathcal{D}\varphi. \tag{2.13}$$

Neither $\mathcal{D}\varphi$ nor the action make sense, since typical φ are distributions. The Gaussian part, however, defines the Gaussian free field μ_m of Section 3, and one is led to define $\mu \propto \mathrm{e}^{-\int V(\varphi)}\mu_m$ with a renormalised V, at least in finite volume in $d = 2$. In $d = 3$ this is possible only as the limit of ultraviolet-regularised measures of this form, and the limit is singular with respect to μ_m. Which measures on $\mathcal{S}'(\mathbb{R}^d)$ correspond to relativistic QFTs? Let $\hat\mu(f) := \int \mathrm{e}^{\mathrm{i}\varphi(f)}\mu(\mathrm{d}\varphi)$ be the characteristic functional and $\Theta(\tau, \mathbf{x}) = (-\tau, \mathbf{x})$, $\Theta f = f \circ \Theta$.

(OS0) Analyticity.
For real $f_1, \dots, f_n \in C_c^\infty(\mathbb{R}^d)$, the map $z \mapsto \hat\mu(\sum_j z_j f_j)$ is entire on $\mathbb{C}^n$.

(OS1) Regularity.
For some $p \in [1, 2]$ and $c > 0$, $|\hat\mu(f)| \leqslant \exp\,(c\,(\|f\|_{L^1} + \|f\|_{L^p}^p))$ for all complex-valued $f \in C_c^\infty(\mathbb{R}^d)$, with $\hat\mu$ extended by (OS0); if $p = 2$, the two-point function is in addition required to be locally integrable.

(OS2) Euclidean invariance.
$\hat\mu(f \circ g) = \hat\mu(f)$ for every Euclidean motion g of $\mathbb{R}^d$.

(OS3) Reflection positivity.
For real $f_1, \dots, f_n$ supported in $\{\tau > 0\}$, the matrix $(\hat\mu\,(f_i - \Theta f_j))_{i,j}$ is positive semidefinite.

(OS4) Ergodicity.
The time translations act ergodically on $(\mathcal{S}'(\mathbb{R}^d), \mu)$.

Theorem 2.6. (Osterwalder–Schrader reconstruction [OS73, OS75, GJ87]) *If μ satisfies (OS0)–(OS3), its moments are the Schwinger functions of a unique Wightman theory satisfying (W1)–(W4) and (W0) except possibly uniqueness of the vacuum. (OS4) holds iff the vacuum is unique.*

Idea. Let $\mathcal{E}_+$ be the functions of φ that depend only on its restriction to $\{\tau > 0\}$, with $(F, G) := \mathbb{E}_\mu[\overline{\Theta F}\, G]$. By (OS3) this is a positive semi-definite form; quotienting by null vectors and completing gives $\mathcal{H}$. Time translations by $t \geqslant 0$ map $\mathcal{E}_+$ to itself and descend to a self-adjoint contraction semigroup e^{-tH}, so $H \geqslant 0$: *positivity of the energy is recovered from reflection positivity*. The vacuum is the class of 1, and the analytic continuation $\tau \to it$ yields the Wightman functions. □

Remark 2.7. (Positivity, clustering and uniqueness of the vacuum) Let $H \geqslant 0$ be self-adjoint, $\Omega \in \ker H$, and let P_0 be the orthogonal projection onto $\ker H$. Positivity of the energy turns uniqueness of the vacuum into a clustering statement, in three steps.

(i) *Imaginary time: no averaging is needed.* By the spectral theorem and dominated convergence, $\mathrm{e}^{-tH} = \int_{[0,\infty)} \mathrm{e}^{-t\lambda} \mathrm{d}E(\lambda) \to E(\{0\}) = P_0$ strongly as $t \to \infty$, because $\mathrm{e}^{-t\lambda} \to 1_{\lambda=0}$ for $\lambda \geqslant 0$. In the reconstruction above, for $F, G \in \mathcal{E}_+$ and τ_t the time translation,

$$\mathbb{E}_\mu[\overline{\Theta F}\, \tau_t G] = (F, \mathrm{e}^{-tH} G) \xrightarrow{t\to\infty} (F, P_0 G).$$

So the Euclidean correlations cluster, $\mathbb{E}_\mu[\overline{\Theta F}\, \tau_t G] \to \overline{\mathbb{E}_\mu F}\, \mathbb{E}_\mu G$, iff P_0 is the projection onto $\mathbb{C}\,\Omega$, i.e. iff the vacuum is unique. With a mass gap the convergence is exponential, at rate $\inf(\mathrm{spec} H \setminus \{0\})$ (Exercise 2.6). Moreover, the spectrum condition $H \geqslant |\mathbf{P}|$ implies that a vector with $H\psi = 0$ also has $\mathbf{P}\psi = 0$: invariance under time translations already gives invariance under all translations. For reflection positive measures, ergodicity (OS4) (by the reconstruction theorem) and clustering in imaginary time (by (i)) are both equivalent to uniqueness of the vacuum.

(ii) *Real time: only averages converge.* The unitary group $\mathrm{e}^{-\mathrm{i}tH}$ does not converge as $t \to \infty$ in general. Only its time averages do: $\frac{1}{T}\int_0^T \mathrm{e}^{-\mathrm{i}tH}\mathrm{d}t \to P_0$ strongly (von Neumann's mean ergodic theorem). If H has an eigenvalue $E > 0$ with eigenvector ψ, then $\langle \psi, \mathrm{e}^{-\mathrm{i}tH}\psi\rangle = \mathrm{e}^{-\mathrm{i}tE}\|\psi\|^2$ oscillates forever, while $\langle \psi, \mathrm{e}^{-tH}\psi\rangle = \mathrm{e}^{-tE}\|\psi\|^2 \to 0$. This is the situation of the lattice systems of Section 2 in finite volume: their real-time correlations never decay, although the vacuum is unique and the Euclidean correlations cluster exponentially.

(iii) *Relativistic theories.* Here Lorentz covariance removes the eigenvalues. If $U(a, 1)\,\psi = \mathrm{e}^{\mathrm{i}p\cdot a}\psi$ with $p \neq 0$, then $U(0, \Lambda)\,\psi$ is an eigenvector with eigenvalue Λp. The orbit of p is uncountable and eigenvectors for distinct eigenvalues are orthogonal, which contradicts the separability of $\mathcal{H}$. So, if the vacuum is unique, the spectral measures of the translations on $\Omega^\perp$ have no atoms, and by Wiener's theorem correlations cluster in mean:

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} |\langle \chi, U(a, 1)\,\psi\rangle - \langle \chi, \Omega\rangle\langle \Omega, \psi\rangle|^2 \mathrm{d}a = 0.$$

Locality then upgrades this to the pointwise cluster property (e) in spacelike directions [SW64]. With a mass gap, the truncated functions decay faster than any power (Ruelle [Rue62]), and in fact exponentially (Araki, Hepp and Ruelle [AHR62]); see [Haa96, §II.4]. Conversely, if $\ker H$ contains a unit vector $\psi \perp \Omega$, then by cyclicity (W4) some product of fields has a nonzero component along ψ, and the corresponding correlations do not factorise. In Euclidean terms the measure is then a mixture of several ergodic components, or phases, as for φ^4_2 at strong coupling with its symmetric mixture of the two pure phases [GJS75].

The Euclidean strategy. (1) Regularise: lattice spacing ε (or another UV cutoff), finite volume. (2) Renormalise: choose ε-dependent counterterms. (3) Prove uniform estimates and remove the cutoffs. (4) Verify the OS axioms: reflection positivity survives limits of lattice approximations; Euclidean invariance must be recovered in the limit; (OS1) needs uniform bounds; (OS4) often follows from a mass gap. (5) Show the limit is not Gaussian. In the next lectures we follow this route using stochastic analysis and PDE methods, starting from the Gaussian free field.

Exercises

Exercise 2.1. (Lattice equations of motion) Derive (2.2) from (2.1). For $V=0$, diagonalise H_ε in Fourier modes $k\in\frac{2\pi}{L}\mathbb{Z}\cap(-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}]$ and show that the frequencies are $\omega_\varepsilon(k)^2=m^2+\frac{4}{\varepsilon^2}\sin^2\frac{k\varepsilon}{2}$. Deduce that $\langle\Omega_\varepsilon,\varphi_x^2\Omega_\varepsilon\rangle=\frac{1}{L}\sum_k\frac{1}{2\omega_\varepsilon(k)}=\frac{1}{2\pi}\log\frac{1}{\varepsilon m}+O(1)$.

Exercise 2.2. (Harmonic oscillator and Ornstein–Uhlenbeck) Let $H=-\frac12\partial_q^2+\frac12 m^2q^2$ on $L^2(\mathbb{R})$. (a) Show that $\Omega(q)=(m/\pi)^{1/4}\mathrm{e}^{-mq^2/2}$ and $E_0=m/2$. (b) Show that $\langle\Omega,q\,\mathrm{e}^{-\tau\bar H}q\,\Omega\rangle=\mathrm{e}^{-m\tau}/(2m)$. (c) Let $(Uf)(q)=\Omega(q)\,f(q)$, unitary from $L^2(\Omega^2\mathrm{d}q)$ to $L^2(\mathrm{d}q)$. Show that $U^{-1}\bar H U=-\frac12\partial_q^2+mq\,\partial_q$, minus the generator of $\mathrm{d}X=-mX\mathrm{d}t+\mathrm{d}B$. (d) Conclude that the Schwinger functions are the moments of the stationary Gaussian process with covariance $\mathrm{e}^{-m|\tau-\sigma|}/(2m)=(-\partial_\tau^2+m^2)^{-1}(\tau,\sigma)$: the GFF in $d=1$.

Exercise 2.3. (Lattice action in general dimension) Verify (2.6)–(2.10). In spacetime dimension d (lattice $\varepsilon\,\mathbb{Z}^{d-1}$ in space, $\delta=\varepsilon$) show that $S_\varepsilon=\varepsilon^{d-2}\sum_{\langle u,v\rangle}\frac12(\varphi_u-\varphi_v)^2+\varepsilon^d\sum_u[\frac12 m^2\varphi_u^2+V(\varphi_u)]$, so that $\varepsilon^{(d-2)/2}\varphi$ is the natural dimensionless spin variable.

Exercise 2.4. (Correlation length and criticality) On $\varepsilon\,\mathbb{Z}^d$ consider the lattice Green function

$$\Gamma_\varepsilon(x)=\int_{[-\pi/\varepsilon,\pi/\varepsilon]^d}\frac{\mathrm{e}^{\mathrm{i}k\cdot x}}{m^2+4\,\varepsilon^{-2}\sum_j\sin^2(\varepsilon k_j/2)}\,\frac{\mathrm{d}k}{(2\pi)^d}.$$

(a) Show that $\Gamma_\varepsilon(x)\to G_m(x)$ for $x\neq0$ (with $x\in\varepsilon\,\mathbb{Z}^d$ along a sequence $\varepsilon\to0$, e.g. dyadic), where G_m is the Green function of $m^2-\Delta$ on $\mathbb{R}^d$. (b) Show that in lattice units $n=x/\varepsilon$ along a coordinate axis, Γ_ε decays like $\mathrm{e}^{-\mu_\varepsilon|n|}$ with $\cosh\mu_\varepsilon=1+\frac12(\varepsilon m)^2$. Deduce that the lattice correlation length is $\xi=\mu_\varepsilon^{-1}\sim(\varepsilon m)^{-1}\to\infty$ and interpret.

Exercise 2.5. (Reflection positivity on the lattice) Let $\Lambda=(\varepsilon\,\mathbb{Z}/T\,\mathbb{Z})\times\Lambda_\varepsilon$ with $T=2K\varepsilon$, let μ_ε have action (2.12), and let Θ be the reflection $\tau\mapsto-\tau$, which fixes the planes $\tau=0$ and $\tau=T/2$. Show that $\mathbb{E}\,[F\cdot\Theta F]\geqslant0$ for every bounded real F depending only on $\{\varphi_u\colon\tau(u)\in[0,T/2]\}$. *Hint:* write $S_\varepsilon=S_+ +\Theta S_+$, with the terms living on the fixed planes split evenly, and integrate first over the sites off the fixed planes.

Exercise 2.6. (Vacuum, mass gap and clustering) Let H be self-adjoint with e^{-tH} trace class for $t>0$, simple ground state Ω, and $E_1:=\min(\mathrm{spec}H\setminus\{E_0\})$. (a) Show that $\mathrm{Tr}(A\mathrm{e}^{-TH})/\mathrm{Tr}(\mathrm{e}^{-TH})=\langle\Omega,A\,\Omega\rangle+O(\mathrm{e}^{-T(E_1-E_0)})$ for bounded A. (b) Show that

$$|\langle\Omega,A\mathrm{e}^{-\tau\bar H}B\,\Omega\rangle-\langle\Omega,A\,\Omega\rangle\langle\Omega,B\,\Omega\rangle|\leqslant\|A\|\|B\|\mathrm{e}^{-\tau(E_1-E_0)}:$$

a mass gap gives exponential clustering of Schwinger functions.

Exercise 2.7. (A toy OS reconstruction) Let μ be a probability measure on $\mathbb{R}^{\varepsilon\mathbb{Z}\times\Lambda_\varepsilon}$, invariant under time translations and reflection positive with respect to Θ. On bounded functions F of $\{\varphi_u\colon\tau(u)\geqslant0\}$ put $(F,G)=\mathbb{E}\,[\overline{\Theta F}\,G]$ and $(\mathsf{T}F)(\varphi)=F(\varphi(\cdot+\varepsilon,\cdot))$. Show that $(F,\mathsf{T}G)=(\mathsf{T}F,G)$ and, by iterating Cauchy–Schwarz, $|(F,\mathsf{T}F)|\leqslant(F,F)$. Deduce that T induces a self-adjoint contraction on the reconstructed Hilbert space, and that $H:=-(2\varepsilon)^{-1}\log\mathsf{T}^2\geqslant0$ on $(\ker\mathsf{T})^\perp$.

Exercise 2.8. (Positivity, clustering and uniqueness) (a) Let $H\geqslant0$ be self-adjoint. Show that $\mathrm{e}^{-tH}\to P_0$ strongly as $t\to\infty$, where P_0 is the projection onto $\ker H$, and that $\frac1T\int_0^T\mathrm{e}^{-\mathrm{i}tH}\mathrm{d}t\to P_0$ strongly. Give an example where $\mathrm{e}^{-\mathrm{i}tH}$ has no weak limit. (b) In the setting of Exercise 2.7, show that $\mathbb{E}[\overline{\Theta F}\,\mathsf{T}^{2n}G]\to\overline{\mathbb{E}\,F}\;\mathbb{E}\,G$ for all F,G iff the T^2-invariant vectors of the reconstructed Hilbert space are the multiples of the vacuum. Why T^2? Find a reflection positive, translation invariant measure for which T has the eigenvalue -1. (c) Under (W0), show that $H\psi=0$ implies $U(a,1)\,\psi=\psi$ for all a. (d) Under (W0), show that the translations have no eigenvector with eigenvalue $p\neq0$. *Hint:* boost it. (e) Let $\mu_\pm$ be two distinct, ergodic, translation invariant, reflection positive measures and $\mu=\frac12(\mu_+ +\mu_-)$. Show that μ is reflection positive but not ergodic, and that its reconstructed Hilbert space contains two orthogonal invariant vectors.

3 The course

These notes cover the first of seven lectures given at SISSA, Trieste, in October–November 2026, for graduate students in analysis, probability or mathematical physics. Each lecture has its own notes. Lectures 1–5 develop the general framework; Lectures 6–7 give a construction of Φ^4_3 on the torus via the FBSDE of Lecture 4. Lectures 1–3 were delivered in October 2026; Lectures 4–7 are provisional, and Sections 16–18 of Lecture 7 will not be delivered in class.

Prerequisites. Measure-theoretic probability and Itô calculus; the spectral theorem for self-adjoint operators; distributions and Fourier analysis. No previous knowledge of quantum field theory is assumed.

Approach. The course follows the Euclidean strategy of Section 2. After the Gaussian free field, interacting theories are studied through the scale decomposition of the field: the Polchinski equation, its probabilistic form as a forward–backward stochastic differential equation (FBSDE) for the field at increasing resolution, and the associated variational problem. Stochastic quantisation (Lecture 3) gives a first dynamical description of the free field; cluster expansions and correlation inequalities appear as complementary approaches.

	Lecture	Main results
1	Relativistic quantum fields and imaginary time	No fields at a point; Källén–Lehmann; Wick rotation; Feynman–Kac; OS axioms
2	The Gaussian free field	Construction; reflection positivity; Wick powers; DS characterisation
3	Stochastic quantisation and stochastic observables	Parabolic and elliptic SQ; scale decomposition; stochastic observables; products and OPE
4	Forward–backward equations	Polchinski equation and FBSDE; force coordinates and the Wetterich equation; Boué–Dupuis formula

Table 3.1. Lectures 1–4.

Lecture 1. Relativistic quantum fields and imaginary time. Sections 1 and 2: Wightman axioms; no fields at a point; Källén–Lehmann representation; quantum wave equation and its problems; lattice Hamiltonian on a circle; Wick rotation from positivity of the energy; Feynman–Kac formula; lattice spin systems, continuum limit = critical point; OS axioms and reconstruction (statement and idea). *Proofs in class:* Proposition 1.6, Theorem 1.9 (sketch), Proposition 2.1. *Exercises:* Exercises 1.3, 1.6, 2.2, 2.4 and 2.5. *Reading:* [SW64, BSZ92, GJ87, Sim79].

Lecture 2. The Gaussian free field. Section 3: Construction on $\mathbb{T}^d$ and $\mathbb{R}^d$; reflection positivity and the free Markov field; Euclidean Källén–Lehmann; Wick powers; Gaussian integration by parts; DS characterisation. *Proofs in class:* Theorem 3.2, Proposition 3.4, Theorem 3.9, Theorem 3.16. *Exercises:* Exercises 3.3, 3.6, 3.7, 3.8 and 3.2. *Reading:* [She07, Jan97].

Lecture 3. Stochastic quantisation and stochastic observables. Separate notes: Parabolic and elliptic stochastic quantisation of the GFF; scale decomposition of the GFF and stochastic observables; Wick powers as observables in every dimension and the operator product expansion. *Proofs in class:* Theorem 4.2, Theorem 4.4. *Exercises:* Exercises 4.1, 4.4, 5.1, 5.3 and 6.2. *Reading:* [DPZ14, ADG20, BCG26].

Lecture 4. Forward–backward equations. Separate notes: Effective potentials and the Polchinski equation; the Polchinski process as an FBSDE whose force is a martingale; the FBSDE in force coordinates, where the transport term is integrated exactly, and the Wetterich equation for the effective average action; the Boué–Dupuis formula, with the Polchinski equation as its Hamilton–Jacobi–Bellman equation. *Reading:* [Pol84, BBD24, BD98, BG20].

Lecture 5. Observables and field equations. Separate notes: observables of the interacting theory; the Euclidean field equation and Nelson's Euclidean noise; the field equation at every resolution, with a sub-Gaussian noise; Dyson–Schwinger equations at every scale, also in pathwise form; infinite volume. Additional material: the algebra of the Polchinski flow, perturbation theory and counterterms, and functional inequalities from the stochastic localisation.

Lectures 6–7. Construction of Φ^4_3 on the torus (separate notes). The Wetterich FBSDE, two renormalisations and the large-field problem (Lecture 6); a priori estimates for the forward–backward system (Lecture 7, Sections 13–15); UV stability, compactness and the limiting FBSDE (Lecture 7, Sections 16–18, not delivered in class). *Reading:* [BG20, BCG26].

Homework and projects. Problem sets can be drawn from the exercises at the end of each section. The computer experiments of Section 3 (lattice GFF, lattice stochastic quantisation) and the zero-dimensional Polchinski process of Lecture 4 can be extended to interacting models and are suitable as small projects. Possible reading projects: the Griffiths–Simon construction [GS73]; dimensional reduction [KLP84, ADG20]; the Euclidean approach to the particle structure of $P(\varphi)_2$ [GJS74].

References and further reading

Guide. *Axioms and general structure:* [SW64, BSZ92, Haa96, RS75]; the point-field result is from [Wig64]; the spectral representation of the two-point function from [Kal52, Leh54]; particle structure of $P(\varphi)_2$ in [GJS74]. *Hamiltonian construction of $P(\varphi)_2$ in finite volume:* [Nel66, GJ68, Seg70, BSZ92]. *Euclidean field theory and reconstruction:* [Sym69, Nel73a, Nel73b, OS73, OS75]; the textbooks [GJ87, Sim74, Sim79] are the standard references for everything in Section 2. *Lattice approximations and statistical mechanics:* [GRS75, GS73, WK74, Bry09, BBS19, ADC21]. *Surveys of the constructive programme:* [Jaf00, Sum12].

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