A Constructive Introduction to Quantum Field Theory

Lecture 2: The Gaussian free field

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Abstract

We study the first example of a Euclidean field theory: the Gaussian free field (GFF) on the torus $\mathbb{T}^d$ and on $\mathbb{R}^d$. We construct it, prove reflection positivity and recover the free relativistic field, discuss the Euclidean Källén–Lehmann representation, and define its local (Wick) operators. We prove that the Dyson–Schwinger equation characterises the GFF among probability measures on distributions. Stochastic quantisation is treated in Lecture 3. Exercises are collected at the end. These notes follow the slides of Lecture 2.

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Introduction

In Lecture 1 we saw that, after the passage to imaginary time, a relativistic quantum field is encoded in a probability measure on distributions satisfying the Osterwalder–Schrader axioms, formally $\mu(\mathrm{d}\varphi) \propto \exp\left(-\int [\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi)]\right) \mathcal{D}\varphi$ (Eq. (2.13)). In this lecture we treat its Gaussian part rigorously. Conventions are those of Lecture 1. Sections, equations and exercises keep the numbering of the notes, so that §1–2 refer to the notes of Lecture 1.

3 The Gaussian free field

3.1 Definition and construction

Throughout, $\mathcal{M}$ is either the torus $\mathbb{T}^d_L = (\mathbb{R}/L\mathbb{Z})^d$ or $\mathbb{R}^d$, $d \geqslant 1$. We write $\mathcal{S}(\mathcal{M})$ for $C^\infty(\mathbb{T}^d_L)$, resp. the Schwartz space, and $\mathcal{S}'(\mathcal{M})$ for its dual. We equip $\mathcal{S}'(\mathcal{M})$ with the σ-algebra generated by the maps $\varphi \mapsto \varphi(f)$; it coincides with the Borel σ-algebra [GV64, Bog98]. Test functions are real unless stated otherwise. Let $m > 0$ and

$$A := m^2 - \Delta, \qquad C := A^{-1}. \tag{3.1}$$

C is a positive bounded operator on $L^2(\mathcal{M})$ with $\|C\| \leqslant m^{-2}$. Both A and C are continuous bijections of $\mathcal{S}(\mathcal{M})$, being Fourier multipliers by $(|k|^2 + m^2)^{\pm 1}$.

Definition 3.1. *The* (massive) Gaussian free field *on $\mathcal{M}$ is the centred Gaussian probability measure $\mu = \mu_m$ on $\mathcal{S}'(\mathcal{M})$ with covariance C:*

$$\int \varphi(f)\,\varphi(g)\ \mu(\mathrm{d}\varphi) = \langle f, Cg\rangle = \iint f(x)\,G(x,y)\,g(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.2}$$

equivalently $\int e^{i\varphi(f)}\mu(d\varphi) = e^{-\frac{1}{2}\langle f,Cf\rangle}$ *for all* $f \in \mathcal{S}(\mathcal{M})$. *Here* G *is the Green function of* A. *On* $\mathbb{R}^d$, $G(x,y) = G_m(x-y)$ *with* $G_m(x) = \int \frac{e^{ik\cdot x}}{|k|^2+m^2}\frac{dk}{(2\pi)^d}$. *On* $\mathbb{T}_L^d$, $G(x,y) = \sum_{n\in\mathbb{Z}^d} G_m(x-y+nL)$ (*method of images*).

Formally $\mu_m(d\varphi)$ "$= Z^{-1}e^{-\frac{1}{2}\langle\varphi,A\varphi\rangle}\mathcal{D}\varphi$", the Gaussian part of (2.13). The Green function satisfies (Exercise 3.1): $G_m(x) \sim \frac{1}{2\pi}\log\frac{1}{|x|}$ as $x\to 0$ if $d=2$; $G_m(x) \sim \frac{\Gamma(d/2-1)}{4\pi^{d/2}}|x|^{2-d}$ if $d\geqslant 3$; $G_m(x) = e^{-m|x|}/(2m)$ if $d=1$; and G_m decays like $|x|^{-(d-1)/2}e^{-m|x|}$ at infinity. Uniqueness of μ_m is clear, since a probability measure on $\mathcal{S}'(\mathcal{M})$ is determined by its characteristic functional.

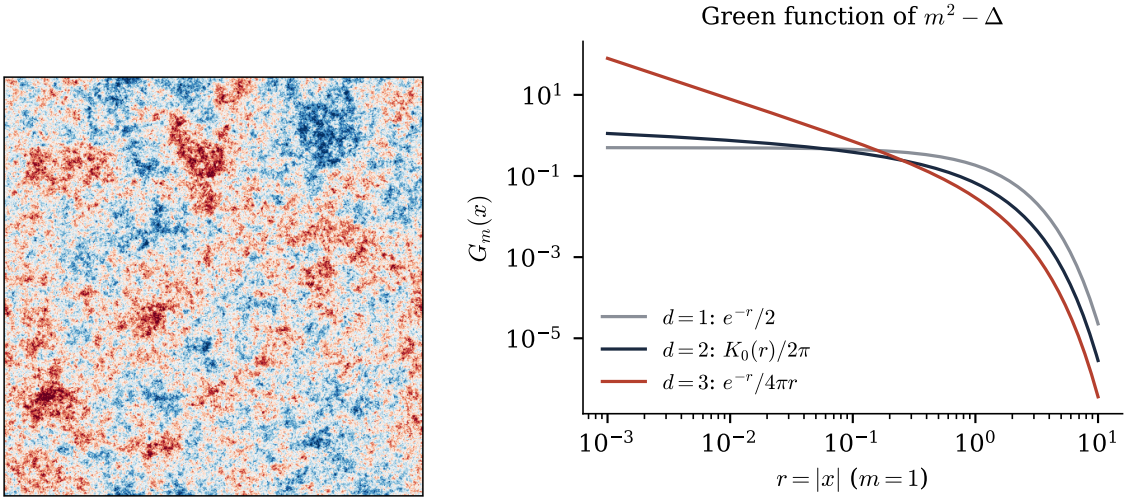


Figure 3.1. Left: a sample of the massive GFF on $\mathbb{T}^2$ ($m=4$, Fourier cutoff 512^2, zero mode removed). It is not a function: refining the cutoff adds fluctuations at all scales. Right: the Green function G_m in $d=1$, 2, 3: bounded, logarithmic and $|x|^{-1}$ singularities at the origin, and exponential decay at infinity.

Theorem 3.2. (Existence and regularity) (*a*) *Let* $\mathcal{M} = \mathbb{T}_L^d$, *let* $(e_n)_{n\geqslant 0}$ *be a real orthonormal basis of* $L^2(\mathbb{T}_L^d)$ *of eigenfunctions of* $-\Delta$, $-\Delta e_n = \kappa_n e_n$, $\lambda_n := m^2+\kappa_n$, *and let* (g_n) *be i.i.d.* $N(0,1)$. *For every* $s > \frac{d}{2}-1$ *the series*

$$\varphi = \sum_{n\geqslant 0} \lambda_n^{-1/2} g_n\, e_n \tag{3.3}$$

converges a.s. and in $L^2(\mathbb{P}; H^{-s}(\mathbb{T}_L^d))$, *and its law is* μ_m. *In particular* $\mu_m(H^{-s}) = 1$ *for* $s > \frac{d}{2}-1$. (*b*) *On* $\mathbb{R}^d$, μ_m *exists by the Bochner–Minlos theorem.*

Proof. (a) With $\|\varphi\|_{H^{-s}}^2 = \sum_n (1+\kappa_n)^{-s}\langle e_n,\varphi\rangle^2$,

$$\mathbb{E}\|\varphi\|_{H^{-s}}^2 = \sum_n \frac{1}{(1+\kappa_n)^s\lambda_n} \asymp \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-s-1} < \infty \iff 2s+2 > d.$$

The partial sums form an L^2-bounded martingale with values in the Hilbert space H^{-s}, so they converge a.s. and in L^2. For $f\in\mathcal{S}(\mathcal{M})$, $\varphi(f) = \sum_n \lambda_n^{-1/2} g_n \langle e_n, f\rangle$ is centred Gaussian with variance $\sum_n \lambda_n^{-1}\langle e_n,f\rangle^2 = \langle f, Cf\rangle$.

(b) Bochner–Minlos [GV64, GJ87]: if $\Psi\colon \mathcal{S}(\mathbb{R}^d)\to\mathbb{C}$ is continuous, positive definite and $\Psi(0)=1$, there is a unique probability measure on $\mathcal{S}'(\mathbb{R}^d)$ with characteristic functional Ψ. For $\Psi(f) = e^{-\frac{1}{2}\langle f,Cf\rangle}$, continuity follows from $\langle f,Cf\rangle \leqslant m^{-2}\|f\|_{L^2}^2$. For positive definiteness, write $\Psi(f_j - f_k) = a_j a_k e^{Q_{jk}}$ with $a_j = e^{-\frac{1}{2}Q_{jj}}$ and $Q_{jk} = \langle f_j, Cf_k\rangle$ positive semidefinite. Then $(e^{Q_{jk}})$ is positive semidefinite by the Schur product theorem. □

Remark 3.3. The Cameron–Martin space of μ_m is $H^1(\mathcal{M})$ with norm $\langle h, A\,h\rangle^{1/2}$: the fields of finite action. It has measure zero, and typical samples are much rougher (Exercise 3.3). In $d=1$ the GFF on $\mathbb{R}$ is the stationary Ornstein–Uhlenbeck process. In $d=2$ it lies in $H^{-\kappa}$ for every $\kappa>0$ but is not a function. In $d=3$ it lies in $H^{-1/2-\kappa}$. On $\mathbb{R}^d$ the samples live in weighted Sobolev spaces.

3.2 Reflection positivity and the free field

Let $\mathcal{M}=\mathbb{R}\times N$ with $N=\mathbb{R}^{d-1}$ or $\mathbb{T}_L^{d-1}$, points $(\tau,\mathbf{x})$, $\Theta(\tau,\mathbf{x})=(-\tau,\mathbf{x})$. Then $A=-\partial_\tau^2+B$ with $B:=m^2-\Delta_N$ on $L^2(N)$, and we set $\omega:=B^{1/2}\geqslant m$.

Proposition 3.4. *(a) As an operator-valued kernel in the time variables, $C(\tau,\tau')=\mathrm{e}^{-|\tau-\tau'|\omega}(2\omega)^{-1}$. (b) For $f\in\mathcal{S}(\mathcal{M})$ with $\operatorname{supp} f\subset\{\tau>0\}$,*

$$\langle \Theta f, Cf\rangle = \Big\| (2\omega)^{-1/2}\int_0^\infty \mathrm{e}^{-\tau\omega} f(\tau,\cdot)\,\mathrm{d}\tau\Big\|^2_{L^2(N)}\geqslant 0. \tag{3.4}$$

(c) μ_m on $\mathbb{R}^d$ satisfies (OS0)–(OS4).

Proof. (a) follows from the partial Fourier transform in τ, the spectral theorem for B, and the scalar identity $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kt}(k^2+w^2)^{-1}\frac{\mathrm{d}k}{2\pi}=\mathrm{e}^{-w|t|}/(2w)$, $w>0$. (b) Since Θf is supported in $\{\tau<0\}$,

$$\begin{aligned}\langle \Theta f, Cf\rangle &= \iint_{\tau<0<\tau'} \langle f(-\tau), \mathrm{e}^{-(\tau'-\tau)\omega}(2\omega)^{-1} f(\tau')\rangle\,\mathrm{d}\tau\,\mathrm{d}\tau' \\ &= \int_0^\infty\int_0^\infty \langle \mathrm{e}^{-s\omega} f(s), (2\omega)^{-1}\mathrm{e}^{-\tau'\omega} f(\tau')\rangle\,\mathrm{d}s\,\mathrm{d}\tau',\end{aligned}$$

which is (3.4). (c) (OS0)–(OS2) and (OS4) follow from the explicit form of $\hat\mu(f)$ and the decay of G_m. For (OS3), C commutes with Θ, so $\hat\mu(f_i-\Theta f_j)=a_i a_j \mathrm{e}^{\langle \Theta f_i, Cf_j\rangle}$ with $a_i=\mathrm{e}^{-\frac12\langle f_i, Cf_i\rangle}$. The matrix $(\langle \Theta f_i, Cf_j\rangle)_{ij}$ is positive semidefinite by (b), hence so is its entrywise exponential (Schur). □

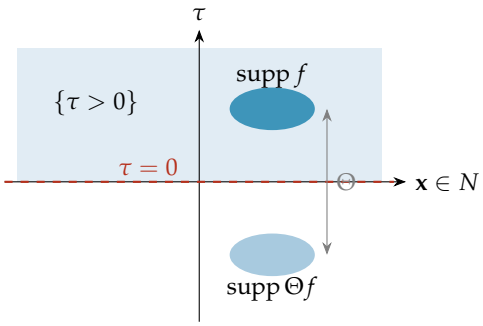


Figure 3.2. Reflection positivity (Proposition 3.4). For f supported in $\{\tau>0\}$ the covariance between f and its mirror image factorises through the time-zero plane: $\langle \Theta f, Cf\rangle = \|(2\omega)^{-1/2}\int_0^\infty \mathrm{e}^{-\tau\omega} f(\tau)\mathrm{d}\tau\|^2\geqslant 0$.

By the OS reconstruction theorem the GFF yields a Wightman theory, the free field of mass m of Section 1.3. Concretely, the reconstructed Hilbert space is the Fock space over $L^2(N)$ with $H=\mathrm{d}\Gamma(\omega)$ (Nelson's free Markov field [Nel73a, Nel73b]). The time-zero restriction of the GFF is Gaussian with covariance $(2\omega)^{-1}$, the ground state measure of the free Hamiltonian in the Schrödinger representation (Exercise 4.4 of Lecture 3). The GFF is also the continuum limit of the lattice model (2.12) with $V=0$ (for $d=2$; in general d see Exercise 2.3 and Exercise 3.10).

Remark 3.5. (Nelson's Markov property) For $U\subset\mathbb{R}^d$ open let Σ_U be the σ-algebra generated by the $\varphi(f)$ with $\operatorname{supp} f\subset U$, and for K closed let $\Sigma_K=\bigcap_{U\supset K}\Sigma_U$ (the *germ* σ-algebra of K). A random field is *Markov* for an open set Λ if $\Sigma_{\bar\Lambda}$ and Σ_{Λ^c} are conditionally independent given $\Sigma_{\partial\Lambda}$: the field inside and the field outside interact only through the field near the boundary.

(i) *The GFF is Markov for every open* Λ [Nel73a]. For the GFF, Σ_K is generated by the $\varphi(g)$ with $g \in H^{-1}$ supported in K [Nel73a]. In Gaussian terms, conditional expectations are then orthogonal projections in the Hilbert space of the $\varphi(f)$, with $\mathbb{E}[\varphi(f)\,\varphi(g)] = \langle f, C\,g\rangle$: $\mathbb{E}[\varphi(f) \mid \Sigma_{\Lambda^c}] = \varphi(\Pi f)$, where Π projects H^{-1} onto the distributions supported in Λ^c. Let $\operatorname{supp} f \subset \Lambda$ and $u = C\,\Pi f$. Orthogonality gives $u = Cf$ on the interior of Λ^c, hence $A\,u = f = 0$ there, while $A\,u = \Pi f = 0$ in Λ. So $\Pi f = A\,u$ is supported on $\partial\Lambda$: this is where the *locality* of $A = m^2 - \Delta$ enters. Equivalently, given the outside, the field in Λ is the A-harmonic extension of its boundary values plus an independent GFF with Dirichlet conditions on $\partial\Lambda$ (the domain Markov property of [She07]). The generalised free fields of the two-mass example below are not Markov, because their C^{-1} is nonlocal.

(ii) *For a* Θ*-invariant field, Markov with a sharp time-zero* σ*-algebra implies reflection positivity.* Take $\Lambda = \{\tau > 0\}$ and let Σ_0 be the germ σ-algebra of $\{\tau = 0\}$. For the GFF it is generated by the time-zero field (*sharp* Markov property), so Θ acts trivially on it. If F is $\Sigma_{\{\tau \geqslant 0\}}$-measurable, then ΘF is $\Sigma_{\{\tau \leqslant 0\}}$-measurable and $\mathbb{E}[\Theta F \mid \Sigma_0] = \mathbb{E}[F \mid \Sigma_0]$, so the Markov property gives

$$\mathbb{E}[\overline{\Theta F}\, F] = \mathbb{E}[\overline{\mathbb{E}[\Theta F \mid \Sigma_0]}\, \mathbb{E}[F \mid \Sigma_0]] = \mathbb{E}[|\mathbb{E}[F \mid \Sigma_0]|^2] \geqslant 0.$$

Sharpness is essential: the fourth-order field with covariance P^{-1} in the discussion of positivity versus locality below is Markov but not reflection positive, because its Σ_0 also contains $\partial_\tau \varphi$, which Θ reverses. Moreover $P_t = \mathbb{E}[\,U_t \cdot \mid \Sigma_0]$, with U_t the time translation, is a semigroup on $L^2(\Sigma_0)$: by the Markov property for $\{\tau > t\}$, $P_t P_s = P_{t+s}$. It is self-adjoint (by Θ-invariance), positive ($P_t = P_{t/2}^2$), contractive and strongly continuous, so $P_t = \mathrm{e}^{-tH}$, and the physical Hilbert space is L^2 of the time-zero field, i.e. the Schrödinger representation. This is Nelson's reconstruction of quantum fields from Euclidean invariant Markov fields [Nel73b], which preceded the Osterwalder–Schrader axioms, and it underlies the Feynman–Kac–Nelson formula (Proposition 2.3).

(iii) *Relevance.* A local perturbation $\mathrm{e}^{-\int_\Lambda V(\varphi)}$ of μ_m factorises over regions and preserves the Markov property. This makes the $P(\varphi)_2$ measures in finite volume Markov fields, and it gives the classical statistical mechanics picture of [GRS75]: conditional distributions given the outside, DLR equations, correlation inequalities. The Markov property is not stable under weak limits, however, and is hard to establish directly for infinite-volume or renormalised theories. Osterwalder and Schrader replaced it by reflection positivity, which neither implies nor is implied by the Markov property (generalised free fields are reflection positive but not Markov, the P^{-1} field is Markov but not reflection positive), but is preserved under weak limits. See [Sim74, GJ87, Gue06] for more.

Remark 3.6. (**Symanzik, Brownian paths and loop soups**) The Euclidean programme goes back to Symanzik [Sym66, Sym69], who proposed to study QFT through its Schwinger functions, viewed as the moments of a classical (formally, Markov) random field, before Nelson and Osterwalder–Schrader made this rigorous. In [Sym69] he also introduced a representation of the field in terms of Brownian paths. Since

$$C(x,y) = \int_0^\infty \mathrm{e}^{-m^2 t}\, p_t(x,y)\, \mathrm{d}t,$$

with p_t the heat kernel of Δ, the two-point function of the GFF is the Green function of a Brownian motion (with generator Δ) killed at rate m^2. For $\mathrm{e}^{-\int V(\varphi)}\mu_m$, the correlations become averages over a gas of Brownian paths and loops with an interaction that depends on their local times. For φ^4 the interaction penalises intersections. On the lattice this is the random walk representation of Brydges, Fröhlich and Spencer [BFS82], the basis of one of the proofs of triviality of φ^4_d for $d > 4$. Heuristically, the interaction becomes ineffective in the continuum for $d \geqslant 4$ because two independent Brownian paths do not intersect there.

The modern continuum version is the *Brownian loop soup* of Lawler and Werner [LW04], a Poisson process of unrooted Brownian loops with intensity α times the loop measure. On a finite graph the occupation field L of the random-walk loop soup satisfies $\mathbb{E}[e^{-\langle\chi,L\rangle}] = \det(1+C\chi)^{-\alpha}$, while $\mathbb{E}[e^{-\frac{1}{2}\langle\chi,\varphi^2\rangle}] = \det(1+C\chi)^{-1/2}$. So for $\alpha=\frac{1}{2}$, L has the law of $\frac{1}{2}\varphi^2$: this is *Le Jan's isomorphism* [LJ10, LJ11], a loop version of Dynkin's isomorphism [Dyn84]. In the continuum, in $d=2,3$, the renormalised occupation field of the soup corresponds to $\frac{1}{2}{:}\varphi^2{:}$, the Wick square of Section 3.3. Lupu [Lup16] showed that on the cable graph the clusters of the $\alpha=\frac{1}{2}$ soup are exactly the sign clusters of the GFF, which is recovered from the soup by tossing independent signs. Qian and Werner [QW19] decomposed the clusters of the two-dimensional continuum soup, and Werner [Wer16] proved a spatial Markov property for loop soups that parallels Nelson's Markov property of the GFF. For a survey see [WP21].

Remark 3.7. (Euclidean Källén–Lehmann representation) The computation in Proposition 3.4(a) says that $\int \mathrm{d}\Omega_M(\mathbf{p})\, e^{-\omega_M(\mathbf{p})|\tau|+i\mathbf{p}\cdot\mathbf{x}} = G_M(\tau,\mathbf{x})$: the Wick rotation of the free two-point function $W_2^{(M)}$ is the Green function G_M of $M^2-\Delta$. Rotating (1.4) term by term, the Schwinger two-point function of any theory satisfying the Wightman (or OS) axioms is

$$S_2(x) = v^2 + \int_{[0,\infty)} \rho(\mathrm{d}M^2)\; G_M(x), \qquad x\neq 0, \qquad \text{i.e.} \qquad \hat{S}_2^T(k) = \int_{[0,\infty)} \frac{\rho(\mathrm{d}M^2)}{|k|^2+M^2}, \tag{3.5}$$

a Stieltjes transform in $|k|^2$. (The Fourier form holds up to contact terms, i.e. a polynomial in $|k|^2$ invisible at $x\neq 0$, and provided $\int\rho(\mathrm{d}M^2)\,(1+M^2)^{-1}<\infty$; otherwise subtractions are needed.) The GFF is the case $\rho=\delta_{m^2}$. Three consequences follow.

(i) *Short distances.* $G_M(x)$ is decreasing in M, and for $d\geqslant 2$ we have $G_M(x)\,/\,G_m(x)\to 1$ as $x\to 0$, for every M. Hence, for every K, $S_2^T(x)\geqslant \rho([0,K^2])\; G_K(x)$, and, if $\rho\neq 0$,

$$\liminf_{x\to 0} \frac{S_2^T(x)}{G_m(x)} \geqslant \rho([0,\infty)) \in (0,\infty].$$

The two-point function of any nontrivial theory is at least as singular as the free one. In particular a scalar field cannot have scaling dimension below the free value $\frac{d-2}{2}$ (the *unitarity bound*, Exercise 3.2).

(ii) *Long distances.* If $\operatorname{supp}\rho\subset[m_0^2,\infty)$, then $S_2^T(x)$ decays at least like $e^{-m_0|x|}$, up to subexponential factors, and exactly at this rate if $m_0^2=\inf\operatorname{supp}\rho$. Conversely, exponential decay at rate m_0 forces $\operatorname{supp}\rho\subset[m_0^2,\infty)$. So a gap in the spectral measure of φ is the same as exponential decay of S_2^T; a mass gap of H corresponds to exponential clustering of all truncated correlations (compare Exercise 2.6).

(iii) *Generalised free fields.* For any positive measure ρ on $(0,\infty)$ with $\int\rho(\mathrm{d}M^2)\,(1+M^2)^{-1}<\infty$ and $\int\rho(\mathrm{d}M^2)\,M^{-2}<\infty$, the Gaussian measure on $\mathcal{S}'(\mathbb{R}^d)$ with covariance $C_\rho=\int\rho(\mathrm{d}M^2)\,(M^2-\Delta)^{-1}$ satisfies the OS axioms: $\hat{C}_\rho\leqslant\hat{C}_\rho(0)=\int\rho(\mathrm{d}M^2)\,M^{-2}$ is bounded, which gives (OS1) with $p=2$, and reflection positivity follows from Proposition 3.4 for each mass $M>0$ and $\rho\geqslant 0$. (If only the first condition holds, the measure may still exist and satisfy the other axioms, but (OS1) in the form of Lecture 1 can fail: for the massless field in $d\geqslant 3$, $f=ig$ with $g\geqslant 0$ supported in a ball of radius R gives $|\hat{\mu}(f)| = e^{\frac{1}{2}\langle g,Cg\rangle}$ with $\langle g,Cg\rangle\sim R^{d+2}$.) It reconstructs the generalised free field of Exercise 1.6. Its Dyson–Schwinger equation $\mathbb{E}[D_hF]=\mathbb{E}[F\,\varphi(C_\rho^{-1}h)]$ still characterises it: under mild conditions on ρ (e.g. ρ finite with a mass gap), C_ρ is a bijection of $\mathcal{S}(\mathbb{R}^d)$, which is all the proof of Theorem 3.16 uses. However C_ρ^{-1} is *nonlocal* unless ρ is a single atom, as the following example shows.

Example: two masses. Let $0<m_1<m_2$, $c_1,c_2>0$, and

$$\rho = c_1^2\,\delta_{m_1^2} + c_2^2\,\delta_{m_2^2}, \qquad C_\rho = c_1^2\,(m_1^2-\Delta)^{-1} + c_2^2\,(m_2^2-\Delta)^{-1}.$$

Euclidean theory. The Gaussian measure μ_ρ with covariance C_ρ is the law of $c_1\varphi_1 + c_2\varphi_2$, where φ_1,φ_2 are independent GFFs on $\mathbb{R}^d$ of masses m_1,m_2. It satisfies (OS0)–(OS4); reflection positivity follows from Proposition 3.4 applied to each term.

The Minkowski theory. Since a Wightman theory is determined up to unitary equivalence by its Wightman functions, which are in turn determined by the Schwinger functions, μ_ρ reconstructs to exactly the following theory. Let $\mathfrak{h}_m = L^2(\mathbb{R}^{d-1}, \mathrm{d}\Omega_m)$ be the one-particle space of mass m and spin 0, carrying the irreducible positive-energy representation u_m of $\mathcal{P}_+^\uparrow$. Then

$$\mathcal{H} = \Gamma(\mathfrak{h}_{m_1}\oplus\mathfrak{h}_{m_2}) \simeq \Gamma(\mathfrak{h}_{m_1})\otimes\Gamma(\mathfrak{h}_{m_2}), \qquad U = \Gamma(u_{m_1}\oplus u_{m_2}), \qquad \Omega = \Omega_1\otimes\Omega_2,$$

$$\varphi(x) = c_1\,\varphi_{m_1}(x)\otimes 1 + c_2\,1\otimes\varphi_{m_2}(x),$$

with φ_m the free field (1.2) of mass m. The Wightman functions are given by Wick's theorem with $W_2 = c_1^2 W_2^{(m_1)} + c_2^2 W_2^{(m_2)}$, and all truncated functions of order $\geqslant 3$ vanish. Physically, these are *two species of free scalar particles*, of masses m_1 and m_2, both created from the vacuum by the same field: $\varphi(f)\,\Omega = c_1\,a_1^*(\tilde f|_{H^+_{m_1}})\,\Omega + c_2\,a_2^*(\tilde f|_{H^+_{m_2}})\,\Omega$. They do not scatter. The mass spectrum consists of the two shells m_1, m_2 and of the multi-particle continuum from $2m_1$. If $m_2 > 2m_1$, the second shell lies inside the two-particle continuum, but the heavy particle is stable because there is no interaction. This is the generalised free field of Exercise 1.6.

What is lost. (1) There is no field equation of second order: $(\Box+m_1^2)(\Box+m_2^2)\,\varphi = 0$, but φ solves no Klein–Gordon equation. The time-zero fields $\varphi(0,\cdot)$, $\pi(0,\cdot)$ satisfy $[\varphi(0,\mathbf{x}),\pi(0,\mathbf{y})] = \mathrm{i}(c_1^2+c_2^2)\,\delta(\mathbf{x}-\mathbf{y})$, which is canonical iff $c_1^2+c_2^2=1$ (the sum rule of Exercise 1.5). Even when they are canonical, they are not a complete set of Cauchy data: $\varphi(t,\cdot)$ depends on $c_1\varphi_{m_1}(0,\cdot)$ and $c_2\varphi_{m_2}(0,\cdot)$ separately, not only on their sum. (2) In Euclidean terms, the DS equation $\mathbb{E}[D_h F] = \mathbb{E}[F\,\varphi(C_\rho^{-1}h)]$ involves the nonlocal operator

$$C_\rho^{-1} = P R^{-1}, \qquad P := (m_1^2-\Delta)(m_2^2-\Delta), \qquad R := c_1^2(m_2^2-\Delta) + c_2^2(m_1^2-\Delta).$$

Replacing h by Rh gives the equivalent local identity $\mathbb{E}[D_{Rh}F] = \mathbb{E}[F\,\varphi(Ph)]$: a fourth-order Euclidean field equation whose contact term is itself differentiated, the Euclidean form of $(\Box+m_1^2)(\Box+m_2^2)\,\varphi = 0$. Correspondingly, φ has no Markov property: for stationary Gaussian fields with a rational spectral density, such as $\hat C_\rho = \hat R/\hat P$, the Markov property holds iff the inverse covariance is a differential operator, i.e. iff $1/\hat C$ is a polynomial [Roz82], which is not the case here. The pair (φ_1,φ_2) is Markov, but its sum is not: this is the Euclidean counterpart of (1).

Positivity versus locality. Conversely, the local fourth-order action $\frac12\langle\varphi, P\varphi\rangle$ defines a Markov Gaussian field with covariance

$$P^{-1} = \frac{(m_1^2-\Delta)^{-1} - (m_2^2-\Delta)^{-1}}{m_2^2-m_1^2}, \qquad \text{i.e.} \qquad \rho = \frac{\delta_{m_1^2}-\delta_{m_2^2}}{m_2^2-m_1^2},$$

with a *negative* weight at m_2^2. By (3.5) and the uniqueness of the Stieltjes transform it is not reflection positive (Exercise 3.12). It reconstructs at best a theory with an indefinite metric, in which the particle of mass m_2 is a ghost of negative norm [PU50, LW69]. More generally, call the Euclidean field equation *local* if C^{-1} is a differential operator, i.e. $\hat C = 1/p(|k|^2)$ with p a polynomial; local field equations give Markov fields, and for rational spectral densities the converse holds [Roz82]. The Fourier transform of a covariance satisfying the OS axioms

is $q(|k|^2) + \int \rho(\mathrm{d}M^2)\,(|k|^2+M^2)^{-1}$, with $\rho \geqslant 0$ and q a polynomial coming from contact terms invisible to ρ. If $\deg p \geqslant 1$, then $q=0$, since $1/p(s) \to 0$ and the Stieltjes term tends to 0 as $s \to \infty$. Moreover $s\int \rho(\mathrm{d}M^2)\,(s+M^2)^{-1}$ increases to $\rho([0,\infty))$ as $s\to\infty$, while $s/p(s)\to 0$ if $\deg p \geqslant 2$; hence $\rho = 0$, a contradiction. If $\deg p = 0$, $\hat{C}$ is constant. So a centred Gaussian measure satisfying the OS axioms has a local Euclidean field equation iff its covariance is $c\,(M^2-\Delta)^{-1}$ with $M>0$, or a multiple of the identity (white noise, whose reconstructed Hilbert space is $\mathbb{C}\,\Omega$). Note that it is the covariance, not ρ, that matters: $a\,1 + c\,(m^2-\Delta)^{-1}$ with $a>0$ is reflection positive and has $\rho = c\,\delta_{m^2}$, but its field equation is nonlocal. *Positivity and locality of the Euclidean field equation together single out the free field* (up to the trivial white noise); locality is also behind Nelson's Markov property of the GFF [Nel73a].

Physical meaning of the nonlocal theory. In Minkowski space the two-mass theory is a perfectly good one. It satisfies all the Wightman axioms, including locality (W3): $[\varphi(x),\varphi(y)]$ is a sum of free-field commutators and vanishes at spacelike separation. The energy is positive and the vacuum unique. It is trivial (no interaction) and redundant, but not inconsistent. "Nonlocal" refers only to the Euclidean field equation for φ alone, equivalently (for rational spectral densities, as here) to the absence of the Markov property. On the Minkowski side this means that φ alone is not a complete set of local degrees of freedom. Its time-zero data do not determine its evolution, and there is no Lagrangian or Hamiltonian written in terms of φ only. The cure is to enlarge the field content: the pair (φ_1,φ_2) has a local action and is Markov, and φ is a linear combination of its components.

Generalised free fields become genuinely unphysical when ρ has a continuous part, i.e. (heuristically) infinitely many free species with masses in a bounded interval. With finitely many masses, φ solves a hyperbolic equation of finite order, and the fields in an arbitrarily thin time slice generate all the observables: the *time-slice axiom*, or primitive causality [Haa96, §II.1]. With a continuum of masses there is no dynamical law of finite order, and this property is not expected to hold. More decisively, the theory violates the phase-space requirements that express "finitely many degrees of freedom per unit volume of phase space", namely the compactness criterion of Haag and Swieca [HSw65] and its sharpening, the nuclearity condition of Buchholz and Wichmann [BW86]. For a system of free fields the nuclearity condition holds iff $\sum_i n_i \mathrm{e}^{-\beta m_i} < \infty$ for all $\beta>0$, where n_i is the multiplicity of the mass m_i [Haa96, §V.5], and this fails when infinitely many species have masses in a bounded interval. Since the nuclearity index of a bounded region plays the role of its partition function, the theory has no sensible thermodynamics. This does not affect interacting theories, whose spectral measure also has a continuous part, the multi-particle continuum. There the continuum is produced by a *local* interaction, and the Euclidean field equation remains local, e.g. $\mathbb{E}\,[D_h F] = \mathbb{E}[F((m^2-\Delta)\,\varphi + 4\,\lambda{:}\,\varphi^3{:})(h)]$ for Φ^4_2, although the inverse two-point function is nonlocal. Thus, for a single field: positive and nonlocal means an incomplete description, which is unphysical for continuous ρ; local and not positive means ghosts; local, positive and with a nontrivial continuum in ρ requires interactions.

3.3 Local operators: Wick products

Interactions $\int V(\varphi(x))\mathrm{d}x$ and nonlinear field equations require local functions of $\varphi(x)$, but φ is not a function for $d \geqslant 2$. We regularise, subtract, and remove the regularisation. The subtraction is encoded in Hermite polynomials with variance $c \geqslant 0$,

$$\mathrm{e}^{ta-ct^2/2} = \sum_{n\geqslant 0} \frac{t^n}{n!} H_n(a;c), \qquad H_2 = a^2 - c, \quad H_3 = a^3 - 3ca, \quad H_4 = a^4 - 6ca^2 + 3c^2. \tag{3.6}$$

Lemma 3.8. *If (X,Y) is a centred Gaussian vector with $\mathbb{E}\,X^2=a$ and $\mathbb{E}\,Y^2=b$, then $\mathbb{E}\,[H_n(X;a)\,H_k(Y;b)]=\delta_{nk}\,n!\,(\mathbb{E}\,[XY])^n$.*

Proof. $\mathbb{E}[\mathrm{e}^{sX-s^2a/2}\mathrm{e}^{tY-t^2b/2}]=\mathrm{e}^{\frac12\mathbb{E}(sX+tY)^2-\frac12 s^2a-\frac12 t^2b}=\mathrm{e}^{st\mathbb{E}[XY]}$. Expand both sides in powers of s, t using (3.6) (the exchange of sum and expectation is justified by Gaussian integrability) and compare coefficients. □

Fix an even mollifier $0\leqslant\eta\in C_c^\infty(B(0,1))$ with $\int\eta=1$, $\eta_\varepsilon=\varepsilon^{-d}\eta\,(\cdot/\varepsilon)$ (periodised on the torus), and let $\varphi_\varepsilon=\eta_\varepsilon*\varphi$, a smooth function. Then $\mathbb{E}\,[\varphi_\varepsilon(x)\,\varphi_\delta(y)]=G_{\varepsilon,\delta}\,(x-y)$ with $G_{\varepsilon,\delta}=\eta_\varepsilon*\eta_\delta*G$. Set $c_\varepsilon=G_{\varepsilon,\varepsilon}(0)$ and

$$:\varphi_\varepsilon^n:(x):=H_n(\varphi_\varepsilon(x);c_\varepsilon),\qquad :\varphi_\varepsilon^n:(f):=\int f(x):\varphi_\varepsilon^n:(x)\,\mathrm{d}x. \tag{3.7}$$

Theorem 3.9. (Wick powers) *Let $n\geqslant1$ with $n\,(d-2)<d$, i.e. every n if $d\leqslant2$, $n\leqslant2$ if $d=3$, $n=1$ if $d\geqslant4$. Then for every $f\in\mathcal{S}(\mathcal{M})$, $:\varphi^n:(f):=\lim_{\varepsilon\to0}:\varphi_\varepsilon^n:(f)$ exists in $L^2(\mu_m)$, is independent of η, and*

$$\mathbb{E}\,[:\varphi^n:(f):\varphi^k:(g)]=\delta_{nk}\,n!\iint f(x)\,G\,(x-y)^n g(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.8}$$

If instead $n\,(d-2)\geqslant d$ and $0\leqslant f\not\equiv0$, then $\mathbb{E}[:\varphi_\varepsilon^n:(f)^2]\to\infty$.

Proof. *Step 1: a uniform bound.* Let $K(z)=1+\log_+(1/|z|)$ if $d=2$, $K(z)=1+|z|^{2-d}$ if $d\geqslant3$, $K\equiv1$ if $d=1$. Then $0<G\leqslant C_0K$, and we claim $G_{\varepsilon,\delta}\leqslant CK$ uniformly in $\varepsilon,\delta\in(0,1]$. Let $r=\max(\varepsilon,\delta)$. Then $\operatorname{supp}(\eta_\varepsilon*\eta_\delta)\subset B(0,2r)$, $\int\eta_\varepsilon*\eta_\delta=1$ and $\|\eta_\varepsilon*\eta_\delta\|_\infty\leqslant r^{-d}\|\eta\|_\infty$. If $|z|\geqslant4r$, then $G_{\varepsilon,\delta}(z)\leqslant\sup_{|w|\geqslant|z|/2}G(w)\leqslant C_0K(z/2)\leqslant CK(z)$. If $|z|<4r$, then

$$G_{\varepsilon,\delta}(z)\leqslant\|\eta\|_\infty\,r^{-d}\int_{B(0,6r)}G\leqslant C\frac{1}{|B(0,6r)|}\int_{B(0,6r)}K\leqslant CK(r)\leqslant C'K(z).$$

Step 2: convergence. By Lemma 3.8 and Fubini,

$$\mathbb{E}\,[:\varphi_\varepsilon^n:(f):\varphi_\delta^k:(g)]=\delta_{nk}\,n!\iint f(x)\,g(y)\;G_{\varepsilon,\delta}\,(x-y)^n\,\mathrm{d}x\,\mathrm{d}y.$$

As $\varepsilon,\delta\to0$, $G_{\varepsilon,\delta}(z)\to G(z)$ for $z\neq0$. Moreover $K^n\in L^1_{\mathrm{loc}}$ iff $n\,(d-2)<d$, and K is bounded on $\{|z|\geqslant1\}$. Since $f,g\in L^1\cap L^\infty$, dominated convergence gives $\mathbb{E}[(:\varphi_\varepsilon^n:(f)-:\varphi_\delta^n:(f))^2]=n!\iint ff\,(G_{\varepsilon,\varepsilon}^n-2G_{\varepsilon,\delta}^n+G_{\delta,\delta}^n)\to0$: the sequence is Cauchy in L^2. The limit covariance is (3.8). Running the same computation with two different mollifiers shows independence of η. *Step 3: divergence.* If $n\,(d-2)\geqslant d$, then $G^n\geqslant c|z|^{-n(d-2)}$ is not locally integrable, and Fatou's lemma gives $\liminf\mathbb{E}[:\varphi_\varepsilon^n:(f)^2]\geqslant n!\iint ffG^n=\infty$. □

Remark 3.10. (i) The subtraction of $c_\varepsilon\to\infty$ is the simplest instance of renormalisation: $\varphi_\varepsilon(x)^n$ itself has no limit when $d\geqslant2$. (ii) The variables $:\varphi^n:(f)$ belong to the n-th Wiener chaos, and $L^2(\mu_m)\simeq\bigoplus_n\mathcal{H}_n$ is the symmetric Fock space over $H^{-1}(\mathcal{M})$ [Jan97]. By Nelson's hypercontractivity, $\|X\|_{L^p}\leqslant(p-1)^{n/2}\|X\|_{L^2}$ for $X\in\mathcal{H}_n$, so the convergence holds in every L^p. A Kolmogorov-type argument gives convergence in $C^{-\kappa}$ for $d=2$. (iii) In $d=2$ on $\mathbb{T}^2$, the interaction $\int:\varphi^4:\mathrm{d}x$ lies in every $L^p(\mu_m)$, and Nelson's estimate gives $\mathrm{e}^{-\lambda\int:\varphi^4:}\in L^1(\mu_m)$ [Nel66, Sim74, GJ87]. This defines the Φ^4_2 measure on the torus. (iv) Other local operators: derivatives $\partial^\alpha\varphi$, and in $d=2$ the Wick exponentials $:\mathrm{e}^{\mathrm{i}\beta\varphi}:$ for $\beta^2<4\pi$ (in L^2, Exercise 3.6), relevant for the sine-Gordon model, and the real exponentials $:\mathrm{e}^{\alpha\varphi}:$ of the Høegh-Krohn model.

Beyond L^2: no random distribution. When $n\,(d-2)\geqslant d$ the divergence in Theorem 3.9 is not an artefact of the L^2 norm: no other topology, and no other choice of probability space, can rescue the limit. Albeverio and Liang [AL07] showed this for the Wick square in $d=4$, and remarked that the argument extends to $d\geqslant 5$: $:\varphi^2:(f)$ is well defined as a Hida distribution, i.e. as a generalised functional of white noise, but for $f>0$ the characteristic functions of its Fourier-cutoff approximations do not converge to a characteristic function (the natural candidate vanishes), so it is not a random variable. For $d\leqslant 3$, instead, it is a random variable, and an infinitely divisible one. The following argument is independent and covers all Wick powers.

Proposition 3.11. *Let $n\,(d-2)\geqslant d$ and $0\leqslant f\not\equiv 0$. Then the family $(:\varphi^n_\varepsilon:(f))_{\varepsilon\in(0,1]}$ is not tight. In particular there is no random distribution Ψ, on any probability space, such that $:\varphi^n_{\varepsilon_k}:\to\Psi$ in law in $\mathcal{S}'$ along some sequence $\varepsilon_k\to 0$.*

Proof. Let $X_\varepsilon=:\varphi^n_\varepsilon:(f)\in\mathcal{H}_n$ and $\sigma_\varepsilon^2=\mathbb{E}\,X_\varepsilon^2$, so that $\sigma_\varepsilon\to\infty$ by Theorem 3.9. The *Paley–Zygmund inequality* [PZ32] states that for $Z\geqslant 0$ with $\mathbb{E}\,Z^2<\infty$ and $\theta\in[0,1]$,

$$\mathbb{P}\,(Z>\theta\;\mathbb{E}\,Z)\geqslant(1-\theta)^2\,\frac{(\mathbb{E}\,Z)^2}{\mathbb{E}\,Z^2}.$$

(Proof: $\mathbb{E}\,Z\leqslant\theta\,\mathbb{E}\,Z+\mathbb{E}\,[Z\,1_{Z>\theta\mathbb{E}Z}]\leqslant\theta\,\mathbb{E}\,Z+(\mathbb{E}\,Z^2)^{1/2}\,\mathbb{P}\,(Z>\theta\,\mathbb{E}\,Z)^{1/2}$ by Cauchy–Schwarz.) Apply it to $Z=X_\varepsilon^2$ with $\theta=\frac14$. Hypercontractivity with $p=4$ gives $\mathbb{E}\,X_\varepsilon^4\leqslant 9^n\,\sigma_\varepsilon^4$, so

$$\mathbb{P}\left(|X_\varepsilon|>\tfrac12\sigma_\varepsilon\right)\geqslant\frac{9}{16}\,9^{-n}=:c_n>0.$$

Given $R>0$, for ε small enough $\frac12\sigma_\varepsilon>R$ and hence $\mathbb{P}\,(|X_\varepsilon|>R)\geqslant c_n$: the family is not tight. If $:\varphi^n_{\varepsilon_k}:\to\Psi$ in law in $\mathcal{S}'$, then $X_{\varepsilon_k}\to\Psi(f)$ in law, and a sequence converging in law is tight, a contradiction. □

Remark 3.12. (The law of a bounded chaos is controlled by its L^2 norm) Let X be in the bounded chaos $\bigoplus_{k\leqslant n}\mathcal{H}_k$, e.g. a polynomial of degree $\leqslant n$ in finitely many Gaussian variables. Three inequalities control its law in terms of $\sigma=\|X\|_{L^2}$ alone, with constants that depend only on n:

- *upper tails* (hypercontractivity [Jan97]): $\|X\|_{L^p}\leqslant(p-1)^{n/2}\,\sigma$, hence $\mathbb{P}\,(|X|>t\,\sigma)\leqslant\exp\,(-c\,t^{2/n})$ for $t\geqslant t_0$;
- *no escape to zero* (Paley–Zygmund): $\mathbb{P}\,(|X|>\frac12\sigma)\geqslant c_n>0$;
- *anti-concentration* (Carbery–Wright [CW01]): $\mathbb{P}\,(|X|\leqslant t\,\sigma)\leqslant C\,n\,t^{1/n}$ for $t>0$.

Consequently, for sequences in a fixed bounded chaos, tightness is equivalent to boundedness in L^2, and convergence in probability is equivalent to convergence in L^2 (and in every L^p): apply the Paley–Zygmund bound to X_k-X_l. So for Wick powers there is a sharp dichotomy: either convergence in L^2, or not even tightness. Nontrivial limits of the divergent objects require a change of normalisation: $\sigma_\varepsilon^{-1}:\varphi^n_\varepsilon:$ is bounded in L^2, and typically converges in law to a *Gaussian* white noise, which retains no information on the n-th power. The fourth moment theorem [NP05] characterises such Gaussian limits within a fixed chaos, and a central limit theorem of this kind for the lattice $:\varphi^4:$ in $d\geqslant 3$ is proved in [AZ96]. In particular, in $d\geqslant 4$ no Wick power with $n\geqslant 2$ exists as a random distribution, so under the GFF the interaction $\int:\varphi^4:$ cannot even be written down.

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3.4 Gaussian integration by parts and the Dyson–Schwinger equation

Let $\mathcal{F}C_b^\infty$ be the space of cylinder functions $F(\varphi) = g(\varphi(f_1), \dots, \varphi(f_n))$ with $g \in C_b^\infty(\mathbb{R}^n)$ and $f_i \in \mathcal{S}(\mathcal{M})$. For $h \in \mathcal{S}(\mathcal{M})$ define

$$D_h F(\varphi) := \frac{\mathrm{d}}{\mathrm{d}t} F(\varphi + t h)\Big|_{t=0} = \sum_i \partial_i g(\varphi(f_1), \dots, \varphi(f_n)) \langle f_i, h\rangle = \langle DF(\varphi), h\rangle, \tag{3.9}$$

with $DF = \sum_i \partial_i g(\dots)\, f_i$, formally $DF(x) = \delta F / \delta\varphi(x)$.

Lemma 3.13. (Gaussian integration by parts) *If $(Y, X_1, \dots, X_n)$ is a centred Gaussian vector and $g \in C_b^\infty(\mathbb{R}^n)$, then $\mathbb{E}[Y g(X)] = \sum_i \mathbb{E}[Y X_i]\ \mathbb{E}[\partial_i g(X)]$.*

Proof. For $g = \mathrm{e}^{\mathrm{i}\langle u, \cdot\rangle}$: $\mathbb{E}[Y \mathrm{e}^{\mathrm{i}\langle u, X\rangle}] = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta Y + \mathrm{i}\langle u, X\rangle}]\Big|_{\theta=0} = \mathrm{i}\sum_i u_i\, \mathbb{E}[Y X_i]\, \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle u, X\rangle}]$, which is the claim. The general case follows by Fourier inversion for $g \in \mathcal{S}(\mathbb{R}^n)$ and then by approximation. □

Proposition 3.14. (Dyson–Schwinger equation for the GFF) *For all $F \in \mathcal{F}C_b^\infty$ and $h \in \mathcal{S}(\mathcal{M})$,*

$$\mathbb{E}_{\mu_m}[D_h F] = \mathbb{E}_{\mu_m}[F\ \varphi(A h)], \qquad \textit{formally} \quad \mathbb{E}\left[\frac{\delta F}{\delta\varphi(x)}\right] = \mathbb{E}[F\ (m^2 - \Delta)\,\varphi(x)]. \tag{3.10}$$

Proof. Under μ_m, $(\varphi(A h), \varphi(f_1), \dots, \varphi(f_n))$ is centred Gaussian with $\mathbb{E}[\varphi(A h)\, \varphi(f_i)] = \langle A h, C f_i\rangle = \langle h, f_i\rangle$. Apply Lemma 3.13. □

Remark 3.15. (DS = Euclidean field equation) (i) Choosing $F = \varphi(f_2) \cdots \varphi(f_n)$ (allowed by approximation, since all moments are finite) gives the hierarchy for the Schwinger functions $S_n(x_1, \dots, x_n) = \mathbb{E}[\varphi(x_1) \cdots \varphi(x_n)]$:

$$(m^2 - \Delta_{x_1})\, S_n(x_1, \dots, x_n) = \sum_{j=2}^{n} \delta(x_1 - x_j)\ S_{n-2}(x_2, \dots, \hat{x}_j, \dots, x_n), \tag{3.11}$$

which together with $S_0 = 1$, $S_1 = 0$ determines all moments (Wick's theorem, Exercise 3.7). Compare with (1.5): in imaginary time, $\Box + m^2$ becomes $-\Delta + m^2$, and the contact terms on the right come from the CCR, i.e. from time ordering. (ii) For a formal Gibbs measure $\mu_V \propto \mathrm{e}^{-\int V(\varphi)}\mu_m$ the same computation gives

$$\mathbb{E}_{\mu_V}[D_h F] = \mathbb{E}_{\mu_V}[F\ ((m^2 - \Delta)\,\varphi + V'(\varphi))(h)], \tag{3.12}$$

the Euclidean counterpart of the nonlinear field equation (1.9) (here $V = \lambda\varphi^4$, so the λ of (1.9) equals 4λ here). Here $V'(\varphi)$ must be a well-defined local operator: for Φ^4_2, $V' = 4\lambda{:}\varphi^3{:}$, and the Euclidean field equation for $P(\varphi)_2$ measures was proved by Dimock and Glimm [DG74]; see also the Dirichlet form approach of Albeverio and Röckner [AR91]. For Φ^4_3 the renormalised (relativistic) field equation, and with it the DS equations, were first derived by Feldman and Rączka [FR77], using the constructions of Glimm, Jaffe, Feldman and Osterwalder. The renormalised cubic term is not a random variable under the Φ^4_3 measure [ALZ06]; a description of it as a random distribution, and a proof of (3.12) for all $\lambda > 0$ in infinite volume, are given in [GH21].

(iii) *Point splitting and the operator product expansion.* The nonlinear term in (3.12) is a product of distributions at the same point, and the natural way to define it is by point splitting: $\varphi^3(x)$ is the regular part of $\varphi(y_1)\,\varphi(y_2)\,\varphi(y_3)$ as $y_i \to x$. Under the GFF, the Wick decomposition of products (the multi-point version of Section 3.3, with $:\varphi(y_1)\,\varphi(y_2)\,\varphi(y_3):$ the product minus all its contractions) gives

$$\begin{aligned}\varphi(y_1)\,\varphi(y_2)\,\varphi(y_3) =:& \varphi(y_1)\,\varphi(y_2)\,\varphi(y_3):\\ &+ G(y_1-y_2)\,\varphi(y_3) + G(y_1-y_3)\,\varphi(y_2) + G(y_2-y_3)\,\varphi(y_1),\end{aligned}$$

and the first term tends to $:\varphi^3:(x)$. This is the beginning of an *operator product expansion* (OPE): a product of fields at nearby points is a sum of local operators at x with coefficients that are singular as the points merge, and the composite operator is what remains after the singular terms are subtracted. With a mollifier the same subtraction reads $:\varphi^3:= \lim_\varepsilon (\varphi_\varepsilon^3 - 3\,c_\varepsilon\,\varphi_\varepsilon)$, $c_\varepsilon = \mathbb{E}\,\varphi_\varepsilon(0)^2$. For Φ^4_2 in finite volume the measure is absolutely continuous with respect to the GFF and the Gaussian subtraction suffices. In an interacting theory, however, the OPE coefficients are those of the interacting measure. For Φ^4_3 the cube needs an additional subtraction $\delta_\varepsilon\,\varphi_\varepsilon$, with δ_ε of order $\lambda \log(1/\varepsilon)$, so that $4\lambda\,\delta_\varepsilon \sim \lambda^2 \log(1/\varepsilon)$ is the second-order (sunset) mass counterterm, i.e. the counterterm of the action reappears as an OPE coefficient of the interacting theory. Composite operators defined by the OPE are Wilson's short-distance expansion [Wil69] and Zimmermann's normal products [Zim73], in terms of which the field equation of perturbative φ^4 reads $(\Box + m^2)\,\varphi = -4\lambda N[\varphi^3]$. Constructively, local products and field equations for $P(\varphi)_2$ were obtained by point splitting by Schrader [Sch74]; the renormalised cube of [GH21] is approximated in the spirit of the OPE; convergence of the OPE in perturbation theory is proved in [HK12]. Lecture 3 describes the OPE at the level of stochastic observables, where the singular coefficients are explicit at every scale.

The DS equation is not only satisfied by the GFF; it characterises it.

Theorem 3.16. (DS characterises the GFF) *Let ν be a probability measure on $\mathcal{S}'(\mathcal{M})$ such that $\varphi(f) \in L^1(\nu)$ for all $f \in \mathcal{S}(\mathcal{M})$ and*

$$\mathbb{E}_\nu[D_h F] = \mathbb{E}_\nu[F\,\varphi(Ah)] \qquad \text{for all } F \in \mathcal{F}C_b^\infty, h \in \mathcal{S}(\mathcal{M}). \tag{3.13}$$

Then $\nu = \mu_m$.

Proof. Fix a real $f \in \mathcal{S}(\mathcal{M})$ and let $h = Cf \in \mathcal{S}(\mathcal{M})$, so that $Ah = f$. For $s \in \mathbb{R}$ apply (3.13) to the real and imaginary parts of $F_s(\varphi) = \mathrm{e}^{\mathrm{i}s\varphi(f)}$, which belong to $\mathcal{F}C_b^\infty$, using $D_h F_s = \mathrm{i}s\,\langle f, Cf\rangle\,F_s$. With $\chi(s) := \mathbb{E}_\nu[F_s]$ this gives

$$\mathrm{i}s\,\langle f, Cf\rangle\,\chi(s) = \mathbb{E}_\nu[\varphi(f)\,\mathrm{e}^{\mathrm{i}s\varphi(f)}] = -\mathrm{i}\chi'(s),$$

where we used $\varphi(f) \in L^1(\nu)$ to differentiate under the expectation. Hence $\chi'(s) = -s\,\langle f, Cf\rangle\chi(s)$ with $\chi(0) = 1$, so $\chi(s) = \mathrm{e}^{-\frac{s^2}{2}\langle f, Cf\rangle}$. Taking $s = 1$: $\int \mathrm{e}^{\mathrm{i}\varphi(f)}\mathrm{d}\nu = \mathrm{e}^{-\frac{1}{2}\langle f, Cf\rangle}$ for every f. Since $\sum_j u_j\varphi(f_j) = \varphi(\sum_j u_j f_j)$, this determines the characteristic function of every vector $(\varphi(f_1), \dots, \varphi(f_k))$, hence all finite-dimensional distributions, hence ν on the cylinder σ-algebra. So $\nu = \mu_m$. □

Remark 3.17. The integrability assumption is the minimal one needed for (3.13) to make sense, since $\varphi(Ah)$ ranges over all $\varphi(g)$, $g \in \mathcal{S}(\mathcal{M})$. Only test functions $F = \mathrm{e}^{\mathrm{i}s\varphi(f)}$ were used: DS turns into a first-order linear ODE for the characteristic function. For $m = 0$ on the torus no probability measure solves (3.13) (Exercise 3.9). For interacting theories, (3.12) is a singular nonlinear relation, and uniqueness of its solutions is a hard and largely open problem. In finite dimensions it is easy (Exercise 3.8). For a result in this direction for the exponential interaction in $d = 2$ see [DGT22].

3.5 Outlook

In Lecture 3 we obtain the GFF as the stationary marginal of the parabolic and elliptic stochastic quantisation equations, and we decompose it along a scale parameter: the field at resolution t is a Gaussian martingale, whose terminal value is the GFF. In Lecture 4 interactions are introduced through the Polchinski equation for the effective potential, and through its probabilistic form, a forward–backward stochastic differential equation for the field at increasing resolution. These tools can be used to construct Φ^4_2 and Φ^4_3. On $\mathbb{T}^2$ the Wick-ordered interaction $\lambda\int{:}\varphi^4{:}\,\mathrm{d}x$ suffices, and the measure satisfies the renormalised DS equation (3.12). In $d=3$ Wick ordering no longer suffices.

Exercises

Exercise 3.1. (Green functions) (a) Using $G_m=\int_0^\infty \mathrm{e}^{-m^2t}p_t\,\mathrm{d}t$, with p_t the heat kernel, show that $G_m(x)=(2\pi)^{-d/2}(m/|x|)^{d/2-1}K_{d/2-1}(m|x|)$, and derive the asymptotics stated after Definition 3.1. (b) Check $G_m(x)=\mathrm{e}^{-m|x|}/(2m)$ for $d=1$ and $G_m(x)=\mathrm{e}^{-m|x|}/(4\pi|x|)$ for $d=3$. (c) Show that on $\mathbb{T}^d_L$ the difference $G-G_m$ is smooth near the diagonal.

Exercise 3.2. (Euclidean Källén–Lehmann and the unitarity bound) (a) Show that

$$\int \mathrm{d}\Omega_M(\mathbf{p})\,\mathrm{e}^{-\omega_M(\mathbf{p})|\tau|+\mathrm{i}\mathbf{p}\cdot\mathbf{x}}=G_M(\tau,\mathbf{x}),$$

and derive (3.5) from (1.4). (b) Show that $G_M(x)$ is decreasing in M and, for $d\geqslant 2$, that $G_M(x)/G_m(x)\to 1$ as $x\to 0$. Prove the lower bound in Remark 3.7(i). (c) Let $d\geqslant 3$ and suppose $S_2^T(x)=|x|^{-2\Delta_\varphi}$ (a scale invariant theory; the scaling dimension Δ_φ is not the Laplacian). Using $G_M(x)=M^{d-2}G_1(Mx)$, show that $\rho(\mathrm{d}M^2)=c\,(M^2)^{\Delta_\varphi-d/2}\mathrm{d}M^2$ reproduces S_2^T for some $c>0$, for $\frac{d-2}{2}<\Delta_\varphi<\frac{d}{2}$, and that this is a locally finite measure iff $\Delta_\varphi>\frac{d-2}{2}$ (for $\Delta_\varphi\geqslant\frac{d}{2}$, S_2^T is not locally integrable). Using uniqueness of the Stieltjes transform, deduce the unitarity bound $\Delta_\varphi\geqslant\frac{d-2}{2}$, with equality iff $\rho\propto\delta_0$, the massless free field.

Exercise 3.3. (Regularity is sharp) (a) For $a_n\geqslant 0$ and (g_n) i.i.d. $N(0,1)$, show that $\mathbb{E}\,\mathrm{e}^{-\sum_n a_n g_n^2}=\prod_n(1+2a_n)^{-1/2}$. Deduce that $\sum_n a_n g_n^2<\infty$ a.s. iff $\sum_n a_n<\infty$. (b) Conclude that $\mu_m(H^{-s}(\mathbb{T}^d_L))=1$ iff $s>\frac{d}{2}-1$. In particular $\mu_m(L^2)=0$ for $d\geqslant 2$, and the Cameron–Martin space H^1 is μ_m-null in every dimension. (c*) Using hypercontractivity and a Kolmogorov argument, show that $\varphi\in C^{1-d/2-\kappa}$ a.s. for every $\kappa>0$.

Exercise 3.4. (Hermite calculus and change of renormalisation) (a) Show that $H_n(a;c)=\sum_{k\leqslant n/2}\frac{n!}{k!\,(n-2k)!}(-\frac{c}{2})^k a^{n-2k}$, $\partial_a H_n=nH_{n-1}$ and $aH_n=H_{n+1}+ncH_{n-1}$. (b) Show that $H_n(a+b;c+c')=\sum_k\binom{n}{k}H_k(a;c)\,H_{n-k}(b;c')$. (c) Show that $H_n(a;c')=\sum_k\binom{n}{2k}(2k-1)!!\,(c-c')^k H_{n-2k}(a;c)$. Interpret this as the effect of changing the mollifier on ${:}\varphi^n{:}$: Wick powers are defined up to lower order Wick powers with *finite* coefficients.

Exercise 3.5. (Wick powers: details) (a) Show that $c_\varepsilon=\frac{1}{2\pi}\log\frac{1}{\varepsilon}+O(1)$ for $d=2$ and $c_\varepsilon\asymp\varepsilon^{2-d}$ for $d\geqslant 3$. (b) In $d=3$, show that $\mathbb{E}[{:}\varphi^3_\varepsilon{:}(f)^2]\asymp\log\frac{1}{\varepsilon}$ for $0\leqslant f\not\equiv 0$: the cube is borderline divergent. (c) In $d=4$, show that $\mathbb{E}[{:}\varphi^2_\varepsilon{:}(f)^2]\asymp\log\frac{1}{\varepsilon}$ as well.

Exercise 3.6. (Wick exponentials) In $d=2$ let ${:}\mathrm{e}^{\mathrm{i}\beta\varphi_\varepsilon}{:}(x):=\mathrm{e}^{\beta^2c_\varepsilon/2}\mathrm{e}^{\mathrm{i}\beta\varphi_\varepsilon(x)}$. Show that

$$\mathbb{E}\,[{:}\mathrm{e}^{\mathrm{i}\beta\varphi_\varepsilon}{:}(x)\,\overline{{:}\mathrm{e}^{\mathrm{i}\beta\varphi_\delta}{:}(y)}]=\mathrm{e}^{\beta^2G_{\varepsilon,\delta}(x-y)},$$

and deduce that ${:}\mathrm{e}^{\mathrm{i}\beta\varphi_\varepsilon}{:}(f)$ converges in $L^2(\mu_m)$ for every f if $\beta^2<4\pi$, while its second moment diverges for $0\leqslant f\not\equiv 0$ if $\beta^2\geqslant 4\pi$. What changes for the real exponential ${:}\mathrm{e}^{\alpha\varphi}{:}$? (Gaussian multiplicative chaos exists in L^1 for $\alpha^2<8\pi$, see [BP25].)

Exercise 3.7. (Wick's theorem from DS) (a) Complete the proof of Lemma 3.13. (b) Derive (3.11) from (3.10), justifying the use of polynomial F. (c) Deduce Isserlis–Wick: $S_{2n+1}=0$ and $S_{2n}(x_1,\dots,x_{2n})=\sum_{\text{pairings }P}\prod_{\{i,j\}\in P}G(x_i-x_j)$.

Exercise 3.8. (DS in finite dimensions) Let $S(x)=\frac{1}{2}\langle x,Qx\rangle+V(x)$ on $\mathbb{R}^N$, with $Q>0$ and V smooth with e^{-V} bounded, and $\mu_V(\mathrm{d}x)=Z^{-1}\mathrm{e}^{-S(x)}\mathrm{d}x$. (a) Show that $\mathbb{E}_{\mu_V}[\partial_h F]=\mathbb{E}_{\mu_V}[F\langle\nabla S,h\rangle]$ for $F\in C^\infty_c$. (b) Let ν be a probability measure with $\nabla S\in L^1(\nu)$ satisfying the same identity. Show that $\nu=\mu_V$. *Hint:* the identity says that the distribution $\mathrm{e}^S\nu$ has zero gradient. (c) Why does this argument fail in infinite dimensions? What replaces it in Theorem 3.16?

Exercise 3.9. (The massless case) (a) Show that for $m=0$ on $\mathbb{T}^d$ no probability measure on $\mathcal{S}'(\mathbb{T}^d)$ satisfies (3.13) with $A=-\Delta$. *Hint:* take $h=1$. (b) Show that the massless GFF exists on the quotient $\mathcal{S}'(\mathbb{T}^d)/\mathbb{R}$ (test functions with $\int f=0$), and characterise it by DS there. (c) On $\mathbb{R}^2$, show that $\langle f,(-\Delta)^{-1}f\rangle<\infty$ iff $\hat f(0)=0$: the infrared problem of the two-dimensional massless field.

Exercise 3.10. (Lattice approximation) Let μ_ε be the centred Gaussian measure on $\mathbb{R}^{(\varepsilon\mathbb{Z}/L\mathbb{Z})^d}$ with covariance $(m^2-\Delta_\varepsilon)^{-1}$ (with respect to $\varepsilon^d\sum_u$), embedded in $\mathcal{S}'(\mathbb{T}^d_L)$ via $\varphi(f)=\varepsilon^d\sum_u\varphi_u f(u)$. Show that $\mu_\varepsilon\to\mu_m$ weakly. Relate μ_ε to (2.12) with $V=0$ (for $d=2$; in general d see Exercise 2.3).

Exercise 3.11. (Computer experiments) (a) Sample the lattice GFF of Exercise 3.10 on $\mathbb{T}^2$ with $\varepsilon=1/N$ (i.e. on $(\mathbb{Z}/N\mathbb{Z})^2$ with mass m/N in lattice units) by FFT, for $N=64,256,1024$, and check that the empirical variance of φ_u grows like $\frac{1}{2\pi}\log N$. (b) Implement an explicit Euler scheme for the lattice version of (4.1) in Lecture 3 with $V=0$ (mind the stability constraint $\Delta t\lesssim\varepsilon^2$), and check that the variance of the k-th Fourier mode relaxes to λ_k^{-1} at rate $2\lambda_k$, with λ_k the eigenvalues of $m^2-\Delta_\varepsilon$. (c) Add $V'(\varphi)=4\lambda(\varphi^3-3c_\varepsilon\varphi)$, i.e. $V=\lambda{:}\varphi^4{:}$, and observe that the solutions have a nontrivial limit as $N\to\infty$, while without the counterterm $-12\lambda c_\varepsilon\varphi$ they converge to zero [HRW12].

Exercise 3.12. (Two masses) In the setting of the example after Remark 3.7: (a) Show that μ_ρ satisfies (OS0)–(OS4), and that the theory described there has Schwinger functions equal to the moments of μ_ρ. (b) Show that $[\varphi(0,\mathbf{x}),\partial_t\varphi(0,\mathbf{y})]=\mathrm{i}(c_1^2+c_2^2)\,\delta(\mathbf{x}-\mathbf{y})$, and that $\varphi(t,\cdot)$ is not a linear function of $\varphi(0,\cdot)$ and $\partial_t\varphi(0,\cdot)$. (c) Verify $\mathbb{E}[D_{Rh}F]=\mathbb{E}[F\,\varphi(Ph)]$. (d) For a single spatial Fourier mode $\mathbf{p}$ (on a torus, say, with real test functions $f(\tau,\mathbf{x})=g(\tau)\cos(\mathbf{p}\cdot\mathbf{x})$), with $\omega_i^2=|\mathbf{p}|^2+m_i^2$, show that the time kernel of P^{-1} is $(\mathrm{e}^{-\omega_1|\tau|}/2\omega_1-\mathrm{e}^{-\omega_2|\tau|}/2\omega_2)/(m_2^2-m_1^2)$, and find f supported in $\{\tau>0\}$ with $\langle\Theta f,P^{-1}f\rangle<0$. *Hint:* choose g with $\int_0^\infty\mathrm{e}^{-\omega_1\tau}g(\tau)\,\mathrm{d}\tau=0$. (e) Let p be a real polynomial of degree $\geqslant 2$, positive on $[0,\infty)$. Show that $1/p(|k|^2)$ is not of the form $\int\rho(\mathrm{d}M^2)\,(|k|^2+M^2)^{-1}$ with $\rho\geqslant 0$. *Hint:* $s\int\rho(\mathrm{d}M^2)\,(s+M^2)^{-1}$ increases to $\rho([0,\infty))$ as $s\to\infty$ (monotone convergence), while $s/p(s)\to 0$.

References and further reading

Guide. *Gaussian fields and Wick calculus:* [GV64, Bog98, Jan97, She07, LSSW16, BP25]. *Nonexistence of Wick powers beyond L^2 and chaos inequalities:* [AL07, PZ32, CW01, NP05, AZ96]. *Euclidean free field and Markov property:* [Sym66, Sym69, Nel73a, Nel73b, GRS75, GJ87, Sim74, Gue06]. *Path and loop representations:* [Sym69, BFS82, Dyn84, LW04, LJ10, LJ11, Lup16, Wer16, QW19, WP21]. *Stochastic quantisation (Lecture 3):* reviews [DH87, DH88, Gub25]; parabolic [PW81, JLM85, AR91, DPD03, DPZ14]; singular SPDEs [Hai14, GIP15, CC18, MW17, HRW12]; construction of Φ^4_3 via dynamics and variational methods [GH21, BG20]; Euclidean field equations and DS equations for $P(\varphi)_2$ [DG74, AR91], for Φ^4_3 [FR77, ALZ06, GH21], and for the exponential model in $d=2$ [DGT22]; operator product expansion and composite operators [Wil69, Zim73, Sch74, HK12]; elliptic stochastic quantisation and dimensional reduction [PS79, KLP84, ADG20, ADG21, GH19].

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