A Constructive Introduction to Quantum Field Theory

Lecture 3: Stochastic quantisation and stochastic observables

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SISSA, October 2026

Abstract

We first show that the Gaussian free field is the stationary one-time (resp. one-point) marginal of the parabolic (resp. elliptic) stochastic quantisation equation. We then decompose it along a scale parameter and take as primary objects the *stochastic observables*: martingales in the filtration of finite-resolution measurements. As examples we study Wick powers as observables in every dimension, and their products (operator product expansion). Interactions are introduced in Lecture 4. Exercises are collected at the end of each section. These notes follow the slides of Lecture 3.

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Introduction

We start from stochastic quantisation (Section 4): the GFF is the stationary marginal of a Langevin dynamics in an auxiliary time, and the one-point marginal of an elliptic equation in two additional dimensions. This is a first way to describe a Euclidean field by an equation for the field itself.

We then change the point of view (Section 5): instead of a single random distribution, we look at the field observed at every finite resolution, and take as primary objects the stochastic observables of [BCG26]: martingales in the filtration of finite-resolution measurements.

Sections, equations and exercises keep the numbering of the whole course: references to Sections 1–3 are to the notes of Lectures 1–2.

Structure. Section 4 treats the stochastic quantisation of the GFF. Section 5 introduces covariance profiles, the field at resolution t and stochastic observables, and Section 6 studies the observables of the GFF and their products. Interactions, the Polchinski equation, the forward–backward equations and the Wetterich equation are the subject of Lecture 4.

Conventions. As in Lectures 1–2: $\mathcal{M}=\mathbb{T}_L^d$ or $\mathbb{R}^d$, $A=m^2-\Delta$, $C=A^{-1}$ with kernel G, μ_m the GFF, $\langle f,g\rangle=\int fg$ for real functions. For a functional F on smooth fields, $DF(\varphi)\in\mathcal{S}'(\mathcal{M})$ and $D^2F(\varphi)$ are its first and second derivatives, $D_hF=\langle DF,h\rangle$. $H_n(a;c)$ are the Hermite polynomials of variance c, and μ^Q is the centred Gaussian measure with covariance Q.

4 Stochastic quantisation

Stochastic quantisation describes a Euclidean field as the marginal of a stochastic dynamics. For the GFF the dynamics are linear and everything can be computed explicitly; we give two versions, parabolic and elliptic.

4.1 The GFF as stationary marginal of parabolic stochastic quantisation

Langevin dynamics. If $S:\mathbb{R}^N\to\mathbb{R}$ is smooth with $\mathrm{e}^{-S}\in L^1$, the diffusion $\mathrm{d}X_t=-\nabla S(X_t)\mathrm{d}t+\sqrt{2}\,\mathrm{d}B_t$ has generator $\mathcal{L}=\Delta-\nabla S\cdot\nabla$, and $\mathcal{L}^*\mathrm{e}^{-S}=\nabla\cdot(\nabla\mathrm{e}^{-S}+\mathrm{e}^{-S}\nabla S)=0$: the Gibbs measure $\mathrm{e}^{-S(x)}\mathrm{d}x/Z$ is invariant. Parisi and Wu [PW81] proposed to apply this to the Euclidean action, introducing a fictitious "stochastic time" t:

$$\partial_t\Phi=-(m^2-\Delta)\,\Phi-V'(\Phi)+\sqrt{2}\,\xi\qquad\text{on }\mathbb{R}_+\times\mathcal{M},\tag{4.1}$$

with ξ space-time white noise. This is (*parabolic*) *stochastic quantisation*. For $V=0$ it is the stochastic heat equation, an infinite-dimensional Ornstein–Uhlenbeck process.

History. Parisi and Wu [PW81] introduced stochastic quantisation to do perturbation theory for gauge theories without gauge fixing. The physics literature of the 1980s is reviewed by Damgaard and Hüffel [DH87], whose reprint volume [DH88] collects the original papers. Jona-Lasinio and Mitter [JLM85] gave the first mathematical construction of the dynamics for $P(\varphi)_2$ on the torus, where the Wick-ordered interaction makes (4.1) a singular SPDE. They used Girsanov's theorem for a variant of (4.1) in which the drift is smoothed by $(1-\Delta)^{-\varepsilon}$ and the noise has covariance $(1-\Delta)^{-\varepsilon}$ (a slightly coloured noise), which still has the $P(\varphi)_2$ measure as invariant measure. A recent review of the mathematical theory is [Gub25].

Space-time white noise on $\mathbb{R}\times\mathcal{M}$ is a centred Gaussian family $(\xi(u))_{u\in L^2(\mathbb{R}\times\mathcal{M})}$ with $\mathbb{E}[\xi(u)\,\xi(v)]=\langle u,v\rangle$, formally $\mathbb{E}[\xi(t,x)\,\xi(s,y)]=\delta(t-s)\,\delta(x-y)$. Let $W_t(h)=\xi(1_{[0,t]}\otimes h)$, $t\geqslant 0$, a cylindrical Brownian motion. For a (weak) solution Φ of (4.1) with $V=0$ and $F=g(\varphi(f_1),\dots,\varphi(f_n))\in\mathcal{FC}_b^\infty$, Itô's formula shows that $F(\Phi_t)-\int_0^t\mathcal{L}F(\Phi_r)\mathrm{d}r$ is a martingale, with

$$\mathcal{L}F(\varphi)=\sum_{i,j}\langle f_i,f_j\rangle\,\partial_{ij}g(\dots)-\sum_i\varphi(Af_i)\,\partial_i g(\dots).\tag{4.2}$$

Proposition 4.1. (DS ⇒ invariance) *For $F,G\in\mathcal{FC}_b^\infty$: $\mathbb{E}_{\mu_m}[G\,\mathcal{L}F]=-\mathbb{E}_{\mu_m}[\langle DG,DF\rangle_{L^2(\mathcal{M})}]$. In particular $\mathbb{E}_{\mu_m}[\mathcal{L}F]=0$.*

Proof. We may assume $F=g(\varphi(f_1),\dots)$ and $G=k(\varphi(f_1),\dots)$ with the same f_i. Apply (3.10) of Lecture 2 to the cylinder function $k\,\partial_i g$ with $h=f_i$ and sum over i: $\sum_i\mathbb{E}[k\,\partial_i g\,\varphi(Af_i)]=\sum_{i,j}\langle f_i,f_j\rangle\,\mathbb{E}[\partial_j k\,\partial_i g+k\,\partial_{ij}g]$. Insert this into $\mathbb{E}[G\,\mathcal{L}F]$. □

So the static characterisation (DS) and the dynamic one (invariance under the Langevin flow) are two faces of the same integration by parts formula.

Theorem 4.2. (Parabolic stochastic quantisation of the GFF) *Let ξ be space-time white noise on $\mathbb{R}\times\mathcal{M}$. For $t\in\mathbb{R}$ and $f\in\mathcal{S}(\mathcal{M})$ define*

$$\Phi_t(f) := \sqrt{2}\ \xi(K_{t,f}), \qquad K_{t,f}(r,x) := 1_{\{r<t\}}(\mathrm{e}^{-(t-r)A}f)(x). \tag{4.3}$$

a) Φ is a centred Gaussian process with $\mathbb{E}[\Phi_t(f)\,\Phi_s(g)] = \langle f, \mathrm{e}^{-|t-s|A}Cg\rangle$. In particular $(\Phi_t)_{t\in\mathbb{R}}$ is stationary and, for every t, Φ_t has a version with values in $\mathcal{S}'(\mathcal{M})$ whose law is μ_m.

b) Φ is a mild and weak solution of (4.1) with $V=0$: for $s<t$ and $h\in\mathcal{S}(\mathcal{M})$,

$$\Phi_t(h) = \Phi_s(\mathrm{e}^{-(t-s)A}h) + \sqrt{2}\ \xi(1_{(s,t)}(r)\,\mathrm{e}^{-(t-r)A}h),$$
$$\Phi_t(h) = \Phi_s(h) - \int_s^t \Phi_r(Ah)\,\mathrm{d}r + \sqrt{2}\ \xi(1_{(s,t)}\otimes h).$$

c) Let Ψ_0 be an $\mathcal{S}'(\mathcal{M})$-valued random variable independent of ξ, and let Ψ be the mild solution started from Ψ_0,

$$\Psi_t(f) = \Psi_0(\mathrm{e}^{-tA}f) + \sqrt{2}\ \xi(1_{(0,t)}(r)\,\mathrm{e}^{-(t-r)A}f), \qquad t\geqslant 0.$$

Then $\mathbb{E}[\mathrm{e}^{\mathrm{i}\Psi_t(f)}] \to \mathrm{e}^{-\frac{1}{2}\langle f,Cf\rangle}$ as $t\to\infty$ for every $f\in\mathcal{S}(\mathcal{M})$. Moreover μ_m is invariant ($\Psi_0\sim\mu_m$ implies $\Psi_t\sim\mu_m$ for all t), and it is the unique invariant probability measure of the dynamics.

Proof. (a) Since A is self-adjoint and commutes with e^{-uA},

$$\langle K_{t,f}, K_{s,g}\rangle = \int_{-\infty}^{t\wedge s} \langle f, \mathrm{e}^{-(t+s-2r)A}g\rangle \mathrm{d}r = \langle f, \mathrm{e}^{-|t-s|A}(2A)^{-1}g\rangle.$$

In particular $K_{t,f}\in L^2(\mathbb{R}\times\mathcal{M})$, and multiplying by 2 gives the covariance. Stationarity follows. For fixed t, $f\mapsto\Phi_t(f)$ is linear and $\mathbb{E}[\Phi_t(f)^2] = \langle f, Cf\rangle \leqslant m^{-2}\|f\|_{L^2}^2$, so it is continuous from $\mathcal{S}(\mathcal{M})$ to $L^2(\mathbb{P})$. By the regularisation theorem for random linear functionals on nuclear spaces (Itô–Nawata, Mitoma) it has an $\mathcal{S}'(\mathcal{M})$-valued version, which is centred Gaussian with covariance C, i.e. has law μ_m. On the torus one can see this directly: the coordinates $\Phi_t(e_n)$ are independent stationary OU processes $\mathrm{d}X = -\lambda_n X\mathrm{d}t + \sqrt{2}\mathrm{d}B$ with variance λ_n^{-1}, and (3.3) of Lecture 2 applies.

(b) The mild formula is the identity $K_{t,h} = K_{s,\mathrm{e}^{-(t-s)A}h} + 1_{(s,t)}(r)\mathrm{e}^{-(t-r)A}h$ together with the linearity of ξ. The weak formulation follows from the mild one by a stochastic Fubini argument (Exercise 4.3).

(c) Since Ψ_0 is independent of ξ,

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\Psi_t(f)}] = \mathbb{E}\big[\mathrm{e}^{\mathrm{i}\Psi_0(\mathrm{e}^{-tA}f)}\big]\,\mathrm{e}^{-\frac{1}{2}\langle f,(1-\mathrm{e}^{-2tA})Cf\rangle}.$$

For $m>0$, $\mathrm{e}^{-tA}f\to 0$ in $\mathcal{S}(\mathcal{M})$ as $t\to\infty$, so $\Psi_0(\mathrm{e}^{-tA}f)\to 0$ a.s. because $\Psi_0\in\mathcal{S}'(\mathcal{M})$. By dominated convergence the first factor tends to 1, while the second tends to $\mathrm{e}^{-\frac{1}{2}\langle f,Cf\rangle}$. If $\Psi_0\sim\mu_m$, the first factor equals $\mathrm{e}^{-\frac{1}{2}\langle f,\mathrm{e}^{-2tA}Cf\rangle}$ and the product is exactly $\mathrm{e}^{-\frac{1}{2}\langle f,Cf\rangle}$, which proves invariance. If ν is invariant, start from $\Psi_0\sim\nu$: then $\mathrm{Law}(\Psi_t)=\nu$ for all t, so the characteristic functional of ν equals that of μ_m, and $\nu=\mu_m$. □

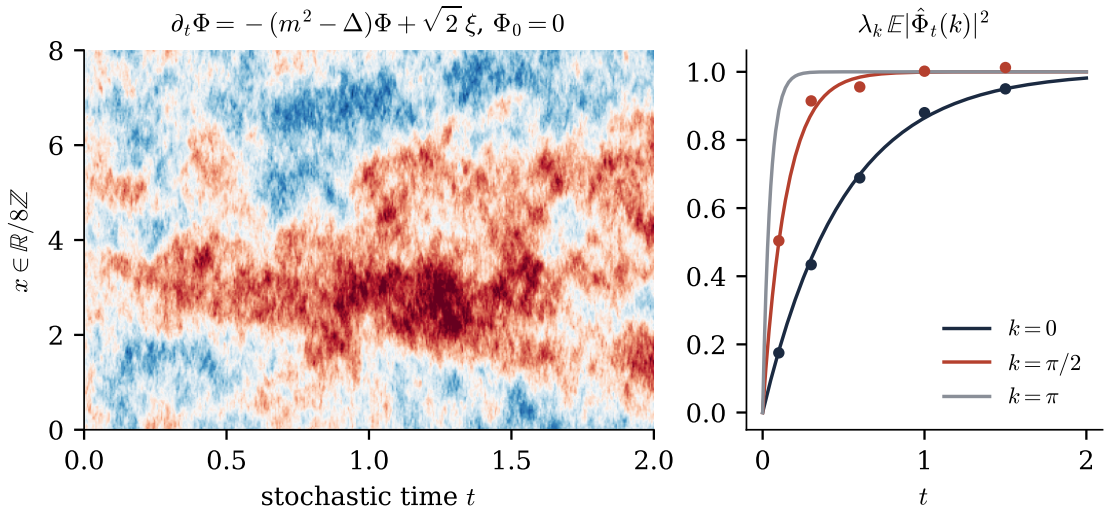


Figure 4.1. Parabolic stochastic quantisation of the GFF on a circle of length 8, $m=1$, started from $\Phi_0=0$. Left: a sample of the solution. Right: each Fourier mode is an Ornstein–Uhlenbeck process, whose variance relaxes to λ_k^{-1} at rate $2\lambda_k$ (Theorem 4.2(c)). Dots: Monte Carlo estimates with 4000 samples.

Remark 4.3. As a Gaussian field on $\mathbb{R}\times\mathcal{M}$, the stationary solution has covariance $2\,(\partial_t+A)^{-1}(-\partial_t+A)^{-1}=2\,(-\partial_t^2+A^2)^{-1}$, and its equal-time slice has covariance $2\int_{\mathbb{R}}(\theta^2+A^2)^{-1}\frac{\mathrm{d}\theta}{2\pi}=A^{-1}$. The stochastic time t is unrelated to the Euclidean time τ. For interacting theories, (4.1) is a singular SPDE. It was solved by Da Prato and Debussche for Φ^4_2 [DPD03] (see also [JLM85, AR91]), and for Φ^4_3 using regularity structures and paracontrolled calculus [Hai14, GIP15, CC18]. Global-in-time bounds [MW17, GH19] then allow constructing the Euclidean Φ^4_3 theory as the invariant measure of the dynamics [GH21].

4.2 The GFF as one-point marginal of elliptic stochastic quantisation

Parisi and Sourlas [PS79] discovered that the elliptic SPDE in $d+2$ dimensions

$$(m^2-\Delta_{\mathbb{R}^2\times\mathcal{M}})\,\Psi+V'(\Psi)=\zeta,\qquad \zeta \text{ white noise on } \mathbb{R}^2\times\mathcal{M}, \tag{4.4}$$

is related to the d-dimensional Gibbs measure: formally, the law of $\Psi(z,\cdot)$, $z\in\mathbb{R}^2$, is $\propto\exp\left(-4\pi\left[\frac{1}{2}\langle\varphi,A\,\varphi\rangle+\int V(\varphi)\right]\right)\mathcal{D}\,\varphi$ (*dimensional reduction*). The underlying mechanism is a hidden supersymmetry [PS79, KLP84]. In [KLP84] the dimensional reduction of the supersymmetric representation is proved rigorously, but not the supersymmetric representation of the solution of the SPDE; the full nonlinear statement was proved for $d=0$ in [ADG20] and for some two-dimensional models (polynomial interactions on $\mathbb{T}^2$, exponential interactions on $\mathbb{R}^2$) in [ADG21]; elliptic Φ^4 equations in 4 and 5 dimensions are solved in [GH19]. For $V=0$ the statement is elementary, and it explains the factor 4π. We normalise the noise so that the slice is exactly μ_m.

Theorem 4.4. (Elliptic stochastic quantisation of the GFF) *Let ζ be white noise on $\mathbb{R}^2\times\mathcal{M}$ (an $\mathcal{S}'(\mathbb{R}^2\times\mathcal{M})$-valued Gaussian variable with $\mathbb{E}[\zeta(u)\,\zeta(v)]=\langle u,v\rangle$) and $\mathsf{L}:=m^2-\Delta_{\mathbb{R}^2\times\mathcal{M}}=-\Delta_{\mathbb{R}^2}+A$.*

a) The equation $\mathsf{L}\Psi=\sqrt{4\pi}\,\zeta$ has a unique $\mathcal{S}'(\mathbb{R}^2\times\mathcal{M})$-valued solution, $\Psi(u)=\sqrt{4\pi}\,\zeta(\mathsf{L}^{-1}u)$.

b) For $z\in\mathbb{R}^2$ and $f\in\mathcal{S}(\mathcal{M})$ the function $K_{z,f}:=\mathsf{L}^{-1}(\delta_z\otimes f)$ lies in $L^2(\mathbb{R}^2\times\mathcal{M})$, and $z\mapsto K_{z,f}$ is continuous. The Wiener integrals $\Psi(z,f):=\sqrt{4\pi}\,\zeta(K_{z,f})$ satisfy $\Psi(v\otimes f)=\int_{\mathbb{R}^2}v(z)\,\Psi(z,f)\,\mathrm{d}z$ a.s. for $v\in\mathcal{S}(\mathbb{R}^2)$.

c) *For all $z\in\mathbb{R}^2$ and $f,g\in\mathcal{S}(\mathcal{M})$, $\mathbb{E}[\Psi(z,f)\,\Psi(z,g)]=\langle f,Cg\rangle$. Hence $\Psi(z,\cdot)$ has an $\mathcal{S}'(\mathcal{M})$-valued version with law μ_m, and the field $(\Psi(z,\cdot))_{z\in\mathbb{R}^2}$ is stationary and isotropic in z.*

Proof. (a) L is a continuous bijection of $\mathcal{S}(\mathbb{R}^2\times\mathcal{M})$, so $\Psi(u):=\sqrt{4\pi}\,\zeta(\mathsf{L}^{-1}u)$ defines an $\mathcal{S}'$-valued variable with $\Psi(\mathsf{L}u)=\sqrt{4\pi}\,\zeta(u)$. If $D\in\mathcal{S}'$ satisfies $D(\mathsf{L}u)=0$ for all $u\in\mathcal{S}$, then $D=0$ since L is onto: the solution is unique.

(b), (c) Take the Fourier transform in z only, with measure $\mathrm{d}q/(2\pi)^2$. The transform of $K_{z,f}$ is the $L^2(\mathcal{M})$-valued function $q\mapsto \mathrm{e}^{-\mathrm{i}q\cdot z}(|q|^2+A)^{-1}f$. By Plancherel, the spectral theorem for A and Tonelli,

$$\langle K_{z,f},K_{z,g}\rangle=\int_{\mathbb{R}^2}\langle f,(|q|^2+A)^{-2}g\rangle\frac{\mathrm{d}q}{(2\pi)^2}=\left\langle f,\left[\int_{\mathbb{R}^2}\frac{\mathrm{d}q}{(2\pi)^2}\,(|q|^2+A)^{-2}\right]g\right\rangle=\frac{1}{4\pi}\langle f,A^{-1}g\rangle,$$

using the scalar identity

$$\int_{\mathbb{R}^2}\frac{\mathrm{d}q}{(2\pi)^2}\frac{1}{(|q|^2+a)^2}=\frac{1}{2\pi}\int_0^\infty\frac{r\,\mathrm{d}r}{(r^2+a)^2}=\frac{1}{4\pi a},\qquad a>0. \tag{4.5}$$

In particular $K_{z,f}\in L^2$, and its continuity in z follows by dominated convergence. Since the transform of $\int v(z)\,K_{z,f}\mathrm{d}z$ is $\hat v(q)\,(|q|^2+A)^{-1}f$, the Bochner integral $\int v(z)\,K_{z,f}\mathrm{d}z$ equals $\mathsf{L}^{-1}(v\otimes f)$. The Wiener integral ζ is an isometry $L^2\to L^2(\mathbb{P})$ and commutes with Bochner integrals, which gives (b). Multiplying by 4π gives the covariance in (c). The $\mathcal{S}'(\mathcal{M})$-valued version is obtained as in Theorem 4.2. Stationarity and isotropy hold because $\mathbb{E}[\Psi(z,f)\,\Psi(z',g)]=4\pi\int\mathrm{e}^{\mathrm{i}q\cdot(z-z')}\langle f,(|q|^2+A)^{-2}g\rangle\frac{\mathrm{d}q}{(2\pi)^2}$ depends only on $|z-z'|$. □

Remark 4.5. (Slices of Gaussian fields) Both stochastic quantisations exhibit the GFF as a *slice* of a Gaussian field in higher dimension. For the field on $\mathbb{R}^k\times\mathcal{M}$ with covariance $(-\Delta_{\mathbb{R}^k}+A)^{-p}$, the slice $\{0\}\times\mathcal{M}$ has covariance $c_{k,p}A^{k/2-p}$ with $c_{k,p}=\Gamma(p-\frac{k}{2})/((4\pi)^{k/2}\Gamma(p))$ (Exercise 4.4). The GFF arises iff $p=1+k/2$: $(k,p)=(2,2)$ is elliptic SQ, while $(k,p)=(1,1)$ gives instead the time-zero field $(2\omega)^{-1}$ of Proposition 3.4 of Lecture 2. When $\mathcal{M}=\mathbb{R}^d$, the elliptic field Ψ is Euclidean invariant in $\mathbb{R}^{d+2}$, and its restriction to *every* codimension-two affine subspace is a GFF.

Exercises

Exercise 4.1. (Infinitesimal invariance characterises μ_m) On $\mathbb{T}^d_L$, let ν be a probability measure on $\mathcal{S}'(\mathbb{T}^d_L)$ with $\varphi(f)\in L^1(\nu)$ and $\mathbb{E}_\nu[\mathcal{L}F]=0$ for all $F\in\mathcal{F}C_b^\infty$. Show that $\nu=\mu_m$. *Hint:* for $f=\sum_{n\leqslant N}a_ne_n$ and $\psi(a)=\mathbb{E}_\nu\mathrm{e}^{\mathrm{i}\varphi(f)}$, show $\sum_n\lambda_na_n\partial_{a_n}\psi=-|a|^2\psi$, and differentiate $s\mapsto\psi(\mathrm{e}^{sD}a)\exp(\frac12\sum_n a_n^2\mathrm{e}^{2\lambda_ns}/\lambda_n)$, where $D=\mathrm{diag}(\lambda_n)$.

Exercise 4.2. (Reversibility) Use Proposition 4.1 to show that $\mathcal{L}$ is symmetric in $L^2(\mu_m)$. Deduce, and check directly from the covariance in Theorem 4.2(a), that the stationary solution is time reversible: $(\Phi_{-t})_{t\in\mathbb{R}}$ and $(\Phi_t)_{t\in\mathbb{R}}$ have the same law.

Exercise 4.3. (Mild implies weak) Prove the weak formulation in Theorem 4.2(b). Show that, as a random distribution on $\mathbb{R}\times\mathcal{M}$, the stationary solution has covariance operator $2(-\partial_t^2+A^2)^{-1}$.

Exercise 4.4. (Slices of Gaussian fields) Let $k\geqslant1$, $p>k/2$, and let Ψ be the centred Gaussian field on $\mathbb{R}^k\times\mathcal{M}$ with covariance $(-\Delta_{\mathbb{R}^k}+A)^{-p}$. (a) Show that $\Psi(0,\cdot)$ is well defined with covariance $c_{k,p}A^{k/2-p}$, $c_{k,p}=\Gamma(p-\frac{k}{2})/((4\pi)^{k/2}\Gamma(p))$. (b) For $(k,p)=(1,1)$ and $\mathcal{M}=N$, show that the time-zero field of the GFF on $\mathbb{R}\times N$ is the ground state measure $\Omega_0(\varphi)^2\,\mathcal{D}\varphi$ of the free Hamiltonian, i.e. Gaussian with covariance $(2\omega)^{-1}$ (compare Exercise 2.2). (c) Show that the restriction of the elliptic field of Theorem 4.4 to a codimension-*one* slice $\mathbb{R}\times\mathcal{M}$ has covariance proportional to $(-\partial^2+A)^{-3/2}$, so it is not a GFF. (d) Show that $\mathbb{E}[\Psi(z,f)\,\Psi(0,g)]=\langle f,|z|A^{-1/2}K_1(|z|A^{1/2})\,g\rangle$ in Theorem 4.4, and recover $\langle f,Cg\rangle$ as $z\to0$.

5 Setting: covariance profiles and observables

In Lectures 1–2 a Euclidean field was a probability measure on $\mathcal{S}'(M)$, i.e. a random distribution φ, and local operators were functions of φ. This viewpoint hides the scale-by-scale organisation of the Wilson–Kadanoff picture [WK74], and forces us to work with singular objects from the start. Already for the GFF, the Wick power $:\varphi^n:$ is not a random distribution when $n(d-2)\geqslant d$, since its regularisations are not even tight (Proposition 3.11 of Lecture 2).

From now on we index everything by a *resolution* $t\in[0,\infty)$. The field observed at resolution t is a smooth function φ_t, and the information available at resolution t is the σ-algebra $\mathcal{F}_t$ generated by $(\varphi_s)_{s\leqslant t}$. A physical quantity is then not a single random variable but a *consistent family* of finite-resolution estimates of it, i.e. a martingale. This is the notion of stochastic observable of [BCG26], on which this lecture is based; see also [Pol84, BK87, Sal99, BBD24] for the renormalisation group side and [BG20, BG23] for the variational side.

In Section 4 the variable t was the auxiliary time of a stochastic dynamics; from now on t is a resolution.

5.1 Covariance profiles

Definition 5.1. *A* covariance profile *is a family $(Q_t)_{t\in[0,\infty)}$ of bounded, symmetric, translation invariant operators on $L^2(M)$ with $Q_0=0$, $t\mapsto Q_t$ smooth, $\dot Q_t:=\partial_t Q_t\geqslant 0$, $Q_t\to C$ as $t\to\infty$, and such that $\dot Q_t$ and Q_t have smooth kernels for every $t<\infty$.*

Our standard example is the *heat profile*

$$Q_t:=C\,\mathrm{e}^{-A/t^2},\qquad \dot Q_t=\frac{2}{t^3}\,\mathrm{e}^{-A/t^2},\tag{5.1}$$

which smooths the Green function with the heat kernel at time t^{-2}. On $\mathbb{R}^d$ the kernels are explicit,

$$\dot Q_t(x-y)=\frac{2\,t^{d-3}\mathrm{e}^{-m^2/t^2}}{(4\pi)^{d/2}}\,\mathrm{e}^{-t^2|x-y|^2/4},\tag{5.2}$$

so $\dot Q_t$ is concentrated on distances $\lesssim 1/t$, and in Fourier space Q_t retains the modes $|k|\lesssim t$: t is a *momentum scale*, or resolution. Other profiles are useful, even when they violate some smoothness requirement of Definition 5.1. A sharp Fourier cutoff is $\hat Q_t(k)=1_{|k|\leqslant t}\,(|k|^2+m^2)^{-1}$, which is not smooth in t. In a *finite range decomposition*, $\dot Q_t(x,y)=0$ for $|x-y|\geqslant 1/t$ (Remark 6.4). We write

$$c_t:=Q_t(x,x),\qquad Q_{T,t}:=Q_T-Q_t\quad(t\leqslant T).$$

For $d\geqslant 2$, $c_t\to\infty$ as $t\to\infty$ (Exercise 5.2).

Let $(W_t)_{t\geqslant 0}$ be a cylindrical Brownian motion on $L^2(M)$, $\mathbb{E}[W_t(x)\,W_s(y)]=(t\wedge s)\,\delta(x-y)$, and

$$X_t:=\int_0^t \dot Q_s^{1/2}\,\mathrm{d}W_s.\tag{5.3}$$

Proposition 5.2. *For every $t<\infty$, X_t is a smooth Gaussian field with law μ^{Q_t}, and $(X_t(x))_{t\geqslant 0}$ is a continuous martingale with $\mathrm{d}\langle X(x),X(y)\rangle_t=\dot Q_t(x,y)\,\mathrm{d}t$. As $t\to\infty$, $X_t\to X_\infty$ in H^{-s}_{loc} for $s>\frac{d}{2}-1$, and $\mathrm{Law}(X_\infty)=\mu_m$. The increments X_T-X_t are independent of $\mathcal{F}_t:=\sigma(X_s:s\leqslant t)$, with law $\mu^{Q_{T,t}}$.*

So the GFF is the terminal value of a Gaussian martingale of smooth fields: the *field at resolution t* (Figure 5.1). The proof is a variant of Theorem 3.2 of Lecture 2 and is left as Exercise 5.1.

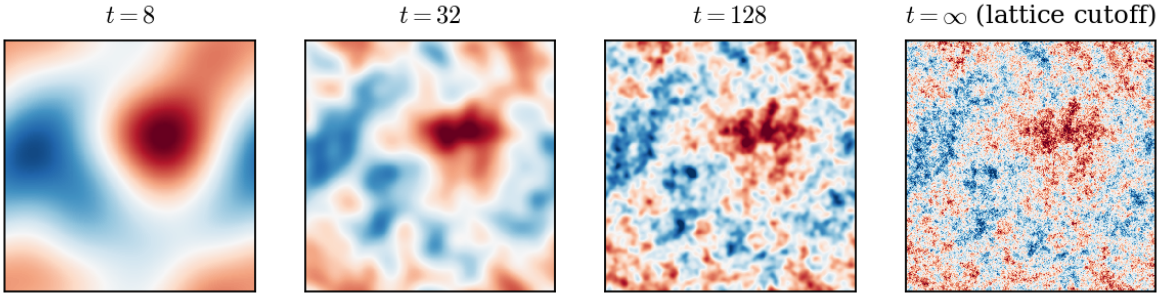


Figure 5.1. The field at resolution t, X_t, for the heat profile on $\mathbb{T}^2$ ($m = 4$, mean subtracted for display), at $t = 8, 32, 128$ and at the lattice cutoff, all built from the same white noise. The martingale X_t reveals finer and finer details, while the large scales are fixed at low resolution.

5.2 Stochastic observables

Fix a filtration $(\mathcal{F}_t)_{t\geqslant 0}$, for the moment that of X, and write $\mathbb{E}_t := \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$.

Definition 5.3. **([BCG26])** *An* observable *is a real or complex $(\mathcal{F}_t)$-martingale $O = (O_t)_{t\geqslant 0}$: $O_s = \mathbb{E}_s[O_t]$ for $s \leqslant t$. Its* expectation *is $\langle O\rangle := \mathbb{E}[O_t]$, which is independent of t. An* observable field *is a family $(O(x))_{x\in M}$ of observables.*

Every integrable $\mathcal{F}_\infty$-measurable random variable Y gives a *closed* observable $O_t = \mathbb{E}_t[Y]$, with $O_t \to Y$. The point of the definition is that interesting observables need *not* be closed: a martingale may diverge as $t \to \infty$ (for instance $X_t(x)$ for $d \geqslant 2$), and still carry a perfectly meaningful family of finite-resolution estimates. The next lemma is elementary but basic.

Lemma 5.4. **(Asymptotic uniqueness)** *(a) If O^1, O^2 are observables with $\|O^1_t - O^2_t\|_{L^1} \to 0$, then $O^1 = O^2$. (b) If (N_t) is any adapted process and O an observable with $\|N_t - O_t\|_{L^1} \to 0$, then $O_s = \lim_{t\to\infty} \mathbb{E}_s[N_t]$ in L^1.*

Proof. (a) $O^1_s - O^2_s = \mathbb{E}_s[O^1_t - O^2_t]$ for every $t \geqslant s$, and conditional expectation is an L^1 contraction. (b) Likewise, $\|O_s - \mathbb{E}_s N_t\|_{L^1} \leqslant \|O_t - N_t\|_{L^1}$. □

Observables as equivalence classes. Call two integrable adapted processes N and N' *asymptotically equivalent*, $N \sim N'$, if $\|N_t - N'_t\|_{L^1} \to 0$ as $t \to \infty$. By Lemma 5.4(a) each equivalence class contains at most one martingale, and by Lemma 5.4(b) this martingale, when it exists, is recovered from any member N of the class by looking back: $O_s = \lim_{t\to\infty} \mathbb{E}_s N_t$ in L^1. So an observable is the same thing as an equivalence class of asymptotically equivalent processes that contains a martingale: every member of the class is a *description* of the observable, and the martingale is its canonical representative. Proposition 5.6 below finds the martingale from a semimartingale description.

Random variables as observables. The map $Y \mapsto (\mathbb{E}_t Y)_{t\geqslant 0}$ sends $L^1(\mathcal{F}_\infty)$ into the observables. It is injective, since $\mathbb{E}_t Y \to Y$ in L^1 (Lévy's upward theorem), and its image is exactly the set of closed observables, i.e. of the uniformly integrable martingales. So the space of observables, a vector space on which $O \mapsto \langle O\rangle$ is a linear form, generalises $L^1(\mathcal{F}_\infty)$: dropping uniform integrability enlarges it, and the new elements are exactly the non-closed observables, those with no terminal value in L^1. For them an almost sure limit, when it exists, does not determine the observable (see Example 5.5).

Example 5.5. **(Brownian observables)** Let B be a standard Brownian motion and $(\mathcal{F}_t)$ its filtration. With $H_n(a; c)$ the Hermite polynomials of variance c, the martingales

$$B_t, \qquad B_t^2 - t, \qquad B_t^3 - 3tB_t, \qquad \dots, \qquad H_n(B_t; t)$$

are observables without a terminal value, and they are not even bounded in L^1: $\mathbb{E}|B_t^2 - t| = t\,\mathbb{E}|Z^2-1| \to \infty$, $Z \sim N(0,1)$. They are the prototypes of the Wick powers of Section 6: at a fixed point, $X_t(x)$ is a continuous Gaussian martingale with quadratic variation c_t, i.e. a Brownian motion run with the clock c_t. The exponential martingale

$$Z_t = \exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right) = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} H_n(B_t; t), \qquad \lambda \in \mathbb{R}, \lambda \neq 0,$$

makes the loss of terminal information explicit: $Z_t \to 0$ almost surely, while $\mathbb{E}\, Z_t = 1$ for every t. The almost sure limit is zero, the observable is not: the mass escapes to rarer and rarer paths, and the limit forgets all the finite-resolution expectations.

Proposition 5.6. (Repair by the future drift) *Let $N_t = N_0 + \int_0^t \beta_s \,\mathrm{d}s + M_t$, with $N_0 \in L^1$, M a martingale and $\mathbb{E}\int_0^\infty |\beta_s|\,\mathrm{d}s < \infty$. Then*

$$O_t := N_t + \mathbb{E}_t\Big[\int_t^\infty \beta_s \mathrm{d}s\Big]$$

is an observable, and it is the only observable with $\|N_t - O_t\|_{L^1} \to 0$.

Proof. $O_t = N_0 + M_t + \mathbb{E}_t[\int_0^\infty \beta_s\,\mathrm{d}s]$ is a sum of martingales, and $\|O_t - N_t\|_{L^1} \leqslant \mathbb{E}\int_t^\infty |\beta_s|\,\mathrm{d}s \to 0$. Uniqueness is Lemma 5.4(a). □

The correction is the expected drift still to come. For instance, for $N_t = B_t + \mathrm{e}^{-t}\sin B_t$ Itô's formula gives $\beta_t = -\frac{3}{2}\mathrm{e}^{-t}\sin B_t$, and since $\mathbb{E}_t \sin B_s = \mathrm{e}^{-(s-t)/2}\sin B_t$,

$$\mathbb{E}_t\Big[\int_t^\infty \beta_s\,\mathrm{d}s\Big] = -\frac{3}{2}\sin B_t \int_t^\infty \mathrm{e}^{-s-(s-t)/2}\mathrm{d}s = -\mathrm{e}^{-t}\sin B_t,$$

so the observable in the class of N is $O_t = B_t$. This correction is the germ of the forward–backward equations of Lecture 4: there the forecast is obtained from the field in the same way, by adding the expected drift still to come.

Example 5.7. (A delta at a Brownian endpoint) Fix $T > 0$, $y \in \mathbb{R}$, let $p_s(w) = (2\pi s)^{-1/2}\mathrm{e}^{-w^2/2s}$ be the heat kernel, and

$$\Pi_t^y := p_{T-t}(B_t - y), \qquad 0 \leqslant t < T.$$

Here the resolution runs over $[0, T)$ (reparametrise by $(T-t)^{-1}$ to get $[T^{-1}, \infty)$). The heat semigroup gives $\mathbb{E}_s\,\Pi_t^y = \Pi_s^y$, and $\int g(y)\,\Pi_t^y\,\mathrm{d}y = \mathbb{E}_t\, g(B_T)$ for bounded g. At fixed y, $\Pi_t^y \to 0$ almost surely as $t \uparrow T$, but $\mathbb{E}\,\Pi_t^y = p_T(y) > 0$. So $\delta(B_T - y)$ is not a random variable, but it is an observable: the family of its conditional densities. The same construction works for any Markov process with a transition density.

So an observable is determined by its asymptotic behaviour. A *germ* for O is a family of functionals $\mathring{O}_t$ on smooth fields such that $\|O_t - \mathring{O}_t(X_t)\|_{L^1} \to 0$. The observable field $O(x)$ is *local* if it admits germs of the form

$$\mathring{O}_t(X_t)(x) = \mathring{O}_t(X_t(x), \nabla X_t(x), \ldots, \nabla^k X_t(x)).$$

Germs are not unique, and this freedom reflects the freedom of choosing counterterms. By Lemma 5.4(b), a germ determines the observable.

Exercises

Exercise 5.1. (The field at resolution t) (a) Prove Proposition 5.2. *Hint:* on the torus, write $X_t = \sum_n (\int_0^t \dot{q}_n(s)^{1/2}\mathrm{d}B_s^n)\, e_n$. (b) For the heat profile, check $\int_0^\infty \dot{Q}_s\,\mathrm{d}s = C$ and $Q_t = C\mathrm{e}^{-A/t^2}$, and derive (5.2). (c) Show that $x \mapsto X_t(x)$ is a.s. real analytic for the heat profile.

Exercise 5.2. (**Divergence of c_t**) For the heat profile on $\mathbb{R}^d$, show that $c_t = \frac{1}{4\pi} E_1(m^2/t^2) = \frac{1}{2\pi}\log\frac{t}{m} + O(1)$ for $d=2$, where E_1 is the exponential integral, and that $c_t \asymp t^{d-2}$ for $d \geqslant 3$.

Exercise 5.3. (**Brownian observables**) (a) Show that $H_n(B_t;t)$ is a martingale and that $\mathbb{E}|H_n(B_t;t)| = t^{n/2}\mathbb{E}|H_n(Z;1)|$. (b) Show that the exponential martingale of Example 5.5 converges to 0 almost surely and is not uniformly integrable. Which observables of Brownian motion are closed? (c) Check the computation of the repaired observable for $N_t = B_t + \mathrm{e}^{-t}\sin B_t$, and find the observable in the class of $N_t = B_t + \mathrm{e}^{-t}B_t^2$.

Exercise 5.4. (**Brownian deltas**) (a) Prove the properties of Π^y stated in Example 5.7. (b) Show that $y \mapsto \Pi_t^y$ is a probability density for every $t < T$: the family $(\Pi^y)_y$ is a measure-valued observable of total mass one, although each Π_t^y tends to zero. (c) Show that $\partial_y^k \Pi^y$ is an observable, with $\int g(y)\,\partial_y^k \Pi_t^y \,\mathrm{d}y = (-1)^k\,\mathbb{E}_t g^{(k)}(B_T)$ for smooth bounded g: the observable of $\partial_y^k \delta(B_T - y)$.

6 Observables of the Gaussian free field and their products

6.1 Wick powers and exponentials

Point evaluations. For every x and every d, $\mathbf{X}(x) := (X_t(x))_{t\geqslant 0}$ is a local observable. For $d \geqslant 2$ it is not closed: $\mathbb{E}\,X_t(x)^2 = c_t \to \infty$.

Wick powers. Recall that $\partial_c H_n(a;c) = -\frac{1}{2}\partial_a^2 H_n(a;c)$ and $\partial_a H_n = n\,H_{n-1}$ (differentiate the generating function $\mathrm{e}^{\tau a - c\tau^2/2}$).

Proposition 6.1. (**Wick observables**) *For every $d \geqslant 1$, $n \geqslant 1$ and $x \in \mathcal{M}$,*

$$:X_t^n:(x) := H_n(X_t(x);c_t)$$

is a continuous martingale, with $\mathrm{d}:X_t^n:(x) = n:X_t^{n-1}:(x)\,\mathrm{d}X_t(x)$ and

$$\mathbb{E}\,[:X_t^n:(x):X_t^k:(y)] = \delta_{nk}\,n!\,Q_t(x,y)^n. \tag{6.1}$$

In particular, for $x \neq y$, $\lim_t \mathbb{E}[:X_t^n:(x):X_t^n:(y)] = n!\,G(x-y)^n$.

Proof. By Itô's formula, using $\mathrm{d}\langle X(x), X(x)\rangle_t = \dot{c}_t\,\mathrm{d}t$,

$$\mathrm{d}H_n(X_t(x);c_t) = \partial_a H_n\,\mathrm{d}X_t(x) + \left(\tfrac{1}{2}\partial_a^2 H_n + \partial_c H_n\right)\dot{c}_t\,\mathrm{d}t = n\,H_{n-1}(X_t(x);c_t)\,\mathrm{d}X_t(x).$$

The covariance is Lemma 3.8 of Lecture 2, applied to the Gaussian vector $(X_t(x), X_t(y))$. □

Compare with Theorem 3.9 of Lecture 2: there $:\varphi^n:$ existed as a random distribution only for $n(d-2) < d$, and by Proposition 3.11 not at all otherwise. As observables, Wick powers exist in *every* dimension and need no smearing. When $n(d-2) \geqslant d$ and $0 \leqslant f \not\equiv 0$, the smeared martingale $t \mapsto :X_t^n:(f)$ is unbounded in L^2 and, by the Paley–Zygmund argument of Lecture 2, not even tight as $t \to \infty$: it has no terminal value, and the observable is the whole family, not a limit. The singularity has not disappeared. It has moved from the observables themselves to their *products* at coincident points, as we now see.

Exponentials. For $\beta \in \mathbb{C}$, $:\mathrm{e}^{\beta X_t}:(x) := \exp\left(\beta\,X_t(x) - \frac{\beta^2}{2}c_t\right)$ is the stochastic exponential of $\beta X(x)$, hence a martingale, and $\mathbb{E}[:\mathrm{e}^{\beta_1 X_t}:(x):\mathrm{e}^{\beta_2 X_t}:(y)] = \mathrm{e}^{\beta_1\beta_2 Q_t(x,y)}$.

6.2 Products of observables

The pointwise product of two martingales is not a martingale. The natural definition of the product keeps the asymptotics.

Definition 6.2. *Two observables O^1, O^2* admit a product *if $O^1_t\,O^2_t \in L^1$ for all t and there is an observable $O^1 \star O^2$ with $\|(O^1\star O^2)_t - O^1_t\,O^2_t\|_{L^1}\to 0$. By Lemma 5.4 it is unique, and $(O^1\star O^2)_s = \lim_t \mathbb{E}_s[O^1_t\,O^2_t]$.*

Proposition 6.3. *Let O^1, O^2 be continuous observables, square integrable on bounded intervals, with*

$$\mathbb{E}\int_0^\infty |\mathrm{d}\langle O^1, O^2\rangle_s| < \infty. \tag{6.2}$$

Then $O^1\star O^2$ exists and

$$(O^1\star O^2)_t = O^1_t\,O^2_t + \mathbb{E}_t\Big[\int_t^\infty \mathrm{d}\langle O^1, O^2\rangle_s\Big]. \tag{6.3}$$

In particular $\langle O^1\star O^2\rangle = \mathbb{E}[O^1_0\,O^2_0] + \mathbb{E}\,\langle O^1, O^2\rangle_\infty$.

Proof. $P_t := O^1_t\,O^2_t - \langle O^1, O^2\rangle_t$ is a martingale on bounded intervals, and $N_t := \mathbb{E}_t\,\langle O^1, O^2\rangle_\infty$ is a closed martingale by (6.2). So $P+N$ is a martingale and equals the right-hand side of (6.3). Its distance from $O^1_t\,O^2_t$ in L^1 is at most $\mathbb{E}\int_t^\infty |\mathrm{d}\langle O^1, O^2\rangle| \to 0$. □

Brownian deltas. The observables Π^y of Example 5.7 show the mechanism of the operator product expansion in its simplest form. Write $\kappa_s = (4\pi s)^{-1/2}$. Completing the square, $(B_t - y)^2 + (B_t - z)^2 = 2\,(B_t - \bar y)^2 + (y-z)^2/2$ with $\bar y = (y+z)/2$, gives the exact identity

$$\Pi^y_t\,\Pi^z_t = \kappa_{T-t}\,\mathrm{e}^{-(y-z)^2/4(T-t)}\,p_{(T-t)/2}(B_t - \bar y), \qquad t<T.$$

Since $B_t \sim N(0,t)$, $\mathbb{E}\,p_{(T-t)/2}(B_t - \bar y) = p_{(T+t)/2}(\bar y)$, and, the product being nonnegative, its expectation is its L^1 norm: for $y \neq z$

$$\|\Pi^y_t\,\Pi^z_t\|_{L^1} = \kappa_{T-t}\,\mathrm{e}^{-(y-z)^2/4(T-t)}\,p_{(T+t)/2}(\bar y) \longrightarrow 0 \qquad (t\uparrow T):$$

so $\Pi^y_t\,\Pi^z_t$ is asymptotically equivalent to the zero observable, and by Definition 6.2 the product exists and $\Pi^y \star \Pi^z = 0$. At $y=z$ the exponential suppression is lost, and $\mathbb{E}\,(\Pi^y_t)^2 = \kappa_{T-t}\,p_{(T+t)/2}(y) \to \infty$: the square does not exist, since a martingale O with $\|O_t - (\Pi^y_t)^2\|_{L^1} \to 0$ would force $\mathbb{E}\,(\Pi^y_t)^2 \to \mathbb{E}\,O_0 < \infty$. The square can, however, be resolved at a fixed coarser resolution. Let $u = T-\varepsilon$ be a fine resolution and $s<u$ a coarser one. Since $p_\varepsilon(w)^2 = \kappa_\varepsilon\,p_{\varepsilon/2}(w)$,

$$\mathbb{E}_s[(\Pi^y_{T-\varepsilon})^2] = \kappa_\varepsilon\,p_{T-s-\varepsilon/2}(B_s - y) = \kappa_\varepsilon\,\Pi^y_s - \frac{\varepsilon\,\kappa_\varepsilon}{4}\,\partial_y^2\Pi^y_s + \frac{\varepsilon^2\,\kappa_\varepsilon}{32}\,\partial_y^4\Pi^y_s + O_{L^1}(\varepsilon^{5/2}),$$

by Taylor expansion in the time variable and $\partial_\tau p_\tau = \frac{1}{2}\partial_y^2 p_\tau$. This is an operator product expansion: singular coefficients ($\kappa_\varepsilon \to \infty$, while $\varepsilon\,\kappa_\varepsilon, \varepsilon^2\,\kappa_\varepsilon \to 0$) times the observables Π^y, $\partial_y^2\Pi^y$, ... The order of the limits matters: the remainder is small only after conditioning at a fixed resolution s. Without conditioning, $(\Pi^y_{T-\varepsilon})^2 - \kappa_\varepsilon\,\Pi^y_{T-\varepsilon} = \kappa_\varepsilon\,[p_{\varepsilon/2} - p_\varepsilon](B_{T-\varepsilon} - y)$ has L^1 norm of order $\varepsilon^{-1/2}$.

Products of point evaluations. Here $\langle \mathbf{X}(x), \mathbf{X}(y)\rangle_t = Q_t(x-y)$ is deterministic and (6.2) reads $\int_0^\infty |\dot Q_s(x-y)|\mathrm{d}s < \infty$. For the heat profile and $d \geqslant 2$ this holds iff $x \neq y$, and then $\dot Q_s \geqslant 0$ and the integral is $G(x-y)$. Hence

$$(\mathbf{X}(x)\star\mathbf{X}(y))_t = X_t(x)\,X_t(y) + G(x-y) - Q_t(x-y), \qquad \langle \mathbf{X}(x)\star\mathbf{X}(y)\rangle = G(x-y), \tag{6.4}$$

which is the correct two-point function. We never had to discuss the regularity of X_∞. For $x=y$ and $d\geqslant 2$ the product does not exist: its expectation would be $\lim_t \mathbb{E}\,X_t(x)^2 = \lim_t c_t = \infty$.

Operator product expansion. Formula (6.4) has an exact short-distance expansion at every resolution: for fixed t, as $y \to x$,

$$(\mathbf{X}(x)\star\mathbf{X}(y))_t - G(x-y) \;\longrightarrow\; X_t(x)^2 - c_t =: X_t^2:(x). \tag{6.5}$$

Since both sides are martingales, this reads $\mathbf{X}(x) \star \mathbf{X}(y) = G(x-y)\,1 + :\mathbf{X}^2:(x) + o(1)$, the remainder tending to zero at each fixed resolution t: the Wick square is the regular part of the product, and $G(x-y)$ is its singular coefficient. More generally (Exercise 6.2), for $x \neq y$,

$$:\mathbf{X}^n:(x) \star :\mathbf{X}^k:(y) = \sum_{j \leqslant n \wedge k} j! \binom{n}{j}\binom{k}{j} G(x-y)^j :\mathbf{X}^{n-j}(x)\mathbf{X}^{k-j}(y):, \tag{6.6}$$

where $:\mathbf{X}^p(x)\mathbf{X}^q(y):_t := H_p(X_t(x); c_t)\, H_q(X_t(y); c_t)$ minus its contractions at scales $\leqslant t$. This is Wick's theorem written at the level of observables. The coefficients $G(x-y)^j$ carry all the singularities, and the observables on the right have regular limits as $y \to x$.

Higher orders. The expansion (6.5) continues at every fixed resolution. With $x_\pm = x \pm \rho e/2$, $|e| = 1$, $\partial = e \cdot \nabla$ and $\mathcal{W}_t(U, V) := U_t V_t - \mathbb{E}[U_t V_t]$,

$$(X(x_-) \star X(x_+))_t = G(\rho e) + :X_t^2:(x) + \rho^2 C_{2,t}(x) + \rho^4 C_{4,t}(x) + O_{L^1}(\rho^6),$$

$$C_2 = \frac{1}{4}(\mathcal{W}(X, \partial^2 X) - \mathcal{W}(\partial X, \partial X)), \qquad C_4 = \frac{1}{192}(\mathcal{W}(X, \partial^4 X) - 4\mathcal{W}(\partial X, \partial^3 X) + 3\mathcal{W}(\partial^2 X, \partial^2 X)).$$

Subtracting G leaves the bilocal Wick product $X_t(x_-)\,X_t(x_+) - Q_t(x_-, x_+)$, which is smooth in ρ at fixed t, and the symmetry $\rho \mapsto -\rho$ kills the odd orders. Each coefficient is a finite sum of Wick bilinears, hence an observable. The constant in the remainder depends on t; the limit $t \to \infty$ is a separate question. A systematic study of the operator product expansion in this framework is due to P. Paulovics (in preparation).

Locality and factorisation. Let O^1, O^2 be observables with germs supported in $U^1_{1/t}$ and $U^2_{1/t}$, where U^i are sets at positive distance and U_r is the r-neighbourhood of U. The covariation $\mathrm{d}\langle O^1, O^2\rangle_t = \langle D O^1_t, \dot{Q}_t\, D O^2_t\rangle \mathrm{d}t$, with D acting on the germs, involves $\dot{Q}_t$ only between the two supports. For a finite range decomposition it vanishes for large t, so the product exists and is asymptotically the pointwise product. For the heat profile it is exponentially small, by (5.2). Thus, under integrability conditions on the drifts of the germs, observables supported on disjoint regions can be multiplied, while the products of observables at the same point require renormalisation (Figure 6.1). This is the structure of a *factorisation algebra* of observables; the precise axioms are not discussed here; see [BCG26].

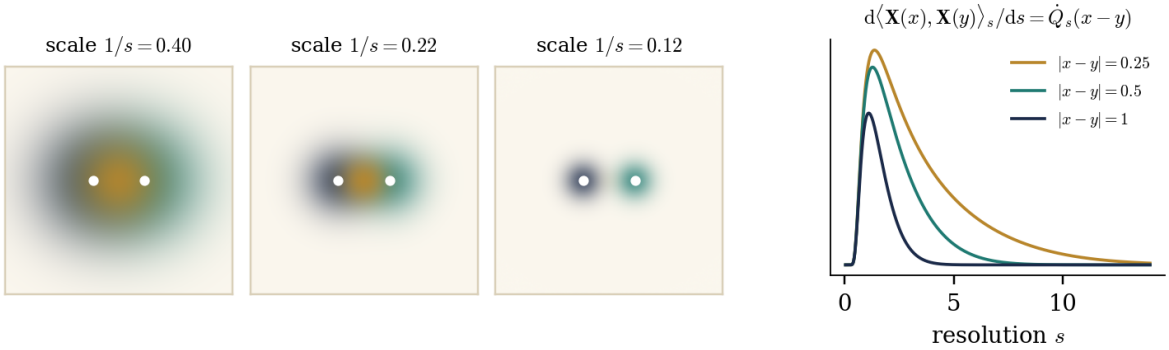


Figure 6.1. Covariation of point observables, heat profile in $d = 2$. Left: the kernels of $\dot{Q}_s^{1/2}$ centred at x (navy) and at y (teal) at three resolutions, with their overlap (gold). The integral of the product of the two kernels is the covariation density $\dot{Q}_s(x-y)$. Right: $s \mapsto \dot{Q}_s(x-y)$ for three distances ($m = 1$). Its integral is $G(x-y)$, and it is negligible for $s \gg 1/|x-y|$. As $y \to x$ the mass moves to $s \to \infty$, which is the divergence of the product at coincident points.

Remark 6.4. A finite range decomposition makes these statements exact: if $\dot{Q}_t(x,y) = 0$ for $|x-y| \geqslant 1/t$, then $X_T - X_t$ restricted to regions at distance $\geqslant 1/t$ are independent. Such decompositions of $(m^2 - \Delta)^{-1}$ exist and are a basic tool of the rigorous renormalisation group [BBS19].

Outlook. Interactions are introduced in Lecture 4. The effective potentials solve the Polchinski equation, and the interacting field seen at increasing resolution solves a forward–backward stochastic differential equation whose force is itself an observable. Products, the operator product expansion and the locality of observables, which we met here for the GFF, are the tools to describe it.

Exercises

Exercise 6.1. **(Hermite calculus along the flow)** (a) Show that $\int H_n(a+b;c)\,\mu^s(\mathrm{d}b) = H_n(a;c-s)$ for $0 \leqslant s \leqslant c$, where $\mu^s = N(0,s)$. (b) Deduce again that $H_n(X_t(x);c_t)$ is a martingale, using only the independence of the increments.

Exercise 6.2. **(Products of Wick observables)** (a) Show that (6.2) holds for $:\mathbf{X}^n:(x)$ and $:\mathbf{X}^k:(y)$ when $x \neq y$ and Q is the heat profile. *Hint:* c_s grows at most polynomially while $\dot{Q}_s(x-y)$ decays like $\mathrm{e}^{-s^2|x-y|^2/4}$. (b) Show that $\langle :\mathbf{X}^n:(x) \star :\mathbf{X}^n:(y)\rangle = n^2 \int_0^\infty (n-1)!\, Q_s^{n-1}\dot{Q}_s\,\mathrm{d}s = n!\, G(x-y)^n$. (c) Prove (6.6).

Exercise 6.3. **(Exponential observables)** For $\beta = \mathrm{i}b$, $b \in \mathbb{R}$, and the heat profile, show that condition (6.2) for $:\mathrm{e}^{\mathrm{i}b\mathbf{X}}:(x) \star \overline{:\mathrm{e}^{\mathrm{i}b\mathbf{X}}:(y)}$ reads $\int^\infty \mathrm{e}^{b^2 c_s}\dot{Q}_s(x-y)\mathrm{d}s < \infty$. Deduce that the product exists for all $x \neq y$ if $d \leqslant 3$, for $|x-y| > |b|/2\pi$ but not for $|x-y| < |b|/2\pi$ if $d = 4$, and not at all if $d \geqslant 5$. For $d \leqslant 3$ show that its expectation is $\mathrm{e}^{b^2 G(x-y)}$, and decide for which b it is locally integrable in y: all b in $d = 1$, $b^2 < 4\pi$ in $d = 2$ (the L^2 threshold of Lecture 2), only $b = 0$ in $d = 3$.

Exercise 6.4. **(The Brownian OPE)** (a) Prove the product identity for $p_\varepsilon(B-y)\,p_\varepsilon(B-z)$ and deduce $\Pi^y \star \Pi^z = 0$ for $y \neq z$. (b) Derive the expansion of $\mathbb{E}_s[(\Pi^y_{T-\varepsilon})^2]$ to all orders in ε, and bound the remainder in L^1 at fixed s. (c) Show that the unconditioned remainder has L^1 norm $\sim \kappa_\varepsilon p_T(y) \int |p_{1/2} - p_1|$.

Exercise 6.5. **(Point splitting)** (a) Derive the coefficients C_2 and C_4 above by Taylor expanding $X_t(x \pm \rho e/2)$. (b) Show that $\mathbb{E}\, C_{2,t} = 0$ and compute $\mathbb{E}[C_{2,t}(x)^2]$ for the heat profile. How does it grow as $t \to \infty$?

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