A Constructive Introduction to Quantum Field Theory

Lecture 5: Observables and field equations

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Abstract

We continue Lecture 4 with the observables of the interacting theory. Rotating the Minkowski field equation to Euclidean space gives the Dyson–Schwinger equation, whose noise, *Nelson's Euclidean noise*, is the transpose of the functional derivative and is fixed by the law of the field. Conditioning on the scales rather than on the neighbours, the field equation holds at every resolution, with a noise which is a conditional expectation of Nelson's noise. This noise is sub-Gaussian whatever the interaction (an observation which, as far as we know, is new, and which is the key input of the a priori estimates of Lecture 7), and the Dyson–Schwinger equation takes a pathwise form, an identity between stochastic observables. We then discuss the passage to infinite volume. The appendices contain the algebra of the flow, perturbation theory and counterterms, and the functional inequalities that follow from the localisation. The treatment is formal: we compute as if all objects were smooth and integrable, and we indicate where the real difficulties lie.

The results presented in Lectures 4–7 were obtained recently, with the use of AI, in collaboration with Sarah-Jean Meyer.

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Introduction

Lecture 4 generated the interacting measure in three ways, renormalisation, optimal control and filtering, and found that they produce the same forward–backward system (7.7) for the field Φ and the force F. In this lecture we study the observables of the interacting theory and its field equation. We use the notation of Lecture 4 throughout: the interaction (7.1) with its counterterm γ_T, the effective potential V_t (7.3) and the force $F_t = -\mathrm{D}V_t(\Phi_t)$, the effective average action Γ_t of Section 8.6 and its Wetterich equation (8.13), $S_t = Q_T - Q_t = Q_{T,t}$, $K = Q_T^{-1}$, $J_t = KQ_t$, the regulator $\mathcal{R}_t = S_t^{-1} - K$, the bridge representation (7.14), the forecast $\psi_t = \mathbb{E}_t[\Phi_T]$ and the conditional covariance Cov_t. Sections 7 and 8 are in Lecture 4, and references to them point there.

Structure. Section 9 treats the observables of the interacting theory, the Euclidean field equation and Nelson's Euclidean noise, the field equation at every resolution, the sub-Gaussian bound for its noise, and infinite volume. Appendices A and B are additional material, referred to from the main text where needed.

What is formal. Rigorous statements require a finite-dimensional setting: a finite lattice, or a Fourier cutoff on the torus, together with a finite terminal resolution T. There, all the identities below are theorems under mild growth assumptions on the interaction, and their proofs are the computations we give. The hard problems are the limits $T \to \infty$ (ultraviolet) and infinite volume (infrared), together with the locality of the limiting objects. We mark with (formal) the statements that we only derive at this level.

Conventions. As in Lectures 1–3: $\mathcal{M} = \mathbb{T}_L^d$ or $\mathbb{R}^d$, $A = m^2 - \Delta$, $C = A^{-1}$ with kernel G, μ_m the GFF, $\langle f, g\rangle = \int fg$ for real functions. For a functional F on smooth fields, $\mathrm{D}F(\varphi) \in \mathcal{S}'(\mathcal{M})$ and $\mathrm{D}^2F(\varphi)$ are its first and second derivatives, $\mathrm{D}_hF = \langle \mathrm{D}F, h\rangle$. $H_n(a;c)$ are the Hermite polynomials of variance c, and μ^Q is the centred Gaussian measure with covariance Q.

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Main lecture

9 Observables and field equations

In Lecture 3 the primary objects were the stochastic observables, martingales in the filtration of finite-resolution measurements [BCG26]. The filtering point of view makes this natural: an observable is a posterior mean $\mathbb{E}_t[G(\Phi_T)]$. We have already met the force F_t and the forecast ψ_t, and the field Φ_t is not one of them. This section develops the language for the interacting theory, and then draws the main consequence of the filter: a field equation at every resolution, with a noise whose moments are controlled a priori.

9.1 Observables of the interacting theory

For a functional G of the terminal field, the observable $O_t = \mathbb{E}_t^{\mathbb{P}}[G(\Phi_T)]$ is $O_t = G_t(\Phi_t)$, with

$$G_t(\phi) = \mu_t^{\phi}(G) = \mathrm{e}^{V_t(\phi)} \int G(\phi + \zeta)\, \mathrm{e}^{-V_T(\phi+\zeta)} \mu^{Q_{T,t}}(\mathrm{d}\zeta). \tag{9.1}$$

Formally, G_t solves the backward Kolmogorov equation of the interacting dynamics,

$$\partial_t G_t + \tfrac{1}{2}\mathrm{Tr}(\dot{Q}_t \mathrm{D}^2 G_t) - \langle \dot{Q}_t \mathrm{D}V_t, \mathrm{D}G_t\rangle = 0, \qquad G_T = G, \tag{9.2}$$

which is the Doob transform of the Gaussian heat equation by e^{-V_t} (Exercise 9.1). Conversely, any solution of (9.2) yields an observable $G_t(\Phi_t)$; at finite T it is closed, but as $T \to \infty$ it need not be. The force is the case $G = -\mathrm{D}V_T$, and is special because it also drives the dynamics.

Tangent observables. Let V_T^θ depend on a coupling θ, and let $U_t := \partial_\theta V_t^\theta$. Differentiating (7.3) gives $U_t(\Phi_t) = \mathbb{E}_t^\theta[\partial_\theta V_T^\theta(\Phi_T)]$, an observable, and

$$\partial_\theta \mathbb{E}^{\nu_T^\theta}[G] = -\mathrm{Cov}^{\nu_T^\theta}(G, \partial_\theta V_T^\theta). \tag{9.3}$$

Thus the tangent observables, such as the running $:\varphi^4:$ for $\theta = \lambda$ (together with the derivative of the counterterm $\partial_\lambda \gamma_T \int :\varphi^2:$), describe the response of the theory to a change of coupling. They are the observables conjugate to the coordinates of theory space.

9.2 Field equations, from Minkowski to Euclidean

The Minkowski field equation. In Lecture 1 the quantum Φ^4 theory was introduced through the operator field equation (1.9) together with the canonical commutation relations (1.6). In the normalisation (7.1) it reads

$$(\Box + m^2)\,\varphi + V'(\varphi) = 0,$$

with $V'(\varphi) = \frac{\lambda}{3}\varphi^3$ suitably renormalised. What it says about correlation functions is best seen on the time-ordered functions

$$\tau(x, x_1, \dots, x_n) = \langle \Omega, \mathrm{T}\,\varphi(x)\,\varphi(x_1)\cdots\varphi(x_n)\,\Omega\rangle.$$

When ∂_t^2 acts on the time ordering, it also differentiates the step functions $\theta(t - t_j)$. The first derivative produces the equal-time commutators $[\varphi(t,\mathbf{x}), \varphi(t,\mathbf{x}_j)]$, which vanish, and the second the commutators

$$[\partial_t \varphi(t,\mathbf{x}), \varphi(t,\mathbf{x}_j)] = -\mathrm{i}\,\delta(\mathbf{x} - \mathbf{x}_j).$$

Formally,

$$(\Box_x + m^2)\,\tau(x, x_1, \dots, x_n) = -\tau(V'(\varphi)(x), x_1, \dots, x_n) - \mathrm{i}\sum_{j=1}^{n} \delta(x - x_j)\,\tau(x_1, \dots, \hat{x}_j, \dots, x_n), \tag{9.4}$$

where $\hat{x}_j$ means that x_j is omitted. The field equation holds as an identity between operators, and the *contact terms* are the trace of the canonical commutator. Since $\mathrm{i}\,[\partial_t \varphi(t,\mathbf{x}), \cdot\,]$ acts on functions of the fields at time t as the functional derivative $\delta / \delta\varphi(t,\mathbf{x})$, the contact terms are a derivative acting on the other fields.

Euclidean rotation. Under $t = -\mathrm{i}\,\tau$ the time-ordered functions continue to the Schwinger functions $S(x, x_1, \dots, x_n)$, the moments of the Euclidean field (Lecture 1; not to be confused with the covariances S_t below). Since $\partial_t^2 = -\partial_\tau^2$, the operator $\Box + m^2$ becomes $A = m^2 - \Delta$, with Δ the Laplacian in all d variables. Formally

$$\delta(t - t_j) = \mathrm{i}\,\delta(\tau - \tau_j),$$

so the factor $-\mathrm{i}$ of the contact terms becomes $+1$:

$$A_x S(x, x_1, \dots, x_n) = -S(V'(\varphi)(x), x_1, \dots, x_n) + \sum_{j=1}^{n} \delta(x - x_j)\,S(x_1, \dots, \hat{x}_j, \dots, x_n).$$

This is the Dyson–Schwinger hierarchy of Lecture 2 (Remark 3.15). For a random field φ with law ν, passing from products $G = \varphi(x_1)\cdots\varphi(x_n)$ to cylinder functionals by linearity and density, it reads

$$\mathbb{E}^\nu[G\,\langle h,\omega\rangle] = \mathbb{E}^\nu[\mathrm{D}_h G], \qquad \omega := A\varphi + V'(\varphi). \tag{9.5}$$

The noise is the transpose of the derivative. The Euclidean field does *not* solve $A\varphi + V'(\varphi) = 0$. Already for the free field, if it did, elliptic regularity would make φ real analytic, which it is not, as Nelson observed [Nel75]. The defect ω is the *noise* of the Euclidean field equation

$$A\varphi + V'(\varphi) = \omega,$$

and (9.5) says what it is. Let $\mathrm{D}_h^\top$ be the transpose of D_h in $L^2(\nu)$. Applying (9.5) to a product of two functionals gives

$$\mathrm{D}_h^\top G = -\mathrm{D}_h G + \langle h,\omega\rangle\, G, \qquad \langle h,\omega\rangle = \mathrm{D}_h^\top 1.$$

The noise is the transpose of the functional derivative applied to the constant 1, that is minus the score of ν: for

$$\nu(\mathrm{d}\varphi) \propto \mathrm{e}^{-\frac{1}{2}\langle\varphi,A\varphi\rangle - V(\varphi)}\mathrm{d}\varphi$$

in finitely many dimensions, $\omega = -\mathrm{D}\log(\mathrm{d}\nu/\mathrm{d}\varphi)$. This is how the canonical commutator survives the rotation. In Minkowski space the contact terms come from a commutator, which acts as a derivative, and the field equation holds exactly. In Euclidean space they come from integration by parts, and the field equation acquires the term that turns the derivative into its transpose. So the noise is not an independent input, as in a stochastic PDE: it is a functional of the field, fixed by its law. For the free field

$$\omega = A\varphi$$

is the Gaussian field with covariance A, i.e. $\omega = A^{1/2}\xi$ for a white noise ξ: one derivative of white noise. For an interacting field it is not Gaussian.

Nelson's Markov field equation. Nelson [Nel75] proposed to take the Euclidean field equation, with $V'(\varphi) = g\varphi^3 - \alpha\varphi$ and α a counterterm, as the *definition* of its noise, which he called ω: $\omega := A\varphi + V'(\varphi)$, and to characterise it by conditioning in real space. On the lattice $\varepsilon\,\mathbb{Z}^d$, with the nearest-neighbour Laplacian, the conditional law of $\varphi(x)$ given the field at all other sites depends only on the nearest neighbours: this is the Markov property. An integration by parts in this one-site conditional law shows that $\omega(x)$ is conditionally centred, $\mathbb{E}[\omega(x)|\varphi(y),\ y\neq x] = 0$, that it satisfies the Dyson–Schwinger hierarchy, and that its covariance is

$$\mathbb{E}[\omega(x)\,\omega(y)] = (-\Delta + \tilde{m}^2)\,\delta(x-y), \qquad \tilde{m}^2 = m^2 + \mathbb{E}[V''(\varphi(x))],$$

while its fourth moment shows that it is not Gaussian. In fact $\varepsilon^d\,\omega(x)$ is minus the score of the conditional law of $\varphi(x)$ given its neighbours: the Markov property localises the noise. Nelson argued, heuristically, that a smeared ω is a random variable if and only if $\mathbb{E}[V''(\varphi(x))]$ stays finite in the continuum limit, that is if the cube can be Wick ordered with respect to the interacting vacuum, and he judged this unlikely in $d = 3$. This agrees with Remark 9.5: the regularised logarithmic derivatives of Φ^4_3 do not converge in L^2 [ALZ06]. The noise of (9.5) is exactly Nelson's noise, which is why we use his letter ω and call it *Nelson's Euclidean noise*. As far as we know, apart from Nelson's original work these notes are the first place where this noise is studied as the noise of a field equation. Its pairings $\langle h,\omega\rangle$ are, up to sign, the logarithmic derivatives of the measure, which appear in the integration by parts formulae for Φ^4_3 [ALZ06]; what is new here is the noise at every resolution, ω_t, and its sub-Gaussian bound (Section 9.4).

A Gaussian noise gives the wrong field. Albeverio and Yoshida [AY02] took the opposite route and prescribed the noise. They solve the elliptic stochastic PDE on $\mathbb{R}^d$

$$A\,\psi + V'(\psi) = A^{1/2}\,\xi,$$

with ξ a Gaussian white noise, for a Wick cube with a spatial cutoff and a small coupling (chiefly in $d=2$), and for Wick exponentials. The method is a nonlinear change of variables on the abstract Wiener space of the free field: $\mathcal{T}(\psi) := \psi + A^{-1}V'(\psi)$ is a free field. Formally, changing variables in the free field measure $\mu^{A^{-1}}$,

$$\mathrm{Law}(\psi) \propto \left|\det(1 + A^{-1}V''(\psi))\right| \mathrm{e}^{-\langle\psi, V'(\psi)\rangle - \frac{1}{2}\langle V'(\psi), A^{-1}V'(\psi)\rangle}\,\mu^{A^{-1}}(\mathrm{d}\psi).$$

This is not $\mathrm{e}^{-V}\mu^{A^{-1}}$, as the authors point out for the cutoff Φ^4_2 measure ([AY02], Remark 2.3): the second term of the exponent and the determinant are nonlocal, the Markov property is lost, and the construction does not yield a reflection positive field. Reflection positivity is obtained only on the subspace of functionals of $\mathcal{T}(\psi)$, which is a free field ([AY02], Theorem 4.1 and Remark 4.1). The noise of a Euclidean field theory cannot be prescribed: it is the transpose of the derivative for the very measure one is constructing, and this forces it to be non-Gaussian.

Conditioning on the scales. The filter gives a third way. Instead of conditioning on the neighbours, condition on the coarse scales. The field equation then holds at every resolution, with a noise which is the conditional expectation of Nelson's Euclidean noise ω. This noise is not Gaussian, but at each resolution it is a genuine random variable, with Gaussian tails uniform in the interaction and in the cutoff (Section 9.4).

9.3 The field equation at every resolution

By (7.14) the field at resolution t is a blurred copy of the terminal field, $\Phi_t = J_t\Phi_T + \zeta_t$. The noise ζ_t is not observed, but its posterior mean is, $\mathbb{E}_t\zeta_t = \Phi_t - J_t\psi_t$. Since $\psi_t = \Phi_t + S_tF_t$ and $1 - J_t = KS_t$, its rescaling $S_t^{-1}\mathbb{E}_t\zeta_t = K\,\Phi_t - J_tF_t$ is the **noise observable** ω_t below. This gives a field equation at every resolution.

Theorem 9.1. (DS equations at resolution t) (formal) *The* noise observable

$$\omega_t := K\,\Phi_t - J_tF_t \tag{9.6}$$

is a centred $\mathbb{P}$-martingale, with

$$\mathrm{d}\omega_t = K\dot{Q}_t^{1/2}\mathrm{d}W_t - J_t\,\mathrm{d}F_t.$$

For every $t \leqslant T$, every h and every functional G,

$$\mathbb{E}^{\mathbb{P}}\left[\mathrm{D}_{J_th}G(\Phi_t)\right] = \mathbb{E}^{\mathbb{P}}\left[G(\Phi_t)\,\langle h, \omega_t\rangle\right] = \mathbb{E}^{\mathbb{P}}\left[G(\Phi_t)\,\langle h, K\,\Phi_t - J_tF_t\rangle\right]. \tag{9.7}$$

At $t = T$ this is the Dyson–Schwinger equation of ν_T,

$$\mathbb{E}^{\nu_T}\left[\mathrm{D}_hG\right] = \mathbb{E}^{\nu_T}\left[G\,\langle h, K\varphi + \mathrm{D}V_T(\varphi)\rangle\right]. \tag{9.8}$$

Proof. From (7.6) and $\dot{J}_t = K\dot{Q}_t$,

$$\mathrm{d}\omega_t = K\left(\dot{Q}_tF_t\mathrm{d}t + \dot{Q}_t^{1/2}\mathrm{d}W_t\right) - K\dot{Q}_tF_t\mathrm{d}t - J_t\mathrm{d}F_t.$$

The drift cancels, F is a martingale and $\omega_0 = 0$. For (9.7) use Proposition 7.2: $\Phi_t \sim \nu_t \propto \mathrm{e}^{-V_t}\mu^{Q_t}$. Gaussian integration by parts for μ^{Q_t} in the direction Q_tk gives

$$\mathbb{E}^{\nu_t}\left[\mathrm{D}_{Q_tk}G\right] = \mathbb{E}^{\nu_t}\left[G\left(\langle k, \varphi\rangle + \langle Q_tk, \mathrm{D}V_t(\varphi)\rangle\right)\right].$$

Take $k = Kh$, so that $Q_t k = J_t h$, and recall $F_t = -\mathrm{D}V_t(\Phi_t)$. □

At $t = T$, $J_T = 1$ and $F_T = -\mathrm{D}V_T(\Phi_T)$, so

$$\omega_T = K\,\Phi_T + \mathrm{D}V_T(\Phi_T)$$

is Nelson's Euclidean noise (9.5) of ν_T, the transpose of the derivative applied to 1. Since ω is a martingale,

$$\omega_t = \mathbb{E}_t[K\,\Phi_T + \mathrm{D}V_T(\Phi_T)]: \tag{9.9}$$

the noise at resolution t is what an observer at resolution t can know of the Euclidean noise. Nelson's $\omega(x)$ is centred given the field at the neighbouring sites; here $\omega_T - \omega_t$ is centred given the field at the coarser scales.

Proposition 9.2. (Dyson–Schwinger, pathwise) (formal) *Let G be a functional of the terminal field, $G_t(\Phi_t) = \mathbb{E}_t[G(\Phi_T)]$ its observable (9.1), and h a test function. Then*

$$N_t := \mathrm{D}_{J_t h} G_t(\Phi_t) - G_t(\Phi_t)\,\langle h, \omega_t\rangle \tag{9.10}$$

is a $\mathbb{P}$-martingale on $[0,T]$ with $N_0 = 0$. Equivalently, $N_t = \mathbb{E}_t[\mathrm{D}_h G(\Phi_T) - G(\Phi_T)\,\langle h, \omega_T\rangle]$.

Proof. $N_0 = 0$ because $J_0 = 0$ and $\omega_0 = 0$. Differentiate (9.2) in ϕ and apply Itô's formula: since $F_t = -\mathrm{D}V_t$, the drift of $\mathrm{D}G_t(\Phi_t)$ is $-\mathrm{D}F_t\dot{Q}_t\mathrm{D}G_t$, and with $\dot{J}_t = K\dot{Q}_t$ the drift of $\langle J_t h, \mathrm{D}G_t(\Phi_t)\rangle$ is $\langle \mathrm{D}G_t, \dot{Q}_t\,(K - \mathrm{D}F_t J_t)\,h\rangle$, by the symmetry of $\mathrm{D}F_t = -\mathrm{D}^2 V_t$. On the other hand $\mathrm{d}G_t(\Phi_t) = \langle \mathrm{D}G_t, \dot{Q}_t^{1/2}\mathrm{d}W_t\rangle$ and, by Theorem 9.1 with $\mathrm{d}F_t = \mathrm{D}F_t\,\dot{Q}_t^{1/2}\mathrm{d}W_t$, the covariation of $G_t(\Phi_t)$ and $\langle h, \omega_t\rangle$ has the same density $\langle \mathrm{D}G_t, \dot{Q}_t\,(K - \mathrm{D}F_t J_t)\,h\rangle$. Both $G_t(\Phi_t)$ and ω_t are martingales, so by the product rule the drift of their product is this covariation, and the drift of N vanishes. At $t = T$, $G_T = G$ and $J_T = 1$. □

The DS equation as an identity between observables. Taking expectations in $\mathbb{E}\,N_t = N_0 = 0$ gives (9.7) for the function G_t, and at $t = T$ the DS equation (9.8). Proposition 9.2 says more, and fits the framework of stochastic observables of Lecture 3. N is the observable of the terminal random variable $\mathrm{D}_h G - G\langle h, \omega_T\rangle$, the defect of the DS equation, and the DS equation is the statement that this observable starts at 0. At resolution t, however, N_t is computed from objects at resolution t only: the observable $G_t(\Phi_t)$ of G, its derivative in the smoothed direction $J_t h$, and the noise ω_t. The terminal value is not needed. This matters as $T \to \infty$: for Φ^4_3 the terminal noise ω_T is not expected to converge to a random variable, so the terminal defect has no limit, while every N_t can. The Dyson–Schwinger equation of the limit is then the statement that, for every terminal functional G and every h, the process N is a martingale null at 0: an identity between observables, with no terminal representative needed. Pathwise, it says that the product $G_t(\Phi_t)\langle h, \omega_t\rangle$ of two martingales differs from a martingale by $\mathrm{D}_{J_t h} G_t(\Phi_t)$, just as the contact terms of (9.4) come from the derivative.

Formally letting $T \to \infty$, $K \to A$, and (9.6) becomes an exact identity between observables at every resolution:

$$A\,\Phi_t = A\,Q_t\,F_t + \omega_t, \tag{9.11}$$

with $A\,Q_t = \mathrm{e}^{-A/t^2}$ for the heat profile. This is the Euclidean field equation $A\varphi = -V'(\varphi) + \text{noise}$ of Lectures 1–2, seen at resolution t. It needs no products of distributions: the nonlinearity enters only through the force *observable* F_t, smeared at scale $1/t$. For the GFF ($F = 0$), (9.7) is Gaussian integration by parts for μ^{Q_t}. For Φ^4_3, where F_t plays the role of the renormalised cube, the limit $t \to \infty$ of (9.11) corresponds to the renormalised DS equation of [GH21]. Lecture 7 proves the field equation at every resolution for the limit (Theorem 18.9).

The field equation for the forecast. The same identity can be written for the forecast. Substituting $\Phi_t = \psi_t - S_t F_t$ (8.4) in (9.6), and using $K S_t = 1 - J_t$, the smoothing operator disappears:

$$K\psi_t = F_t + \omega_t, \qquad F_t = -\mathrm{D}\Gamma_t(\psi_t) = -\mathrm{D}V_t(\Phi_t). \tag{9.12}$$

As $T \to \infty$ the two forms of the field equation at resolution t are

$$A\,\Phi_t + \mathrm{e}^{-A/t^2}\mathrm{D}V_t(\Phi_t) = \omega_t \quad \text{(Polchinski process)}, \qquad A\,\psi_t + \mathrm{D}\Gamma_t(\psi_t) = \omega_t \quad \text{(forecast)}.$$

In the first, the force is smeared by e^{-A/t^2} so the nonlinearity is seen only through a smoothing operator. In the second it acts on the same variable as the linear part, with no smoothing: this is the Euler–Lagrange equation of the action

$$\frac{1}{2}\langle \psi, A\,\psi\rangle + \Gamma_t(\psi)$$

forced by the noise ω_t. For Φ^4, at finite T and to first order in λ, the Wetterich equation (8.13) gives

$$\Gamma_t \approx V_T + \frac{1}{2}\mathrm{Tr}(S_t \mathrm{D}^2 V_T),$$

and since $\partial_c H_4 = -6H_2$ and $c_T - S_t(x,x) = c_t$, the one loop moves the Wick ordering from the cutoff to the resolution:

$$K\psi_t + \frac{\lambda}{3}H_3(\psi_t; c_t) + (\text{mass counterterm and higher orders}) = \omega_t.$$

The cube acts directly on the forecast, and pairing with ψ_t gives

$$\langle \psi_t, K\psi_t\rangle + \frac{\lambda}{3}\int (\psi_t^4 - 3c_t\psi_t^2) = \langle \psi_t, \omega_t\rangle + \dots,$$

where $a^4 - 3c_t a^2 \geqslant \frac{1}{2}a^4 - \frac{9}{2}c_t^2$ and c_t is finite at fixed t: heuristically, a coercive bound up to a volume constant, driven by the sub-Gaussian noise of Section 9.4. With the bare Wick ordering c_T the quadratic term would diverge; in $d = 3$ only the mass counterterm γ_T, of order λ^2, still does. In the form for Φ_t the same positivity is hidden behind e^{-A/t^2}.

The integration by parts formula (9.7) also transfers: a functional of ψ_t is a functional of Φ_t, and $\mathrm{D}_x\psi_t = \mathrm{Cov}_t S_t^{-1}$ (Proposition 8.3), so

$$\mathbb{E}^{\mathbb{P}}[G(\psi_t)\langle h, K\psi_t - F_t\rangle] = \mathbb{E}^{\mathbb{P}}[\mathrm{D}_{\mathrm{Cov}_t \mathcal{R}_t h} G(\psi_t)].$$

The direction of differentiation is now random, the conditional covariance applied to $\mathcal{R}_t h$.

Remark 9.3. (Comparison with stochastic quantisation) In parabolic or elliptic stochastic quantisation (Lecture 3, §4) the field equation is imposed in an auxiliary time, and the field at fixed time has the right law. Here the field equation (9.11) holds at each resolution, and the auxiliary parameter is the resolution itself. Taking conditional expectations of an SPDE solution with respect to a scale filtration gives identities of the same shape, but with a different noise covariance: the martingale $J_t\,\mathrm{d}F_t$ in $\mathrm{d}\omega_t$ is essential [BCG26].

Corollary 9.4. (Dyson–Schwinger along the scales, formal) *Let $t_1 \leqslant \dots \leqslant t_k \leqslant t < T$, $h, h_1, \dots, h_k$ be test functions and $G = g(\langle \Phi_{t_1}, h_1\rangle, \dots, \langle \Phi_{t_k}, h_k\rangle)$. Then*

$$\mathbb{E}[G\langle \omega_t, h\rangle] = \mathbb{E}\sum_{i=1}^{k} (\partial_i g)(\cdots)\langle J_{t_i} h, h_i\rangle, \tag{9.13}$$

that is, the derivative of G in the direction which shifts the field at each scale s by $J_s h$. The increments are consistent: $\mathbb{E}[G\langle\omega_t-\omega_s,h\rangle]=0$ for $t_k\leqslant s\leqslant t$.

Proof. G is $\mathcal{F}_t$-measurable, so $\mathbb{E}[G\langle\omega_t,h\rangle]=\mathbb{E}[G\langle S_t^{-1}\zeta_t,h\rangle]$. By (7.14), $\Phi_s=J_s\Phi_T+\zeta_s$, the bridge covariance is $\mathbb{E}[\zeta_s\otimes\zeta_t]=Q_sK\,S_t$ for $s\leqslant t$, so the covariance of $\langle S_t^{-1}\zeta_t,h\rangle$ and $\langle\zeta_s,k\rangle$ is $\langle J_sh,k\rangle$. Apply Gaussian integration by parts in ζ, given Φ_T. The last claim is the martingale property of ω. □

Remark 9.5. (Closed versus local) At finite T the force is a closed observable, $F_t=\mathbb{E}_t[F_T]$. For Φ^4, $F_T(x)=-\frac{\lambda}{3}H_3(\Phi_T(x);c_T)-2\gamma_T H_1(\Phi_T(x);c_T)$ is local. As $T\to\infty$ one expects the force to converge as an observable. In $d=2$ the limit is closed, $F_\infty=-\frac{\lambda}{3}{:}\Phi_\infty^3{:}$. In $d=3$ it is expected not to be. If it were, $F_t=\mathbb{E}_t[F_\infty]$ with $F_\infty(h)\in L^2$, the DS equation of Theorem 9.1 would pass to the limit ($T\to\infty$, then $t\to\infty$), with F_∞ paired only with functionals of Φ_∞. Conditioning on $\sigma(\Phi_\infty)$ would then give a logarithmic derivative of the Φ^4_3 measure that is a square-integrable function of the field, $-\langle Ah,\Phi_\infty\rangle+\mathbb{E}[F_\infty(h)|\Phi_\infty]$. This is not expected: the regularised logarithmic derivatives do not converge in L^2 [ALZ06]. The limiting force would then be an observable in the proper sense, a compatible family with no terminal representative. Locality of the limiting force is the crux of the construction [BCG26].

9.4 Gaussian bounds for the noise

The noise observable ω_t is not Gaussian: it depends on the unknown law of the interacting field. It is a conditional expectation in two ways. In (9.9) it is the projection of the terminal noise ω_T, whose law depends on the interaction and which, for Φ^4_3, is not expected to have a limit as a random variable when $T\to\infty$. The filter gives a second representation, as the projection of a Gaussian whose law does not depend on the interaction. Conditioning can only reduce the spread, so the moments of ω_t are bounded by those of an explicit Gaussian, uniformly in the interaction.

One mode: the Brownian bridge. Consider a single mode with $Q_t=t$, so that X is a Brownian motion, $K=1/T$, $J_t=t/T$ and $S_t=T-t$. If the endpoint $\Phi_T=y$ were known, Φ would be a Brownian bridge from 0 to y, with Doob drift

$$F_t=\frac{y-\Phi_t}{T-t}.$$

By (7.14) its position at time t is a point of the straight line from 0 to y, plus a Gaussian deviation:

$$\Phi_t=\frac{t}{T}y+\zeta_t,\qquad \zeta_t\sim N\Big(0,\frac{t\,(T-t)}{T}\Big).$$

Then $K\,\Phi_t-J_tF_t=\zeta_t/(T-t)$: with a known endpoint the noise is the rescaled deviation from the straight line, a centred Gaussian with variance $t/(T\,(T-t))=(T-t)^{-1}-T^{-1}=\mathcal{R}_t$, whatever y. Now let the endpoint be random, with law ν_T. The observer at resolution t sees Φ_t, but not the line the path is following. The drift becomes the posterior mean of the bridge drift, and the noise the posterior mean of the rescaled deviation,

$$\omega_t=\mathbb{E}\left[\frac{\Phi_t-\frac{t}{T}\Phi_T}{T-t}\middle|\Phi_t\right],$$

an average of Gaussian deviations, weighted by the posterior law of the endpoint (Figure 9.1, Exercise 9.6). Equivalently $J_tF_t=K\,\Phi_t-\omega_t$, i.e. $F_t=\Phi_t/t-(T/t)\,\omega_t$. The dictionary with the field is:

	one mode	field
resolved covariance	t	Q_t
prior precision	$1/T$	$K=Q_T^{-1}$
attenuation	t/T	$J_t=KQ_t$
unresolved covariance	$T-t$	$S_t=Q_T-Q_t$
bridge deviation	$\zeta_t\sim N(0,t(T-t)/T)$	$\zeta_t\sim N(0,Q_t(1-J_t))$
regulator	$\frac{t}{T(T-t)}$	$\mathcal{R}_t=S_t^{-1}-K$
force	$\mathbb{E}_t[(\Phi_T-\Phi_t)/(T-t)]$	$F_t=\mathbb{E}_t[S_t^{-1}(\Phi_T-\Phi_t)]$
noise	$\mathbb{E}_t[(\Phi_t-\frac{t}{T}\Phi_T)/(T-t)]$	$\omega_t=\mathbb{E}_t[S_t^{-1}\zeta_t]$

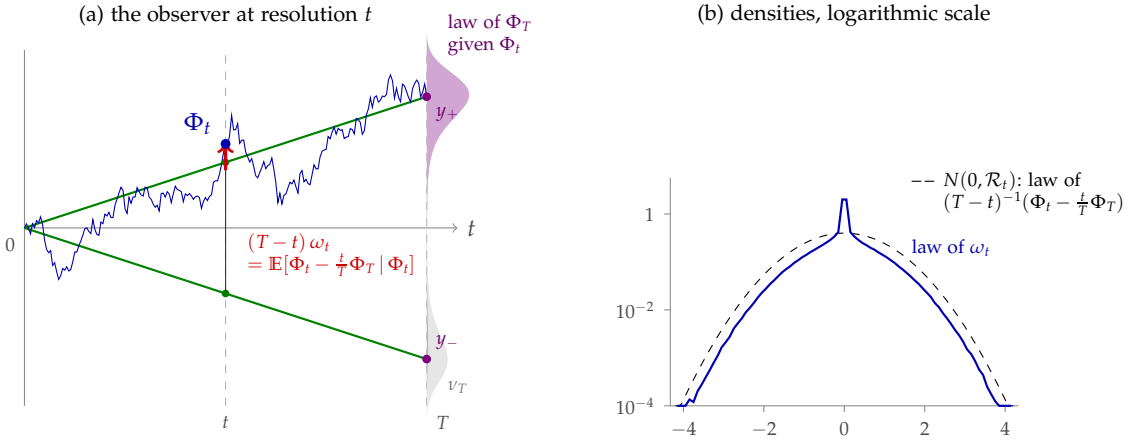


Figure 9.1. The noise as a posterior mean, for one mode with $Q_t=t$, $T=1$, $t=1/2$, and the double-well terminal law $\nu_T(\mathrm{d}y)\propto \mathrm{e}^{-2y^4+6y^2}N(0,1)(\mathrm{d}y)$ (grey, at time T). (a) A path observed up to time t (the faded part is not seen). For each possible endpoint $y_\pm$ the bridge follows the straight line $s\mapsto s\,y_\pm/T$ (green), and Φ_t deviates from it by a Gaussian amount. The observer weighs the endpoints by their posterior law given Φ_t (violet), and $(T-t)\,\omega_t$ is the posterior mean of the deviation (thick arrow). (b) The law of ω_t (solid) has thinner tails than the Gaussian $N(0,\mathcal{R}_t)$ of the deviation with a known endpoint (dashed): averaging thins the tails. The peak at 0 comes from the fields between the wells, where a small change of Φ_t moves the posterior from one well to the other and ω_t hardly changes, since the jump of the posterior mean compensates the motion of Φ_t: a fold of the map $\Phi_t\mapsto\omega_t$ (Exercise 9.6(c)).

The general case. Work at finite cutoff, where Φ_t has a Lebesgue density p_t, and let $0<t<T$. By Proposition 7.2,

$$-\mathrm{D}\log p_t(a)=Q_t^{-1}a-F_t(a),$$

and since $J_tQ_t^{-1}=K$,

$$\omega_t=-J_t\,\mathrm{D}\log p_t(\Phi_t). \tag{9.14}$$

The noise is minus the score of the law of the field at resolution t, seen through J_t.

Only deterministic covariance operators, all functions of A, are commuted below. The variance that appears is the regulator (8.6) of the Wetterich equation: since $S_t=Q_T(1-J_t)$,

$$J_t^2\,(Q_t(1-J_t))^{-1}=KJ_t(1-J_t)^{-1}=S_t^{-1}-Q_T^{-1}=\mathcal{R}_t. \tag{9.15}$$

Proposition 9.6. (Sub-Gaussian noise) *Let $0<t<T$. Then*

$$\omega_t=\mathbb{E}\big[S_t^{-1}\zeta_t|\Phi_t\big],$$

and ω_t is dominated in the convex order by $N(0,\mathcal{R}_t)$:

$$\mathbb{E}\,\Psi(\omega_t)\leqslant\mathbb{E}\,\Psi(\Xi),$$

$\Xi \sim N(0, \mathcal{R}_t)$, *for every convex* Ψ *for which both sides are defined. In particular, for every* h *and* $u \in \mathbb{R}$,

$$\mathbb{E}\, e^{u\langle h, \omega_t\rangle} \leqslant \exp\left(\frac{u^2}{2}\langle h, \mathcal{R}_t h\rangle\right): \tag{9.16}$$

every smeared noise observable is centred and sub-Gaussian, with the regulator as variance and with no convexity assumption on the interaction.

Proof. Write $\Sigma_t = Q_t(1 - J_t)$, so that $J_t \Sigma_t^{-1} = S_t^{-1}$, and g for the density of $N(0, \Sigma_t)$. Then

$$p_t(a) = \mathbb{E}\, g(a - J_t \Phi_T)$$

and $\mathrm{D}g(z) = -\Sigma_t^{-1} z\, g(z)$, so

$$-\mathrm{D}\log p_t(\Phi_t) = \mathbb{E}[\Sigma_t^{-1}\zeta_t | \Phi_t];$$

apply J_t and use (9.14). By conditional Jensen,

$$\Psi(\omega_t) \leqslant \mathbb{E}[\Psi(S_t^{-1}\zeta_t)|\Phi_t],$$

and $S_t^{-1}\zeta_t$ is centred Gaussian with covariance $S_t^{-1}\Sigma_t S_t^{-1} = \mathcal{R}_t$. □

What the observer cannot resolve. The proof is the one-mode picture. Given the terminal field, $S_t^{-1}\zeta_t$ is the free bridge noise, Gaussian with covariance $\mathcal{R}_t$ and independent of Φ_T: the interaction does not enter. The observer at resolution t does not know Φ_T and replaces the bridge noise by its posterior mean. The interaction enters only through the posterior law $\mu_t^{\Phi_t}$ of Φ_T, the weights of Figure 9.1. Since $\zeta_t = \Phi_t - J_t\Phi_T$, the part that is averaged out is $S_t^{-1} J_t$ applied to the posterior fluctuation of the terminal field, and the decomposition of the variance gives the exact identity

$$\mathrm{Cov}(\omega_t) = \mathcal{R}_t - S_t^{-1} J_t\, \mathbb{E}[\mathrm{Cov}_t]\, J_t S_t^{-1}, \tag{9.17}$$

with $\mathrm{Cov}_t = \mathrm{Cov}(\Phi_T|\mathcal{F}_t)$. The noise is the free bridge noise minus what the observer cannot resolve. The two extremes are the free field, where $\mathrm{Cov}_t = S_t$ and (9.17) gives KJ_t, the covariance of KX_t, and a strong pinning potential, where $\mathrm{Cov}_t \to 0$ and the bound is attained (Exercise 9.5). By Proposition 8.3, $\mathrm{Cov}_t = S_t - S_t \mathrm{D}^2 V_t(\Phi_t)\, S_t$, so that

$$\mathrm{Cov}(\omega_t) = KJ_t + J_t\, \mathbb{E}[\mathrm{D}^2 V_t(\Phi_t)]\, J_t:$$

the free value, corrected by the mean curvature of the effective potential. This is the analogue at resolution t of Nelson's covariance $-\Delta + m^2 + \mathbb{E}[V''(\varphi)]$, and here it is finite at every resolution. Proposition 9.6 is the same comparison for all convex functions, rather than for the variance alone.

For the heat profile (compare (8.15)), $K = A\, e^{A/T^2}$ and $J_t = e^{-A(t^{-2} - T^{-2})}$ give

$$\mathcal{R}_t = \frac{A\, e^{A/T^2}}{e^{A(t^{-2} - T^{-2})} - 1} \leqslant \frac{4}{3} t^2 \quad (T \geqslant 2t), \qquad \mathcal{R}_t = \frac{A}{e^{A/t^2} - 1} \leqslant t^2 \quad (T = \infty)$$

(Exercise 9.5). At a finite spatial cutoff all of this holds with terminal resolution $T = \infty$, before any continuum measure is constructed.

A new observation. The bound (9.16) is, as far as we know, a new observation of these notes. Its proof is elementary, conditional Jensen applied to a Gaussian, but its consequence is not: at every resolution the field solves an elliptic equation (Theorem 9.1) whose noise has the tails of a Gaussian with covariance the regulator $\mathcal{R}_t$, uniformly in the interaction and in the cutoff, without any convexity or smallness assumption. The interaction enters the noise only through the conditioning, and conditioning cannot make Gaussian tails worse. This is what makes the field equation usable beyond perturbation theory, and it will be crucial for the a priori estimates of the FBSDE in Lecture 7: there the large fields of the forward equation are controlled by the coercivity of the cube against a noise whose tails are known in advance.

Tails of the force. The noise is the score of the law of Φ_t. The force $F_t = -\mathrm{D}V_t(\Phi_t) = -\mathrm{D}\Gamma_t(\psi_t)$ of (9.12) is the gradient of the effective potential, and its tails can be read directly from the measure at resolution t, $\nu_t \propto \mathrm{e}^{-V_t}\mu^{Q_t}$ (Proposition 7.2), when Γ_t is convex.

Proposition 9.7. (Sub-Gaussian force) *At finite cutoff, let $0<t<T$, let V_T be even, and assume that Γ_t is convex. Then for every h and $u\in\mathbb{R}$,*

$$\mathbb{E}\,\mathrm{e}^{u\langle h,F_t\rangle} \leqslant \exp\left(\frac{u^2}{2}\langle h, S_t^{-1}h\rangle\right). \tag{9.18}$$

Proof. Two properties of V_t are used. (a) *Semiconcavity.* By Proposition 8.3, $\mathrm{D}^2V_t = S_t^{-1} - S_t^{-1}\mathrm{Cov}_t S_t^{-1} \leqslant S_t^{-1}$ for every interaction, so

$$V_t(x+a) \leqslant V_t(x) + \langle \mathrm{D}V_t(x), a\rangle + \frac{1}{2}\langle a, S_t^{-1}a\rangle.$$

(b) *Convexity.* By (8.12), V_t is the infimal convolution of Γ_t with the convex quadratic $\frac{1}{2}\langle\cdot, S_t^{-1}\cdot\rangle$, hence convex. It is even, since V_T is and the Polchinski semigroup commutes with $x\mapsto -x$.

Step 1. By (a) with $a = u\,h$,

$$\mathrm{e}^{u\langle h,F_t(x)\rangle - V_t(x)} \leqslant \mathrm{e}^{\frac{u^2}{2}\langle h,S_t^{-1}h\rangle - V_t(x+uh)}.$$

Integrate against μ^{Q_t} and shift by the Cameron–Martin formula, $\int G(x+a)\,\mu^{Q_t}(\mathrm{d}x) = \mathrm{e}^{-\frac{1}{2}\langle a,Q_t^{-1}a\rangle}\int G(y)\,\mathrm{e}^{\langle Q_t^{-1}a,y\rangle}\mu^{Q_t}(\mathrm{d}y)$:

$$\mathbb{E}\,\mathrm{e}^{u\langle h,F_t\rangle} \leqslant \mathrm{e}^{\frac{u^2}{2}\langle h,(S_t^{-1}-Q_t^{-1})h\rangle}\;\mathbb{E}\,\mathrm{e}^{u\langle Q_t^{-1}h,\Phi_t\rangle}. \tag{9.19}$$

Step 2. For every k the same shift gives $\mathbb{E}\,\mathrm{e}^{\langle k,\Phi_t\rangle} = \mathrm{e}^{\frac{1}{2}\langle k,Q_tk\rangle}m(Q_tk)\,/\,m(0)$, with $m(a) = \int \mathrm{e}^{-V_t(x+a)}\mu^{Q_t}(\mathrm{d}x)$. By (b) and Prékopa's theorem m is log-concave: it is the marginal of the log-concave function $(x,a)\mapsto \mathrm{e}^{-V_t(x+a)-\frac{1}{2}\langle x,Q_t^{-1}x\rangle}$. It is also even, so $\log m(0) \geqslant \frac{1}{2}\log m(a) + \frac{1}{2}\log m(-a) = \log m(a)$. Hence $\mathbb{E}\,\mathrm{e}^{\langle k,\Phi_t\rangle} \leqslant \mathrm{e}^{\frac{1}{2}\langle k,Q_tk\rangle}$: the field at resolution t is no wider than the free field X_t. With $k = u\,Q_t^{-1}h$ in (9.19) the two terms with Q_t^{-1} cancel. □

Remark 9.8. (Where convexity enters) That convexity of Γ_t makes the force sub-Gaussian was remarked by Marcello Porta during the lectures. Step 1 uses only $\mathrm{D}^2V_t \leqslant S_t^{-1}$, which holds for every interaction. Convexity of Γ_t enters only in Step 2, to compare Φ_t with the free field. It is equivalent to $\mathrm{Cov}_t \leqslant S_t$ (Section 8.5): it holds at every t when V_T is convex, but for a double well, and for the Wick-ordered Φ^4 interaction, it can fail at low resolution, where the law of Φ_t is wider than μ^{Q_t}. Without convexity, use the bridge (7.14) instead, $\Phi_t = J_t\,\Phi_T + \zeta_t$ with $\zeta_t \sim N(0,\Sigma_t)$ independent of Φ_T. Since $J_tQ_t^{-1} = K$ and $Q_t^{-1}\Sigma_tQ_t^{-1} = Q_t^{-1} - K$, (9.19) becomes

$$\mathbb{E}\,\mathrm{e}^{u\langle h,F_t\rangle} \leqslant \exp\left(\frac{u^2}{2}\langle h, \mathscr{R}_t h\rangle\right)\;\mathbb{E}_{\nu_T}\mathrm{e}^{u\langle Kh,\Phi_T\rangle},$$

which also follows from $F_t = \mathbb{E}[K\,\Phi_T - S_t^{-1}\,\zeta_t \,\big|\, \Phi_t]$ (by (9.12) and Proposition 9.6) and conditional Jensen. The force is the noise plus the terminal field seen through K: unlike the noise, it is sub-Gaussian uniformly in the interaction only when the terminal field is. For convex even V_T, Step 2 at $t=T$ bounds the last factor by $\mathrm{e}^{\frac{u^2}{2}\langle h,Kh\rangle}$, and since $\mathscr{R}_t + K = S_t^{-1}$ this recovers (9.18). For one mode with $V_T(y) = \kappa y^2/2$, $F_t = -\kappa_t\Phi_t$ with $\kappa_t = \kappa/(1+\kappa S_t)$, and $\mathrm{Var}F_t = \kappa_t^2/(Q_t^{-1}+\kappa_t)$ increases to $\mathscr{R}_t$ as $\kappa\to\infty$: the bound $S_t^{-1} = \mathscr{R}_t + K$ is attained up to the term K.

Convexity of the effective average action $\Gamma_t^{\mathrm{EAA}} = \Gamma_t + \frac{1}{2}\langle \cdot, K \cdot \rangle$ (Section 8.6) is not enough. Its gradient at the forecast is the noise, $\mathrm{D}\Gamma_t^{\mathrm{EAA}}(\psi_t) = K\psi_t - F_t = \omega_t$ by (9.12), which is sub-Gaussian with no assumption at all (Proposition 9.6); the force differs from it by $K\psi_t$. Since $\mathrm{D}^2\Gamma_t^{\mathrm{EAA}} = \mathrm{Cov}_t^{-1} - \mathscr{R}_t$ and $\mathscr{R}_t = S_t^{-1} J_t$, Γ_t^{EAA} is convex if and only if $\mathrm{Cov}_t \leqslant J_t^{-1} S_t$, i.e. $\mathrm{D}^2 V_t \geqslant -Q_t^{-1}$: the law of Φ_t is log-concave. This holds at $t = 0$, where Γ_0^{EAA} is a Legendre transform, but a log-concave law may be much wider than μ^{Q_t}, and Step 2 needs $\mathrm{D}^2 V_t \geqslant 0$. For one mode with $Q_T = 1$, $Q_t = q$ and the negative mass $V_T(y) = -(1-\varepsilon)\, y^2/2$, every ν_t is Gaussian, so Γ_t^{EAA} is convex, but $\mathrm{Var} F_t = (1-\varepsilon)^2 q / (\varepsilon\,(\varepsilon + q - q\varepsilon)) \sim 1/\varepsilon$ while $S_t^{-1} = 1/(1-q)$.

The roughness of the noise. Compare spectral densities for the heat profile with $T = \infty$ and $A = |k|^2 + m^2$: white noise ξ has density 1, the GFF has A^{-1}, the GFF noise $\omega_t = A X_t$ has $A\,\mathrm{e}^{-A/t^2}$, and the bound of Proposition 9.6 is

$$\mathscr{R}_t(k) = \frac{A}{\mathrm{e}^{A/t^2} - 1} \approx \begin{cases} t^2, & |k| \ll t, \\ A\,\mathrm{e}^{-A/t^2}, & |k| \gg t. \end{cases}$$

Since $x/(\mathrm{e}^x - 1) \leqslant \mathrm{e}^{-x/2}$, $\mathscr{R}_t \leqslant t^2 \mathrm{e}^{-A/2t^2}$: at resolution t the noise is no larger than t times a white noise smoothed at scale $1/t$. Pointwise, $\mathrm{Var}\ \omega_t(x) \leqslant \mathscr{R}_t(x,x) \lesssim t^{d+2}$, i.e. $\omega_t(x) = O(t^{1+d/2})$. This is the size at scale $1/t$ of a distribution of regularity $-1 - d/2$ ($-5/2$ in $d = 3$): seen at its own resolution, the noise is one derivative rougher than white noise ($-d/2$) and two rougher than the GFF ($1 - d/2$), the same as the GFF noise $A\varphi = A^{1/2}\xi$ of the Dyson–Schwinger equation, whatever the interaction. It is not a bound on a limit distribution: ω_t is a martingale in t, not the smoothing of a fixed distribution, and on a fixed test function (9.16) only gives $\langle h, \omega_t \rangle = O(t)$, while the free noise stays bounded. For Φ_3^4 no L^2 limit of the logarithmic derivative is expected [ALZ06]. Compared with the free noise, the bound is sharp at and above the resolution, where $\mathscr{R}_t/(A\,\mathrm{e}^{-A/t^2}) = (1 - \mathrm{e}^{-A/t^2})^{-1} \to 1$, and larger by the factor t^2/A on the resolved modes. Uniformly in the interaction it cannot be improved: for a pinning potential $V_T = \frac{\kappa}{2}\|\varphi\|^2$, the covariance of ω_t tends to $\mathscr{R}_t$ as $\kappa \to \infty$ (Exercise 9.5).

The noise as a stochastic observable. In the language of Lecture 3, the results of this section say the following. The family $(\omega_t)_{t \leqslant T}$ is an observable: a martingale in the filtration of the measurements, whose terminal value is Nelson's Euclidean noise $\omega_T = K\Phi_T + \mathrm{D}V_T(\Phi_T)$ of (9.9). It is expected to be a *local* observable in the sense of Lecture 3 (Section 5.2). Formally, as $T \to \infty$, $K \to A$ and the terminal value becomes $(A\varphi)(x) + v'(\varphi(x))$, a function of the field and of its second derivatives at x; one expects germs of the same form, $A\Phi_t(x) + w_t(\Phi_t(x))$ with a renormalised local nonlinearity w_t (the Wick cube $\frac{\lambda}{3} H_3(\cdot; c_t)$ and a mass counterterm). Since $\omega_t + J_t F_t = K\Phi_t$, whose limit $A\Phi_t$ is local and linear, the noise is local exactly when the force is, and for Φ_3^4 this is the crux of the construction (Remark 9.5). At finite T, and for Φ_2^4, the noise is also closed; for Φ_3^4 it is expected to be local but not closed, like the point evaluations of the free field in Lecture 3. At each resolution it is a random variable, indeed a smooth random field, and every smeared $\langle h, \omega_t \rangle$ is centred and sub-Gaussian, with variance at most $\langle h, \mathscr{R}_t h \rangle$, uniformly in the interaction and in the cutoff (9.16). Paired with functionals of the field up to resolution t, it acts as the transpose of the smoothed derivative, $\langle h, \omega_t \rangle = (\mathrm{D}_{J_t h})^{\top} 1$ in $L^2(\nu_t)$ (9.7, 9.13). At its own resolution it has the size of $A^{1/2}\xi$: formally it behaves like one derivative of white noise, *whatever the interaction*. What may be lost as $T \to \infty$ is only the terminal value. For Φ_3^4 the Euclidean noise $A\varphi + V'(\varphi)$ is not expected to be a random variable, as Nelson anticipated [Nel75, ALZ06]; but its conditional expectations at every finite resolution are, and they form a consistent family, $\omega_s = \mathbb{E}_s \omega_t$ for $s \leqslant t$.

This is the sense in which the Euclidean field equation of Φ_3^4 has a well-defined sub-Gaussian noise: it exists as a stochastic observable, without a terminal representative.

Remark 9.9. (Pathwise bounds) (sketch) Work at finite spatial cutoff, with $T=\infty$ or $t\leqslant T/2$, so that $\mathcal{R}_t$ obeys the bounds below uniformly in the cutoff. The norm

$$\|f\|_\varepsilon = \sup_x (1+|x|)^{-\varepsilon}|f(x)|$$

is convex, so Proposition 9.6 dominates $\|\omega_t\|_\varepsilon$ by $\|\Xi\|_\varepsilon$, with $\Xi\sim N(0,\mathcal{R}_t)$. For this Gaussian field, $\mathcal{R}_t(x,x)\lesssim t^{d+2}$ and

$$\langle \delta_x-\delta_y, \mathcal{R}_t(\delta_x-\delta_y)\rangle \lesssim t^{d+4}|x-y|^2,$$

so Dudley's bound and Gaussian concentration give Gaussian tails for $\|\Xi\|_\varepsilon$ at the level $t^{1+d/2}\sqrt{\log(2+t)}$, uniformly in L. Since ω is a martingale, $\|\omega_s\|_\varepsilon$ is a submartingale, and Doob's inequality extends the tails to $\sup_{s\leqslant t}$. Summing over dyadic t, almost surely, with a random constant $\tilde{C}$,

$$\|\omega_t\|_\varepsilon \leqslant \tilde{C}\, t^{1+d/2}\sqrt{\log(2+t)} \qquad \text{for all } t\geqslant 1 \quad (1\leqslant t\leqslant T/2 \text{ if } T<\infty),$$

with $\tilde{C}$ having Gaussian tails uniformly in the interaction and in the spatial cutoff. This is a pathwise bound at each resolution, in the sense above.

Control of the force. Write $\mathrm{d}F_s = \mathrm{Y}_s\,\mathrm{d}W_s$ as in Section 8. By Theorem 9.1,

$$\mathrm{d}\omega_s = (K\dot{Q}_s^{1/2} - J_s\mathrm{Y}_s)\,\mathrm{d}W_s,$$

so

$$\int_0^t J_s\mathrm{Y}_s\,\mathrm{d}W_s = K\int_0^t \dot{Q}_s^{1/2}\,\mathrm{d}W_s - \omega_t.$$

Itô's isometry, $(a+b)^2\leqslant 2a^2+2b^2$ and (9.16) give

$$\mathbb{E}\int_0^t \|\mathrm{Y}_s^* J_s h\|^2\,\mathrm{d}s \leqslant 2\langle h,\mathcal{R}_t h\rangle + 2\langle h, KQ_tKh\rangle. \tag{9.20}$$

For the heat profile with $T=\infty$, $KQ_tK = A\,\mathrm{e}^{-A/t^2}\leqslant t^2/\mathrm{e}$, so the right-hand side is at most $2(1+\mathrm{e}^{-1})\,t^2\|h\|_{L^2}^2$. This controls a smoothed derivative of the force, $\mathrm{Y}_s = \mathrm{D}F_s(\Phi_s)\,\dot{Q}_s^{1/2}$, without any estimate on the terminal force.

Consequence for Φ^4_3. In the field equation (9.11) the noise is therefore controlled a priori at each fixed resolution, uniformly in the ultraviolet cutoff. A bound on Φ_t uniform in the ultraviolet cutoff then follows from a coercivity estimate for the renormalised effective force F_t, which converts the control of $A\Phi_t - AQ_tF_t$ into control of Φ_t. This is the remaining analytic input, and it is where the large-field problem of Lecture 6 enters. Lecture 7 carries this out: its forward estimate is the forecast form (9.12) of this field equation, with the explicit force of Lecture 6 in place of the exact one, and with the bound of Proposition 9.6.

9.5 Infinite volume

On $\mathbb{T}_L^d$ the free energy $V_0(0)$ and the effective potential V_t are extensive: they grow like L^d, and (7.9) has no limit as $L\to\infty$. Three objects are intensive and survive formally:

i. *the force.* For a local interaction $V_T(\varphi)=\int v_T(\varphi(x))\mathrm{d}x$, $\mathrm{D}V_T(\varphi)(x) = v_T'(\varphi(x))$, and (7.7) reads, on $\mathbb{R}^d$,

$$\Phi_t = \int_0^t \dot{Q}_s F_s\,\mathrm{d}s + \int_0^t \dot{Q}_s^{1/2}\mathrm{d}W_s, \qquad F_t(x) = -\mathbb{E}_t[v_T'(\Phi_T(x))]. \tag{9.21}$$

Here every quantity is local, and the equation makes sense without a volume cutoff. Infinite-volume limits of $\nu_T^{(L)}$ are expected to be laws of solutions of (9.21). Uniqueness of such solutions (e.g. for small coupling, or by contraction in weighted spaces) is a uniqueness statement for the infinite-volume theory. Non-uniqueness is expected in the presence of phase transitions, e.g. for Φ^4_2 with a deep double-well potential [GJS75];

ii. *the Burgers flow* (8.1) of the force, for the same reason;

iii. *differences of free energies.* For a local f, $\mathcal{W}_T^{(L)}(f)$ in (7.12) is a difference of two extensive infima, and it stays bounded as $L \to \infty$. Its variational characterisation, through the infinite-volume Euler–Lagrange equation (9.21), is the basis of the infinite-volume analysis of Φ_2^4 and $\exp(\Phi)_2$ in [BG23].

Differences of free energies. Fix the volume, and let u^* be the optimal drift of (7.9), so that $X^{u^*} = \Phi$ is the Polchinski process (Theorem 7.6). In (7.12) the infimum may be taken over perturbations $u = u^* + w$ of the optimal drift. The drift enters linearly, so $X^{u^*+w} = \Phi + I(w)$ with $I_t(w) = \int_0^t \dot{Q}_s^{1/2} w_s \, \mathrm{d}s$, and by the Euler–Lagrange equation (7.11) the cross term of the energy is $\mathbb{E}\int_0^T \langle u_s^*, w_s \rangle \, \mathrm{d}s = -\mathbb{E}\langle \mathrm{D}V_T(\Phi_T), I_T(w) \rangle$. The two extensive infima then cancel exactly, and what is left is a variational problem for the perturbation alone.

Proposition 9.10. (Variational principle relative to the FBSDE) (*formal*) *Let Φ be the Polchinski process, i.e. the solution of the FBSDE (7.7) driven by the Brownian motion W, and ν_T the law of Φ_T. For every f with $\mathrm{e}^{-f} \in L^1(\nu_T)$ and $f(\Phi_T + I_T(w))$ integrable for the admissible w below (for instance f bounded from below, or linear),*

$$-\log \int \mathrm{e}^{-f} \, \mathrm{d}\nu_T = \inf_w \mathbb{E}\Big[f(\Phi_T + I_T(w)) + B_{V_T}(\Phi_T, I_T(w)) + \frac{1}{2}\int_0^T \|w_s\|^2 \mathrm{d}s \Big], \tag{9.22}$$

where the infimum runs over adapted w with $\mathbb{E}\int_0^T \|w_s\|^2 \mathrm{d}s < \infty$, and

$$B_V(\phi, h) := V(\phi + h) - V(\phi) - \langle \mathrm{D}V(\phi), h \rangle$$

is the remainder of the first-order Taylor expansion of V. The minimiser satisfies $w_s = -\dot{Q}_s^{1/2}\, \mathbb{E}_s[\mathrm{D}f(\Phi_T^f) + \mathrm{D}V_T(\Phi_T^f) - \mathrm{D}V_T(\Phi_T)]$, with $\Phi^f = \Phi + I(w)$: the perturbed field solves the FBSDE with terminal potential $V_T + f$, and its force is that of Φ corrected by $F_s^f - F_s = -\mathbb{E}_s[\mathrm{D}f(\Phi_T^f) + \mathrm{D}V_T(\Phi_T^f) - \mathrm{D}V_T(\Phi_T)]$.

Proof. By (7.12), the left-hand side is $\inf_u J_f(u) - J_0(u^*)$, with

$$J_f(u) = \mathbb{E}\Big[f(X_T^u) + V_T(X_T^u) + \frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s \Big].$$

Writing $u = u^* + w$ and expanding $\frac{1}{2}\|u^* + w\|^2$, the terms $V_T(\Phi_T)$ and $\frac{1}{2}\int \|u^*\|^2$ cancel against $J_0(u^*)$, and the cross term is computed above. The first-order condition in w is obtained as in (7.11). □

For a local interaction $V_T(\varphi) = \int v_T(\varphi(x)) \, \mathrm{d}x$,

$$B_{V_T}(\phi, h) = \int b_T(\phi(x), h(x)) \, \mathrm{d}x, \qquad b_T(a, h) = v_T(a + h) - v_T(a) - v_T'(a)\, h,$$

and $b_T(a, 0) = 0$. So every term of (9.22) is local: there is no free energy and no extensive quantity, only the field Φ and the perturbation that f induces on it. For f supported in a bounded region one expects the optimal w, and with it $I(w)$, to decay away from the support, at the rate of the decay of correlations of the measure. Both sides of (9.22) then make sense in infinite volume, with Φ a solution of the infinite-volume FBSDE (9.21). This suggests the following formal description of the infinite-volume measures (at a fixed ultraviolet cutoff T):

an infinite-volume measure is the law of Φ_T for a solution Φ of (9.21) which satisfies (9.22), with $B = \int b_T \, \mathrm{d}x$, for all local f.

The Laplace functionals in (9.22) determine the measure, and the condition involves only intensive quantities. When the FBSDE has several solutions, as is expected in the presence of phase transitions, each solution gives a measure with its own variational principle; the perturbations w of finite energy only probe a given solution locally. For Φ^4_3 this remainder term contains the divergent counterterms, and, as for (7.12), the change of variables of [BG20] is needed before (9.22) has bounds uniform in T; in two dimensions, compare the infinite-volume analysis of [BG23].

The coercive estimates must then be *local*: one uses weighted norms and needs bounds whose constants do not depend on the volume. This infrared problem is the analogue, for the scale dynamics, of the "coming down from infinity" estimates for the parabolic dynamics.

Exercises

Exercise 9.1. **(Doob transform)** Show that G_t in (9.1) solves (9.2), by writing $\mathrm{e}^{-V_t}G_t = \mathrm{e}^{\frac{1}{2}\Delta_{Q_{T,t}}}(\mathrm{e}^{-V_T}G)$. Prove (9.3).

Exercise 9.2. **(The relative variational principle)** (a) Derive (9.22) and the equation for the minimiser. (b) For a Gaussian ν_T (quadratic V_T) and a linear $f = \langle g, \cdot\rangle$, compute the remainder term, solve (9.22), and recover $-\log \int \mathrm{e}^{-\langle g,\cdot\rangle}\,\mathrm{d}\nu_T = \langle g, m\rangle - \frac{1}{2}\langle g, \Sigma g\rangle$, with m and Σ the mean and covariance. (c) For a local interaction and f supported in a ball, show that B_{V_T} only sees the region where $I(w) \neq 0$, and discuss why (9.22) is bounded uniformly in the volume.

Exercise 9.3. **(DS from the dynamics)** Fill in the Itô computations in the proof of Proposition 9.2, and deduce (9.7) without using Proposition 7.2, for G of the form $G_t(\Phi_t)$ with G_t solving (9.2). Where is the symmetry of $\mathrm{D}F_t$ used? Compute the quadratic variation of N for the free field and G linear.

Exercise 9.4. **(Gaussian check)** For $V_T = 0$ show that (9.7) reduces to $\mathbb{E}\,[\mathrm{D}_{Q_t Kh} G(X_t)] = \mathbb{E}[G(X_t)\langle Kh, X_t\rangle]$, and compare with the DS characterisation of the GFF in Lecture 2.

Exercise 9.5. **(Sub-Gaussian noise)** (a) Check (9.14) and the bridge representation, and show that the Girsanov density depends only on X_T. (b) Compute $\mathcal{R}_t$ for the heat profile, and show that $\mathcal{R}_t \leqslant \frac{4}{3}t^2$ for $T \geqslant 2t$ and $\mathcal{R}_t \leqslant t^2$ for $T = \infty$. (c) Compute the law of ω_t for $V_T = 0$ and for $V_T = \frac{\kappa}{2}\|\varphi\|^2$, and show that its covariance tends to $\mathcal{R}_t$ as $\kappa \to \infty$. In what sense is (9.16) sharp? (d) Show that $\mathcal{R}_t(x,x) \asymp t^{d+2}$ for $t \gg m$, and complete the argument of Remark 9.9.

Exercise 9.6. **(The noise of a two-point endpoint)** Take one mode with $Q_t = t$, and let the endpoint have the law $\nu_T = \frac{1}{2}(\delta_a + \delta_{-a})$ (a limit of deep double wells). (a) Show that $\mathbb{E}[\Phi_T|\Phi_t = x] = a\tanh(a\,x/(T-t))$ and compute ω_t as a function of Φ_t. (b) Check (9.17), and that $\mathrm{Var}(\omega_t) \leqslant \mathcal{R}_t$. (c) Show that ω_t is not monotone in Φ_t when $t\,a^2 > T\,(T-t)$, and relate this to the peak in Figure 9.1.

Exercise 9.7. **(Contact terms)** Verify (9.4) for $n = 1$ and the free field, using (1.6), and check the Euclidean rotation of the contact term.

Appendices: additional material

Appendix A Algebraic structure of the Polchinski flow

A.1 A semigroup of renormalisation maps

For a positive covariance S define the renormalisation map

$$\mathfrak{R}_S(V) := -\log\,(\mu^S * \mathrm{e}^{-V}), \qquad (\mu^S * u)(\phi) = \int u\,(\phi+\zeta)\,\mu^S(\mathrm{d}\zeta). \tag{A.1}$$

Since $\mu^S * \mu^{S'} = \mu^{S+S'}$,

$$\mathfrak{R}_S \circ \mathfrak{R}_{S'} = \mathfrak{R}_{S+S'} = \mathfrak{R}_{S'} \circ \mathfrak{R}_S, \qquad V_t = \mathfrak{R}_{Q_{T,t}}(V_T). \tag{A.2}$$

So the effective potentials are an orbit of an *abelian* semigroup, indexed by the cone of positive covariances, acting nonlinearly on interactions and linearly on Boltzmann factors e^{-V}. Its infinitesimal generators are the vector fields

$$\mathcal{D}_B(V) := \tfrac{1}{2}\mathrm{Tr}(B\,\mathrm{D}^2 V) - \tfrac{1}{2}\langle \mathrm{D}V, B\,\mathrm{D}V\rangle, \qquad \tfrac{\mathrm{d}}{\mathrm{d}s}\mathfrak{R}_{sB}(V)\Big|_{s=0} = \mathcal{D}_B(V), \tag{A.3}$$

and (7.8) reads $\partial_t V_t = -\mathcal{D}_{\dot{Q}_t}(V_t)$.

Proposition A.1. (Flatness) (formal) *(a)* $[\mathcal{D}_B, \mathcal{D}_{B'}] = 0$ *for all positive* B, B'. *(b)* V_t *depends on the profile only through* $Q_{T,t}$. *In particular the free energy* $V_0(0)$ *and the law* ν_T *depend only on* Q_T, *not on the path* $t \mapsto Q_t$.

Proof. (a) Differentiate $\mathfrak{R}_{sB}\mathfrak{R}_{rB'} = \mathfrak{R}_{rB'}\mathfrak{R}_{sB}$ at $s = r = 0$, or compute directly (Exercise A.1). (b) is (A.2). □

Different profiles with the same endpoints are different *coordinate systems* on the same renormalisation group trajectory. They change the process Φ and the observables at intermediate resolution, but not the terminal theory. The flow has no holonomy [BCG26].

A.2 Perturbation theory and counterterms

Expanding $V_t = \mathfrak{R}_{Q_{T,t}}(V_T)$ in cumulants of the Gaussian increment $\zeta \sim \mu^{Q_{T,t}}$ gives

$$V_t(\phi) = \mathbb{E}_\zeta[V_T(\phi+\zeta)] - \tfrac{1}{2}\mathrm{Var}_\zeta(V_T(\phi+\zeta)) + \dots \tag{A.4}$$

For (7.1), Exercise 6.1 of Lecture 3 shows that Gaussian convolution simply *lowers the Wick ordering*: since $Q_{T,t}(x,x) = c_T - c_t$,

$$V_t(\phi) = \int \left[\frac{\lambda}{12} H_4(\phi(x); c_t) + \gamma_T H_2(\phi(x); c_t)\right] \mathrm{d}x + O(\lambda^2). \tag{A.5}$$

Here $\gamma_T = O(\lambda^2)$ is kept for later use. At first order the effective potential is the Wick-ordered potential at the current resolution. This is the dual statement of Proposition 6.1 of Lecture 3: $\int H_4(X_t; c_t)$ is an observable.

Proposition A.2. (Second order) (formal) *Write* $S = Q_{T,t}$ *and* $:\phi^p:_t = H_p(\phi; c_t)$. *Then*

$$V_t(\phi) = \int \left[\frac{\lambda}{12} :\phi^4:_t + \gamma_T :\phi^2:_t\right] - \frac{\lambda^2}{288}\sum_{k=1}^{4} \binom{4}{k}^2 k! \iint S(x,y)^k :\phi^{4-k}:_t(x) :\phi^{4-k}:_t(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{A.6}$$

up to $O(\lambda^3)$.

Proof. By the binomial formula for Hermite polynomials,

$$H_4(\phi+\zeta; c_t + s) = \sum_{k=0}^{4} \binom{4}{k} H_k(\zeta; s)\, H_{4-k}(\phi; c_t), \qquad s = S(x,x).$$

Then compute the variance in (A.4) with Lemma 3.8 of Lecture 2. □

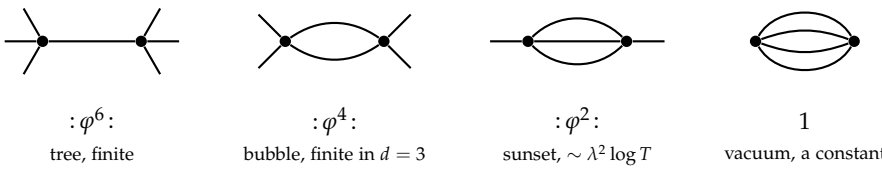


Figure A.1. The terms $k = 1,2,3,4$ of (A.6), with k lines S joining two Wick-ordered vertices; Wick ordering has removed the lines from a vertex to itself. As $T \to \infty$ in $d = 3$, only the sunset, $k = 3$, is a non-integrable divergence for the $:\varphi^2:$ term; it is cancelled by γ_T.

Now let $T \to \infty$ at fixed t, so that $S \to C - Q_t$, which near the diagonal is as singular as G. In $d = 2$ every S^k is locally integrable, so with $\gamma_T = 0$ every term has a limit: Wick ordering suffices. In $d = 3$, $G(z) \sim (4\pi|z|)^{-1}$:

- $k = 1, 2$: S and S^2 are locally integrable. These terms are finite and nonlocal.
- $k = 3$ (the *sunset*): $S^3 \sim |z|^{-3}$ is not integrable. Writing $\phi(x)\,\phi(y) = \frac{1}{2}(\phi(x)^2 + \phi(y)^2) - \frac{1}{2}(\phi(x) - \phi(y))^2$, up to a field-independent constant the divergent part is local:

$$-\frac{\lambda^2}{3}\Big(\int S^3\Big) \int :\phi^2:_t + \text{(finite)},$$

  with $\int S^3 \sim \frac{1}{16\pi^2} \log T$. It is cancelled by

$$\gamma_T := \frac{\lambda^2}{3} \int_{\mathbb{T}_L^3} Q_T(z)^3 \, \mathrm{d}z = \frac{\lambda^2}{48\pi^2} \log T + O(1), \tag{A.7}$$

  since $\gamma_T - \frac{\lambda^2}{3} \int S^3$ converges (Exercise A.2).
- $k = 4$: a divergent constant $-\frac{\lambda^2}{12} L^3 \int S^4 \sim T$, the vacuum energy. It cancels in ν_T but not in the free energy.

Power counting (Exercise A.3) shows that no further field-dependent counterterms are needed in $d = 3$: Φ_3^4 is superrenormalisable. Thus renormalisation appears as a property of the flow: one chooses the terminal data V_T so that the effective potentials V_t at *fixed* resolution have limits. The divergences at second order are also related to the singularity of Φ_3^4 with respect to the GFF [BG21]: the measures ν_t at fixed t are expected to stay absolutely continuous with respect to μ^{Q_t}, while the terminal one is not.

Exercises

Exercise A.1. (Commuting flows) Show by direct computation that $\mathrm{D}\mathcal{P}_B(V)[\mathcal{P}_{B'}(V)] = \mathrm{D}\mathcal{P}_{B'}(V)[\mathcal{P}_B(V)]$. Which property of $\mathrm{D}^2 V$ and $\mathrm{D}^3 V$ is used?

Exercise A.2. (The sunset) (a) Show that $\int_{|z| \leqslant 1} G(z)^3 \, \mathrm{d}z = \infty$ in $d = 3$, and that $\int Q_T^3 = \frac{1}{16\pi^2} \log T + O(1)$ for the heat profile. (b) Show that $Q_T^3 - (Q_T - Q_t)^3$ is integrable uniformly in T, and deduce that the counterterm (A.7) makes the $:\phi^2:$ coefficient in (A.6) converge. (c) Show that the remaining part $\iint S^3(x,y)\,(\phi(x) - \phi(y))^2$ converges for smooth ϕ.

Exercise A.3. (Power counting) Show that the superficial degree of divergence of a connected graph of Φ_d^4 with n vertices and E external legs is $D_G = d - \frac{d-2}{2} E + (d-4)\,n$. Deduce that in $d = 3$ the only divergent graphs with loops are the two-point graphs with $n \leqslant 2$ (the tadpole and double tadpole, removed by Wick ordering, and the sunset) and vacuum graphs with $n \leqslant 3$.

Appendix B Stochastic localisation

Proposition 7.2 and Section 7.3 say that the Polchinski process is a stochastic localisation of ν_T. This appendix records why that decomposition is useful, and what it proves.

Why it is useful. Three features make this decomposition a tool:

- *It is a martingale.* $\nu_T = \mathbb{E}\,\mu_t^{\Phi_t}$, and the expectation of any test function under $\mu_t^{\Phi_t}$ is a martingale. Itô calculus then splits variance and entropy exactly into the part already revealed and the part still hidden.
- *The pieces are simpler.* The Gaussian factor adds $\mathcal{R}_t$ to the Hessian of the potential. If ν_T is log-concave, μ_t^{ϕ} is $\mathcal{R}_t$-uniformly log-concave, so Bakry–Émery and Brascamp–Lieb apply to it even when their constants are useless for ν_T itself.

- *One quantity controls the loss.* What is revealed per unit resolution is governed by Cov_t, whose evolution (8.9) is explicit. An inequality for ν_T reduces to showing that the localisation does not happen too fast.

The process can also be run as an algorithm: with an approximate force it is a sampler, as in the algorithmic stochastic localisation of [EMS22], and denoising diffusions are of the same kind [Mon23]. By (9.14), the noise of the field equation is, up to J_t, the score that such algorithms estimate.

B.1 Log-Sobolev inequalities along the localisation

Bauerschmidt and Bodineau [BB21] proved log-Sobolev inequalities by running the Polchinski flow (see the survey [BBD24]). In their language the renormalised measure at a given scale is the law of Φ_t, and the fluctuation measure is the conditional law $\mu_t^{\Phi_t}$ of Proposition 7.2, i.e. the stochastic localisation of ν_T [Eld23]. (Their scale parameter runs opposite to our resolution: their initial scale is our T, where the renormalised measure is ν_T and the fluctuation measure is trivial, and their integrated covariance is our S_t.) In this dictionary their argument is an entropy computation along our filtration.

Entropy as quadratic variation. Let $\rho \geqslant 0$ with $\mathbb{E}_{\nu_T}\rho = 1$, and let $M_t = \mathbb{E}_t[\rho(\Phi_T)] = G_t(\Phi_t)$, with $G_t(\phi) = \int \rho \, \mathrm{d}\mu_t^{\phi}$. The martingale M runs from $M_0 = 1$ to $M_T = \rho(\Phi_T)$, and its noise is that of the field: $\mathrm{d}M_t = \langle \mathrm{D}G_t(\Phi_t), \dot{Q}_t^{1/2}\mathrm{d}W_t\rangle$. Itô's formula for $M\log M$ gives

$$\mathrm{Ent}_{\nu_T}(\rho) = \frac{1}{2}\,\mathbb{E}\int_0^T \frac{\mathrm{d}\langle M\rangle_t}{M_t} = 2\,\mathbb{E}\int_0^T \|\dot{Q}_t^{1/2} D\sqrt{G_t}(\Phi_t)\|^2 \,\mathrm{d}t,$$

and, stopping at resolution t, the chain rule

$$\mathrm{Ent}_{\nu_T}(\rho) = \mathrm{Ent}(G_t(\Phi_t)) + \mathbb{E}\big[\mathrm{Ent}_{\mu_t^{\Phi_t}}(\rho)\big]:$$

the entropy already revealed at resolution t plus the expected entropy still hidden. The increments are controlled by the conditional law: differentiating the density of μ_t^{ϕ} in ϕ, $\mathrm{D}G_t(\phi) = S_t^{-1}\mathrm{Cov}_{\mu_t^{\phi}}(\rho, y)$.

Theorem B.1. (Multiscale Bakry–Émery [BB21, BBD24]) *(formal in infinite dimensions) Suppose that there are real numbers (possibly negative) κ_t such that, for all $t \in [0,T]$ and all ϕ, as quadratic forms,*

$$\dot{Q}_t\,\mathrm{D}^2 V_t(\phi)\,\dot{Q}_t + \frac{1}{2}\ddot{Q}_t \geqslant \kappa_t \dot{Q}_t.$$

Then ν_T satisfies the log-Sobolev inequality

$$\mathrm{Ent}_{\nu_T}(\rho) \leqslant \frac{2}{\varrho}\,\mathbb{E}_{\nu_T}[\|\dot{Q}_T^{1/2} D\sqrt{\rho}\|^2], \qquad \frac{1}{\varrho} = \int_0^T \exp\Big(-2\int_t^T \kappa_r \mathrm{d}r\Big)\,\mathrm{d}t.$$

The hypothesis gives, at each resolution, the gradient bound

$$\|\dot{Q}_t^{1/2} D\sqrt{G_t}\|^2 \leqslant \mathrm{e}^{-2\int_t^T \kappa_r \mathrm{d}r} \int \|\dot{Q}_T^{1/2} D\sqrt{\rho}\|^2 \,\mathrm{d}\mu_t^{\phi};$$

inserted in the entropy formula it gives the theorem [BBD24]. For $V_T = 0$ and a linear profile $Q_t = t\,Q_T/T$ one may take $\kappa = 0$, so $1/\varrho = T$ and the theorem is the Gaussian log-Sobolev inequality $\mathrm{Ent}(\rho) \leqslant 2\,\mathbb{E}\langle D\sqrt{\rho}, Q_T D\sqrt{\rho}\rangle$. The hypothesis is a convexity condition on the effective potentials at every scale, not on V_T, corrected by the scale dependence $\ddot{Q}_t$ of the decomposition; κ_t may be negative at some scales: only the weights $\mathrm{e}^{-2\int_t^T \kappa_r \mathrm{d}r}$ enter ϱ.

In terms of the conditional covariance. By Proposition 8.3, $\mathrm{D}^2 V_t = S_t^{-1} - S_t^{-1}\mathrm{Cov}_t S_t^{-1}$, and $S_t^{-1}\dot{Q}_t S_t^{-1} = \dot{\mathcal{R}}_t$ is the derivative of the regulator (8.6). Since $\dot{Q}_t S_t^{-1} = S_t \dot{\mathcal{R}}_t$, the hypothesis reads

$$S_t \dot{\mathcal{R}}_t \mathrm{Cov}_t \dot{\mathcal{R}}_t S_t \leqslant \dot{Q}_t S_t^{-1} \dot{Q}_t + \frac{1}{2}\ddot{Q}_t - \kappa_t \dot{Q}_t.$$

If $\dot{Q}_t$ is invertible, multiplying on the left by $S_t \dot{Q}_t^{-1} = \dot{\mathcal{R}}_t^{-1} S_t^{-1}$ and on the right by its transpose $\dot{Q}_t^{-1} S_t$ shows that it is equivalent to the explicit bound

$$\mathrm{Cov}_{\mu_t^\phi} \leqslant S_t - S_t \dot{Q}_t^{-1}\Big(\kappa_t \dot{Q}_t - \frac{1}{2}\ddot{Q}_t\Big)\dot{Q}_t^{-1} S_t \qquad \forall \phi. \tag{B.1}$$

The second derivative $\ddot{Q}_t$ is not estimated: it is a property of the Gaussian decomposition, and it enters only through the combination $\kappa_t \dot{Q}_t - \frac{1}{2}\ddot{Q}_t$. When Q_t and S_t are functions of one operator (Fourier multipliers, say), the right-hand side is $S_t - S_t^2 \dot{Q}_t^{-2}(\kappa_t \dot{Q}_t - \frac{1}{2}\ddot{Q}_t)$ mode by mode. If $\kappa_t \dot{Q}_t > \frac{1}{2}\ddot{Q}_t$ (for the linear profile: $\kappa_t > 0$), every conditional law is strictly more concentrated than the unresolved Gaussian fluctuation; if not, Cov_t may exceed S_t, as in a double well at low resolution. A concave profile ($\ddot{Q} \leqslant 0$) makes the bound smaller but forces κ down by the same amount: for the Gaussian with $Q_t = 1 - \mathrm{e}^{-t}$ one has $\ddot{Q} = -\dot{Q}$, the hypothesis holds with $\kappa = -\frac{1}{2}$, and $\mathrm{Cov}_t = S_t$. The hypothesis is thus an upper bound on the conditional covariance, uniform in the conditioning, measured in the metric $\dot{\mathcal{R}}_t$. Closely related is the *entropic stability* of Chen and Eldan [CE22] (see also [BBD24]): if, for all $r \leqslant t$ and all ϕ, $\dot{\mathcal{R}}_r \mathrm{Cov}_{\mu_r^\phi} \dot{\mathcal{R}}_r \leqslant \alpha_r \dot{\mathcal{R}}_r$, then

$$\mathrm{Ent}_{\nu_T}(\rho) \leqslant \exp\Big(\int_0^t \alpha_r \,\mathrm{d}r\Big)\, \mathbb{E}\big[\mathrm{Ent}_{\mu_t^{\Phi_t}}(\rho)\big]:$$

the total entropy is controlled by the entropy that is still hidden at resolution t. The low-resolution part of the localisation is where the measure may be multimodal, and the condition says that it does not hide too much. With the Brascamp–Lieb bounds of Section 8.4 the conditional covariance is controlled by the Hessian of V_T when this is convex; for Φ_3^4 it is not, and the condition is a statement about the effective potentials, i.e. about the renormalisation group flow itself.

Proposition B.2. (Poincaré from the localisation) *Suppose that, for all $r \leqslant t$ and all ϕ, $\dot{\mathcal{R}}_r \mathrm{Cov}_{\mu_r^\phi} \dot{\mathcal{R}}_r \leqslant \alpha_r \dot{\mathcal{R}}_r$. Then, for every f,*

$$\mathrm{Var}_{\nu_T}(f) \leqslant \exp\Big(\int_0^t \alpha_r \,\mathrm{d}r\Big)\, \mathbb{E}\,\mathrm{Var}_{\mu_t^{\Phi_t}}(f).$$

If moreover ν_T is log-concave, then

$$\mathrm{Var}_{\nu_T}(f) \leqslant \exp\Big(\int_0^t \alpha_r \,\mathrm{d}r\Big)\, \mathbb{E}_{\nu_T}\langle \mathrm{D}f, \mathcal{R}_t^{-1}\mathrm{D}f\rangle.$$

Proof. Let $M_r = \mathbb{E}_r f(\Phi_T)$ and $\chi_r = \mathrm{Cov}_{\mu_r^{\Phi_r}}(f, y)$. As in Section B.1, $\mathrm{d}\langle M\rangle_r = \langle \chi_r, \dot{\mathcal{R}}_r \chi_r\rangle\,\mathrm{d}r$, so $v(r) := \mathbb{E}\,\mathrm{Var}_{\mu_r^{\Phi_r}}(f)$ has $v' = -\mathbb{E}\langle \chi_r, \dot{\mathcal{R}}_r \chi_r\rangle$ and $v(0) = \mathrm{Var}_{\nu_T}(f)$. By Cauchy–Schwarz, with $h = \dot{\mathcal{R}}_r \chi_r$, $\langle \chi_r, h\rangle^2 \leqslant \mathrm{Var}_{\mu_r^{\Phi_r}}(f)\,\langle h, \mathrm{Cov}_r h\rangle \leqslant \alpha_r \mathrm{Var}_{\mu_r^{\Phi_r}}(f)\,\langle \chi_r, \dot{\mathcal{R}}_r \chi_r\rangle$. Hence $v' \geqslant -\alpha_r v$, which gives the first bound. For the second, the Hessian of the potential of μ_t^ϕ is at least $\mathcal{R}_t$: apply Brascamp–Lieb to μ_t^ϕ and take the expectation. □

For $V_T = 0$ one has $\mathrm{Cov}_r = S_r$. In one mode, or for the linear profile, where $\dot{Q}_r S_r^{-1}$ is a number, $\alpha_r = \dot{Q}_r S_r^{-1}$ and $\exp\int_0^t \alpha_r \,\mathrm{d}r = Q_T S_t^{-1}$: the first bound is an equality for linear f, and the second tends to the Gaussian Poincaré inequality as $t \to T$ (Exercise B.2).

An application: the KLS conjecture. Take an isotropic log-concave measure on $\mathbb{R}^n$ (mean zero, covariance 1) and the identity control, $\mathscr{R}_t = t$. The second bound reads $\mathrm{Var}(f) \leqslant t^{-1}\,\mathrm{e}^{\int_0^t \alpha}\,\mathbb{E}|\mathrm{D}f|^2$, so a Poincaré inequality follows if Cov_r stays bounded in operator norm up to some time t. The Kannan–Lovász–Simonovits conjecture asks for a Poincaré constant independent of n, and hence for a time t independent of n. Along the path, the bound holds only with high probability, not uniformly in ϕ, so the argument has to be adapted, and proving that the covariance stays bounded is the heart of the matter. Following Eldan [Eld13], this has given the bounds of Chen [Che21] and Klartag–Lehec [KL22], and then Klartag's bound [Kla23]: the KLS isoperimetric constant is at most $C\sqrt{\log n}$, i.e. the Poincaré constant is $O(\log n)$. The related slicing conjecture has since been proved [KL24].

The connection with Bauerschmidt–Bodineau. The multiscale Bakry–Émery criterion of Theorem B.1 and the localisation study the same measure-valued martingale from two sides. Bauerschmidt and Bodineau [BB21] (see [BB19] for an earlier decomposition argument, and [BBD24] for a survey) start from the renormalisation group: integrate out the fine scales and follow the potential. Eldan's approach follows, along one path, the conditional law. The dictionary is:

these notes	Bauerschmidt–Bodineau	stochastic localisation
resolution t increasing	their scale, reversed	time t increasing
law ν_t of Φ_t	renormalised measure	law of the tilt
μ_t^ϕ	fluctuation measure	tilted measure
S_t, $\mathscr{R}_t$	integrated covariance, —	—, t
$G_t = \int \rho \,\mathrm{d}\mu_t^\phi$	Polchinski semigroup on ρ	martingale $\int \rho \,\mathrm{d}\mu_t$
$\mathrm{D}^2 V_t = S_t^{-1} - S_t^{-1}\mathrm{Cov}_t S_t^{-1}$	Hessian of the potential	covariance of the tilted measure

Proposition 8.3 is the bridge: convexity of the renormalised potentials is an upper bound on the conditional covariance, namely (B.1). The two methods then use this information differently.

- *Bauerschmidt–Bodineau.* The Kolmogorov equation (9.2) transports G_t by the drift $\dot{Q}_t F_t = -\dot{Q}_t \mathrm{D}V_t$. Its Jacobian plays the role of the curvature in the Bakry–Émery computation, and the scale dependence of the metric $\dot{Q}_t$ adds $\frac{1}{2}\ddot{Q}_t$. This yields the gradient bound for G_t at each scale, which is integrated in the entropy formula. The output is a log-Sobolev inequality with an explicit constant, directly at the terminal scale, also far from log-concavity. The hypothesis is uniform in ϕ, which suits field theories where the renormalisation group controls V_t for all fields, as for sine-Gordon at $\beta < 6\pi$ [BB21].
- *Localisation.* One bounds what is lost on $[0,t]$ (the proposition above, or entropic stability [CE22]) and uses a known inequality for μ_t^ϕ. It works directly with moments of the conditional law, and can use bounds that hold only with high probability along the path, as in the KLS problem.

Exercises

Exercise B.1. (Entropy along the localisation) (a) Derive the formula for $\mathrm{Ent}_{\nu_T}(\rho)$ as the integrated quadratic variation of M, and the chain rule. (b) Show that $\mathrm{D}G_t(\phi) = S_t^{-1}\mathrm{Cov}_{\mu_t^\phi}(\rho, y)$. (c) For $V_T = 0$ and a linear profile, recover the Gaussian log-Sobolev inequality from Theorem B.1. (d) In zero dimensions, with $Q_t = 1 - \mathrm{e}^{-t}$ and a convex V_T, find κ_t and the resulting constant, and compare with the Bakry–Émery criterion.

Exercise B.2. (**Poincaré from the localisation**) (a) Check that the measurement noise $S_t^{-1}\zeta_t$ of (7.18) has covariance $\mathcal{R}_{s\wedge t}$. (b) For $V_T = 0$ and one mode, compute α_r and verify the Gaussian check above. (c) In zero dimensions, with $\mathcal{R}_r = r$ and $\nu = \frac{1}{2}N(a,\sigma^2) + \frac{1}{2}N(-a,\sigma^2)$, compute $\mathrm{Cov}_{\mu_r^\phi}$ and its supremum over ϕ. Using the first bound as $t \to \infty$, deduce the Poincaré constant $\leqslant \sigma^2 \mathrm{e}^{a^2/\sigma^2}$. Compare with the true order $\mathrm{e}^{a^2/2\sigma^2}$: why is the exponent off by a factor 2?

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