A Constructive Introduction to Quantum Field Theory

Lecture 7: Φ^4_3 on the torus
II. A priori estimates, UV stability and the limiting FBSDE

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Abstract

We prove the central estimate of the construction: bounds, uniform in the lattice, on the error Z of the approximate forecast and on the time the forecast spends at large fields. The two halves of the FBSDE give two inequalities which close only together, with small coefficient $C\lambda^2/q_*$, for every coupling and every finite mass. The same bounds hold for the flow restarted from any field at any scale. So the exact flow is an explicit flow with a control of finite relative entropy. This writes the limit measure as an explicit density over an explicit SDE, for the whole lattice sequence, and it gives the FBSDE on the continuum torus, the field equation at every resolution, uniqueness in law for the scalar form of the FBSDE, a renormalised variational formula with no infinite constant, and quartic exponential tails.

The results presented in Lectures 4–7 were obtained recently, with the use of AI, in collaboration with Sarah-Jean Meyer.

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Introduction

Notation from Lecture 6. We recall the symbols used throughout; the definitions are in Sections 10.1–10.5, 11 and 12 of Lecture 6.

- *Lattice.* Λ_N is the periodic lattice of mesh N^{-1} in the unit torus, $H_N = \mathbb{R}^{\Lambda_N}$, $\int f = N^{-3}\sum_x f(x)$, $\langle u, v\rangle = \int u\, v$, and $\|\cdot\|_p$ is the L^p norm for $\int$. For a kernel, $\|C\|_{\mathrm{row}} = \sup_x \int |C(x,z)|\mathrm{d}z$ is the row norm, which bounds $\|C\|_{p\to p}$ for every p when C is symmetric. $\Delta = \Delta_N$ is the nearest-neighbour Laplacian.

- *Resolution and free operators* (10.1)–(10.2). The resolution $q \in [q_0, \infty)$ is the squared mass of the regulator, and $q_0 > 0$ is the squared mass of the Gaussian reference. $K_q = q - \Delta$ is the precision and $C_q = K_q^{-1}$ the covariance of the scales not yet resolved at resolution q; $\|C_q\|_{p\to p} = q^{-1}$. In particular $C = C_{q_0}$ is the covariance of the Gaussian reference and $K_{q_0} = q_0 - \Delta$ the free field operator. The resolved covariance is $Q_q = C - C_q$, with $\partial_q Q_q = C_q^2$, and the bubble $a_2(q) = (C_q^2)(x,x) \lesssim q^{-1/2}$ is the variance of the shell at resolution q, by (10.4).

- *Couplings* (11.1). λ is the quartic coupling, $U(\varphi) = \frac{\lambda}{12}\int \varphi^4$, so that $D^2U(y) = \lambda\, y^2$ and the gradient of the interaction is $\frac{\lambda}{3}\, y^3$. $g_q = \lambda\, q^{-1/2}$ is the dimensionless coupling and $r_q = (q/\lambda)^{1/2}$ the field size at which the cube balances K_q. q_* is the auxiliary scale above which g_q is small.

- *Processes* (Section 10.3). Φ_q is the Polchinski process, with $\Phi_{q_0} = 0$ and $\Phi_\infty \sim \nu_N$; W is the Brownian motion on H_N driving it, B the Brownian motion of the bridge (10.7), and $\mathbb{E}_q$ the conditional expectation given the past of Φ. The forecast is $\psi_q = \mathbb{E}_q \Phi_\infty$, and Cov_q is the conditional covariance of Φ_∞. The filtration is $\mathcal{F}_q = \sigma(\Phi_s \colon s \leqslant q)$ (the same letter as the exact force, as in Lecture 6; the meaning is always clear from the context). The forecast is a function of the field, $\psi_q = m_q(\Phi_q)$, and so is the approximate forecast, $Y_q = y_q(\Phi_q)$, where $y = y_q(x)$ solves $K_q y + \bar{\mathcal{F}}_q(y) = K_q x$.

- *Forces and coefficients* (10.9)–(10.12), (12.5). $\mathcal{F}_q$ is the exact force in the forecast coordinate, $K_q\Phi_q = K_q\psi_q + \mathcal{F}_q(\psi_q)$, and $\bar{\mathcal{F}}_q(y) = \frac{\lambda}{3}\, y^3 + M_q y + \tilde{\ell}_q(y)$ the explicit approximate force (12.1): the cube, a linear operator $M_q = \mu_q I + \Theta_q$, made of the exact running scalar mass and the centred kernel $\Theta_q = \Theta_q^{[3]}$ produced by the iteration, and the clipped corrections $\tilde{\ell}_q$. $H_q(y) = (K_q + D\bar{\mathcal{F}}_q(y))^{-1}$ is the approximate response, $b_q = -H_q\mathcal{D}_q$ the drift produced by the force defect $\mathcal{D}_q$, and $\Xi_q = \mathrm{Cov}_q - H_q(Y_q)$. In the coordinate equation, $\varrho_q(y) = C_q\,(M_q y + \tilde{\ell}_q(y))$ collects the linear and the clipped parts of the explicit force.

At the end of Lecture 6 we had, at each lattice cutoff N, the exact system (Theorem 10.6)

$$\begin{aligned} \mathrm{d}Y_q &= [b_q(Y_q) + H_q(Y_q)\, Z_q]\mathrm{d}q + H_q(Y_q)\,\mathrm{d}W_q, \\ \mathrm{d}Z_q &= -[b_q(Y_q) + H_q(Y_q)\, Z_q]\mathrm{d}q + \Xi_q\,\mathrm{d}W_q, \qquad q \geqslant q_0, \end{aligned} \tag{1}$$

where Y is the approximate forecast built from the explicit force $\bar{\mathcal{F}}_q$, $Z = \psi - Y$ is its error, and $\psi = Y + Z$ is the exact forecast, a martingale. The inputs from Lecture 6, all for $q \geqslant q_*$, are:

A) *Global coercivity* (Lemma 12.1): $H_q(y)$ is comparable to $(K_q + \lambda\, y^2)^{-1}$ and

$$\|H_q(y)\|_{p\to p} \leqslant (1 + \varepsilon_q)\,/\,q,$$

$\varepsilon_q = C\,(g_q + g_q^2)$.

B) *The source* (Proposition 12.2): $b = b^{\mathrm{b}} + b^{\mathrm{l}}$ with $\|b_q^{\mathrm{b}}\|_\infty \leqslant C\,\lambda^{7/2}\, q^{-5/2}$ and

$$\|b_q^{\mathrm{l}}(y)\|_2^2 \leqslant C\,\lambda^4\, q^{-5}\|t_q(y)\|_6^6,$$

where $t_q(y) = y\,\mathbb{1}_{|y|>r_q}$.

C) *The coordinate equation* (12.5):

$$K_q\,(Y_q + \varrho_q(Y_q)) + \frac{\lambda}{3}\, Y_q^3 = K_q\,\Phi_q,$$

with a small Lipschitz correction, $\|D\,\varrho_q\|_{p\to p} \leqslant \varepsilon_q$.

D) *Further bounds*, used from Section 16.3 on: the global linear bound $\|b_q(y)\|_p \leqslant C\,\lambda\, q^{-3/2}\|y\|_p$ of Proposition 12.2; the bound $\mathrm{diag}\, H_q(y)^2 \leqslant C\, a_2(q)$ of (12.3); below q_*, the frozen force (12.7)–(12.9); and, for the density, the defect bound (11.14) together with the bound (21.11) of the Supplement on the large-field defect $\mathfrak{D}$ before multiplication by H.

The goal is to show that Z is small and that Y rarely visits large fields, *uniformly in* N. This is UV stability. Neither equation can be estimated alone: the forward equation contains Z, and the source of the backward equation is controlled by the large fields of Y.

How a priori estimates for FBSDEs are usually obtained. For an FBSDE with Lipschitz coefficients one estimates $\mathbb{E}\|Z_q\|^2$ by Itô's formula and Gronwall, and closes either on a short time interval or under a monotonicity condition [MY99]. Here the time interval is infinite, the coefficients are not uniformly Lipschitz in the relevant norms, and the coupling is not small in a Lipschitz sense. What we have instead is structure: the backward equation is linear in Z, the forward equation is a memoryless cubic elliptic equation at each resolution, and its noise is sub-Gaussian whatever the interaction (Proposition 9.6 of Lecture 5, as far as we know a new observation of these notes).

Where the system starts. The system starts at q_0 with $Y_{q_0} = Z_{q_0} = 0$. The energies below are integrated from q_* (Section 12.4), and no value of Y at q_* is used: this is what the memoryless field equation makes possible. Below q_* everything follows from the martingale property of the forecast (Section 16.3).

Sections 13–15 prove these estimates at a fixed lattice and a finite upper scale. Sections 16–18 remove the cutoffs and identify the limit.

The role of the estimates. Each estimate has one job.

- *Noise bound* (Lemma 13.2): the noise ζ has free-field tails whatever the interaction. This is where the posterior structure enters.
- *Forward coercivity* (Proposition 14.3): pathwise, at each scale, the cube bounds the large-field part of Y by the noise tails and Z. It controls where the explicit force is inaccurate.
- *Backward representation* (Lemma 15.1, Proposition 15.2): Z is the transported defect, as the equation is linear in it. Its energy is bounded by the bounded-field source, a boundary term, and $C\lambda^2/q_*$ times the large-field action. This factor is the only smallness that closes the loop.
- *Boundary* (Lemma 16.1): qualitative, at fixed N. It removes the boundary term, and it is why the upper scale goes first.
- *UV stability* (Theorem 16.2): the result of these four, $\mathbb{E}\|Z_q\|_2^2 \leqslant C\, q^{-5/2}$ uniformly in N. It says that the control turning the explicit flow into the exact one has finite entropy.
- *Restarted laws* (Proposition 16.4): the same from any field at any scale, and below q_* by the martingale property of the forecast.
- *Defect on all fields and Lipschitz coordinate* (Lemmas 18.1, 16.3): the defect is integrable along both flows, and the explicit flow is a strong continuum SDE, continuous in the cutoff.
- *Terminal tail* (Section 17): tightness in H^{-s}, $s > \frac{1}{2}$.
- *Sources* (Proposition 17.3, Theorems 17.4–17.5): quartic tails by a linear source, a restart at $(q_0, h\, C_{q_0} f)$, and Gaussian tails by a quadratic tilt. These are end results, and nothing else depends on them.
- *Pointwise bounds* (Proposition 14.4, Lemmas 14.5, 17.1): after closure, large fields occur only at finitely many dyadic scales. Neither the loop nor the identification of the limit uses them.

So the formula for ν (Theorem 18.5) needs the first five at the start of the flow, and if $q_0 < q_*$ the deduction below q_* of Proposition 16.4 there, all with first moments of the energies; plus the defect, the coordinate, the tail, the consistency of the coefficients, and moments of the explicit flow, which are Itô's formula. The FBSDE then follows by Girsanov's theorem (Theorem 18.6), with no further estimate on the flow, only the convergence of H and b and lower semicontinuity. The restart at every (q, x) gives the kernel Π_q: the formula (18.18), the statement that the forecast is a continuous function of the field, its convergence at each scale, and Section 18.6; the sources use it at $(q_0, h\, C_{q_0} f)$. Higher moments of the energies serve only for the stretched exponential bound, the higher moments in Corollary 17.2 and in Lemma 17.1, and (16.10) with $p > 1$.

Detailed proofs of what these lectures cite or condense are in the Supplement.

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13 The energies

13.1 Moments at fixed cutoff

All estimates below are first proved on a finite interval $[q_*, T]$ and at fixed N. There, everything is finite.

Lemma 13.1. (Preliminary moments) *At fixed N and T, the maxima over $[q_*, T]$ of Φ, Y, ψ, Z have all moments, and so does $\int_{q_*}^{T}(\|\mathrm{Cov}_q\|_{\mathrm{HS}}^2 + \|\Xi_q\|_{\mathrm{HS}}^2)\mathrm{d}q$. Consequently the energies E_T, J_T and G_T defined below have all moments at fixed N, T.*

Proof. Φ has all maximal moments, since ν_N has all moments. Global coercivity, $K_q + D\bar{\mathcal{F}}_q \geqslant c\,q$ and $\bar{\mathcal{F}}_q(0) = 0$, gives $\|Y_q\| \leqslant C\|K_q\Phi_q\| / q$. Doob's inequality for the martingale $\psi_q = \mathbb{E}_q\,\Phi_\infty$ gives its maximal moments, and $Z = \psi - Y$. For the integrands, $\|\mathrm{Cov}_q\|_{\mathrm{HS}} \leqslant \mathrm{TrCov}_q \leqslant \mathbb{E}_q\|\Phi_\infty\|^2$, and $\|H_q(Y_q)\|_{\mathrm{HS}} \leqslant C\sqrt{\dim H_N} / q$. The energies involve, besides Y and Z, the field ζ_q of Section 13.2 and $\varrho_q(Y_q)$; finite-dimensional norm equivalence and $\|\varrho_q(y)\| \leqslant \varepsilon_q\|y\|$ give their moments. □

These constants depend on N and T. They are used only to justify the absorptions below, never in the final bounds.

13.2 The field equation

The forecast equation of Lecture 5. In the profile (10.2), where $S_q = C_q$ and the regulator is $q - q_0$, the noise observable of Theorem 9.1 of Lecture 5 is

$$\omega_q := K_q\Phi_q - (q - q_0)\,\psi_q = \mathbb{E}_q[B_{q-q_0}], \tag{13.1}$$

the second form by the bridge (10.7),

$$K_q\Phi_q = (q - q_0)\Phi_\infty + B_{q-q_0}.$$

Since $F_q = K_q(\psi_q - \Phi_q) = -\mathcal{F}_q(\psi_q)$ and $K_q = K_{q_0} + q - q_0$, the field equation for the forecast (9.12) of Lecture 5 reads

$$K_{q_0}\psi_q + \mathcal{F}_q(\psi_q) = \omega_q:$$

an elliptic equation for the forecast at resolution q, in which the exact force $\mathcal{F}_q$ acts with no smoothing operator, and whose noise is controlled whatever the interaction (Proposition 9.6 of Lecture 5, here in one line).

Lemma 13.2. (The noise) *For every convex Ψ on H_N, $\mathbb{E}\,\Psi(\omega_q) \leqslant \mathbb{E}\,\Psi(B_{q-q_0})$, whatever the interaction. In particular $\zeta_q := C_q\,\omega_q$ satisfies, for all x and $u \in \mathbb{R}$,*

$$\mathbb{E}\,\mathrm{e}^{u\zeta_q(x)} \leqslant \exp\left(\frac{u^2}{2}\,(q - q_0)\,a^2(q)\right), \qquad (q - q_0)\,a^2(q) \leqslant C\sqrt{q}, \tag{13.2}$$

uniformly in N.

Proof. Conditional Jensen in (13.1), with $\Psi(f) = \mathrm{e}^{u\langle C_q\delta_x, f\rangle}$, and (10.4). □

The explicit force. The exact force $\mathcal{F}_q$ is unknown, and so is its coercivity. We replace it by the explicit force of (12.5),

$$\bar{\mathcal{F}}_q(y) = K_q \varrho_q(y) + \frac{\lambda}{3} y^3,$$

and the forecast by the approximate forecast, $\psi_q = Y_q + Z_q$. The coordinate equation $K_q Y_q + \bar{\mathcal{F}}_q(Y_q) = K_q \Phi_q$ and (13.1) give the exact identity

$$K_{q_0} Y_q + \bar{\mathcal{F}}_q(Y_q) = \omega_q + (q - q_0)\, Z_q. \tag{13.3}$$

This is the forecast equation of Lecture 5 with the explicit force in place of the exact one, and it is the equation we estimate:

- it is an elliptic equation at each resolution, with *no memory*: the past enters only through the noise ω_q;
- the coercivity comes from the cube of $\bar{\mathcal{F}}_q$, which is explicit;
- the price of the replacement is the error $(q - q_0)\, Z_q$, the only trace of the backward equation, at the same resolution;
- the identity is pathwise: no expectation has been taken.

The smoothed noise. The noise ω_q is rough, with covariance dominated by that of B_{q-q_0}, a white noise; pointwise estimates need a noise which is a function. So we use $\zeta_q = C_q \omega_q = \mathbb{E}_q[C_q B_{q-q_0}]$, the filtered version of the Gaussian bridge noise ζ_t of Section 9.3 of Lecture 5, which by (13.2) has the tails of the free field at resolution q. It is not Gaussian, and it depends on the interaction through ψ, but its bounds do not. Since $K_{q_0}\zeta_q = \omega_q - (q - q_0)\zeta_q$, subtracting it from (13.3) and writing

$$V_q := Y_q + \varrho_q(Y_q) - \zeta_q, \tag{13.4}$$

(V_q is not to be confused with the potentials $V_t(x)$, V_N of Lectures 4–6 and of Sections 16.3 and 18.2; the potentials are functions of a field, V_q here is a random field) we obtain (13.3) in the form in which it is estimated,

$$K_{q_0} V_q + \frac{\lambda}{3} Y_q^3 = (q - q_0)\,(\zeta_q - \varrho_q(Y_q) + Z_q), \qquad Y_q = V_q + \zeta_q - \varrho_q(Y_q). \tag{13.5}$$

The mass of (13.5) is only q_0, so at large resolution all the coercivity must come from the cube. The next section shows that it does.

13.3 Thresholds and energies

Define the signed excess of V over three quarters of the threshold, the large-field part of Y, and the large values of the noise,

$$w_q = \operatorname{sgn}(V_q)\,\big(|V_q| - \tfrac{3}{4} r_q\big)_+, \qquad t_q = Y_q \mathbb{1}_{|Y_q| > r_q}, \qquad \xi_q = \zeta_q \mathbb{1}_{|\zeta_q| > r_q/32}, \tag{13.6}$$

all pointwise in space, with $r_q = (q/\lambda)^{1/2}$. The three random energies are

$$\begin{aligned} E_T &= \frac{1}{\lambda} \int_{q_*}^{T} q^{1/2} \|Z_q\|_2^2 \, \mathrm{d}q && \text{(backward energy)}, \\ J_T &= \int_{q_*}^{T} q^{-3/2} \left[q\, \|w_q\|_4^4 + \lambda \|w_q\|_6^6 + \lambda \|t_q\|_6^6 \right] \mathrm{d}q && \text{(large-field action)}, \\ G_T &= \int_{q_*}^{T} \left[q^{-1/2}\, \|\xi_q\|_4^4 + \lambda\, q^{-3/2} \|\xi_q\|_6^6 \right] \mathrm{d}q && \text{(noise tails)}. \end{aligned} \tag{13.7}$$

Only *thresholded* sixth powers are integrated over all scales: $\mathbb{E}\|\zeta_q\|_6^6$ can be of order $q^{3/2}$, so an unthresholded noise contribution to J would not be integrable uniformly in N. The weights are matched to each other and to the source bounds:

quantity	weight	reason
$\|Z_q\|_2^2$	$q^{1/2}$	forcing: $q^{-3/2}\cdot\lambda^{-1}\,q^2\|Z\|_2^2$
$\lambda\|t_q\|_6^6$	$q^{-3/2}$	source: $q^{7/2}\|b^1\|^2\leqslant C\lambda^3\cdot\lambda\,q^{-3/2}\|t\|_6^6$
$\|b^1_q\|_2^2$	$q^{7/2}$	backward propagator s/q and $\int_q^\infty s^{-3/2}\mathrm{d}s=2\,q^{-1/2}$

Exercises

Exercise 13.1. (Matched weights) Check the three weights in the table. Show from (13.5) that the forward estimate, with the weight $q^{-3/2}$ on the large-field action, needs the weight $q^{1/2}$ on $\|Z\|_2^2$. Show that the backward estimate of Section 15 also holds with weight $q^{1/2+\sigma}$ for $0<\sigma<\frac12$, keeping the other weights of the table ($q^{-3/2}$ on the large-field action, $q^{7/2}$ on $\|b^1\|_2^2$) as in the proof. Which term breaks at $\sigma=\frac12$, and how does the feedback coefficient depend on σ? (Hint: in Proposition 15.2 the boundary integral is taken up to ∞.)

Exercise 13.2. (Moments of the noise) Show from Lemma 13.2 that $\mathbb{E}\|\zeta_q\|_6^6\leqslant 15\,((q-q_0)\,a^2(q))^3\leqslant C\,q^{3/2}$ and $\mathbb{E}\|\xi_q\|_6^6\leqslant C\,q^{3/2}\exp(-c\sqrt{q}/\lambda)$. For $V_N=0$, compute the covariance of ζ_q and compare it with the bound. On which modes is the bound sharp?

Exercise 13.3. (The memory form) Let $X_q=\int_{q_0}^q C_s\,\mathrm{d}W_s$, so that $K_q\,X_q=W_q+\int_{q_0}^q X_s\,\mathrm{d}s$, and $V_q'=Y_q+\varrho_q(Y_q)-X_q$. Using $K_q\Phi_q=W_q+\int_{q_0}^q\psi_s\,\mathrm{d}s$, derive the equation with memory

$$K_q\,V_q'+\frac{\lambda}{3}\,Y_q^3=\int_{q_0}^q(V_s'-\varrho_s(Y_s)+Z_s)\,\mathrm{d}s.$$

Below the auxiliary scale ϱ comes from the frozen force, and is not small. Here the noise is exactly Gaussian, but the memory has the size of the mass term. Why would testing it with w^3 need the Hardy inequality $\|q^{-1}\int_0^q f\|_{L^4(q^{-1/2}\mathrm{d}q)}\leqslant\frac87\|f\|_{L^4(q^{-1/2}\mathrm{d}q)}$, and where would the margin come from? What does (13.5) trade to avoid this?

14 Forward coercivity

14.1 The scalar model

On one site the Laplacian disappears. We also drop the small Lipschitz correction ϱ (restored in Proposition 14.3) and treat the noise and the error as given numbers. Then (13.5) is a scalar equation at each q, and all the constants can be checked by hand.

Lemma 14.1. (Scalar forward coercivity) *Let* $\lambda,\ q>0,\ a\geqslant 0$, $0\leqslant\varpi\langle leqslant>q\rangle$ and $v,g,z\in\mathbb{R}$ with

$$a\,v+\frac{\lambda}{3}\,(v+g)^3=\varpi\,(g+z).\qquad(14.1)$$

Let $r=(q/\lambda)^{1/2}$, $w=\mathrm{sgn}(v)\,(|v|-\frac34\,r)_+$, $\xi=g\,\mathbb{1}_{|g|>r/32}$. Then

$$\frac{q}{8}\,w^4+\frac{\lambda}{16}\,w^6\leqslant 54\,q\,\xi^4+\frac{4225}{6}\,\lambda\,\xi^6+\frac{12}{\lambda}\,q^2\,z^2.\qquad(14.2)$$

Moreover $t=(v+g)\,\mathbb{1}_{|v+g|>r}$ satisfies $|t|\leqslant\frac{32}{7}\,(|w|+|\xi|)$.

Proof. *Multiply* (14.1) *by* w^3, *where* $w\neq0$; *there* $|v|=|w|+\frac34\,r$ *and* v,w *have the same sign, so* $a\,v\,w^3\geqslant0$. *Since* $|g|\leqslant\frac{r}{32}+|\xi|$, $\mathrm{sgn}(w)\,(v+g)\geqslant|w|+\frac{11}{16}\,r-|\xi|$ *(we give up* $\frac{1}{32}\,r$ *to leave room, on the lattice, for* ϱ*). The cube being increasing,* (14.3) *and* $\lambda\,r^2=q$ *give*

$$\frac{\lambda}{3}\,w^3\,(v+g)^3\geqslant\frac{\lambda}{12}\,w^6+\frac{q}{4}\,w^4+\frac{q\,r}{13}|w|^3-\frac{4225}{6}\,\lambda\,\xi^6.$$

This is the large-field coercivity: above the threshold the cube provides a sixth power, a quartic term of the size of a mass q, *and a cubic term which dominates the noise below the threshold. On the right side,* $\varpi\,g\,w^3\leqslant\frac{q\,r}{32}|w|^3+q|\xi||w|^3$, *and the first term is absorbed. By Young's inequality,* $q|\xi||w|^3\leqslant\frac{q}{8}\,w^4+54\,q\,\xi^4$ *and* $q|z||w|^3\leqslant\frac{\lambda}{48}\,w^6+\frac{12}{\lambda}\,q^2\,z^2$. *This gives* (14.2). *For the last claim,* $|v+g|\leqslant|w|+\frac34\,r+\frac{r}{32}+|\xi|$, *so on* $\{|v+g|>r\}$ *we have* $\frac{7}{32}|v+g|\leqslant|v+g|-\frac{25}{32}\,r\leqslant|w|+|\xi|$. $\square$

***Lemma* 14.2. (Two elementary inequalities)** *For $x, \rho \geqslant 0$, $(x-\rho)^3 \geqslant \frac{3}{4}x^3 - 65\,\rho^3$. For u, ρ, $r \geqslant 0$,*

$$\frac{1}{3}u^3\left(u+\frac{11}{16}r-\rho\right)^3 \geqslant \frac{1}{12}u^6+\frac{1}{4}r^2u^4+\frac{1}{13}r^3u^3-\frac{4225}{6}\rho^6. \tag{14.3}$$

Proof. *$\frac{1}{4}x^3+64\,\rho^3-3\,x^2\,\rho=\frac{1}{4}(x-8\,\rho)^2(x+4\,\rho)\geqslant 0$, so $(x-\rho)^3\geqslant x^3-3\,x^2\,\rho-\rho^3\geqslant\frac{3}{4}x^3-65\,\rho^3$. Apply it with $x=u+b$, $b=\frac{11}{16}r$: $\frac{1}{3}u^3(x-\rho)^3\geqslant\frac{1}{4}u^3x^3-\frac{65}{3}u^3\rho^3$, and $x^3\geqslant u^3+3\,b^2\,u+b^3$ with $\frac{3}{4}b^2=\frac{363}{1024}r^2>\frac{1}{4}r^2$ and $\frac{1}{4}b^3=\frac{1331}{16384}r^3>\frac{1}{13}r^3$. Finally $\frac{65}{3}u^3\rho^3\leqslant\frac{1}{6}u^6+\frac{4225}{6}\rho^6$, and $\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$.* □

What the scalar lemma says. *The excess of the forecast over the threshold is controlled, at each resolution separately, by two things only: how large the noise is, whose tails are known and exponentially small at large q, and the error Z. There is no smallness in the coupling: the estimate is uniform in λ once the threshold scales like $r_q=(q/\lambda)^{1/2}$. It is pathwise, and it needs no integration in q.*

14.2 The lattice version

***Proposition* 14.3. (Forward coercivity)** *For q_* large (depending on λ), at every cutoff N, pathwise and for every $q\geqslant q_*$,*

$$q\|w_q\|_4^4+\lambda\|w_q\|_6^6+\lambda\|t_q\|_6^6\leqslant C\,(q\|\xi_q\|_4^4+\lambda\|\xi_q\|_6^6+\lambda^{-1}\,q^2\|Z_q\|_2^2). \tag{14.4}$$

Consequently, for every T,

$$J_T\leqslant C\,(G_T+E_T),\qquad \|G_T\|_{L^p(\mathbb{P})}\leqslant C\,\lambda^3\,p^5\quad(p\geqslant 1), \tag{14.5}$$

with C independent of N, T and p.

Proof: the changes from the scalar case. *Test (13.5) with w_q^3 in $L^2(\Lambda_N)$ and proceed as in Lemma 14.1, with Hölder's inequality in space. Three new terms appear.*

The elliptic term. *$\langle w^3, K_{q_0}V\rangle\geqslant 0$. Indeed $w^3=\phi'(V)$ for the convex function $\phi(v)=\frac{1}{4}(|v|-\frac{3}{4}r)_+^4$, $Vw^3\geqslant 0$, and for any nondecreasing ϕ',*

$$\langle\phi'(V),-\Delta_N V\rangle=N^{-1}\sum_{x\sim x'}(\phi'(V(x))-\phi'(V(x')))\,(V(x)-V(x'))\geqslant 0,$$

summing over unordered nearest-neighbour pairs. So the elliptic term only helps.

The coordinate correction. *Split $\varrho_q(y)=\varrho_q^0+\varrho_q^1$ according to $|y|\leqslant r_q$ or not (writing $\varrho_q(y)=L_q(y)\,y$ with L the averaged derivative). Then $\|\varrho_q^0\|_\infty\leqslant\varepsilon_q\,r_q\leqslant r_q/32$ shifts the threshold, and $\|\varrho_q^1\|_j\leqslant\varepsilon_q\|t_q\|_j$, $j=4,6$. Then $|Y|\leqslant|w|+\frac{13}{16}r_q+|\xi|+|\varrho^1|$; on $\{|Y|>r_q\}$ this gives $\frac{3}{16}|Y|\leqslant|w|+|\xi|+|\varrho^1|$, so for small ε*

$$\|t_q\|_j\leqslant C\,(\|w_q\|_j+\|\xi_q\|_j),\qquad \|\varrho_q^1\|_j\leqslant C\,\varepsilon_q\,(\|w_q\|_j+\|\xi_q\|_j).$$

This also supplies the t part of (14.4). These are operator bounds: ϱ is nonlocal, but small in every L^p. In the cube and on the right side of (13.5) the error is $\rho=|\xi|+|\varrho^1|$. The bounded parts on the right cost $\frac{q\,r}{16}\|w\|_3^3$, which is absorbed by $\frac{q\,r}{13}\|w\|_3^3$ from the cube. The cube error is at most $C\,\lambda\,(\|\xi\|_6^6+\varepsilon^6\|w\|_6^6)$, and Young's inequality gives $C\,q\,(\|\xi\|_4^4+\varepsilon^4\|w\|_4^4)$. The w parts are absorbed for small ε: these are deterministic smallness conditions on g_{q_*}, independent of p.

The error. *$|\langle w^3,(q-q_0)\,Z\rangle|\leqslant\frac{\lambda}{48}\|w\|_6^6+\frac{12}{\lambda}\,q^2\|Z\|_2^2$. Multiplying (14.4) by $q^{-3/2}$ and integrating gives $J_T\leqslant C\,(G_T+E_T)$.*

Tails of the noise. *By Minkowski's inequality in x, for $j=4,6$, $\|\|\xi_q\|_j^j\|_{L^p(\mathbb{P})}\leqslant\sup_x\||\xi_q(x)|^j\|_{L^p(\mathbb{P})}$, and by Hölder and (13.2), with threshold $r_q/32=\frac{1}{32}(q/\lambda)^{1/2}$, this is at most*

$$(\mathbb{E}|\zeta_q(x)|^{2jp})^{\frac{1}{2p}}\,\mathbb{P}(|\zeta_q(x)|>r_q/32)^{\frac{1}{2p}}\leqslant C_j\,p^{j/2}\,q^{j/4}\,\mathrm{e}^{-c\sqrt{q}/(p\lambda)},$$

and integrating in q with $u=\sqrt{q}/(p\lambda)$ gives $\|G_T\|_p\leqslant C\lambda^3 p^5$. □

No information on the interacting field has been used, apart from the equation itself and the posterior structure of the noise. The tails of ζ are those of the free field whatever the interaction: this replaces the subtraction of a Gaussian field.

14.3 A pointwise bound

The maximum principle gives more than the integrated estimate: a bound in L^∞ driven by the cube alone.

***Proposition* 14.4. (L^∞ coercivity)** *For q_* large, at every cutoff, pathwise and for every $q\geqslant q_*$,*

$$\|Y_q\|_\infty\leqslant C\Big[\Big(\frac{q}{\lambda}\Big)^{1/3}\|\zeta_q\|_\infty^{1/3}+\|\zeta_q\|_\infty+q^{1/4}+q\|Z_q\|_2\Big]. \tag{14.6}$$

In particular there is $c>0$ such that $t_q=0$, i.e. $\|Y_q\|_\infty\leqslant r_q$, whenever $\|\zeta_q\|_\infty+q\|Z_q\|_2\leqslant c\,r_q$: large fields occur only when the noise or the error is large.

Proof. *Let $\varpi=q-q_0$, $d=\zeta_q-\varrho_q(Y_q)$, and remove the error by $\mathcal{H}=\varpi\,C_{q_0}Z_q$. Then $\|\mathcal{H}\|_\infty\leqslant\varpi\,a_2(q_0)^{1/2}\|Z_q\|_2\leqslant C\,q\|Z_q\|_2$, since the kernel of C_{q_0} is in L^2 uniformly in N by (10.4). By (13.5), $W=V-\mathcal{H}$ and $D=\mathcal{H}+d$ satisfy*

$$K_{q_0}W+\frac{\lambda}{3}(W+D)^3=\varpi\,d,\qquad Y_q=W+D.$$

At a positive maximum of W, $K_{q_0}W\geqslant 0$, so $W+D\leqslant(3\,\varpi\|d\|_\infty/\lambda)^{1/3}$ there; similarly at a negative minimum. Hence $\|Y_q\|_\infty\leqslant 2\|D\|_\infty+(3\,q\|d\|_\infty/\lambda)^{1/3}$. With $\|\varrho_q(Y_q)\|_\infty\leqslant\varepsilon_q\|Y_q\|_\infty$, the term $2\,\varepsilon_q\|Y_q\|_\infty$ is absorbed, and by Young's inequality $(q\varepsilon_q\|Y_q\|_\infty/\lambda)^{1/3}\leqslant\frac14\|Y_q\|_\infty+C\,(q\varepsilon_q/\lambda)^{1/2}$, where $q\,\varepsilon_q/\lambda\leqslant C\sqrt{q}$ since $g_q=\lambda\,q^{-1/2}$. This proves (14.6). For the last claim use $q/\lambda=r_q^2$: the terms of (14.6) are at most $C\,(c^{1/3}+c+g_q^{1/2})\,r_q$, and $g_q^{1/2}\leqslant g_{q_*}^{1/2}$ is small. □

***Lemma* 14.5. (Supremum of the noise)** *Uniformly in N and in the interaction,*

$$\|\mathcal{N}\|_{L^p(\mathbb{P})}\leqslant C\sqrt{p},\qquad \mathcal{N}:=\sup_{q\geqslant q_*}q^{-1/4}\,(\log(2+q))^{-1/2}\|\zeta_q\|_\infty.$$

Proof. *At fixed q the norm $f\mapsto\|C_q f\|_\infty$ is convex, so by Lemma 13.2 $\|\zeta_q\|_\infty$ is dominated by the supremum of the Gaussian field $C_q B_{q-q_0}$. Its variance is at most $C\sqrt{q}$, and for $\theta<1$ its increments have variance at most $C\sqrt{q}\,(\sqrt{q}|x-y|)^\theta$ uniformly in N, so Dudley's bound gives the mean $\lesssim q^{1/4}\sqrt{\log(2+q)}$ and Gaussian concentration gives fluctuations at the scale $q^{1/4}$. For the supremum over $s\in[Q,2\,Q]$, write $C_s=C_{2Q}\,K_{2Q}\,C_s$, where $\|K_{2Q}\,C_s\|_{\infty\to\infty}\leqslant 2$. Since $\mathrm{d}\omega_q=(I-(q-q_0)\mathrm{Cov}_q)\,\mathrm{d}W_q$ by (10.6) and (10.8), ω is a martingale, $\|C_{2Q}\,\omega_s\|_\infty$ is a submartingale, and Doob's inequality gives, for $p\geqslant 2$, $\|\sup_{s\in[Q,2Q]}\|\zeta_s\|_\infty\|_{L^p}\leqslant C\,Q^{1/4}\,(\sqrt{\log(2+Q)}+\sqrt{p})$. Markov's inequality with $p\asymp\vartheta^2\log(2+Q)$ bounds the probability that the normalised supremum over $[Q,2\,Q]$ exceeds ϑ by $(2+Q)^{-c\vartheta^2}$, for $\vartheta\geqslant\vartheta_0$. This is summable over dyadic Q, and $\|\mathcal{N}\|_p\leqslant C\sqrt{p}$ follows.* □

Exercises

***Exercise* 14.1. (Without the cube)** *Replace $\frac{\lambda}{3}(v+g)^3$ by 0 in (14.1). Show that $v=\varpi\,a^{-1}(g+z)$, so that the estimate fails as $q\to\infty$ at fixed mass a. Which term of (14.2) replaces the missing mass, and how does the threshold $r=(q/\lambda)^{1/2}$ enter?*

***Exercise* 14.2. (Discrete convexity)** *Prove that $\langle\phi'(u),-\Delta_N u\rangle\geqslant 0$ for every convex $\phi\in C^1$ and $u\in H_N$, and that equality holds iff $\phi'(u(x))=\phi'(u(x'))$ on every edge $x\sim x'$ with $u(x)\neq u(x')$.*

***Exercise* 14.3. (Numerical check)** *Solve (14.1) numerically for random λ,q,a,ϖ,g,z, and check (14.2). Take g Gaussian with variance $\sqrt{q}$, $z=0$, $\lambda=1$, and compare $|v+g|$ with $(q\,|g|/\lambda)^{1/3}$ and with r. Is the exponent $5/12$ of $(q\,|g|)^{1/3}\approx q^{5/12}$ attained on one site? What should improve it on the lattice?*

15 Backward feedback and closure

15.1 The backward representation

The backward equation in (1) is linear in Z. Its propagator is the adapted operator solution of

$$\partial_s \mathcal{P}_{q,s} = \mathcal{P}_{q,s}\, H_s(Y_s), \qquad \mathcal{P}_{q,q} = I, \qquad s \geqslant q. \tag{15.1}$$

By (A), $\|\mathcal{P}_{q,s}\|_{2\to 2} \leqslant \exp \int_q^s \frac{1+\varepsilon_u}{u} \mathrm{d}u \leqslant L_0\, \frac{s}{q}$, with $L_0 = \exp\left(\int_{q_*}^{\infty} \varepsilon_u \frac{\mathrm{d}u}{u}\right) \leqslant \exp\left(C\,(g_{q_*} + g_{q_*}^2)\right)$.

***Lemma* 15.1. (Conditional representation)** *For $q_* \leqslant q \leqslant T$, at fixed N,*

$$Z_q = \mathbb{E}_q\left[\mathcal{P}_{q,T} Z_T + \int_q^T \mathcal{P}_{q,s}\; b_s(Y_s)\, \mathrm{d}s\right]. \tag{15.2}$$

Proof. *By the product rule and (1), $\mathrm{d}_s(\mathcal{P}_{q,s} Z_s) = \mathcal{P}_{q,s} H_s Z_s \mathrm{d}s - \mathcal{P}_{q,s}\,(b_s + H_s Z_s)\mathrm{d}s + \mathcal{P}_{q,s}\,\Xi_s \mathrm{d}W_s = -\mathcal{P}_{q,s}\, b_s \mathrm{d}s + \mathcal{P}_{q,s}\,\Xi_s \mathrm{d}W_s$. The stochastic integral is a true martingale by Lemma 13.1 and the deterministic bound on $\mathcal{P}$. Integrate from q to T and take $\mathbb{E}_q$.* □

So Z is the expected source accumulated on the unresolved scales, transported back to q, plus the boundary error. This is the conditional-expectation form of a backward equation, as in $F_t = \mathbb{E}_t[F_T]$ for the force (Theorem 7.3 of Lecture 4); here the source is the defect.

15.2 Three contributions

Split $Z = Z^{\mathrm{t}} + Z^{\mathrm{b}} + Z^{\mathrm{l}}$ according to the terminal term and the two parts of the source. We estimate each in the scalar manner: norms of the propagator times norms of the source.

Bounded fields. *By (B),*

$$\|Z_q^{\mathrm{b}}\|_2 \leqslant \int_q^{\infty} L_0\, \frac{s}{q}\; C\,\lambda^{7/2}\, s^{-5/2}\, \mathrm{d}s = 2\, L_0\, C\, \lambda^{7/2}\, q^{-3/2}, \qquad \frac{1}{\lambda}\int_{q_*}^{\infty} q^{1/2} \|Z_q^{\mathrm{b}}\|_2^2\, \mathrm{d}q \leqslant C\,\lambda^6\, q_*^{-3/2}.$$

This is where the two correction steps of Lecture 6 are used: with one step the source would be $O(s^{-2})$ and the first integral would diverge logarithmically.

Boundary. *$q\|Z_q^{\mathrm{t}}\|_2 \leqslant L_0\; \mathbb{E}_q\, \|T Z_T\|_2$, so, with $\delta_{N,T,p} := \|T Z_T\|^2_{L^{2p}(\mathbb{P};L^2)}$,*

$$\left|\frac{1}{\lambda}\int_{q_*}^{T} q^{1/2} \|Z_q^{\mathrm{t}}\|_2^2 \mathrm{d}q\right|_{L^p(\mathbb{P})} \leqslant \frac{L_0^2}{\lambda}\int_{q_*}^{\infty} q^{-3/2}\mathrm{d}q\;\; \delta_{N,T,p} = \frac{2\, L_0^2}{\lambda\, q_*^{1/2}}\; \delta_{N,T,p}.$$

At fixed N, $\delta_{N,T,p} \to 0$ as $T \to \infty$ (Lemma 16.1).

Large fields. *Let $A^{\mathrm{l}} := \int_{q_*}^{T} s^{7/2} \|b_s^{\mathrm{l}}(Y_s)\|_2^2 \mathrm{d}s \leqslant C\lambda^4 \int s^{-3/2} \|t_s\|_6^6 \leqslant C\lambda^3\, J_T$ by (B). By Cauchy–Schwarz in s,*

$$\|Z_q^{\mathrm{l}}\|_2 \leqslant \frac{L_0}{q}\; \mathbb{E}_q \int_q^T s\, \|b_s^{\mathrm{l}}\|_2 \mathrm{d}s \leqslant \frac{L_0}{q}\left(\int_q^{\infty} s^{-3/2} \mathrm{d}s\right)^{1/2} \mathbb{E}_q[(A^{\mathrm{l}})^{1/2}],$$

so that, by Jensen's inequality, $\|Z_q^{\mathrm{l}}\|_2^2 \leqslant 2\, L_0^2\, q^{-5/2}\; \mathbb{E}_q\, A^{\mathrm{l}}$. Integrating, and using that $\mathbb{E}_q$ is a contraction on $L^p(\mathbb{P})$,

$$\left|\frac{1}{\lambda}\int_{q_*}^{T} q^{1/2} \|Z_q^{\mathrm{l}}\|_2^2 \mathrm{d}q\right|_{L^p} \leqslant \frac{2\, L_0^2}{\lambda}\int_{q_*}^{\infty} q^{-2}\mathrm{d}q\, \|A^{\mathrm{l}}\|_{L^p} \leqslant \frac{C\,\lambda^2}{q_*}\, \|J_T\|_{L^p}.$$

***Proposition* 15.2. (Backward feedback)** *At every cutoff,*

$$\|E_T\|_{L^p} \leqslant C\,\lambda^6\, q_*^{-3/2} + C\,\lambda^{-1}\, q_*^{-1/2}\; \delta_{N,T,p} + C\,\lambda^2\, q_*^{-1} \|J_T\|_{L^p}, \tag{15.3}$$

with C independent of N, T and p.

Proof. *$\|Z\|^2 \leqslant 3\,(\|Z^{\mathrm{t}}\|^2 + \|Z^{\mathrm{b}}\|^2 + \|Z^{\mathrm{l}}\|^2)$ and the three estimates above.* □

15.3 Closing the loop

Theorem 15.3. (Cutoff-uniform energy estimate) *There is q_*, depending on λ, such that for every N, T and $p \geqslant 1$,*

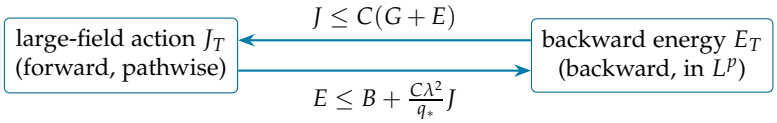
$$\|E_T\|_{L^p} + \|J_T\|_{L^p} \leqslant C_\lambda\,(1 + p^5 + \delta_{N,T,p}). \tag{15.4}$$

Proof. *Let $x = \|E_T\|_p$ and $y = \|J_T\|_p$. By Propositions 14.3 and 15.2,*

$$y \leqslant A_0\ (C\,\lambda^3\,p^5 + x), \qquad x \leqslant B_0 + B_1\,\delta_{N,T,p} + \theta\ y, \qquad \theta = \frac{C\,\lambda^2}{q_*},$$

with A_0, B_0, B_1 and the constant in θ independent of N, T, p, and bounded for all q_* beyond the thresholds already required. Choose q_* such that also $A_0\,\theta \leqslant \frac{1}{2}$. Then $y \leqslant A_0\,(C\,\lambda^3\,p^5 + B_0 + B_1\,\delta_{N,T,p}) + \frac{1}{2}\,y$. Since $y < \infty$ by Lemma 13.1, it can be absorbed, and x follows. □

The structure of the argument is the point:



The smallness is in the feedback, *λ^2 / q_*, not in the coupling of either equation, and it is independent of the moment order, so a single q_* closes the loop for every p. The growth p^5 is then inherited from the tails of the noise, G_T, and gives stretched exponential moments $\mathbb{E}\exp\{c\,(E_\infty + J_\infty)^{1/5}\} < \infty$ once the boundary term is removed. Note that $\lambda^2 / q_* = g_{q_*}^2$.*

15.4 Consequences

The boundary term $\delta_{N,T,p}$ is removed in Section 16, at each fixed N, by letting $T \to \infty$. Assuming this for the moment, we obtain

$$\sup_N \|E_\infty^N\|_{L^p} + \sup_N \|J_\infty^N\|_{L^p} \leqslant C_\lambda\,(1 + p^5), \qquad \sup_N \mathbb{E}\|Z_q^N\|_2^2 \leqslant C_\lambda\,q^{-5/2}. \tag{15.5}$$

The second bound follows from the representation formula on $[q, \infty)$: the three contributions are, up to the factor 3, $O(q^{-3})$, 0 and, by the computation above, $2\,L_0^2\,q^{-5/2}\,\mathbb{E}\,A^{\mathrm{l}}$. So the approximate forecast differs from the exact one by $O(q^{-5/4})$ in $L^2(\mathbb{P}; L^2)$ (root mean square), uniformly in the lattice. Since $\|H_q\| \leqslant C\,/\,q$, the coupling term $H_q\,Z_q$ in the forward equation is $O(q^{-9/4})$ in $L^2(\mathbb{P}; L^2)$, integrable in q: the forward equation is, up to an integrable drift, the explicit SDE $\mathrm{d}Y = b(Y)\,\mathrm{d}q + H(Y)\,\mathrm{d}W$. The same computation holds along the path. At $T = \infty$ the boundary part vanishes (Lemma 16.1) and, with A^{l} now integrated up to ∞ (it has all moments, since $A^{\mathrm{l}} \leqslant C\,\lambda^3\,J_\infty$), $q^{5/2}\|Z_q\|_2^2 \leqslant 3\,(C\,\lambda^7\,q_*^{-1/2} + 2\,L_0^2\,\mathbb{E}_q\,A^{\mathrm{l}})$. Since $\mathbb{E}_q\,A^{\mathrm{l}}$ is a martingale, Doob's inequality gives, for $p \geqslant 2$,

$$\Big\| \sup_{q \geqslant q_*} q^{5/2}\|Z_q\|_2^2 \Big\|_{L^p(\mathbb{P})} \leqslant C_\lambda\,(1 + p^5). \tag{15.6}$$

Exercises

***Exercise* 15.1. (The scalar backward estimate)** *On one site, with $H_q(y) = (q + \lambda\,y^2)^{-1}$, show that*

$$\mathcal{P}_{q,s} = \frac{s}{q}\exp\left(-\int_q^s \frac{\lambda\,Y_u^2}{u\,(u + \lambda\,Y_u^2)}\mathrm{d}u\right) \leqslant \frac{s}{q}.$$

In which sense do large fields make the backward equation more contractive? Redo the three estimates of Section 15 on one site.

***Exercise* 15.2. (Absorption needs finiteness)** *Give an example of nonnegative numbers x, y (possibly infinite) with $y \leqslant 1 + x$, $x \leqslant 1 + \frac{1}{2}\,y$ and $y = \infty$. Where is Lemma 13.1 used in the proof of Theorem 15.3?*

***Exercise* 15.3. (Moment growth)** *From $\|E_\infty + J_\infty\|_p \leqslant C\,(1 + p^5)$ for all $p \geqslant 1$, deduce $\mathbb{E}\exp\{c\,(E_\infty + J_\infty)^{1/5}\} < \infty$ for small c.*

Removing the cutoffs

Sections 13–15 proved (15.4) at every lattice cutoff N and every upper scale T. Three things remain.

i. *Show that the boundary term vanishes: $T Z_T \to 0$ as $T \to \infty$ at fixed N (Section 16).*

ii. *Turn the uniform energy bounds into tightness of the measures (Section 17).*

iii. *Identify the limit: the deterministic coefficients; then, from the restarted flow, the limit as an explicit density over an explicit flow, along the whole sequence; then the equations, read off by Girsanov's theorem (Section 18).*

16 The boundary at infinite resolution

16.1 The boundary lemma

At a fixed lattice, the terminal resolution is $q = \infty$, and the field at resolution q, Φ_q, converges to the terminal field Φ_∞. The approximate force $\bar{\mathcal{F}}_q$ was built to converge to the bare force: $\mu_q \to \mu_N$, $\Theta_q \to 0$, and the nonlinear corrections were integrated from $q = \infty$, so that they vanish there. This is the boundary condition of the backward equation.

***Lemma* 16.1. (Boundary lemma)** *At fixed N, for every $p < \infty$, as $q \to \infty$,*

$$\|Y_q - \Phi_\infty\|_{L^p(\mathbb{P};H_N)} + \|q\, Z_q\|_{L^p(\mathbb{P};H_N)} \leqslant C_{N,p}\; q^{-1/2}, \qquad \text{hence} \qquad \delta_{N,T,p} \leqslant C_{N,p}\, T^{-1}. \tag{16.1}$$

The constants depend on N; this is not a uniform estimate.

Proof. *All norms on H_N are equivalent, and we do not track the dependence on N.*

The field at resolution q. *By (10.7), $\Phi_q = (q - q_0)\, C_q\, \Phi_\infty + C_q\, B_{q-q_0}$, and $(q - q_0)\, C_q - I = -C_q\, K_{q_0} = O(q^{-1})$, $\|C_q\, B_{q-q_0}\|_{L^p} = O(q^{-1/2})$. So $\|\Phi_q - \Phi_\infty\|_{L^p} \leqslant C_{N,p}\, q^{-1/2}$.*

The approximate forecast. *Global coercivity gives $\|Y_q\| \leqslant C\, \|K_q \Phi_q\| \,/\, q$, bounded in every L^p uniformly in q. The coordinate equation $\Phi_q = Y_q + C_q\, \bar{\mathcal{F}}_q(Y_q)$ and the bounds $|\mu_q| \leqslant C_N$, $\|\Theta_q\|_{\mathrm{row}} = O_N(q^{-1/2})$ and $\|\tilde{\ell}_q(y)\|_\infty \leqslant C \mathsf{S}_4(\ell_q^{[3]}) \|y\|_\infty^3 \leqslant C\, \lambda^2\, q^{-1/2} \|y\|_\infty^3$ (Taylor from the four-leg vertex, since the low jet vanishes) show that $\bar{\mathcal{F}}_q(Y_q)$ is bounded in L^p, so $\|Y_q - \Phi_q\|_{L^p} = O(q^{-1})$. Moreover $|\mu_q - \mu_N| = O_N(q^{-1})$, and therefore*

$$\|\bar{\mathcal{F}}_q(Y_q) - D\, V_N(\Phi_\infty)\|_{L^p} \leqslant C_{N,p}\, q^{-1/2}. \tag{16.2}$$

Thus the explicit force at the approximate forecast converges to the bare force at the terminal field.

The error. *Subtract the approximate coordinate equation $K_q \Phi_q = K_q\, Y_q + \bar{\mathcal{F}}_q(Y_q)$ from the exact one ((10.6) and Proposition 10.4(b)), $K_q \Phi_q = K_q\, \psi_q + \mathbb{E}_q\, [D\, V_N(\Phi_\infty)]$:*

$$q\, Z_q = q\, C_q\, [\bar{\mathcal{F}}_q(Y_q) - \mathbb{E}_q\, D\, V_N(\Phi_\infty)]. \tag{16.3}$$

Let $G = D\, V_N(\Phi_\infty)$ and $G_q^0 = D\, V_N(\Phi_q)$, which is $\mathcal{F}_q$-measurable. Then $\|\mathbb{E}_q\, G - G\|_{L^p} \leqslant \|\mathbb{E}_q\, (G - G_q^0)\|_{L^p} + \|G_q^0 - G\|_{L^p} \leqslant 2\, \|G - G_q^0\|_{L^p} = O(q^{-1/2})$, by the cubic Lipschitz bound for $D\, V_N$ and the first step. With (16.2) and $\|q\, C_q\| \leqslant 1$ this proves the claim. □

The proof uses no estimate uniform in N, and no concentration of the posterior: only that the Polchinski process Φ_q, which is $\mathcal{F}_q$-measurable, converges to the hidden field. In Lecture 4 language, the forecast and the field at resolution q have the same limit at infinite resolution, and the force $F_q = K_q\, (\psi_q - \Phi_q)$ converges to the bare force $-D\, V_N(\Phi_\infty)$ (Exercise 16.1; recall $F_q = \mathbb{E}_q\, [-D\, V_N(\Phi_\infty)]$ by (10.6), so this is the conditional-expectation step of the proof).

16.2 UV stability

Now let $T \to \infty$ at fixed N in (15.4): the paths and the law ν_N do not depend on T, E_T and J_T increase to E_∞ and J_∞, and $\delta_{N,T,p} \to 0$ by Lemma 16.1. Only then take the supremum over N.

***Theorem* 16.2. (UV stability)** *For q_* large enough, depending on λ,*

$$\sup_N(\|E_\infty^N\|_{L^p}+\|J_\infty^N\|_{L^p})\leqslant C_\lambda\,(1+p^5),\qquad \sup_N \mathbb{E}\|Z_q^N\|_2^2\leqslant C_\lambda\, q^{-5/2}, \tag{16.4}$$

and

$$\sup_N \mathbb{E}\exp\{c\,(E_\infty^N+J_\infty^N)^{1/5}\}<\infty$$

for small $c>0$.

Proof. *Monotone convergence in (15.4), then the representation formula (15.2) on $[q,\infty)$, whose terminal term vanishes by Lemma 16.1. The exponential moment follows from the polynomial growth by Markov's inequality with $p\asymp R^{1/5}$.* □

16.3 Restarted laws

The coordinate is Lipschitz for the uniform norm, by the maximum principle.

***Lemma* 16.3. (Lipschitz coordinate)** *Uniformly in N, $\|y_s(x)-y_s(x')\|_\infty\leqslant 3\|x-x'\|_\infty$ for $s\geqslant q_*$ and with the constant 6 for $s<q_*$. So the drift in (18.4) is Lipschitz on L^∞ with constant $7/s$.*

Proof. *With $\delta=y-y'$, $h=x-x'$, $a=\delta+\varrho_s(y)-\varrho_s(y')$, (12.5) gives $K_s(a-h)+\frac{\lambda}{3}(y^3-(y')^3)=0$. At a positive maximum of $a-h$, $K_s(a-h)\geqslant s(a-h)>0$, so $y^3<(y')^3$ there, i.e. $\delta<0$ and $a-h\leqslant\|\varrho_s(y)-\varrho_s(y')\|_\infty+\|h\|_\infty$; similarly at a negative minimum. Since ϱ_s is ε_s-Lipschitz on L^∞, $\|\delta\|_\infty\leqslant 2\|h\|_\infty+2\varepsilon_s\|\delta\|_\infty$. For $s<q_*$ use (12.7) and $\|C_{q_*}K_s\|_{\infty\to\infty}\leqslant 2$. Finally $\|C_s\|_{\infty\to\infty}=1/s$.* □

The estimates above use the start of the flow only through the noise. So they hold, with the changes below, for the flow restarted from any field at any scale, and these estimates are the input of Section 18.2. Let $\mathrm{e}^{-V_q(x)}=\mathbb{E}\mathrm{e}^{-V_N(x+\int_q^\infty C_s\,\mathrm{d}W_s)}$, and define the exact force $F_q=-DV_q$, and let $\mathbb{P}^{q,x}$ be the exact flow conditioned on $\Phi_q=x$, $\mathrm{d}\Phi=-C_s^2DV_s(\Phi)\,\mathrm{d}s+C_s\,\mathrm{d}W$. Fix N, $q\geqslant q_0$ and $x\in H_N$, $R=\|x\|_\infty$. By the Markov property of the flow, $\mathbb{P}^{q,x}$ is the law of $\Phi_t=C_t(K_qx+(t-q)\hat\Phi+B'_{t-q})$, $t\geqslant q$, where B' is a Brownian motion independent of $\hat\Phi$, and $\hat\Phi$ has the law

$$\nu^{[q,x]}(\,\mathrm{d}z)\propto\exp\Big(\langle z,K_qx\rangle-\frac{1}{2}\langle z,K_qz\rangle-V_N(z)\Big)\,\mathrm{d}z. \tag{16.5}$$

Note that $\nu^{[q_0,0]}=\nu_N$, so $\mathbb{P}^{q_0,0}$ is the flow itself. The posterior given Φ_t is $\propto\exp\big(\langle z,K_t\Phi_t\rangle-\frac{1}{2}\langle z,K_tz\rangle-V_N(z)\big)\,\mathrm{d}z$, the same function of Φ_t as for ν_N. So $\psi_t=m_t(\Phi_t)$, and Y_t, $Z_t=\psi_t-Y_t$ and the system (1) are unchanged, with the same coefficients and with $Y_q=y_q(x)$, $\|Y_q\|_\infty\leqslant 6R$ by Lemma 16.3. Let $\omega_t=K_t\Phi_t-(t-q)\psi_t=K_qx+\mathbb{E}_tB'_{t-q}$, $\zeta_t=C_t\omega_t$. Then

$$\zeta_t=d_t+\zeta_t^0,\qquad d_t=C_tK_qx,\qquad \zeta_t^0=C_t\mathbb{E}_tB'_{t-q}. \tag{16.6}$$

Here $C_tK_q=I-(t-q)C_t$ and $(t-q)C_t$ has a positive kernel of total mass at most 1, so $\|d_t\|_\infty\leqslant 2R$. By conditional Jensen, ζ^0 satisfies Lemma 13.2, with $(t-q)a_2(t)\leqslant C\sqrt{t}$. With the field $V_t=Y_t+\varrho_t(Y_t)-\zeta_t$ of (13.4) (not the potential $V_t(x)$, always written with its argument), the computation of (13.5) with q_0 replaced by q gives the field equation

$$K_qV_t+\frac{\lambda}{3}Y_t^3=(t-q)(\zeta_t-\varrho_t(Y_t)+Z_t). \tag{16.7}$$

It has the form of (13.5), with mass $K_q\geqslant 0$ and $0\leqslant t-q\leqslant t$, and it holds for $t\geqslant q_*$ also when $q<q_*$, as it uses only $K_t=K_q+t-q$ and the coordinate equation at t.

***Proposition* 16.4. (Restarted UV stability)** *Let $q^\vee=\max(q,q_*)$. With energies on $[q^\vee,\infty)$ and E,J as in (13.7), for $p\geqslant 1$, with C independent of N,q,x,p,*

$$\|E_\infty\|_{L^p(\mathbb{P}^{q,x})}+\|J_\infty\|_{L^p(\mathbb{P}^{q,x})}\leqslant C(1+p^5+p^3R^6), \tag{16.8}$$

$$\mathbb{E}^{q,x}\|Z_s\|_2^2\leqslant C(1+R^6)s^{-5/2},\quad \mathbb{E}^{q,x}\|Y_s\|_2^2\leqslant C(1+R^6)(R^2+s^{1/2}),\quad s\geqslant q^\vee. \tag{16.9}$$

Below the auxiliary scale, if $q<q_*$, for $p\geqslant 1$,

$$\sup_{q\leqslant s\leqslant q_*}\mathbb{E}^{q,x}(\|\psi_s\|_2^{2p}+\|Y_s\|_\infty^{2p}+\|Z_s\|_2^{2p})\leqslant C_{p,R}. \tag{16.10}$$

Proof. Forward. *Proposition 14.3 applies pathwise to (16.7): its proof uses only a nonnegative mass, a right-side coefficient $t-q\in[0,t]$, the positivity of $\langle w^3,K_qV\rangle$ and the bounds on ϱ. So $J_T\leqslant C(G_T+E_T)$ with G_T built on $\xi_t=\zeta_t1_{|\zeta_t|>r_t/32}$.*

The noise. *Let $q_R=(128)^2\lambda R^2$, so that $2R\leqslant r_t/64$ for $t\geqslant q_R$. For such t, $|\zeta|>r_t/32$ forces $|\zeta^0|>r_t/64$ and $|\zeta|\leqslant 2|\zeta^0|$, and the proof of (14.5) applies to ζ^0 with this threshold. For $q\leqslant t<q_R$ use $\|\xi_t\|_j^j\leqslant C(R^j+\|\zeta_t^0\|_j^j)$ and $\|\|\zeta_t^0\|_j^j\|_{L^p}\leqslant Cp^{j/2}t^{j/4}$. Then the part of G on $[q^\vee,q_R)$ has L^p norm at most*

$$C\int_{q^\vee}^{q_R}(t^{-1/2}(R^4+p^2t)+\lambda t^{-3/2}(R^6+p^3t^{3/2}))\,\mathrm{d}t\leqslant Cp^3(1+R^6),$$

the constant depending on λ. So $\|G_T\|_p\leqslant C(p^5+p^3R^6)$.

Backward and closure. *Lemma 13.1 holds as $\nu^{[q,x]}$ has all moments. Lemma 15.1 and Proposition 15.2 hold on $[q^\vee,T]$, as they use only the propagator (15.1), the deterministic bounds (B) on $b^{\mathrm{b}},b^{\mathrm{l}}$ and the conditional expectations; the feedback coefficient is $C\lambda^2/q^\vee\leqslant C\lambda^2/q_*$. No initial value of Y at $q^\vee$ is used. So Theorem 15.3 holds with $\delta_{T,p}$ computed under $\mathbb{P}^{q,x}$.* ***Boundary:*** *the proof of Lemma 16.1 applies at fixed N, with $\Phi_t\to\hat\Phi$ and the coordinate equation $K_t\Phi_t=K_t\psi_t+\mathbb{E}_tD\,V_N(\hat\Phi)$, so $\delta_{T,p}\to 0$. This proves (16.8). The bound on Z follows from the representation (15.2) as in (15.5). The bound on Y follows from Itô's formula for $\|Y\|_2^2$ in the system (1), with $\|Y_q\|_2\leqslant 3R$ if $q\geqslant q_*$ and the moments of Y_{q_*} below if $q<q_*$, the linear bound on b (Proposition 12.2), $\|H\|\leqslant C/s$ and $\operatorname{Tr}H^2\leqslant Cs^{-1/2}$ ((12.3)). The coupling term needs care: the naive bound $2\|Y\|_2\|Z\|_2C/s$ would give the Gronwall factor $\exp\int Cs^{-1}\mathrm{d}s$, which is not bounded. Split instead $2|\langle Y_s,H_sZ_s\rangle|\leqslant s^{-3/2}\|Y_s\|_2^2+Cs^{-1/2}\|Z_s\|_2^2$: the first part gives the bounded factor $\exp\int s^{-3/2}\mathrm{d}s$, and by the bound on Z the second contributes at most $C(1+R^6)\int s^{-3}\mathrm{d}s<\infty$.*

Below the auxiliary scale: the deduction. *Let $q<q_*$. By the representation (15.2) at q_* and conditional Jensen, $\|Z_{q_*}\|_2$ has all moments. By (14.4) at the scale q_*, pathwise, $\lambda\|t_{q_*}\|_6^6\leqslant C(q_*\|\xi\|_4^4+\lambda\|\xi\|_6^6+\lambda^{-1}q_*^2\|Z_{q_*}\|_2^2)$, and $\|Y_{q_*}\|_2\leqslant r_{q_*}+\|t_{q_*}\|_6$. So $\psi_{q_*}=Y_{q_*}+Z_{q_*}$ has all moments in L^2. As ψ is a martingale, $\mathbb{E}\|\psi_s\|_2^{2p}\leqslant\mathbb{E}\|\psi_{q_*}\|_2^{2p}$ for $s\leqslant q_*$, by Jensen. Now $Y_s=y_{q_*}(x'_s)$ with $x'_s=C_{q_*}K_s\Phi_s=C_{q_*}K_qx+(s-q)C_{q_*}\psi_s+C_{q_*}\mathbb{E}_sB'_{s-q}$. By Lemma 16.3, $\|Y_s\|_\infty\leqslant 3\|x'_s\|_\infty$, as $y_{q_*}(0)=0$, and $C_{q_*}\colon L^2\to L^\infty$ is bounded uniformly in N, while $\|C_{q_*}\mathbb{E}_sB'_{s-q}\|_\infty$ is dominated in every convex moment by the supremum of a Gaussian field at the fixed scale q_*, by conditional Jensen as in Lemma 13.2. This bounds $\mathbb{E}\|Y_s\|_\infty^{2p}$. Then $Z_s=\psi_s-Y_s$. This proves (16.10).* □

Exercises

***Exercise* 16.1. (The force at infinite resolution)** *Show that, at fixed N, $F_q=K_q(\psi_q-\Phi_q)\to -D\,V_N(\Phi_\infty)$ in every L^p, and that $F_q=K_qZ_q-\bar{\mathcal{F}}_q(Y_q)$. Interpret the last identity: the force observable of Lecture 4 is the explicit force at the approximate forecast, corrected by K_qZ_q.*

***Exercise* 16.2. (Why the stronger boundary)** *Using the bound on the boundary term in the proof of Proposition 15.2, explain why $Z_T\to 0$ alone would not make it vanish, and why the rate $\|TZ_T\|=o(1)$ of Lemma 16.1 is what is needed.*

17 Tightness and tails

17.1 One Hilbert space

To compare lattices, partition $\mathbb{T}^3$ into cubic cells $\square_x$ of side N^{-1} centred at $x\in\Lambda_N$, and let

$$(\iota_Nu)(z)=u(x)\,(z\in\square_x),\qquad (j_Nf)(x)=N^3\int_{\square_x}f,\qquad \mathsf{P}_N=\iota_Nj_N.$$

$\iota_N\colon H_N \to H = L^2(\mathbb{T}^3)$ is an isometry, $j_N = \iota_N^*$, and $\mathsf{P}_N \to I$ strongly. Fields are embedded by ι_N, operators by $\iota_N \cdot j_N$. The embedded innovation has covariance P_N; in the limit it becomes a cylindrical Brownian motion on H, obtained as the limit of the lattice innovations.

17.2 Spatial suprema by a maximum principle

The energy bounds control L^4 and L^6 norms of large fields. Spatial suprema at finite resolution come from the L^∞ coercivity of Proposition 14.4.

***Lemma* 17.1. (Supremum after closure)** *Let $\mathcal{Y} = \sup_{q \geqslant q_*} q^{-5/12} (\log(2+q))^{-1/6} \|Y_q\|_\infty$ and $\mathcal{Z} = \sup_{q \geqslant q_*} q^{5/4} \|Z_q\|_2$. For q_* large, with $\mathcal{N}$ the supremum of the noise of Lemma 14.5,*

$$\mathcal{Y} \leqslant C_\lambda (1 + \mathcal{N} + \mathcal{Z}), \qquad \|\mathcal{N}\|_{L^p} \leqslant C\sqrt{p}, \qquad \|\mathcal{Z}^2\|_{L^p} \leqslant C_\lambda (1 + p^5), \tag{17.1}$$

uniformly in N.

Proof. *By the definitions, $\|\zeta_q\|_\infty \leqslant q^{1/4} (\log(2+q))^{1/2} \mathcal{N}$ and $q\|Z_q\|_2 \leqslant q^{-1/4} \mathcal{Z}$. Inserting these in (14.6), the first term gives $\lambda^{-1/3} q^{5/12} (\log(2+q))^{1/6} \mathcal{N}^{1/3}$, and the others are of lower order. The bound on $\mathcal{Z}$ is (15.6), whose boundary term vanishes by Lemma 16.1.* □

Any exponent below $\frac{1}{2}$ keeps Y below the threshold r_q at large q. The exponent $5/12$ comes from the cube alone. The memory form of Exercise 13.3, with a weight in q absorbing the memory, gives every exponent above $\frac{1}{4}$, at the price of that absorption.

At the terminal resolution there is no such bound: Φ_∞ is a distribution.

What the pointwise bound gives, and what it does not. *The L^∞ bound cannot replace Proposition 14.3 in the loop: it is linear in $q\|Z_q\|_2$, so its sixth power is not controlled by the energy E, while the test with w^3 gives a bound quadratic in Z. Once the loop has closed it does give more. Almost surely $\|Y_q\|_\infty \ll r_q$ past a random scale, so large fields occur only at finitely many dyadic scales, and $\mathcal{D}_q(Y_q)$ is then the bounded-field defect alone. Suprema over finite intervals of scales $\geqslant q_*$ have all moments uniformly in N, which is the boundedness part of the hypothesis of Section 18.1 along the exact flow, though Section 18.1 does not need it, as it passes through the explicit flow. The same maximum principle, in the form of Lemma 16.3, gives the estimate below q_* in Proposition 16.4.*

17.3 The terminal field

The approximate forecast converges at infinite resolution in negative Sobolev spaces, uniformly in N (to the same limit as the forecast ψ). The martingale part of Y has bracket $\int H^2$, and the positive inequality $H_q^2 \leqslant \frac{C}{q} C_q$ gives

$$\int_Q^\infty \|(1-\Delta)^{-s/2} H_q\|_{\mathrm{HS}}^2 \mathrm{d}q \leqslant C \int_Q^\infty \frac{1}{q} \sum_k \frac{(1+|k|^2)^{-s}}{q + c|k|^2} \mathrm{d}q \leqslant C_s\, Q^{-(s-\frac{1}{2})}, \qquad \tfrac{1}{2} < s < \tfrac{3}{2}, \tag{17.2}$$

where $q + c|k|^2$ bounds the lattice symbol from below (aliases of the cell embedding only increase the frequency). The drifts are integrable at infinity in H: the bounded-field drift is $O(q^{-5/2})$, the large-field drift is controlled by $\int q^{7/2} \|b^{\mathrm{l}}\|^2 \leqslant C\lambda^3 J_\infty$ and Cauchy–Schwarz, and $\int \|H_q Z_q\| \mathrm{d}q \leqslant C \int q^{-1} \|Z_q\| \leqslant C\lambda^{1/2} E_\infty^{1/2}$. More precisely, the squared drift tails beyond Q are $O(Q^{-3})$ (bounded-field), $O(Q^{-5/2})\lambda^3 J_\infty$ (large-field) and $O(Q^{-3/2})\lambda E_\infty$ (the HZ term), all at most $Q^{-(s-\frac{1}{2})}$ for $s < \frac{3}{2}$. By Doob's inequality, for $\frac{1}{2} < s < \frac{3}{2}$,

$$\sup_N \mathbb{E} \sup_{t \geqslant Q} \|Y_t - Y_Q\|_{H^{-s}}^2 \leqslant C_s\, Q^{-(s-\frac{1}{2})}. \tag{17.3}$$

Letting $t \to \infty$ at fixed N (Lemma 16.1) this holds with Y_t replaced by Φ_∞.

The field at the lower scale. *As in the proof of Proposition 16.4, $\|Y_{q_*}\|_2$ has all moments, uniformly in N. So $\sup_N \mathbb{E}\|\iota_N \Phi^N_\infty\|^2_{H^{-s'}} < \infty$ for $\frac{1}{2} < s' < \frac{3}{2}$, and these spaces embed compactly in H^{-s} for $s > s'$:*

***Corollary* 17.2. (Tightness of the lattice measures)** *For every $s > \frac{1}{2}$ the embedded laws $\nu_N \circ \iota_N^{-1}$ are tight in $H^{-s}(\mathbb{T}^3)$, and $\sup_N \mathbb{E}\,\|\iota_N \Phi^N_\infty\|^p_{H^{-s}} < \infty$ for every p.*

For the moments: the drift integrals have all moments by (16.4), and the H^{-s} martingale has a deterministic bracket bound by (17.2) at $Q = q_*$, the field at the lower scale has all moments by the paragraph above, so the Burkholder–Davis–Gundy inequality gives all moments.

The threshold $s > \frac{1}{2}$ is that of the free field in $d = 3$: the interacting field is no rougher than the Gaussian one. At every finite resolution its part V is smoother (H^1).

17.4 Exponential moments

Theorem 16.2 gives only stretched exponential moments of the energies. Hairer and Steele [HSt22] showed that the measure has quartic exponential tails, as its formal density suggests, by tilting it and using a priori bounds for the tilted dynamics. Here is a proof in the language of this lecture, through linear sources: the filter is available for every prior, and the noise bound does not depend on the prior.

*Let $f \in H_N$, $h \in \mathbb{R}$, and let $\hat\nu \propto \mathrm{e}^{h\langle \varphi, f\rangle}\nu_N$, which has all moments at fixed N. Let Φ_q be the Polchinski process with terminal law $\hat\nu$, i.e. the bridge (10.7) built on $\hat\Phi \sim \hat\nu$, $K_q\Phi_q = (q - q_0)\hat\Phi + B_{q-q_0}$, and $\hat\psi_q = \mathbb{E}_q\hat\Phi$. **Keep the force, and shift the field in the coordinate equation:***

$$K_q Y_q + \bar{\mathcal{F}}_q(Y_q) = K_q\Phi_q + h\,f, \qquad Z_q = \hat\psi_q - Y_q. \tag{17.4}$$

***Proposition* 17.3. (The source changes the field equation, not the system)** *For $q \geqslant q_*$: the pair (Y, Z) solves (1) with the same b, H; $TZ_T \to 0$ in every L^p at fixed N; the field equation (13.5) holds with the extra term $h\,f$ on the right side. If $q_1 \geqslant q_*$ and $q_1^{3/2} \geqslant 70\lambda^{1/2}|h|\|f\|_\infty$, then the conclusion of Theorem 16.2 holds on $[q_1, \infty)$, with constants independent of N, h, f, q_1.*

Proof. *This is the flow restarted at (q_0, x) with $x = h\,C_{q_0}f$: then $\nu^{[q_0,x]} = \hat\nu$ and $K_q\Phi_q + h\,f = K_{q_0}x + (q - q_0)\hat\Phi + B_{q-q_0}$. So the system and the boundary are as in Section 16.3; the extra term in the field equation is $(q - q_0)d_q + K_{q_0}d_q = h\,f$ with $d_q = h\,C_q f$. Since $\|d_q\|_\infty \leqslant |h|\|f\|_\infty/q \leqslant r_q/64$ for $q \geqslant q_1$, as $64 < 70$, the proof of Proposition 16.4 applies with no range below the threshold. The backward estimate and the closure only improve with q_1.* □

There is also an exact identity with no cube and no derivative. As $\omega_q = K_q\Phi_q - (q - q_0)\hat\psi_q$ and $K_q\Phi_q = (q - q_0)\hat\Phi + B_{q-q_0}$ (here ω is the unshifted noise, $\omega_q = \mathbb{E}_q[B_{q-q_0}]$), for every prior and $q > q_0$,

$$\hat\Phi = Y_q + Z_q + \frac{\omega_q - B_{q-q_0}}{q - q_0}. \tag{17.5}$$

***Theorem* 17.4. (Quartic tails)** *Let $\|f\|_* = \|f\|_1 + \|f\|_2 + \|f\|_\infty$. There are β, C independent of N and f such that*

$$\mathbb{E}|\langle \Phi^N_\infty, f\rangle| \leqslant C\|f\|_*, \quad \mathbb{E}\exp(\beta\langle \Phi^N_\infty, f\rangle^4/\|f\|_*^4) \leqslant C. \tag{17.6}$$

Consequently, $\sup_N \mathbb{E}\exp(\beta_\rho\|\iota_N\Phi^N_\infty\|^4_{H^{-\rho}}) < \infty$ for $\rho > \frac{3}{2}$. The same bounds hold for the limit, by Theorem 18.5.

Proof. *Let $\|f\|_* = 1$ and $q_h = \max(q_*, 2q_0 + 1, (70\lambda^{1/2}|h|)^{2/3})$. Put $X = \langle\hat\Phi, f\rangle$. By (17.5), $|Y| \leqslant r_q + |t_q|$ and $q - q_0 \geqslant q/2$,*

$$|X| \leqslant r_q\|f\|_1 + \|t_q\|_6\|f\|_{6/5} + \|Z_q\|_2\|f\|_2 + \frac{2}{q}|\langle \omega_q - B_{q-q_0}, f\rangle|.$$

Average over $[q_h,\,2q_h]$. By Proposition 17.3, $\hat{\mathbb{E}}J_\infty \leqslant C$ on $[q_h,\,\infty)$; since $\lambda(2q_h)^{-3/2}\int_{q_h}^{2q_h}\|t_q\|_6^6\,\mathrm{d}\,q \leqslant J_\infty$, Hölder's inequality gives the average of $\hat{\mathbb{E}}\|t_q\|_6$ at most $C(q_h^{1/2}/\lambda)^{1/6}$. Also $\hat{\mathbb{E}}\|Z_q\|_2 \leqslant C\,q^{-5/4}$, and the last term has expectation at most $C\,q^{-1/2}$ by Lemma 13.2. So $\hat{\mathbb{E}}|X| \leqslant C\,r_{q_h} + C \leqslant C(1+|h|)^{1/3}$. With $L(h) = \log\mathbb{E}\mathrm{e}^{h\langle\Phi_\infty,\,f\rangle}$ we have $L'(h) = \hat{\mathbb{E}}X$ at fixed N, so $L(h) \leqslant C(|h| + |h|^{4/3})$, and Chernoff's inequality gives $\mathbb{P}(|\langle\Phi_\infty,\,f\rangle| > R) \leqslant 2\mathrm{e}^{-cR^4}$ for $R \geqslant R_0$.

For the norm, take a real Fourier basis f_k and $w_k = (1+|k|^2)^{-\rho}$, with $W = \sum w_k < \infty$. As $\langle\iota_N\varphi,\,f_k\rangle = \langle\varphi,\,j_N f_k\rangle$ and $\|j_N f_k\|_* \leqslant C$, the bound applies to $z_k = \langle\iota_N\Phi_\infty,\,f_k\rangle$. Cauchy–Schwarz gives $(\sum w_k z_k^2)^2 \leqslant W\sum w_k z_k^4$, and Jensen's inequality concludes. □

The scale $q_h \sim |h|^{2/3}$ is where the cube at its threshold, $q\,r_q$, dominates the source; it plays the role of the time in coming down from infinity. On one site $L(h) \sim \frac{3}{4}(\lambda/3)^{-1/3}h^{4/3}$, so the powers are sharp there. The moments with $\rho > \frac{3}{2}$ are at a regularity far from the free threshold. At the threshold we have the Gaussian order, by a quadratic tilt.

***Theorem* 17.5. (Gaussian tails at the free threshold)** *For $s > \frac{1}{2}$ there is $\beta_s > 0$ with $\sup_N \mathbb{E}\exp(\beta_s\|\iota_N\Phi_\infty^N\|_{H^{-s}}^2) < \infty$.*

Proof. *Let $\frac{1}{2} < s < \frac{3}{2}$, $W = \frac{\beta}{2}\|\iota_N\varphi\|_{H^{-s}}^2$, $\hat{\nu} \propto \mathrm{e}^W\nu_N$, with no shift. As in the proof of Lemma 16.1, with $V_N - W$ in place of V_N, (16.3) gives $TZ_T \to DW(\hat{\Phi})$ at fixed N, with $\|DW\|_2 \leqslant \beta\|\iota_N\varphi\|_{H^{-s}}$. Let $x = \hat{\mathbb{E}}\|\iota_N\hat{\Phi}\|_{H^{-s}}^2 < \infty$. Then (15.4) with $p=1$ gives $\hat{\mathbb{E}}(E_\infty + J_\infty) \leqslant C(1+\beta^2 x)$, and the argument for the field at the lower scale gives $\hat{\mathbb{E}}\|Y_{q_*}\|_2^2 \leqslant C(1+\beta^2 x)$. Integrating the forward equation on $[q_*,\infty)$, with (17.2) for the martingale and Cauchy–Schwarz in q for the drifts, $x \leqslant C_s(1+\beta^2 x)$. So $x \leqslant 2C_s$ for small β. Thus $\hat{\nu}(A) \geqslant \frac{1}{2}$ for the ball A of radius K, $K^2 = 4C_s$, in H^{-s}. Since $\hat{\nu}(A) \leqslant \mathcal{Z}^{-1}\mathrm{e}^{\beta K^2/2}$, this bounds $\mathcal{Z} = \mathbb{E}\mathrm{e}^W \leqslant 2\mathrm{e}^{\beta K^2/2}$.* □

Exercises

***Exercise* 17.1. (One site: the sharp threshold)** *On one site, with $\nu \propto \exp\!\big(-\frac{\lambda}{12}x^4 - \frac{q_0}{2}x^2\big)$, show that $\mathbb{E}\exp\!\big(\frac{\beta}{12}x^4\big)$ is finite if and only if $\beta \leqslant \lambda$. Compute $L(h)$ asymptotically and compare with Theorem 17.4.*

***Exercise* 17.2. (The shift)** *Show directly on one site that $q(\hat{\psi}_q - y_q) \to h$ if y_q solves $q\,y + \frac{\lambda}{3}y^3 = q\,\Phi_q$, and that $q\,Z_q \to 0$ with (17.4). Why does a quartic tilt have no such shift, and which power of the field does its boundary term produce?*

***Exercise* 17.3. (Moment growth and determinacy)** *Deduce Carleman's condition for $\langle\Phi,\,f\rangle$ from Theorem 17.4. Why does the theorem exclude a nondegenerate Gaussian law for $\langle\Phi,\,f\rangle$?*

***Exercise* 17.4. (Regularity at finite resolution)** *On $I = [q_*, Q]$, test $K_{q_0}V_q = (q-q_0)(\zeta_q - \varrho_q(Y_q) + Z_q) - \frac{\lambda}{3}Y_q^3$ with V_q, and show from the energy bounds that $\sup_N \mathbb{E}\int_I \|V_q\|_{H^1}^2\,\mathrm{d}\,q < \infty$. In which sense is the field at finite resolution smoother than the free field?*

18 Identifying the limit

18.1 The deterministic coefficients

The coefficients M_q^N, $\ell_q^{[j],N}$, $H_q^N(y)$, $b_q^N(y)$ are constructed at each lattice. Their uniform bounds (Proposition 11.2) do not by themselves identify a limit: a weakly convergent vertex cannot in general be contracted against a varying covariance. The three tools are:

- Weak vertex measures. *Derivative kernels with bounded rooted norms are signed measures with bounded mass; they converge weakly along subsequences, and the centred kernel Θ_q converges in a weak topology controlled by its first moment. The limiting scalar mass solves the same scalar fixed-point problem, whose solution is unique in its ball, and the centred kernels are explicit outputs of the iteration; so the whole sequence converges.*

- Compact sandwich. *If $G_N \to G$ in Hilbert–Schmidt norm, and B_N are symmetric, uniformly bounded and converge weakly to B, with $\sup_N \|G_N \mathsf{B}_N\| \leqslant \theta < 1$ and $\|G\mathsf{B}\| \leqslant \theta$, then $(I + G_N\mathsf{B}_N)^{-1} G_N \to (I + G\mathsf{B})^{-1} G$ in Hilbert–Schmidt norm. The free resolvent is the compact factor that turns weak convergence of the derivative of the force into strong convergence of the response.*

- Contraction against a moving covariance. *The rows of the inverses are Hölder (any exponent $a < \frac{1}{2}$); with the Hilbert–Schmidt convergence given by the sandwich, the squared kernels $(H_q^N)^2$ converge uniformly to a continuous kernel. Weakly convergent vertices of bounded variation can therefore be contracted against them, and the loop steps of the recursion pass to the limit.*

With these, if $Y_q^N \to Y_q$ uniformly on $\mathbb{T}^3$ for almost every $q \in I$, with $\sup_I \|Y^N\|_\infty$ bounded, in probability, then $H_q^N(Y_q^N) \to H_q(Y_q)$ and $b_q^N(Y_q^N) \to b_q(Y_q)$ in $L^2(I;\mathrm{HS})$ and $L^2(I;H)$, in probability. By the resolvent identity $H^N = (I - H^N D\bar{\mathcal{F}}^N)C^N$, for bounded y, H^N is small in Hilbert–Schmidt norm on high modes, uniformly in N; so the Fourier projection and j_N give the same limit, and $\int_I \iota_N H^N(Y^N)\,\mathrm{d}W^N \to \int_I H(Y)\,\mathrm{d}W$ in H, in probability, when W^N is the projection of W, and the processes are adapted to a filtration for which W is a Brownian motion. For $q < q_*$ this holds with the frozen coefficients, as $\mathcal{U}$ in (12.8) is the loop part of the defect, and Y is continuous across q_* although b jumps.

18.2 The exact flow as a controlled explicit flow

Compactness would give limit points, and nothing in the FBSDE selects among them. A change of measure does better, following the idea of Gubinelli and Meyer for sine–Gordon [GM24]: the exact flow is an explicit flow with control Z, and that control has finite relative entropy.

Let $y = y_q(x)$ solve $K_q y + \bar{\mathcal{F}}_q(y) = K_q x$, let $\bar{f}_q(x) = K_q(y - x)$ be the explicit force in the field variable, and recall the exact force $F_q = -D\,V_q$ of Section 16.3. Differentiating the coordinate equation in q and x gives the Burgers equation with a source,

$$\partial_q \bar{f} + \frac{1}{2}\mathrm{Ctr}_{C_q^2} D^2 \bar{f} + D\bar{f}[C_q^2 \bar{f}] = -K_q H_q \mathcal{D}_q \circ y. \tag{18.1}$$

Up to here we have needed only forces. To compare the laws of the two flows we need, for the first time, a scalar. Since $D\bar{\mathcal{F}}$ is symmetric, $\bar{\mathcal{F}}$ is a gradient, and so are $\bar{f}_q$ and the defect $\mathcal{D}_q$ (as $\frac{1}{2}\mathrm{Ctr}_{H^2} D^2\bar{\mathcal{F}} = -\frac{1}{2} D \operatorname{Tr} H$). Let

$$e_q(y) = \int_0^1 \langle \mathcal{D}_q(\tau y), y\rangle\,\mathrm{d}\tau, \quad \bar{V}_q(x) = -\int_0^1 \langle \bar{f}_q(\tau x), x\rangle\,\mathrm{d}\tau, \tag{18.2}$$

so that $D\,e_q = \mathcal{D}_q$, $D\bar{V}_q = -\bar{f}_q$, $e_q(0) = \bar{V}_q(0) = 0$. Then (18.1), together with $D\,y = H_q K_q$, is the gradient of

$$\partial_q \bar{V} + \frac{1}{2}\mathrm{Tr}(C_q^2 D^2 \bar{V}) - \frac{1}{2}\|C_q D\bar{V}\|_2^2 = e_q \circ y + c(q), \quad c(q) = \frac{1}{2}\mathrm{Tr}(C_q - H_q(0)), \tag{18.3}$$

where the constant is read off at $x = 0$, as $D^2\bar{V}(0) = K_q - K_q H_q(0) K_q$. Here $\bar{V}$ is the value function of the explicit flow, in the sense of Section 8.5 of Lecture 4, and e is its running cost. This is the only place where a potential enters, and it is unavoidable: Girsanov alone gives a density in terms of the unknown Z, and only the identity $\mathcal{D} = D\,e$ makes it explicit.

*So $\bar{V}$ solves the Polchinski equation up to its defect. Since $C_s^2 D\bar{V}_s(x) = C_s(x - y_s(x))$, the **explicit flow** $\mathbb{R}^{q,x}$ from x at scale q is*

$$\mathrm{d}x = -C_s(x - y_s(x))\,\mathrm{d}s + C_s\,\mathrm{d}W, \quad x_q = x. \tag{18.4}$$

Its coordinate $Y=y_s(x_s)$ solves $\mathrm{d}Y=b(Y)\,\mathrm{d}s+H(Y)\,\mathrm{d}W$, $b=-H\mathcal{D}$: indeed $\mathrm{d}(K_sx)=Y\,\mathrm{d}s+\mathrm{d}W$, and one applies Itô's formula to $K_sY+\bar{\mathcal{F}}_s(Y)=K_sx$. It contains no Z. All this holds for all $q\geqslant q_0$; below q_* the defect is (12.8).

***Lemma* 18.1. (The defect on all fields)** *Let π_s be the pointwise projection on $[-r_s,r_s]$. For all y and $s\geqslant q_*$,*

$$|e_s(y)|\leqslant C\lambda^5 s^{-3}\|\pi_s y\|_4^4+C\lambda^4 s^{-2}\|\pi_s y\|_2^2+C\lambda^3 s^{-5/2}\|t_s(y)\|_6^6. \tag{18.5}$$

Proof. *On the ball $\mathcal{D}_s=\mathcal{D}_s^o$, so $|e_s(\pi_s y)|\leqslant C\lambda^5 s^{-3}\|\pi_s y\|_4^4+C\lambda^4 s^{-2}\|\pi_s y\|_2^2$ by the primitive formula and (11.14): the quartic term comes from the cubic part of the defect, the quadratic one from its centred linear part, as $\frac12|\langle y,\mathcal{E}_s^c y\rangle|\leqslant\frac12\|\mathcal{E}_s^c\|_{\mathrm{row}}\|y\|_2^2$. With $z_\theta=\pi_s y+\theta(y-\pi_s y)$, $e_s(y)-e_s(\pi_s y)=\int_0^1\langle\mathcal{D}_s(z_\theta),y-\pi_s y\rangle\,\mathrm{d}\theta$, and $y-\pi_s y$ is supported on the set $\mathcal{A}$ where $|y|>r_s$, with norm at most $\|t_s\|_2$. Write $\mathcal{D}_s(z)=D\kappa_s(z)\,\mathcal{D}_s^o(\kappa_s z)+\mathfrak{D}_s(z)$. Then $|\mathcal{D}^o|\leqslant\mathsf{S}_4(\mathcal{D}^o)(2r_s)^3/6+\|\mathcal{E}_s^c\|_{\mathrm{row}}2r_s\leqslant C\lambda^{7/2}s^{-3/2}$ and $|\mathcal{A}|^{1/2}\leqslant r_s^{-1}\|t_s\|_2$, so the first term contributes at most $C\lambda^4 s^{-2}\|t_s\|_2^2$. By the term-by-term bounds behind Proposition 12.2 (the four groups of Section 12.3 of Lecture 6, proved in Section 21.3 of the Supplement), which bound $\mathfrak{D}$ before multiplication by H (the bound stated for $b^{\mathrm{l}}=-H\mathfrak{D}$ would not suffice, as H^{-1} is unbounded), $\|\mathfrak{D}_s(z)\|_2\leqslant C\,g_s(\|t_s(z)\|_2+r_s^{-1}\|z^2\chi_s(z)\|_2)$; since $|z_\theta|\leqslant|y|$ and z_θ has size above r_s only on $\mathcal{A}$, the second term contributes at most $C\,g_s(\|t_s\|_2^2+r_s^{-1}\|t_s\|_4^2\|t_s\|_2)$. On the support of t_s we have $|t_s|\geqslant r_s$, so $\|t_s\|_2\leqslant r_s^{-2}\|t_s\|_6^3$ and $\|t_s\|_4^2\leqslant r_s^{-1}\|t_s\|_6^3$. All these are at most $C\,g_s r_s^{-4}\|t_s\|_6^6=C\lambda^3 s^{-5/2}\|t_s\|_6^6$.* □

With $\Delta=V-\bar V$ (Δ is not the Laplacian in this section), $\mathcal{C}_N(q)=\int_q^\infty c(s)\,\mathrm{d}s$ and $A_N(q,x)=V_q(x)-\bar V_q(x)-\mathcal{C}_N(q)$, subtracting (18.3) from the Polchinski equation gives

$$\partial_s\Delta+\frac12\mathrm{Tr}(C_s^2D^2\Delta)-\langle C_s^2D\bar V_s,D\Delta\rangle-\frac12\|C_sD\Delta\|_2^2+e_s\circ y_s+c(s)=0. \tag{18.6}$$

This is a Hamilton–Jacobi–Bellman equation with the control u, added as a drift C_su_s to the explicit flow, $\mathrm{d}Y=(b(Y)+H(Y)u)\,\mathrm{d}s+H(Y)\,\mathrm{d}W$, and running cost $e+c+\frac12\|u\|^2$, with optimal feedback $u^*=-C_sD\Delta_s$. Moreover $-C_sD\Delta_s=C_s(F_s-\bar f_s)=Z_s$, since $F=K(\psi-x)$ and $\bar f=-D\bar V=-K(x-y)$. So the exact flow is the explicit flow with control Z.

***Theorem* 18.2. (Density and sandwich)** *At fixed N, $\mathbb{P}^{q,x}$ and $\mathbb{R}^{q,x}$ are equivalent on the paths on $[q,\infty)$, and*

$$\frac{\mathrm{d}\mathbb{P}^{q,x}}{\mathrm{d}\mathbb{R}^{q,x}}=\mathrm{e}^{A_N(q,x)}\exp\Big(-\int_q^\infty e_s(Y_s)\,\mathrm{d}s\Big),\quad \mathrm{e}^{-A_N(q,x)}=\mathbb{E}^{\mathbb{R}^{q,x}}\exp\Big(-\int_q^\infty e_s(Y_s)\,\mathrm{d}s\Big). \tag{18.7}$$

Further,

$$\mathrm{Ent}(\mathbb{P}^{q,x}|\mathbb{R}^{q,x})=\frac12\int_q^\infty\mathbb{E}^{q,x}\|Z_s\|_2^2\,\mathrm{d}s, \tag{18.8}$$

$$A_N(q,x)=\mathrm{Ent}(\mathbb{P}^{q,x}|\mathbb{R}^{q,x})+\mathbb{E}^{\mathbb{P}^{q,x}}\int_q^\infty e_s(Y_s)\,\mathrm{d}s, \tag{18.9}$$

and so

$$\mathbb{E}^{\mathbb{P}^{q,x}}\int_q^\infty e_s(Y_s)\,\mathrm{d}s\leqslant A_N(q,x)\leqslant\mathbb{E}^{\mathbb{R}^{q,x}}\int_q^\infty e_s(Y_s)\,\mathrm{d}s. \tag{18.10}$$

Uniformly in N, for $\|x\|_\infty\leqslant R$ and $T\geqslant q^\vee$,

$$\mathbb{E}^{\mathbb{P}^{q,x}}\int_T^\infty|e_s(Y_s)|\,\mathrm{d}s+\mathbb{E}^{\mathbb{R}^{q,x}}\int_T^\infty|e_s(Y_s)|\,\mathrm{d}s\leqslant C_RT^{-1/2}. \tag{18.11}$$

The integrals over $[q, q_*]$, if $q < q_*$, are at most C_R. In particular $|A_N(q,x)| \leqslant C_R$.

Proof. Density on *$[q,T]$. Both are absolutely continuous with respect to the free flow from x, $\mathrm{d}\xi = C_s\,\mathrm{d}W$. As e^{-V} is harmonic the first density is $\mathrm{e}^{-V_T(\xi_T)+V_q(x)}$. By (18.3) and Itô's formula the second is $\exp\bigl(-\bar V_T(\xi_T)+\bar V_q(x)+\int_q^T (e_s(y_s(\xi_s))+c(s))\,\mathrm{d}s\bigr)$; it is a true martingale since the drift is Lipschitz at fixed N. So*

$$M_T := \frac{\mathrm{d}\mathbb{P}^{q,x}}{\mathrm{d}\mathbb{R}^{q,x}}\bigg|_{[q,T]} = \exp\biggl(\Delta_q(x) - \Delta_T(x_T) - \int_q^T (e_s(Y_s)+c(s))\,\mathrm{d}s\biggr).$$

Entropy on *$[q,T]$. Under $\mathbb{P}^{q,x}$ the explicit flow has the extra drift C_sZ_s, so $\log M_T = \int_q^T \langle Z_s, \mathrm{d}W_s\rangle + \frac12\int_q^T \|Z_s\|_2^2\,\mathrm{d}s$, and $\mathrm{Ent}_T = \frac12\mathbb{E}\int_q^T \|Z\|_2^2$, as the stochastic integral is a true martingale at fixed N. By Proposition 16.4 and $\mathbb{E}\|Z_s\|_2^2 \leqslant C_R s^{-5/2}$, (16.10), $\sup_T \mathrm{Ent}_T \leqslant C_R$. So M_T is uniformly integrable under $\mathbb{R}^{q,x}$, and $\mathbb{P}^{q,x} \ll \mathbb{R}^{q,x}$ on $[q,\infty)$ with density $M_\infty = \lim_T M_T$, and (18.8) holds.*

The limit of *M_T. Both laws have, at fixed N, $x_T \to x_\infty$ almost surely. Under $\mathbb{P}^{q,x}$, $x_T \to \hat\Phi$, from $x_T = C_T(K_q x + (T-q)\hat\Phi + B'_{T-q})$. Under $\mathbb{R}^{q,x}$, $\mathrm{d}Y = b(Y)\,\mathrm{d}s + H(Y)\,\mathrm{d}W$ with $\|b(Y)\|_2 \leqslant C\lambda s^{-3/2}\|Y\|_2$ and $\|H\| \leqslant C/s$, so $\sup_s \|Y_s\|_2$ is finite and Y_s converges almost surely, the drift being integrable and the martingale bounded in L^2; and $x - y = \varrho_s(y) + \frac{\lambda}{3}C_s y^3 = O_N(s^{-1}(1+\|y\|_\infty^3)) \to 0$. Then $\Delta_T(x_T) \to 0$: at fixed N the coefficients converge with rates and $y - x = O_N(T^{-1}(\|x\|_\infty + \|x\|_\infty^3))$, so $\bar V_T \to V_N$, and $V_T \to V_N$ by Gaussian integration, uniformly on bounded sets. By the bounds below $\int_q^\infty |e_s(Y_s)|\,\mathrm{d}s < \infty$ almost surely under both laws, and c is integrable at fixed N. So $M_\infty = \exp\bigl(\Delta_q(x) - \mathcal{C}_N(q) - \int_q^\infty e\bigr) = \mathrm{e}^{A_N}\mathrm{e}^{-\int e}$, which is positive; hence the equivalence and (18.7), the second formula by $\mathbb{E}^{\mathbb{R}} M_\infty = 1$.*

The identity. *Take $\mathbb{E}^{\mathbb{P}}$ of $\log M_\infty$ in (18.7): this is (18.9), the integrability coming from (18.11). The left inequality in (18.10) is $\mathrm{Ent} \geqslant 0$, the right one Jensen's inequality in (18.7). So the exact and explicit potentials are compared using only expectations of the defect and the entropy of the control.*

The expectations. *For $s < q_*$ use (12.9), so $|e_s(y)| \leqslant C\|y\|_2^2$ by (18.2): under $\mathbb{P}^{q,x}$ by (16.10); under $\mathbb{R}^{q,x}$ by Itô's formula for $\|Y\|_2^2$, the bounds (12.9) and $\|Y_q\|_2 \leqslant 6R$. So these contribute at most C_R. For $s \geqslant q_*$ use Lemma 18.1, $|e_s(y)| \leqslant C\lambda^5 s^{-3}\|\pi_s y\|_4^4 + C\lambda^4 s^{-2}\|\pi_s y\|_2^2 + C\lambda^3 s^{-5/2}\|t_s(y)\|_6^6$. Under $\mathbb{P}^{q,x}$: the third term integrates on $[T,\infty)$ to at most $C\lambda^2 T^{-1} J_\infty$. For the first two, with $\|\pi_s Y\|_4^4 \leqslant r_s^2\|Y\|_2^2$, both are at most $C\lambda^4 s^{-2}\|Y\|_2^2$, and (16.9) gives a bound $C_R T^{-1/2}$. Under $\mathbb{R}^{q,x}$: with $\varsigma_s = Y_s - Y_q$, by the linear bound $\|b(Y)\|_p \leqslant C\lambda s^{-3/2}\|Y\|_p$ of Proposition 12.2, valid for every p, $\mathrm{diag} H^2 \leqslant C\,s^{-1/2}$ and Itô's formula, $\mathbb{E}\|\varsigma_s\|_j^j \leqslant C(s^{j/4} + (g_{q\vee}R)^j)$ for $j = 2, 6, 12$, with C depending on q_* if $q < q_*$, by (12.9) on $[q, q_*]$; in that case read g_q below as g_{q_*}, only C_R being claimed. The first two terms give $C\lambda^5 T^{-2}R^4 + C\lambda^4 T^{-1}R^2 + C\lambda^4 T^{-1/2}(1 + g_q^2 R^2 T^{-1/2})$. For the third, when $s \geqslant q_R/C$, $\|t_s\|_6^6 \leqslant C r_s^{-6}\|\varsigma_s\|_{12}^{12}$ as before, giving $C\lambda^6 T^{-3/2}$ and terms in $g_q R$ of higher decay. When $s < q_R/C$, $\|t_s\|_6^6 \leqslant C(R^6 + \|\varsigma_s\|_6^6)$, whose integral with the weight $\lambda^3 s^{-5/2}$ on $[T, q_R/C]$ is at most $C\lambda^3(T^{-3/2}R^6 + (g_q R)^6 T^{-3/2} + \log_+(q_R/T))$. Since $\log_+(q_R/T) \leqslant C_R T^{-1/2}$, this is (18.11).* □

18.3 The limit law

Couple the lattices by common Fourier Gaussians, W^N being the projection of a cylindrical W on a fixed probability space. The consistency of the fixed-scale coefficients of Section 18.1 gives the following, along $b_{*,N} \to b_*$.

Lemma 18.3. (Consistency) *If $x_N \in H_N$, x is continuous, and the interpolations $\iota_N x_N \to x$ uniformly, then for each s, $\iota_N y_s^N(x_N) \to y_s(x)$ uniformly, and $e_s^N(y_s^N(x_N)) \to e_s(y_s(x))$.*

Proof. *For y: boundedness by the maximum principle, equicontinuity by Lemma 16.3 and translation covariance, uniqueness of the coordinate equation among continuous y by testing with the difference, as $y-x\in H^1$. For e: $e_s(0)=0$, so $e_s(y)=\int_0^1\langle\mathcal{D}_s(t\,y),y\rangle\,\mathrm{d}t$. By the consistency of the coefficients (Theorem 31.2 of the Supplement), $\mathcal{D}_s^N(\phi_N)$ converges weakly in L^2 at each s for smooth bounded ϕ, and uniformly in N, $\|\mathcal{D}_s^N(y)-\mathcal{D}_s^N(z)\|_2\leqslant C(1+\|z\|_\infty)(\|y-z\|_2+\|y^2-z^2\|_2)$. A three-term approximation extends the weak convergence to uniformly convergent fields, and dominated convergence in t gives the claim. Moreover $|e_s^N(y)|\leqslant C\,g_s\|y\|_2^2$ by the global linear bound on $\mathcal{D}$ behind Proposition 12.2 (Lemma 21.2 of the Supplement), which dominates on $[q,T]$. For $s<q_*$ the same applies to $\mathcal{U}$ in (12.8), and $\|y\|_2^2$ is continuous; for y_s use (12.7) and the convergence of the kernel of C_{q_*} in L^1.* □

By Lemma 16.3, $x=\xi+v$ with $\xi_t=x_q+\int_q^t C_s\,\mathrm{d}W_s$ and v solving an ODE on $C(\mathbb{T}^3)$, uniquely, for each path $\xi\in C([q,T];L^\infty)$. So the explicit flow $\mathbb{R}^{q,x}$ is a continuum strong solution, adapted to W, for continuous x. By Lemma 18.3 and the continuity of the ODE, if $\iota_N x_N\to x$ uniformly, then along the coupling $\sup_{s\leqslant T}\|\iota_N x_s^N-x_s\|_\infty\to 0$ almost surely, since $\mathbb{E}\sup_{s\leqslant T}\|\iota_N\xi_s^N-\xi_s\|_\infty^p\leqslant C_pN^{-\delta p}$ by Kolmogorov interpolation between the covariance difference and the Hölder bounds, and Borel–Cantelli. Testing, $\langle Y_s,h\rangle$ converges as $s\to\infty$ in L^2 for $h\in L^2$, uniformly in N, since $\|H_sh\|_2\leqslant Cs^{-1}\|h\|_2$ and $\|b\|_2\leqslant C\lambda s^{-3/2}\|Y\|_2$. So $\langle Y_\infty,h\rangle$ is defined; by (17.2) and the Itô bound on $\mathbb{E}\|Y_s\|_2^2$, Y_s converges also in $H^{-s'}$, $s'>\frac12$, in $L^2(\mathbb{R}^{q,x})$, uniformly in N.

***Theorem* 18.4. (Limit of the kernels)** *Let $\iota_N x_N\to x$ uniformly, x continuous. The densities (18.7) converge in L^1 along the coupling, $A_N(q,x_N)\to A(q,x)$, and for bounded continuous cylinder functions O of the terminal field*

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{q,x_N}}O(\hat\Phi)\to\Pi_qO(x)&:=\mathrm{e}^{A(q,x)}\mathbb{E}^{\mathbb{R}^{q,x}}\Big[O(Y_\infty)\exp(-\int_q^\infty e_s(Y_s)\,\mathrm{d}s)\Big],\\ \mathrm{e}^{-A(q,x)}&=\mathbb{E}^{\mathbb{R}^{q,x}}\exp(-\int_q^\infty e_s(Y_s)\,\mathrm{d}s).\end{aligned}\tag{18.12}$$

The same holds for bounded functionals of the path $(x_s)_{s\leqslant T}$ and finitely many $\langle\varphi,h_i\rangle$ jointly, continuous for the uniform norm, and for $O(\varphi)=\langle\varphi,h\rangle$, $h\in L^2$. So A and $\Pi_qO(x)$ are jointly continuous in $(q,x)\in[q_0,\infty)\times C(\mathbb{T}^3)$, also for $O=\langle\cdot,h\rangle$. We write $\langle m_q(x),h\rangle=\Pi_q\langle\cdot,h\rangle(x)$, the ***limit decoupling field****; $\|m_q(x)\|_2\leqslant C_R$ by Proposition 16.4.*

Proof. *Let $a_N=\exp\big(-\int_q^\infty e_s^N(Y_s^N)\,\mathrm{d}s\big)$, $f_N=\mathrm{e}^{A_N}a_N$. On $[q,T]$ the integrals converge almost surely by Lemma 18.3, since the paths are bounded; the tail on $[T,\infty)$ is small in L^1 uniformly in N by (18.11). So $a_N\to a=\exp\big(-\int_q^\infty e\big)$ in probability. By (18.8), (16.9) and (16.10), $\mathbb{E}\,f_N\log f_N\leqslant C_R$, so (f_N) is uniformly integrable by de la Vallée Poussin's criterion, and $|A_N|\leqslant C_R$ by Theorem 18.2. Along any subsequence with $A_N\to A$, $f_N\to\mathrm{e}^Aa$ in L^1, so $\mathrm{e}^{-A}=\mathbb{E}\,a$. Thus the limit is unique and $A_N\to A$. As $\langle Y_\infty^N,h\rangle\to\langle Y_\infty,h\rangle$ in probability, L^1 convergence of f_N gives (18.12) for bounded continuous O and, the paths converging uniformly along the coupling, for the joint functionals. For $\langle\varphi,h\rangle$ use also the uniform second moment of $\langle\hat\Phi,h\rangle$ under $\mathbb{P}^{q,x_N}$, from the system (1) and (16.9). Continuity: the limit of Π^N along every uniformly convergent sequence exists, so the limit function is continuous in x. If $(q_n,x_n)\to(q,x)$, the flows driven by the same W converge uniformly on compact scales by Lemma 16.3, $\int_{q\wedge q_n}^{q\vee q_n}|e_s(Y_s)|\,\mathrm{d}s\to0$ by the bounds on the defect, and the tails and the entropy are uniform; the same argument gives joint continuity.* □

***Theorem* 18.5. (The measure)** *For bounded continuous cylinder O, the whole sequence converges, and with $\mathbb{R}$ the explicit flow $\mathrm{d}Y=b(Y)\,\mathrm{d}s+H(Y)\,\mathrm{d}W$ on $[q_0,\infty)$, $Y_{q_0}=0$,*

$$\int O\,\mathrm{d}\nu=\frac{\mathbb{E}^{\mathbb{R}}\Big[O(Y_\infty)\mathrm{e}^{-\int_{q_0}^\infty e_s(Y_s)\,\mathrm{d}s}\Big]}{\mathbb{E}^{\mathbb{R}}\Big[\mathrm{e}^{-\int_{q_0}^\infty e_s(Y_s)\,\mathrm{d}s}\Big]}.\tag{18.13}$$

With $V_N(0)=0$ and μ_C the Gaussian measure of covariance C_{q_0}, the renormalised free energy converges:

$$-\log\int \mathrm{e}^{-V_N}\,\mathrm{d}\mu_C - \mathcal{C}_N(q_0) \to A(q_0,0) = -\log\mathbb{E}^{\mathbb{R}}\mathrm{e}^{-\int_{q_0}^{\infty} e_s(Y_s)\,\mathrm{d}s}. \tag{18.14}$$

Proof. *This is Theorem 18.4 at $(q,x)=(q_0,0)$, as $\mathbb{P}^{q_0,0}$ is the flow for ν_N, and $\bar{V}_{q_0}(0)=0$, $V_{q_0}(0)=-\log\int \mathrm{e}^{-V_N}\,\mathrm{d}\mu_C$. With the tightness of Corollary 17.2 this identifies the limit along the whole sequence.* □

The divergent constant has not disappeared: it is $\mathcal{C}_N(q_0)$ in (18.14), and it does not enter (18.13). Exercise 18.9 computes it in the Gaussian case.

18.4 The filtering model and the FBSDE

We now read off the limit FBSDE from the density. Let $\mathbb{R}=\mathbb{R}^{q_0,0}$, $\mathbb{P}=f\mathbb{R}$ with $f=\mathrm{e}^{A(q_0,0)}\mathrm{e}^{-\int_{q_0}^{\infty} e}$, on the paths of x. Let $\Phi=Y_\infty$, $\Phi_q=x_q$, tested with smooth functions, and $\mathcal{F}_q=\sigma(x_s\colon s\leqslant q)$.

***Theorem* 18.6. (Construction of Φ^4_3)** *Let $\lambda, q_0>0$, $r\in\mathbb{R}$, with the counterterm of Theorem 11.3 (or q_* large and $b_{*,N}\to b_*$, $|b_{*,N}|\leqslant B_0\lambda q_*^{1/2}$). Under $\mathbb{P}$:*

a) ***The filtering model.*** *$\Phi\sim\nu$, and $x_q=(q-q_0)C_q\Phi+C_qB_{q-q_0}$ with B a cylindrical Brownian motion independent of Φ.*

b) ***The forecast.*** *The conditional law of Φ given $\mathcal{F}_q$ is $\Pi_q(\Phi_q,\cdot)$. So for smooth h,*

$$\langle\psi_q,h\rangle := \mathbb{E}[\langle\Phi,h\rangle|\mathcal{F}_q] = \langle m_q(\Phi_q),h\rangle, \tag{18.15}$$

which is a function of x_q, and ψ_q is the conditional mean of the terminal field.

c) ***The FBSDE.*** *$W_q=K_qx_q-\int_{q_0}^{q}\psi_s\,\mathrm{d}s$ is a cylindrical Brownian motion for $(\mathcal{F}_q)$. With $Y_q=y_q(x_q)$, $Z=\psi-Y$, and for $q\geqslant q_0$,*

$$\begin{aligned} Y_q &= \int_{q_0}^{q}(b_s(Y_s)+H_s(Y_s)Z_s)\,\mathrm{d}s+\int_{q_0}^{q}H_s(Y_s)\,\mathrm{d}W_s\\ Z_q &= -\int_{q_0}^{q}(b_s(Y_s)+H_s(Y_s)Z_s)\,\mathrm{d}s+\int_{q_0}^{q}\Xi_s\,\mathrm{d}W_s, \end{aligned} \tag{18.16}$$

where $\Xi=\mathrm{Cov}-H(Y)$, $\psi=\int\mathrm{Cov}\,\mathrm{d}W$ with no orthogonal part, and Cov is the conditional covariance,

$$\langle\mathrm{Cov}_q h,k\rangle = \mathbb{E}[\langle\Phi,h\rangle\langle\Phi,k\rangle|\mathcal{F}_q]-\langle\psi_q,h\rangle\langle\psi_q,k\rangle. \tag{18.17}$$

Moreover $Y_{q_0}=Z_{q_0}=0$, and $\mathbb{E}\|Z_q\|_2^2\leqslant C\,q^{-5/2}$ for large q.

d) ***The terminal value.*** *$Y_q\to\Phi$ in H^{-s} in mean square, for every $s>\frac{1}{2}$.*

e) ***The lattice.*** *Along the whole sequence, in law and jointly with Φ^N_∞: x^N converges uniformly on compact sets of scales and space, Y^N_q uniformly in space at each q, $\langle\psi^N_q,h\rangle$, $\langle Z^N_q,h\rangle$ at each q, the innovations W^N in $C(I;H^{-\vartheta})$, $\vartheta>\frac{3}{2}$, to W. The energies E_∞, J_∞ satisfy the bounds of Theorem 16.2 in the limit.*

Proof. (a). *By Theorem 18.4 at $(q_0,0)$, the pair (x^N,Φ^N_∞) under the exact law converges in law to (x,Φ) under $\mathbb{P}$. At each N, $K_q\Phi^N_q-(q-q_0)\Phi^N_\infty$ is a Brownian motion independent of Φ^N_∞; test with j_Nh, as $\iota_NK_qj_Nh$ converges in every Sobolev norm. So $\langle B_t,h\rangle$ for smooth h are jointly Gaussian, independent of Φ, with covariance $(s\wedge t)\langle h,k\rangle$; by density B is a cylindrical Brownian motion, with paths in $H^{-\vartheta}$, $\vartheta>\frac{3}{2}$. Moreover $\mathcal{F}_q\subset\sigma(\Phi,B_r\colon r\leqslant q-q_0)$, as $x_r=(r-q_0)C_r\Phi+C_rB_{r-q_0}$.*

(b). *Under* $\mathbb{R}$ *the flow is Markov, and by Theorem 18.4 at* (t, x_t)*,*

$$D_t := \mathbb{E}^{\mathbb{R}}[f\,|\mathcal{F}_t] = \exp(A(q_0,0) - A(t,x_t) - \int_{q_0}^{t} e_s(Y_s)\,\mathrm{d}s), \tag{18.18}$$

and $\mathbb{E}^{\mathbb{P}}[O(\Phi)|\mathcal{F}_t] = \mathbb{E}^{\mathbb{R}}[fO(\Phi)|\mathcal{F}_t]\,/\,D_t = \Pi_t O(x_t)$*. The right side of (18.18) is continuous in* t*, so (18.18) holds for all* t *simultaneously, and* D *is continuous and positive.*

(c): the forward equation. *By Lemma 16.3 the flow is a strong solution, and* x *and* $W^{\mathbb{R}}$ *determine each other; so* $\mathcal{F}$ *is a Brownian filtration under* $\mathbb{R}$*. As* $f>0$ *has finite entropy, by lower semicontinuity from Theorem 18.2, martingale representation gives* $\mathrm{d}D = D\langle Z', \mathrm{d}W^{\mathbb{R}}\rangle$ *with* $\mathbb{E}^{\mathbb{P}}\int \|Z'\|_2^2 = 2\mathrm{Ent}(\mathbb{P}|\mathbb{R})$*, and* $W := W^{\mathbb{R}} - \int Z'$ *is a* $\mathbb{P}$*-Brownian motion for* $\mathcal{F}$ *by Girsanov's theorem. Under* $\mathbb{R}$*, the lattice equation* $\mathrm{d}Y = b(Y)\,\mathrm{d}s + H(Y)\,\mathrm{d}W^{\mathbb{R}}$ *passes to the limit by Section 18.1, along the convergence of the coupling at each scale. So under* $\mathbb{P}$*,* $\mathrm{d}Y = (b(Y) + H(Y)Z')\,\mathrm{d}s + H(Y)\,\mathrm{d}W$*, and* $\mathrm{d}(K_s x_s) = (Y_s + Z'_s)\,\mathrm{d}s + \mathrm{d}W_s$*.*

The drift is the forecast. *By (a), the increments of* B *after* s *are independent of* Φ *and of* $\mathcal{F}_s$*, so* $\mathbb{E}[\langle K_t x_t - K_s x_s, h\rangle|\mathcal{F}_s] = (t-s)\langle \psi_s, h\rangle$ *for smooth* h*. As* ψ *is a martingale,* $\langle K_q x_q, h\rangle - \int_{q_0}^{q}\langle \psi_u, h\rangle\,\mathrm{d}u$ *is an* $\mathcal{F}$*-martingale. So is* $\langle W_q, h\rangle = \langle K_q x_q, h\rangle - \int_{q_0}^{q}\langle Y_u + Z'_u, h\rangle\,\mathrm{d}u$*, where* $Y + Z'$ *is locally in* $L^2(\,\mathrm{d}u\,\mathrm{d}\mathbb{P}; L^2)$*, by (16.9), (16.10) and lower semicontinuity. Their difference is a continuous martingale of finite variation, null at* q_0*, so it vanishes. With a countable dense set of* h*,* $\psi_u = Y_u + Z'_u$ *for almost every* u*, almost surely, where* $\psi_u = m_u(x_u)$ *is defined at every scale. So* $Z := \psi - Y = m(x) - y(x)$ *is defined at every scale,* $Z' = Z$ *almost everywhere, and* W *is the innovation. Moreover* $\mathbb{E}\|\psi_Q\|_2^2 \leqslant \mathrm{liminf}_N \mathbb{E}\|\psi_Q^N\|_2^2 < \infty$*, by Fatou's lemma along (e) and the same bounds, so* ψ *is square integrable.*

Representation, with no orthogonal part. *Let* $S = \langle \Phi, h\rangle$ *and* $M = \mathbb{E}^{\mathbb{P}}[S|\mathcal{F}]$*. Under* $\mathbb{R}$*,* $N = \mathbb{E}^{\mathbb{R}}[fS|\mathcal{F}]$ *is a Brownian integral,* $\mathrm{d}N = \langle n, \mathrm{d}W^{\mathbb{R}}\rangle$*, and* $M = N\,/\,D$*, so Itô's formula gives* $\mathrm{d}M = \langle n\,/\,D - M Z, \mathrm{d}W^{\mathbb{R}} - Z\,\mathrm{d}s\rangle = \langle n\,/\,D - M Z, \mathrm{d}W\rangle$*. This defines* $\mathrm{Cov}h$*. Here* $S \in L^2(\mathbb{P})$ *by Theorem 18.4 and Fatou's lemma, so* M *is a true* L^2*-martingale and* $\mathbb{E}\int \|\mathrm{Cov}_s h\|_2^2\,\mathrm{d}s = \mathrm{Var}S$*. Summing over an orthonormal basis,* $\mathbb{E}\int_{q_0}^{Q} \|\mathrm{Cov}_s\|_{\mathrm{HS}}^2\,\mathrm{d}s = \mathbb{E}\|\psi_Q\|_2^2$*.*

The covariance. *For* G *bounded and* $\mathcal{F}_s$*-measurable and* k *smooth,* $W_t - W_s$ *tested with* k *is* $(t-s)\langle \Phi, k\rangle + \langle B_{t-q_0} - B_{s-q_0}, k\rangle - \int_s^t \langle \psi_u, k\rangle\,\mathrm{d}u$*, and the term in* B *is independent of* (G, S)*. So* $\mathbb{E}[G\,S\langle W_t - W_s, k\rangle] = \int_s^t \mathbb{E}[G\,S(\langle \Phi, k\rangle - \langle \psi_u, k\rangle)]\,\mathrm{d}u$*. By the representation and Itô's isometry the left side is* $\int_s^t \mathbb{E}[G\langle \mathrm{Cov}_u h, k\rangle]\,\mathrm{d}u$*. On the right, condition on* $\mathcal{F}_u \supset \mathcal{F}_s$*. So the integral over* $[q_0, \cdot]$ *of the difference of the two sides of (18.17) is a continuous martingale of finite variation, hence null. This is (18.17) for almost every* q*, almost surely, and* Cov *is symmetric and nonnegative, Hilbert–Schmidt almost everywhere.*

The rest of (c). *The equation for* Z *is the difference of those for* ψ *and* Y*. At* q_0*,* $x_{q_0} = 0$*: so* $Y_{q_0} = 0$ *since* $\bar{\mathcal{F}}$ *is odd, and* $Z_{q_0} = \psi_{q_0} = \mathbb{E}_\nu \Phi = 0$*. The bound on* Z *is weak lower semicontinuity from Theorem 16.2, by (e).*

(d). *(17.3) passes to the limit by lower semicontinuity.*

(e). *For the path and* Φ *this is Theorem 18.4; for* $\psi^N = m^N(x^N)$ *and* Z^N *use continuous convergence in the same theorem and the uniform bound of Proposition 16.4. Then* $W_q^N = K_q x_q^N - \int_{q_0}^{q} \psi_s^N \mathrm{d}s$ *converges tested, and* W^N *is tight in* $C(I; H^{-\vartheta})$ *as a Brownian motion. For the energies,* $\zeta_q = \Phi_q - (q - q_0)C_q\psi_q$ *and* $V_q = C_q\Big[(q-q_0)\psi_q - \frac{\lambda}{3}Y_q^3\Big]$ *converge uniformly, since the rows of the kernel of* C_q^N *converge uniformly in* L^2*, and along the coupling* $\psi_q^N = m_q^N(x_q^N)$ *is bounded in* L^2 *and converges weakly. The integrands of* E *and* J *are lower semicontinuous in these variables, and Fatou's lemma applies.* □

***Remark* 18.7. (Hidden noise)** *Let* B *be a real Brownian motion and* $L_t^n = \int_0^t (1 + \sin(n\,s))\,\mathrm{d}B_s$*. Then* (L^n, B) *converges in law to* $(B + 2^{-1/2}\tilde{B}, B)$*, with* $\tilde{B}$ *an independent Brownian motion: the integrands converge weakly to* 1*, but their squares converge weakly to* $\frac{3}{2}$*. A limiting FBSDE obtained by compactness could in this way have an extra martingale, orthogonal to the driving noise. A compactness proof must exclude this by strong compactness of* Cov^N*, for which the compactness route of Section 32.2 of the Supplement uses the Riccati equation of the conditional covariance (Exercise 18.3). Here the density does it: the filtration is Brownian under the reference. What remains open is whether* $\mathcal{F}$ *is generated by* W *(Section 18.7).*

18.5 The renormalised field equation

The terminal law satisfies an integration by parts formula in which the renormalised cube appears only tested in expectation.

***Theorem* 18.8. (Integration by parts)** *Let $h, h_1, \dots, h_k$ be smooth and $G(\varphi) = g(\langle\varphi, h_1\rangle, \dots, \langle\varphi, h_k\rangle)$ with $g \in C_b^\infty(\mathbb{R}^k)$, and let $D_h G(\varphi) = \frac{\mathrm{d}}{\mathrm{d}a} G(\varphi + a\,h)\Big|_{a=0}$. Along the sequence of Theorem 18.6 (recall $C = (m^2 - \Delta_N)^{-1}$ with $m^2 = q_0$),*

$$\lim_N \mathbb{E}\left[G(\iota_N \Phi_\infty^N)\, \langle \mu_N \Phi_\infty^N + \frac{\lambda}{3}(\Phi_\infty^N)^3, j_N h\rangle_{H_N} \right] = \mathbb{E}_\nu \left[D_h G(\Phi) - G(\Phi)\langle \Phi, (q_0 - \Delta)\, h\rangle \right]. \tag{18.19}$$

Proof. *At fixed N, Gaussian integration by parts in the density $\mathrm{e}^{-V_N - \frac{1}{2}\langle\varphi, K_{q_0}\varphi\rangle}$ gives, with $x = \Phi_\infty^N$ and $h_N = j_N h$,*

$$\mathbb{E}\left[G(\iota_N x)\langle \mu_N x + \frac{\lambda}{3} x^3, h_N\rangle \right] = \mathbb{E}\left[D_{h_N}(G \circ \iota_N)(x) \right] - \mathbb{E}\left[G(\iota_N x)\langle x, K_{q_0} h_N\rangle \right].$$

Cell averaging commutes with lattice translations, so $K_{q_0} j_N h = j_N (q_0 - \Delta_N^{\mathrm{tr}})\, h$ with Δ_N^{tr} the finite-difference Laplacian acting on smooth functions, which converges to $\Delta\, h$ in every Sobolev norm. The right side converges by Corollary 17.2 (weak convergence in H^{-s} and uniform moments), and the derivative term is a finite sum with converging coefficients $\langle j_N h, j_N h_i\rangle \to \langle h, h_i\rangle$. □

This is the terminal case of the Dyson–Schwinger equation of Theorem 9.1, (9.8) of Lecture 5, for the limiting theory. It does not say that $\mu_N \Phi + \frac{\lambda}{3}\Phi^3$ converges as a random distribution: only its expectation against cylinder functions is identified. This is consistent with the expectation of Lecture 5 (Remark 9.5) that the limiting force is an observable without a terminal representative; it is not proved here that no such representative exists. At every finite *resolution, on the other hand, the force is explicit: $F_q = K_q Z_q - \bar{\mathcal{F}}_q(Y_q)$ (Exercises 16.1 and 18.1).*

***Theorem* 18.9. (The field equation in the limit)** *For the limit of Theorem 18.6, Theorem 9.1 and Corollary 9.4 of Lecture 5 hold at every resolution, with $F_q = K_q(\psi_q - \Phi_q)$, $\omega_q = K_q \Phi_q - (q - q_0)\psi_q$ and $J_q = (q - q_0) C_q \to I$. That is, for all smooth h and cylinder G, almost surely as distributions, and with no renormalisation constant,*

$$K_{q_0}\Phi_q = J_q F_q + \omega_q, \tag{18.20}$$

$$\mathbb{E}\, D_{J_q h} G(\Phi_q) = \mathbb{E}[G(\Phi_q)\langle \omega_q, h\rangle] = \mathbb{E}[G(\Phi_q)(\langle \Phi_q, K_{q_0} h\rangle - \langle F_q, J_q h\rangle)], \tag{18.21}$$

and $\langle F, h\rangle, \langle \omega, h\rangle$ are centred martingales, with

$$\langle F_q, h\rangle = \int_{q_0}^{q} \langle (\mathrm{Cov}_s K_s - I)h,\, \mathrm{d}W_s\rangle$$

and $\mathbb{E}\mathrm{e}^{\langle \omega_q, h\rangle} \leqslant \mathrm{e}^{(q - q_0)\|h\|_2^2/2}$. Moreover:

a) ***The renormalised cube as an observable.*** *For large q, with the explicit force*

$$\langle \bar{\mathcal{F}}_q(Y_q), h\rangle := \langle K_q \Phi_q, h\rangle - \langle Y_q, K_q h\rangle,$$

$$\|\langle F_q, h\rangle + \langle \bar{\mathcal{F}}_q(Y_q), h\rangle\|_{L^2(\mathbb{P})} \leqslant C\|h\|_{H^2} q^{-1/4}. \tag{18.22}$$

So $\langle F, h\rangle$ is the unique martingale asymptotically equivalent to the germ

$$-\langle \bar{\mathcal{F}}_q(Y_q), h\rangle,$$

in the sense of Lemma 5.4 of Lecture 3.

b) ***Infinite resolution.*** *As $q \to \infty$,*

$$\lim_{q\to\infty} \mathbb{E}[G(\Phi_q)\langle \bar{\mathcal{F}}_q(Y_q), J_q h\rangle] = \mathbb{E}_\nu[D_h G(\Phi) - G(\Phi)\langle \Phi, (q_0 - \Delta)h\rangle], \tag{18.23}$$

the right side of (18.19).

c) ***Along the scales.*** *For $g\in C_b^\infty(\mathbb{R}^k)$, $G=g(\langle\Phi_{q_1},h_1\rangle,\dots,\langle\Phi_{q_k},h_k\rangle)$, $q_i\leqslant q$,*

$$\mathbb{E}\sum_i \partial_i g\,\langle J_{q_i}h,h_i\rangle=\mathbb{E}[G(\langle\Phi_q,K_{q_0}h\rangle+\langle\bar{\mathcal{F}}_q(Y_q),J_qh\rangle)]+O(C\|g\|_\infty\|h\|_{H^2}q^{-1/4}),\tag{18.24}$$

so at every large resolution the renormalised cube closes the hierarchy, up to an error which vanishes as $q\to\infty$.

Proof. *The algebra and the integration by parts are those of Lecture 5, since by Theorem 18.6(a),(b), $\omega_q=\mathbb{E}_q[B_{q-q_0}]$, with $\Phi_q=J_q\Phi+C_qB_{q-q_0}$ and B independent of Φ. The martingale representation is $\psi=\int\mathrm{Cov}\,\mathrm{d}W$, and the exponential bound is conditional Jensen. For (18.22), $F_q=K_qZ_q-\bar{\mathcal{F}}_q(Y_q)$, so its left side is at most $\|K_qh\|_2(\mathbb{E}\|Z_q\|_2^2)^{1/2}$, and Theorem 18.6(c) applies. For (18.23): by (18.22) with $K_qJ_qh=(q-q_0)h$, replacing $-F_q$ by $\bar{\mathcal{F}}_q(Y_q)$ in (18.21) costs $O(q^{-1/4})$. Moreover $\langle\Phi_q,k\rangle=\langle\psi_q,J_qk\rangle+\langle\zeta_q,k\rangle$ with $\mathbb{E}\langle\zeta_q,k\rangle^2\leqslant\|k\|_2^2/q$, and $\psi_q\to\Phi$ tested in mean square, by the martingale property. So $\Phi_q\to\Phi$ tested with smooth functions, and $J_qh\to h$ in every Sobolev norm. (18.24) follows from (18.22) with J_qh, as $\|J_qh\|_{H^2}\leqslant\|h\|_{H^2}$.* □

This is the renormalised cube: the cube of the forecast with its counterterms, at resolution q. By (18.23) the two orders of limits, $N\to\infty$ at the terminal resolution and $q\to\infty$ after N, give the same right side. What is not proved is that F is closed: (18.22) gives no bound on $\langle F_q,h\rangle$ uniform in q.

18.6 Uniqueness in law and the variational formula

Along the exact flow the value $A_t=A(t,x_t)$ solves a scalar backward equation, the integrated form of the vector one and the analogue of (4.13) of Gubinelli–Meyer. Indeed, by (18.18) and $\mathrm{d}D=D\langle Z,\mathrm{d}W^{\mathbb{R}}\rangle$,

$$A_t=\int_t^\infty(e_s(Y_s)+\tfrac12\|Z_s\|_2^2)\,\mathrm{d}s+\int_t^\infty\langle Z_s,\mathrm{d}W_s\rangle.\tag{18.25}$$

***Definition* 18.10. (Scalar FBSDE)** *Fix $q\geqslant q_0$ and a continuous x. A **solution** is a filtered probability space, of law $\mathbb{Q}$, with an $\mathcal{F}_t$-cylindrical Brownian motion W and adapted processes (x,Y,Z,A), x with paths in $C([q,\infty);C(\mathbb{T}^3))$, with $Y_t=y_t(x_t)$, $x_q=x$, $\mathrm{d}x=-C_s(x-Y)\,\mathrm{d}s+C_s(Z\,\mathrm{d}s+\mathrm{d}W)$, with $\mathbb{E}\int_q^\infty\|Z\|_2^2\,\mathrm{d}s+\mathbb{E}\int_q^\infty|e_s(Y_s)|\,\mathrm{d}s<\infty$, satisfying (18.25), A_q being deterministic, and such that $\mathcal{E}_T=\exp\big(-\int_q^T\langle Z,\mathrm{d}W\rangle-\frac12\int_q^T\|Z\|_2^2\big)$ is a uniformly integrable martingale.*

The forward equation here is $\mathrm{d}(K_sx_s)=(Y_s+Z_s)\,\mathrm{d}s+\mathrm{d}W_s$: the forward equation of (18.16). The limit law is a solution, by the proof of Theorem 18.6, at (q,x) in place of $(q_0,0)$: $\mathcal{E}_t=D_t^{-1}$ is uniformly integrable under the exact law, closed by $\mathrm{d}\mathbb{R}/\mathrm{d}\mathbb{P}$, and $A(t,x_t)\to0$ as $D_t\to f>0$.

***Theorem* 18.11. (Uniqueness in law)** *For every solution, the law of x on paths is the limit of $\mathbb{P}^{q,x_N}$, $A_q=A(q,x)$, and Z is the adapted functional of the path of x of the exact solution. So the terminal law is $\Pi_q(x,\cdot)$.*

Proof. *Let $\mathrm{d}\mathbb{Q}'=\mathcal{E}_\infty\,\mathrm{d}\mathbb{Q}$. Under $\mathbb{Q}'$, $W'=W+\int Z$ is a Brownian motion, and $\mathrm{d}x=-C_s(x-Y)\,\mathrm{d}s+C_s\,\mathrm{d}W'$. By pathwise uniqueness (Lemma 16.3), x under $\mathbb{Q}'$ has law $\mathbb{R}^{q,x}$. By (18.25) at $t=q$, $\mathcal{E}_\infty^{-1}=\exp\big(\int_q^\infty\langle Z,\mathrm{d}W\rangle+\frac12\int_q^\infty\|Z\|_2^2\big)=\exp\big(A_q-\int_q^\infty e_s(Y_s)\,\mathrm{d}s\big)$, a functional of the path of x. So the law of x under $\mathbb{Q}$ is $\mathrm{e}^{A_q}\mathrm{e}^{-\int e}\mathbb{R}^{q,x}$, and normalisation gives $A_q=A(q,x)$. By Theorem 18.4 this is the limit of $\mathbb{P}^{q,x_N}$. For Z: under $\mathbb{Q}'$, for the filtration $\mathcal{F}$, W' is a Brownian motion and x is Markov with respect to $\mathcal{F}$, so the density process on $\mathcal{F}_t$ is $D_t=\mathbb{E}'\big[\mathrm{e}^{A_q-\int_q^\infty e}\big|\mathcal{F}_t\big]$, an adapted functional of the path of x. The drift Z under $\mathbb{Q}$ is determined by the covariation of $\log D$ with W', which is computed from the path of x only, since $C_s\,\mathrm{d}W'$ is a differential of x and its drift.* □

So if the backward equation is written for the scalar value A, the solution is unique in law. Its Z is the feedback $m_s(x_s)-y_s(x_s)$, and by Theorem 18.6 it coincides with $\psi-Y$ of (18.16); differentiability of A in x is not claimed.

***Theorem* 18.12. (Renormalised variational formula)** *For $q \geqslant q_0$, x continuous, and O bounded measurable on paths, under the reference $\mathbb{R}^{q,x}$,*

$$-\log \mathbb{E}^{\mathbb{R}^{q,x}} \mathrm{e}^{-O-\int_q^\infty e_s(Y_s)\,\mathrm{d}s} = \inf_{\mathbb{Q}\ll\mathbb{R}^{q,x},\,\mathrm{Ent}(\mathbb{Q}|\mathbb{R}^{q,x})<\infty} \left[\mathbb{E}^{\mathbb{Q}}\left[O + \int_q^\infty (e_s(Y_s) + \frac{1}{2}\|u_s\|_2^2)\,\mathrm{d}s\right]\right], \tag{18.26}$$

where, for $\mathbb{Q}$ of finite entropy, u is the drift of W under $\mathbb{Q}$ added to (18.4) as $C_s u_s\,\mathrm{d}s$, so that Y solves $\mathrm{d}Y = (b(Y) + H(Y)u)\,\mathrm{d}s + H(Y)\,\mathrm{d}W^{\mathbb{Q}}$ and $\mathrm{Ent}(\mathbb{Q}|\mathbb{R}^{q,x}) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\int \|u\|_2^2$. For $O = 0$ the unique minimiser is $u = Z$, and the value is $A(q,x)$. In particular, at $(q,x) = (q_0, 0)$, for O a bounded continuous cylinder function of the terminal field,

$$-\log \int \mathrm{e}^{-O}\,\mathrm{d}\nu = \mathcal{G}(O) - \mathcal{G}(0), \quad \mathcal{G}(O) = \inf_u \mathbb{E}\left[O(Y_\infty^u) + \int_{q_0}^\infty (e_s(Y_s^u) + \frac{1}{2}\|u_s\|_2^2)\,\mathrm{d}s\right], \quad Y_{q_0}^u = 0, \tag{18.27}$$

the infimum being over the laws of finite entropy with respect to $\mathbb{R}$, and $\mathcal{G}(0) = A(q_0, 0)$ is the renormalised free energy (18.14). The minimiser for $O = 0$ is $u = Z$. No infinite constant occurs in $\mathcal{G}$, and there is no initial law to optimise over.

Proof. *The Gibbs variational principle: for a probability $\mathbb{R}$ and a measurable $\mathcal{S}$ with $\mathbb{E}^{\mathbb{R}}\mathrm{e}^{-\mathcal{S}} < \infty$, for every $\mathbb{Q}$ with $\mathrm{Ent}(\mathbb{Q}|\mathbb{R}) < \infty$, $\mathbb{E}^{\mathbb{Q}}\mathcal{S}^- \leqslant \mathrm{Ent}(\mathbb{Q}|\mathbb{R}) + \log \mathbb{E}^{\mathbb{R}}\mathrm{e}^{\mathcal{S}^-}$; where $\mathrm{e}^{\mathcal{S}^-} \leqslant 1 + \mathrm{e}^{-\mathcal{S}}$. This gives that $\mathbb{E}^{\mathbb{Q}}\mathcal{S}$ is well-defined in $(-\infty, \infty]$ and $\mathbb{E}^{\mathbb{Q}}\mathcal{S} + \mathrm{Ent}(\mathbb{Q}|\mathbb{R}) \geqslant -\log \mathbb{E}^{\mathbb{R}}\mathrm{e}^{-\mathcal{S}}$, with equality exactly at $\mathrm{d}\mathbb{Q} \propto \mathrm{e}^{-\mathcal{S}}\,\mathrm{d}\mathbb{R}$ if $\mathbb{E}^{\mathbb{Q}}|\mathcal{S}| < \infty$ there. Apply this with $\mathcal{S} = O + \int_q^\infty e$. The integrability $\mathbb{E}^{\mathbb{R}}\mathrm{e}^{-\mathcal{S}} < \infty$ is Theorem 18.4. At the minimiser for $O = 0$, $\mathbb{E}|\mathcal{S}| < \infty$ passes to the limit from (18.11) by Fatou; for O bounded the same holds as the tilted law has a bounded density with respect to it. Since $\mathbb{R}^{q,x}$ is the strong solution driven by W, every $\mathbb{Q}$ of finite entropy has a drift u with $\mathrm{Ent} = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\int \|u\|^2$ by Föllmer's theorem. At $O = 0$, the minimiser has control Z, by Theorem 18.11 ($u = Z$ is the Föllmer drift of $\mathbb{P}^{q,x}$). The equation for Y holds under $\mathbb{R}^{q,x}$ by the proof of Theorem 18.6, hence under $\mathbb{Q} \ll \mathbb{R}^{q,x}$ with $\mathrm{d}W = \mathrm{d}W^{\mathbb{Q}} + u\,\mathrm{d}s$. (18.27) is (18.26) at $(q_0, 0)$, with (18.13).* □

***Remark* 18.13. (Coercivity as a corollary)** *(18.26) with $O = 0$ reads, for every control, $\mathbb{E}\int \left(e_s(Y_s^u) + \frac{1}{2}\|u_s\|^2\right)\mathrm{d}s \geqslant A(q,x) \geqslant -C_R$. This is the coercivity $\mathbb{E}\int \left(e_s + \frac{\theta}{2}\|u_s\|^2\right)\mathrm{d}s \geqslant -C_R$ with constant $\theta = 1$. It has been obtained from the exact solution without any estimate on the controlled equation. An independent proof with $\theta < 1$, as suggested by the test with Y^3 in the memory form, would give the exponential moments $\mathbb{E}^{\mathbb{R}}\mathrm{e}^{-p\int e}$, $p > 1$, which are not needed here.*

***Remark* 18.14. (Comparison)** *In the Barashkov–Gubinelli functional the Lagrangian contains $\frac{1}{2}\|u\|^2$ for a control singular at the cutoff, and renormalisation is a change of variables in the cost, subtracting the Wick cube of the free field ($u = -\frac{\lambda}{3}\dot{Q}^{1/2}{:}X^3{:} + r$ in Lecture 4 [BG20]). Here the reference process carries the renormalisation, the control is the remainder Z, it is in L^2 uniformly in N (of size $q^{-5/4}$), and the only interaction left in the cost is the defect e, with $\mathbb{E}\int |e|$ small at large q. The coercivity is used pathwise, the remainder in expectation.*

***Remark* 18.15. (Comparison with sine–Gordon)** *In [GM24] the smoothed force and the defect are bounded uniformly in the field, there is a spatial cut-off, and the same formula follows from boundedness; (18.25) is the analogue of their (4.13). Here the defect is unbounded. What replaces the bound is the entropy of the control: UV stability of the restarted laws makes the densities uniformly integrable, and exponential moments of the defect are not needed. The backward error alone does not give the result: from a given law at scale q, Girsanov's theorem reduces the terminal law to that one, which is the unknown. Only the start at q_0, or the restart from a fixed field, removes it. The limit can also be identified as the invariant measure of the Φ^4_3 dynamics [HSt22, HMt18, HMa18, HSc22, MW17, MW20, GH21], where quartic tails hold also at $H^{-1/2-\vartheta}$ on the literature's lattices; we have not used this.*

18.7 What is not proved, and where to go next

- The vector FBSDE and strong solutions. *Theorem 18.11 gives uniqueness in law for the scalar form. A solution of (18.16) alone need not have Z of gradient form. It would suffice that any solution has a Lipschitz decoupling field, which needs a pointwise bound on $\mathrm{Cov}_q(x)$, or high-scale contraction with non-degeneracy on compact intervals; only $L^2(\mathrm{HS})$ control of* Cov *is available. The same bound is what is missing for the Polchinski process to be adapted to the innovation W: the explicit flow is a strong solution, the exact one is not known to be. The Allinger–Mitter theorem does not apply: its finite-energy hypothesis is on the signal Φ, and $\|\Phi\|_2 = \infty$ here.*
- Quartic tails at the free threshold. *Theorem 17.4 gives quartic moments in $H^{-\rho}$ for $\rho > \frac{3}{2}$, and Theorem 17.5 Gaussian moments for $s > \frac{1}{2}$. The quartic moment at s near $\frac{1}{2}$ would need spatially resolved sources.*
- Axioms. *Reflection positivity holds at every lattice, for reflections in lattice and mid-lattice planes (nearest-neighbour action), and passes to ν for every reflection in a coordinate plane, by approximating the plane with lattice planes (standard; Exercise 18.4). Rotation invariance (absent on the torus) and clustering need infinite volume.*
- Infinite volume. *The force, its Burgers flow, and differences of free energies are intensive (Section 9.5 of Lecture 5). The estimates of this lecture must then be made local, with weights in space and constants per unit volume. A new difficulty appears in the backward equation: its integrand Ξ is a nonlocal operator, and the backward error at a point can receive noise from arbitrarily far away. Uniform-in-volume coefficient bounds and local energy moments are themselves open; they could give an infinite-volume Gibbs subsequence. Controlling the "noise from infinity" (together with local compactness of the blocks of the derivative of the force) is the main additional step needed to retain the full FBSDE without an orthogonal martingale.*

Summary of the course. *The lectures went from the Wightman axioms to the Euclidean formulation (Lecture 1), to the free field (Lecture 2), to stochastic quantisation and the scale decomposition (Lecture 3), and to three views of an interacting field, renormalisation, optimal control and filtering, meeting in the force coordinates (Lecture 4), and to the field equation at every resolution, whose noise, a conditional expectation of Nelson's Euclidean noise, is sub-Gaussian whatever the interaction (Lecture 5). In Lectures 6–7 the FBSDE in forecast coordinates, together with an explicit solution of the force equation up to an integrable defect, gave a construction of Φ^4_3 in which every object has a probabilistic meaning: the forecast is a conditional mean, the response is a conditional covariance, the Brownian motion is an innovation, and UV stability is the statement that an explicit force predicts the conditional mean up to an error that is small at high resolution. In the end the exact flow is the explicit flow with control the error, the measure is an explicit density over the explicit flow, and the renormalised variational problem has that error as its optimal control.*

Exercises

***Exercise* 18.1. (The force observable)** *Show that $\langle F_q^N, j_N h\rangle = \langle \psi_q^N, K_q\, j_N h\rangle - \langle W_q^N, j_N h\rangle - \int_{q_0}^q \langle \psi_s^N, j_N h\rangle \mathrm{d}s$. Deduce that, along the sequence of Theorem 18.6, these converge, with uniformly bounded moments, to $\langle F_q, h\rangle = \int_{q_0}^q \langle (\mathrm{Cov}_s K_s - I)\, h, \mathrm{d}W_s\rangle$, a martingale for the limiting filtration (note $F_{q_0} = 0$, since ν_N is even). (Cross-check: $F_q = K_q Z_q - \bar{\mathcal{F}}_q(Y_q)$, with Theorem 16.2, gives a uniform second-moment bound.) Why does this not give a terminal value F_∞?*

***Exercise* 18.2. (Hidden noise)** *Prove the claim of Remark 18.7. In the proof of Theorem 18.6, which step excludes it?*

***Exercise* 18.3. (Riccati and positivity)** *At fixed N, show that the conditional covariance Cov_q satisfies $\mathrm{d}\mathrm{Cov}_q = -\mathrm{Cov}_q^2\, \mathrm{d}\, q + \mathfrak{k}_q[\mathrm{d}W_q]$, where $\mathfrak{k}_q$ is the third conditional cumulant of Φ_∞ (the conditional law is an exponential family, Proposition 10.4(b) of Lecture 6). For a Gaussian $\nu_N = \mu^C$, solve this and check that $\mathrm{Cov}_q = C_q$. For a general ν_N, show that $\mathrm{Tr}\mathrm{Cov}_q$ is a supermartingale, and compare with (18.17).*

***Exercise* 18.4. (Reflection positivity)** *Show that ν_N is reflection positive for reflections in lattice and mid-lattice planes (nearest-neighbour Gaussian times a local potential). Deduce reflection positivity of ν for every coordinate plane $\{x_1 = a\}$: take cylinder functions of tests supported in $\{a < x_1 < a + \frac{1}{2}\}$ (one of the two components of the torus minus the fixed planes $\{x_1 = a\}$ and $\{x_1 = a + \frac{1}{2}\}$ of θ_a), cell-average them, and use lattice planes $a_N \to a$, so that for large N the averaged tests lie on the correct side. Since lattice and mid-*

lattice reflections map cells to cells, $\theta_{a_N} j_N f = j_N \theta_{a_N} f$, so the reflected tested variables are $\langle \iota_N \Phi, \theta_{a_N} f \rangle$, with $\theta_{a_N} f \to \theta_a f$ in every H^s.

***Exercise* 18.5. (Dyson–Schwinger by Gaussian integration by parts)** *(a) At fixed N, check that (18.21) is Theorem 9.1 of Lecture 5 in the profile of Lecture 6, and compare the proofs of Theorem 9.1 and of Corollary 9.4 ($k=1$) of Lecture 5: score of the law of Φ_q against the filtered noise. (b) For $V_N = 0$, write $F = 0$ and check (18.20) directly. (c) Show that $\langle F, h \rangle$ has bracket $\int \|(\mathrm{Cov}_s K_s - I)\, h\|_2^2 \mathrm{d}s$, and that (18.22) does not bound it uniformly in q. What would a bound imply?*

***Exercise* 18.6. (Negative Sobolev threshold)** *Verify (17.2), and show that at $s = \frac{3}{2}$ the bound becomes $Q^{-1} \log(2+Q)$.*

***Exercise* 18.7. (One site)** *On one site take $K_s = s$, $\bar{\mathcal{F}}(y) = \frac{\lambda}{3} y^3 + a\, y$ independent of s, with $s + a > 0$ on the range considered, and $D V_N = \bar{\mathcal{F}}$. Show that $\mathcal{D}_s(y) = \frac{1}{2} \bar{\mathcal{F}}''(y) (s + \bar{\mathcal{F}}'(y))^{-2}$ and $e_s(y) + c(s) = \frac{1}{2}(s^{-1} - (s + \bar{\mathcal{F}}'(y))^{-1})$. Simulate $\mathrm{d}x = -s^{-1}(x - y_s(x))\,\mathrm{d}s + s^{-1}\,\mathrm{d}W$, and check (18.7) and (18.10) numerically. How small is the entropy?*

***Exercise* 18.8. (Freezing)** *In the setting of the previous exercise freeze the force below q_*, and check (18.13) and (18.14) numerically for several q_*. Which quantities change with q_*, and which do not?*

***Exercise* 18.9. (Gaussian case)** *For $V_N = \frac{\mu}{2} \|\varphi\|^2$ and $\bar{\mathcal{F}}_q = \mu\, y$ for $q \geqslant q_*$, show that $\mathcal{D} = 0$ above q_* and $\mathcal{D} = -y$ below, so $e = -\frac{1}{2}\|y\|^2$. Compute the density (18.7) mode by mode, and show that the constant diverging with N is carried by $\mathcal{C}_N$.*

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