

A CONSTRUCTIVE INTRODUCTION TO QUANTUM FIELD THEORY

Lecture 1: Relativistic quantum fields and imaginary time

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Abstract

We recall the Wightman axioms for a relativistic scalar field, explain why quantum fields must be operator-valued distributions, and discuss the Källén–Lehmann representation of the two-point function. Using a lattice regularisation of the quantum wave equation on a spatial circle we show how positivity of the energy allows the analytic continuation to imaginary time, leading to the Feynman–Kac (path integral) representation and to the identification of Euclidean field theories with continuum limits of short-range lattice spin systems at criticality. We conclude with the Osterwalder–Schrader axioms. Exercises are collected at the end of each section. These notes follow the slides of Lecture 1.

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Introduction

Constructive quantum field theory aims at constructing, with full mathematical rigour, examples of *interacting* relativistic quantum fields, i.e. models satisfying a small set of axioms which encode quantum mechanics, special relativity and causality. In spacetime dimension $d = 2$ and $d = 3$ this programme has been carried out for several models ($P(\varphi)_2$, Yukawa₂, Φ_3^4 , sine-Gordon, ...), while in $d = 4$ no interacting model is known and the Φ_4^4 theory is trivial [ADC21].

The main lesson of the 1970s [Sym69, Nel73a, Nel73b, OS73, OS75, GRS75] is that it is much more efficient to work in *imaginary time*. The relativistic theory is then encoded in a probability measure on distributions on $\mathbb{R}^d$, which looks like the continuum limit of a classical lattice spin system at a critical point. All modern approaches (cluster expansions, renormalisation group, stochastic quantisation, variational methods) work at this Euclidean level.

Plan. Section 1 recalls the Minkowski picture: Wightman axioms, the need for distributions, the Källén–Lehmann representation, the quantum wave equation. Section 2 explains the passage to imaginary time on a lattice regularisation of the field on a circle, the Feynman–Kac formula, the statistical mechanics interpretation and the Osterwalder–Schrader axioms. Lecture 2 (§3 of the notes) studies the first example, the Gaussian free field. Section A outlines the six lectures of the course.

Conventions. $\hbar = c = 1$. Spacetime dimension is $d = 1 + s$, $s \geq 1$ the number of space dimensions. Inner products $\langle f, g \rangle = \int \bar{f}g$ are antilinear in the first slot. For $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f}(k) = \int e^{-ik \cdot x} f(x) dx$. Constants C may change from line to line.

1 Relativistic quantum fields in Minkowski space

1.1 Kinematics

Minkowski space is $\mathbb{R}^{1,s} \simeq \mathbb{R}^d$ with points $x = (x^0, \mathbf{x})$ and bilinear form $x \cdot y = x^0 y^0 - \mathbf{x} \cdot \mathbf{y}$, $x^2 = x \cdot x$. Two points are *spacelike separated* if $(x - y)^2 < 0$. The open forward light cone is $V_+ = \{p : p^0 > |\mathbf{p}|\}$, with closure $\bar{V}_+$. The restricted Lorentz group $\mathcal{L}_+^\uparrow$ is the connected component of the identity of the group of linear maps preserving $x \cdot y$; the restricted Poincaré group is $\mathcal{P}_+^\uparrow = \mathbb{R}^d \rtimes \mathcal{L}_+^\uparrow$, with $(a, \Lambda)(b, \Lambda') = (a + \Lambda b, \Lambda \Lambda')$, acting on spacetime by $x \mapsto \Lambda x + a$.

By Wigner’s theorem a relativistic quantum system carries a (projective) unitary representation of $\mathcal{P}_+^\uparrow$ on its Hilbert space of states. For a scalar field we may assume a true representation.

1.2 The Wightman axioms

Definition 1.1 (Wightman QFT of a hermitian scalar field). A Wightman theory is given by the following data, subject to the following conditions.

(W0) Relativistic quantum theory.

A separable Hilbert space $\mathcal{H}$ and a strongly continuous unitary representation U of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$. Write $U(a, \mathbb{1}) = e^{iP \cdot a} = e^{i(P^0 a^0 - \mathbf{P} \cdot \mathbf{a})}$ for commuting self-adjoint generators $P = (P^0, \mathbf{P})$; $H := P^0$ is the Hamiltonian.

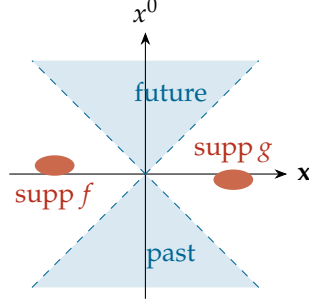


Figure 1: Minkowski space for $s = 1$. The shaded regions are timelike to the origin. The supports of f and g are spacelike separated, so $[\varphi(f), \varphi(g)] = 0$ by (W3). In momentum space the same forward cone $\bar{V}_+$ carries the joint spectrum of $(H, \mathbf{P})$.

Spectrum condition: the joint spectrum of $(P^0, \dots, P^s)$ is contained in $\bar{V}_+$; in particular $H \geq |\mathbf{P}| \geq 0$.

Vacuum: there is a unit vector Ω , unique up to a phase, with $U(a, \mathbb{1})\Omega = \Omega$ for all $a \in \mathbb{R}^d$.

(W1) Fields.

There is a dense subspace $\mathcal{D} \subset \mathcal{H}$ and, for every $f \in \mathcal{S}(\mathbb{R}^d)$, an operator $\varphi(f)$ with domain $\mathcal{D}$ such that $\Omega \in \mathcal{D}$, $\varphi(f)\mathcal{D} \subset \mathcal{D}$, $U(a, \Lambda)\mathcal{D} \subset \mathcal{D}$; for $\chi, \psi \in \mathcal{D}$ the map $f \mapsto \langle \chi, \varphi(f)\psi \rangle$ is a tempered distribution; and $\langle \chi, \varphi(f)\psi \rangle = \langle \varphi(\bar{f})\chi, \psi \rangle$.

(W2) Covariance.

$U(a, \Lambda)\varphi(f)U(a, \Lambda)^{-1} = \varphi(f_{(a, \Lambda)})$ on $\mathcal{D}$, where $f_{(a, \Lambda)}(x) = f(\Lambda^{-1}(x - a))$.

(W3) Locality.

If the supports of f and g are spacelike separated, then $[\varphi(f), \varphi(g)]\psi = 0$ for all $\psi \in \mathcal{D}$.

(W4) Cyclicity of the vacuum.

The span of $\{\varphi(f_1) \cdots \varphi(f_n)\Omega : n \geq 0, f_j \in \mathcal{S}(\mathbb{R}^d)\}$ is dense in $\mathcal{H}$.

Formally $\varphi(f) = \int \varphi(x)f(x)dx$, and $\varphi(x^0, \mathbf{x}) = e^{iHx^0}\varphi(0, \mathbf{x})e^{-iHx^0}$ is the Heisenberg-picture field. The vacuum is automatically Lorentz invariant. The *Wightman functions*

$$W_n(f_1, \dots, f_n) := \langle \Omega, \varphi(f_1) \cdots \varphi(f_n)\Omega \rangle \quad (1.1)$$

extend, by the Schwartz kernel theorem, to tempered distributions $W_n \in \mathcal{S}'(\mathbb{R}^{dn})$. They satisfy: (a) *positivity*: $\sum_{j,k} W_{j+k}(g_j^* \otimes g_k) \geq 0$ for every finite sequence $g_j \in \mathcal{S}(\mathbb{R}^{dj})$, with $g^*(x_1, \dots, x_j) = \overline{g(x_j, \dots, x_1)}$; (b) *Poincaré invariance*; (c) *spectral condition*: $W_n(x_1, \dots, x_n) = w_n(x_1 - x_2, \dots, x_{n-1} - x_n)$ with $\hat{w}_n$ supported in $\bar{V}_+^{n-1}$ (for a suitable sign convention); (d) *locality*: W_n is unchanged under exchange of two adjacent spacelike separated arguments; (e) *cluster property*, equivalent to uniqueness of the vacuum.

Theorem 1.2 (Wightman reconstruction [SW64]). *Any sequence $(W_n)_{n \geq 0}$ of tempered distributions with $W_0 = 1$ satisfying (a)–(d) is the sequence of Wightman functions of a Wightman theory, unique up to unitary equivalence. The vacuum is unique iff (e) holds.*

So constructing a QFT amounts to constructing its Wightman functions. The difficulty is that positivity (a) is a nonlinear, nonlocal constraint which is hard to check directly. For the axioms, the reconstruction theorem and the algebraic viewpoint see [SW64, BSZ92, Haa96]. The Euclidean approach of Section 2 replaces it by reflection positivity, which is stable under the limits one has to take. Note also that the axioms say nothing about dynamics: there is no Lagrangian, and the interaction is hidden in the higher Wightman functions.

1.3 Why operator-valued distributions?

The free field. Let $m > 0$, $\omega(\mathbf{p}) = (|\mathbf{p}|^2 + m^2)^{1/2}$, and let $d\Omega_m(\mathbf{p}) = d\mathbf{p}/((2\pi)^s 2\omega(\mathbf{p}))$ be the Lorentz invariant measure on the mass hyperboloid $H_m^+ = \{p : p^2 = m^2, p^0 > 0\}$, parametrised by $\mathbf{p}$. On the Fock space over $\mathfrak{h} = L^2(\mathbb{R}^s, d\Omega_m)$ let $a(\mathbf{p}), a^*(\mathbf{p})$ be the annihilation/creation operator-valued distributions, $[a(\mathbf{p}), a^*(\mathbf{q})] = (2\pi)^s 2\omega(\mathbf{p})\delta(\mathbf{p} - \mathbf{q})$, $a(\mathbf{p})\Omega = 0$. The free field is

$$\varphi(x) = \int d\Omega_m(\mathbf{p}) [a(\mathbf{p})e^{-ip \cdot x} + a^*(\mathbf{p})e^{ip \cdot x}]_{p^0=\omega(\mathbf{p})}. \quad (1.2)$$

For $f \in \mathcal{S}(\mathbb{R}^d)$ one gets $\varphi(f)\Omega = a^*(\tilde{f})\Omega$ with $\tilde{f}(\mathbf{p}) = \int f(x)e^{ip \cdot x}dx|_{p^0=\omega(\mathbf{p})}$, and

$$W_2(x-y) = \int d\Omega_m(\mathbf{p}) e^{-ip \cdot (x-y)}, \quad \|\varphi(f)\Omega\|^2 = \int d\Omega_m(\mathbf{p}) |\tilde{f}(\mathbf{p})|^2 < \infty. \quad (1.3)$$

But formally $\|\varphi(x)\Omega\|^2 = W_2(0) = \int d\Omega_m = +\infty$ for $s \geq 1$: field fluctuations at a point are infinite. The divergence is ultraviolet (large momenta). It is logarithmic for $s = 1$ and power-like for $s \geq 2$. This is not an artefact of the free field.

Proposition 1.3 (No fields at a point; Wightman [Wig64]). Assume (W0)–(W2) and suppose there is $\psi \in \mathcal{H}$ such that $\varphi(f)\Omega = \int f(x)U(x, \mathbb{1})\psi dx$ for all $f \in \mathcal{S}(\mathbb{R}^d)$, i.e. “ $\varphi(0)\Omega = \psi$ exists”. Then $\psi = \langle \Omega, \psi \rangle \Omega$, hence $\varphi(f)\Omega = \langle \Omega, \psi \rangle (\int f)\Omega$.

Lemma 1.4. Let $d \geq 2$ and let μ be a finite positive Borel measure on $\bar{V}_+$ invariant under $\mathcal{L}_+^\uparrow$. Then $\mu = \mu(\{0\})\delta_0$.

Proof. A boost of rapidity θ in the x^1 direction acts as $p^0 \pm p^1 \mapsto e^{\pm\theta}(p^0 \pm p^1)$, leaving the other components unchanged. Let $S_j^\pm = \{p \in \bar{V}_+ : 2^j \leq p^0 \pm p^1 < 2^{j+1}\}$, $j \in \mathbb{Z}$. The boost with $e^\theta = 2$ maps S_j^+ onto S_{j+1}^+ , so $\mu(S_j^+)$ is independent of j . These sets are disjoint and μ is finite, so $\mu(S_j^+) = 0$. The same argument applies to S_j^- . Finally $\bigcup_j (S_j^+ \cup S_j^-) = \bar{V}_+ \setminus \{0\}$, since $p^0 \pm p^1 \geq 0$ on $\bar{V}_+$ and $p^0 + p^1 = p^0 - p^1 = 0$ forces $p = 0$. $\square$

Proof of Proposition 1.3. By covariance, $U(0, \Lambda)\varphi(f)\Omega = \varphi(f_{(0, \Lambda)})\Omega$. Since $U(0, \Lambda)U(x, \mathbb{1}) = U(\Lambda x, \mathbb{1})U(0, \Lambda)$, after the change of variables $x \mapsto \Lambda x$ this reads

$$\int f(x) U(\Lambda x, \mathbb{1}) (U(0, \Lambda)\psi - \psi) dx = 0 \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d).$$

Letting $f \rightarrow \delta_0$ gives $U(0, \Lambda)\psi = \psi$. Let μ be the spectral measure of ψ for the translations, $\langle \psi, U(a, \mathbb{1})\psi \rangle = \int e^{ip \cdot a} \mu(dp)$. It is finite with mass $\|\psi\|^2$, supported in $\bar{V}_+$ by the spectrum condition, and Lorentz invariant because ψ is. By Lemma 1.4, $\langle \psi, U(a, \mathbb{1})\psi \rangle = \|\psi\|^2$ for all a , hence $U(a, \mathbb{1})\psi = \psi$ (equality in Cauchy–Schwarz), and $\psi \in \mathbb{C}\Omega$ by uniqueness of the vacuum. $\square$

Remark 1.5. So the only theories with point-like fields have $\varphi(f)\Omega \propto \Omega$; using locality and the Reeh–Schlieder theorem one then shows that φ is a multiple of the identity [SW64, Wig64]. Physically (Bohr–Rosenfeld), measuring a field at a sharp point requires infinite energy. Quantum fields are therefore *operator-valued distributions*, and every nonlinear expression in $\varphi(x)$ (interaction terms, field equations, composite operators) requires a definition. This is the root of renormalisation.

1.4 The two-point function: the Källén–Lehmann representation

The free two-point function (1.3) is the Fourier transform of the invariant measure on the mass shell. In every Wightman theory the two-point function is a positive superposition of such free ones. For $M \geq 0$ let $W_2^{(M)}$ be the two-point function (1.3) of the free field of mass M . For $M = 0$ and $d \geq 3$ it is defined with the invariant measure $d\Omega_0 = d\mathbf{p}/((2\pi)^s 2|\mathbf{p}|)$ on the boundary of the cone; for $d = 2$ it is not tempered. By translation invariance of Ω , $\langle \Omega, \varphi(f)\Omega \rangle = v \int f$ for some $v \in \mathbb{R}$.

Theorem 1.6 (Källén–Lehmann [Käl52, Leh54]). *Assume (W0)–(W2). There is a positive Borel measure ρ on $[0, \infty)$, the spectral measure of the field, such that*

$$W_2(x - y) = v^2 + \int_{[0, \infty)} \rho(dM^2) W_2^{(M)}(x - y) \quad (1.4)$$

in the sense of tempered distributions. The measure ρ grows at most polynomially and is such that the right-hand side is tempered; in particular $\rho(\{0\}) = 0$ if $d = 2$.

Sketch. Let $\psi_f := \varphi(f)\Omega - v(\int f)\Omega \perp \Omega$. By hermiticity $\langle \psi_f, \psi_g \rangle = W_2(f \otimes g) - v^2 \int f \int g$, and by covariance $U(a, \mathbb{1})\psi_f = \psi_{f(a, \mathbb{1})}$. Hence the truncated function $W_2^T := W_2 - v^2$ is a distribution of positive type in $x - y$. By the Bochner–Schwartz theorem (equivalently, the spectral theorem for the translations restricted to the closed span of the ψ_f) it is the Fourier transform of a positive tempered measure μ on momentum space. The spectrum condition gives $\text{supp } \mu \subset \bar{V}_+$. Since $\psi_f \perp \Omega$ and Ω is the only translation invariant vector, $\mu(\{0\}) = 0$. Covariance and $U(0, \Lambda)\Omega = \Omega$ make μ invariant under $\mathcal{L}_+^\uparrow$. The $\mathcal{L}_+^\uparrow$ -orbits in $\bar{V}_+ \setminus \{0\}$ are the hyperboloids H_M^+ , $M > 0$, and (for $d \geq 3$) the punctured boundary of the cone, $M = 0$. Each carries a unique invariant measure up to normalisation, namely $d\Omega_M$. Disintegrating μ along $M = (p^2)^{1/2}$ gives $\mu = \int \rho(dM^2) \Omega_M$, which is (1.4). $\square$

Remark 1.7 (Reading the spectrum off ρ). (i) *Free fields.* The free field has $\rho = \delta_{m^2}$. Conversely, if $\rho = \delta_{m^2}$ with $m > 0$ then φ is a free field (Jost–Schroer theorem [SW64]). So an interacting theory must have $\rho \neq \delta_{m^2}$; this is not sufficient, since generalised free fields have arbitrary ρ and no interaction (Exercise 1.6).

(ii) *Particles.* An atom $Z\delta_{m^2}$ of ρ isolated from the rest of its support, $Z > 0$, signals a particle of mass m : an isolated eigenvalue m of the mass operator $(H^2 - \mathbf{P}^2)^{1/2}$. The remaining part of ρ describes multi-particle states. In the absence of bound states it starts at $(2m)^2$, or at $(3m)^2$ if the theory is invariant under $\varphi \mapsto -\varphi$. For weakly coupled $P(\varphi)_2$ models the isolated one-particle shell was established in [GJS74].

(iii) *No point fields, again.* Formally $W_2^T(0) = \int \rho(dM^2) \int d\Omega_M = \infty$ for any $\rho \neq 0$ when $d \geq 2$. The singularity at coincident points of a nontrivial Wightman field is at least as strong as that of the free field. Remark 3.5 gives a precise Euclidean version of this statement.

(iv) *Canonical fields and field-strength renormalisation.* If the sharp-time fields exist and satisfy the CCR (1.6), then $\int \rho(dM^2) = 1$ (Exercise 1.5). Hence $Z = \rho(\{m^2\}) \leq 1$, with equality iff $\rho = \delta_{m^2}$. The time-zero fields of $P(\varphi)_2$ models satisfy the CCR [GJ87]. In $d = 4$, perturbation theory gives $\int \rho = \infty$, i.e. “ $Z = 0$ ”: the renormalised field cannot be canonical, and must be rescaled by a divergent factor (wave-function renormalisation).

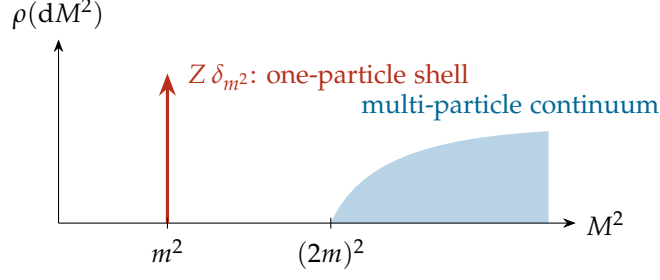


Figure 2: Typical spectral measure of an interacting field with mass gap (Theorem 1.6 and Remark 1.7): an isolated atom at the one-particle mass, $0 < Z \leq 1$, and a continuum starting at $(2m)^2$, or at $(3m)^2$ in the presence of the symmetry $\varphi \mapsto -\varphi$. The free field has $\rho = \delta_{m^2}$.

1.5 The quantum wave equation

Since $(-p^2 + m^2)e^{\pm ip \cdot x} = 0$ on the mass shell, the free field (1.2) solves the Klein–Gordon equation in the sense of distributions:

$$(\square + m^2)\varphi = 0, \quad \square = \partial_t^2 - \Delta_{\mathbf{x}}. \quad (1.5)$$

Moreover, the *sharp-time fields* $\varphi(t, g) = \int \varphi(t, \mathbf{x})g(\mathbf{x})d\mathbf{x}$ and $\pi(t, g)$, with $\pi = \partial_t \varphi$, are well defined for $g \in \mathcal{S}(\mathbb{R}^s)$ (Exercise 1.3), and satisfy the canonical commutation relations (CCR)

$$[\varphi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \quad [\varphi(t, \mathbf{x}), \varphi(t, \mathbf{y})] = [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0. \quad (1.6)$$

One may think of φ as the solution of a *linear wave equation with quantum initial data*: $(\varphi(0, \cdot), \pi(0, \cdot))$ satisfy the CCR and are “distributed” according to the vacuum.

Free field on a spatial circle. For $s = 1$ and space $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$, the momenta are $k \in \frac{2\pi}{L}\mathbb{Z}$, $\omega_k = (k^2 + m^2)^{1/2}$, and

$$\varphi(0, x) = \sum_k (2L\omega_k)^{-1/2} (a_k e^{ikx} + a_k^* e^{-ikx}), \quad [a_k, a_{k'}^*] = \delta_{kk'}, \quad H_0 = \sum_k \omega_k a_k^* a_k. \quad (1.7)$$

The free field is an infinite collection of independent harmonic oscillators, one per mode. With an ultraviolet cutoff $|k| \leq R$,

$$\langle \Omega, \varphi_R(0, x)^2 \Omega \rangle = \sum_{|k| \leq R} \frac{1}{2L\omega_k} = \frac{1}{2\pi} \log \frac{R}{m} + O(1) \xrightarrow{R \rightarrow \infty} \infty. \quad (1.8)$$

Interactions. The classical nonlinear Klein–Gordon equation $(\square + m^2)u + \lambda u^3 = 0$ is Hamiltonian with $H = \int [\frac{1}{2}\pi^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}m^2 u^2 + \frac{\lambda}{4}u^4]d\mathbf{x}$. Its quantum version would be the operator equation

$$(\square + m^2)\varphi + \lambda \varphi^3 = 0, \quad (1.9)$$

together with the CCR (1.6). This immediately runs into three problems.

- (i) $\varphi(x)^3$ has no meaning, being the cube of a distribution. It must be *renormalised*. In $d = 2$ Wick ordering $:\varphi^3:$ suffices (cf. (1.8) and Section 3.3); in $d = 3$ further mass and vacuum energy counterterms are needed; in $d = 4$ the theory is expected to be (and for Φ_4^4 is [ADC21]) trivial.

- (ii) *Haag's theorem* [Haa96]: in infinite volume the interacting field at fixed time cannot be unitarily equivalent to the free one, so there is no interaction picture. The interacting theory lives in a representation of the CCR inequivalent to the Fock one.
- (iii) Even in finite volume, $H = H_0 + \lambda \int_{\mathbb{T}_L} \varphi(0, x)^4 dx$ is the sum of a positive operator and an operator unbounded below. That it is bounded below was Nelson's breakthrough [Nel66], based on hypercontractivity. Its self-adjointness and the construction of the corresponding local Heisenberg dynamics, with finite propagation speed, are due to Glimm–Jaffe [GJ68] and Segal [Seg70]; see [BSZ92] for a textbook account of this Hamiltonian approach. Removing the volume cutoff for φ_2^4 was achieved by Glimm and Jaffe in a series of difficult papers; the full set of Wightman axioms for weakly coupled $P(\varphi)_2$ models was established in [GJS74], already with Euclidean methods. For Φ_3^4 see [GJ73, FO76].

The Euclidean approach (Symanzik, Nelson, Osterwalder–Schrader, Guerra–Rosen–Simon) simplified this enormously. We now explain how it arises.

Exercises

Exercise 1.1 (The free two-point function). (a) Show that W_2 in (1.3) is a Lorentz invariant tempered distribution solving $(\square + m^2)W_2 = 0$, whose Fourier transform is proportional to $\theta(p^0)\delta(p^2 - m^2)$. (b) For $d = 2$ show that $W_2(0, x) = (2\pi)^{-1}K_0(m|x|)$ and deduce the logarithmic singularity at $x = 0$. (c) Show that $[\varphi(x), \varphi(0)] = W_2(x) - W_2(-x)$ vanishes for spacelike x , i.e. the free field satisfies (W3). *Hint*: boost x to the plane $x^0 = 0$ and use the symmetry $\mathbf{p} \mapsto -\mathbf{p}$ of $d\Omega_m$.

Exercise 1.2 (Invariant measures on the hyperboloid). Show that $\int_{\mathbb{R}^d} \theta(p^0)\delta(p^2 - m^2)g(p) dp = \int g(\omega(\mathbf{p}), \mathbf{p}) \frac{d\mathbf{p}}{2\omega(\mathbf{p})}$, that this is the unique $\mathcal{L}_+^\dagger$ -invariant measure on H_m^+ up to constants, and give a direct proof that H_m^+ carries no finite invariant measure.

Exercise 1.3 (Which restrictions make sense?). For a distribution T on $\mathbb{R}^d$, the free field “ $\varphi(T)\Omega$ ” is defined iff $\int d\Omega_m |\tilde{T}|^2 < \infty$. Decide for which of the following this holds: (a) $T = \delta_x$; (b) $T = \delta(t) \otimes g(\mathbf{x})$ with $g \in \mathcal{S}(\mathbb{R}^s)$ (sharp time); (c) $T = h(t) \otimes \delta(\mathbf{x})$ with $h \in \mathcal{S}(\mathbb{R})$; (d) $T = \delta'(t) \otimes g(\mathbf{x})$, which gives $\pi(0, g)$. Verify the CCR (1.6).

Exercise 1.4 (Analyticity from the spectrum condition). (a) Let $H \geq 0$ be self-adjoint. Show that $z \mapsto \langle \psi, e^{-izH} \chi \rangle$ is bounded and analytic on $\{\text{Im } z < 0\}$, continuous up to the boundary, and is determined by its values on the negative imaginary half-line. (b) Using (W0), show that $a \mapsto U(a, \mathbb{1})\psi$ extends to a bounded analytic $\mathcal{H}$ -valued function on the tube $\mathbb{R}^d + iV_+$.

Exercise 1.5 (A canonical sum rule). (a) For the free field of mass M , write $[\varphi(x), \varphi(0)] = W_2^{(M)}(x) - W_2^{(M)}(-x)$ and show that $\partial_t[\varphi(t, \mathbf{x}), \varphi(0, \mathbf{0})]|_{t=0} = -i\delta(\mathbf{x})$, independently of M . (b) Assume a Wightman field has sharp-time fields satisfying (1.6). Using (1.4), show (at least formally) that $\int \rho(dM^2) = 1$. Deduce $0 \leq Z \leq 1$ in Remark 1.7(iv).

Exercise 1.6 (Generalised free fields). Let $\varphi_1, \dots, \varphi_n$ be independent free fields of masses $M_1, \dots, M_n > 0$, on the tensor product of their Fock spaces, and $\varphi = \sum_i c_i \varphi_i$ with $c_i \in \mathbb{R}$. (a) Show that φ satisfies (W0)–(W4) and has $\rho = \sum_i c_i^2 \delta_{M_i^2}$. (b) Show that $\prod_i (\square + M_i^2)\varphi = 0$, but that φ satisfies no Klein–Gordon equation of a single mass if $n \geq 2$. (c) Show that all truncated Wightman functions of order ≥ 3 vanish, so there is no scattering. (d) Extend the construction to an arbitrary polynomially bounded ρ .

Exercise 1.7 (Wick ordering on the circle). Using (1.7), verify (1.5), (1.6) and (1.8). For $g \in C^\infty(\mathbb{T}_L)$ show that $\int g(x) \varphi_R(0, x)^2 dx \Omega$ diverges as $R \rightarrow \infty$, while $\int g(x) : \varphi_R(0, x)^2 : dx \Omega$ converges in Fock space, where $:\varphi_R^2: := \varphi_R^2 - \langle \Omega, \varphi_R^2 \Omega \rangle$.

2 From real to imaginary time

We work with a regularised version of (1.9) with finitely many degrees of freedom, where everything is rigorous, and follow how the quantum evolution turns into a classical probability measure.

2.1 A lattice Hamiltonian on the circle

Fix $L > 0$, $N \in \mathbb{N}$, $\varepsilon = L/N$, and the periodic lattice $\Lambda_\varepsilon = \varepsilon\mathbb{Z}/L\mathbb{Z} \subset \mathbb{T}_L$ (N sites). Configurations are $\varphi = (\varphi_x)_{x \in \Lambda_\varepsilon} \in \mathbb{R}^{\Lambda_\varepsilon}$. In the Schrödinger representation $\mathcal{H}_\varepsilon = L^2(\mathbb{R}^{\Lambda_\varepsilon}, d\varphi)$, φ_x acts by multiplication and $\pi_x := -i\varepsilon^{-1} \partial / \partial \varphi_x$, so that $[\varphi_x, \pi_y] = i\varepsilon^{-1} \mathbb{1}_{x=y}$, the lattice version of $i\delta(x-y)$. Let V be a polynomial bounded below (e.g. $V(u) = \lambda u^4$) and

$$H_\varepsilon := \sum_{x \in \Lambda_\varepsilon} \varepsilon \left[\frac{1}{2} \pi_x^2 + \frac{1}{2} (\nabla_\varepsilon \varphi)_x^2 + \frac{1}{2} m^2 \varphi_x^2 + V(\varphi_x) \right] = -\frac{1}{2\varepsilon} \Delta_{\mathbb{R}^{\Lambda_\varepsilon}} + U_\varepsilon(\varphi), \quad (2.1)$$

where $(\nabla_\varepsilon \varphi)_x = (\varphi_{x+\varepsilon} - \varphi_x)/\varepsilon$ and $U_\varepsilon(\varphi) = \varepsilon \sum_x [\frac{1}{2} (\nabla_\varepsilon \varphi)_x^2 + \frac{1}{2} m^2 \varphi_x^2 + V(\varphi_x)]$. A short computation (Exercise 2.1) shows that the Heisenberg fields $\varphi_x(t) = e^{itH_\varepsilon} \varphi_x e^{-itH_\varepsilon}$ satisfy

$$\partial_t^2 \varphi_x(t) = (\Delta_\varepsilon \varphi(t))_x - m^2 \varphi_x(t) - V'(\varphi_x(t)), \quad (\Delta_\varepsilon \varphi)_x = \frac{\varphi_{x+\varepsilon} + \varphi_{x-\varepsilon} - 2\varphi_x}{\varepsilon^2}, \quad (2.2)$$

a lattice regularisation of the quantum wave equation (1.9), now with a perfectly rigorous meaning.

H_ε is a Schrödinger operator on $\mathbb{R}^N$ with a confining potential ($U_\varepsilon \geq c|\varphi|^2 - C$ since $m > 0$). Hence it is essentially self-adjoint on C_c^∞ , bounded below, with compact resolvent, and e^{-tH_ε} is trace class for $t > 0$. The integral kernel of e^{-tH_ε} is strictly positive, so by the Perron–Frobenius argument the ground state energy $E_0 = \inf \text{spec } H_\varepsilon$ is a simple eigenvalue with a strictly positive eigenfunction Ω_ε , the *lattice vacuum* [RS75, Sim79]. We set

$$\bar{H} := H_\varepsilon - E_0 \geq 0 \quad (\text{positivity of the energy}). \quad (2.3)$$

2.2 Positivity of the energy and analytic continuation

Let $F_1, \dots, F_n : \mathbb{R}^{\Lambda_\varepsilon} \rightarrow \mathbb{C}$ be bounded measurable (acting as multiplication operators) and $F_j(t) = e^{itH_\varepsilon} F_j e^{-itH_\varepsilon}$. Since $\bar{H} \Omega_\varepsilon = 0$,

$$W(t_1, \dots, t_n) := \langle \Omega_\varepsilon, F_1(t_1) \cdots F_n(t_n) \Omega_\varepsilon \rangle = \langle \Omega_\varepsilon, F_1 e^{-i(t_1-t_2)\bar{H}} F_2 \cdots e^{-i(t_{n-1}-t_n)\bar{H}} F_n \Omega_\varepsilon \rangle. \quad (2.4)$$

Proposition 2.1 (Wick rotation). *W depends only on $\zeta_j = t_j - t_{j+1}$ and extends to a function $\mathcal{W}(\zeta_1, \dots, \zeta_{n-1})$ which is bounded and continuous on $\overline{\mathbb{C}}_-^{n-1}$ and analytic on $\mathbb{C}_-^{n-1}$, $\mathbb{C}_- = \{\text{Im } \zeta < 0\}$. At $\zeta_j = -i(\tau_j - \tau_{j+1})$, $\tau_1 \geq \dots \geq \tau_n$, it gives the Schwinger functions*

$$S(\tau_1, \dots, \tau_n) = \langle \Omega_\varepsilon, F_1 e^{-(\tau_1-\tau_2)\bar{H}} F_2 \cdots e^{-(\tau_{n-1}-\tau_n)\bar{H}} F_n \Omega_\varepsilon \rangle, \quad (2.5)$$

and the Schwinger functions determine the Wightman functions W .

Proof. By the spectral theorem, $e^{-i\zeta H} = \int e^{-i\zeta\lambda} dE(\lambda)$ with $|e^{-i\zeta\lambda}| = e^{\lambda \operatorname{Im} \zeta} \leq 1$ for $\lambda \geq 0$, $\operatorname{Im} \zeta \leq 0$. It is therefore a contraction, strongly continuous on $\overline{\mathbb{C}}_-$ and analytic in the interior. Products of such operators give separately analytic, bounded, continuous functions, hence jointly analytic ones. If $\mathcal{W}$ vanishes on $(-i\mathbb{R}_+)^{n-1}$, the identity theorem in ζ_1 (with the other variables on the imaginary axis) shows that it vanishes for all $\zeta_1 \in \mathbb{C}_-$. Proceeding variable by variable, and using continuity up to the boundary, gives $W \equiv 0$. $\square$

Remark 2.2. This is exactly where positivity of the energy enters: it allows rotating $t \mapsto -i\tau$ in all time differences simultaneously. It will be used twice more: $e^{-\tau H}$, $\tau \geq 0$, is a contraction semigroup, which can be represented as a path integral (Feynman–Kac); and $e^{-T\bar{H}}$ projects onto the vacuum as $T \rightarrow \infty$. (That the Feynman–Kac weights are positive is a separate fact, coming from the kinetic term.) In a relativistic theory, the Wightman functions are boundary values of functions analytic in the forward tube (Exercise 1.4). By Lorentz covariance these extend to a domain containing all non-coincident Euclidean points, where they define symmetric, Euclidean invariant Schwinger functions [SW64, OS73].

2.3 Discretising imaginary time: the Feynman–Kac formula

The kinetic part of H_ε generates a Gaussian semigroup with kernel

$$e^{\frac{\delta}{2\varepsilon}\Delta}(\varphi, \varphi') = \left(\frac{2\pi\delta}{\varepsilon}\right)^{-N/2} \exp\left(-\frac{\varepsilon}{2\delta}|\varphi - \varphi'|^2\right).$$

Hence the symmetric Trotter approximant $T_\delta := e^{-\delta U_\varepsilon/2} e^{\frac{\delta}{2\varepsilon}\Delta} e^{-\delta U_\varepsilon/2}$ has the explicit positive kernel

$$T_\delta(\varphi, \varphi') = \left(\frac{2\pi\delta}{\varepsilon}\right)^{-N/2} \exp\left(-\frac{\varepsilon}{2\delta} \sum_x (\varphi_x - \varphi'_x)^2 - \frac{\delta}{2}(U_\varepsilon(\varphi) + U_\varepsilon(\varphi'))\right), \quad (2.6)$$

and $T_\delta^{\lfloor \tau/\delta \rfloor} \rightarrow e^{-\tau H_\varepsilon}$ strongly as $\delta \rightarrow 0$ (Trotter product formula [RS80, Sim79]). Let $T = J\delta$ and consider the space-time lattice

$$\Lambda = \Lambda_{\delta,\varepsilon}^{T,L} := (\delta\mathbb{Z}/T\mathbb{Z}) \times (\varepsilon\mathbb{Z}/L\mathbb{Z}), \quad \varphi = (\varphi_{\tau,x})_{(\tau,x) \in \Lambda}, \quad \varphi_\tau := (\varphi_{\tau,x})_{x \in \Lambda_\varepsilon}. \quad (2.7)$$

Writing the trace of a product of integral operators as an integral over closed chains, we get $\operatorname{Tr}(T_\delta^J) = (2\pi\delta/\varepsilon)^{-NJ/2} \int_{\mathbb{R}^\Lambda} e^{-S_{\delta,\varepsilon}(\varphi)} \prod_{u \in \Lambda} d\varphi_u$ with the *lattice Euclidean action*

$$S_{\delta,\varepsilon}(\varphi) = \sum_{(\tau,x) \in \Lambda} \varepsilon \delta \left[\frac{1}{2} \left(\frac{\varphi_{\tau+\delta,x} - \varphi_{\tau,x}}{\delta} \right)^2 + \frac{1}{2} \left(\frac{\varphi_{\tau,x+\varepsilon} - \varphi_{\tau,x}}{\varepsilon} \right)^2 + \frac{1}{2} m^2 \varphi_{\tau,x}^2 + V(\varphi_{\tau,x}) \right]. \quad (2.8)$$

This discretises $S(\varphi) = \iint_{\mathbb{T}_T \times \mathbb{T}_L} [\frac{1}{2}(\partial_\tau \varphi)^2 + \frac{1}{2}(\partial_x \varphi)^2 + \frac{1}{2}m^2 \varphi^2 + V(\varphi)] d\tau dx$. Imaginary time and space now play symmetric roles: the Minkowski metric has become Euclidean. Define the probability measure

$$\mu_{\delta,\varepsilon}^{T,L}(d\varphi) := Z^{-1} e^{-S_{\delta,\varepsilon}(\varphi)} \prod_{u \in \Lambda} d\varphi_u. \quad (2.9)$$

Inserting multiplication operators between the factors T_δ gives, for $\tau_1 \geq \dots \geq \tau_n$ in $\delta\mathbb{Z}$ with $\tau_1 - \tau_n < T$,

$$\frac{\operatorname{Tr}(F_1 T_\delta^{k_1} F_2 T_\delta^{k_2} \dots F_n T_\delta^{k_n})}{\operatorname{Tr}(T_\delta^J)} = \int F_1(\varphi_{\tau_1}) \dots F_n(\varphi_{\tau_n}) \mu_{\delta,\varepsilon}^{T,L}(d\varphi), \quad (2.10)$$

where $k_j = (\tau_j - \tau_{j+1})/\delta$ for $j < n$ and $k_n = J - \sum_{j < n} k_j$.

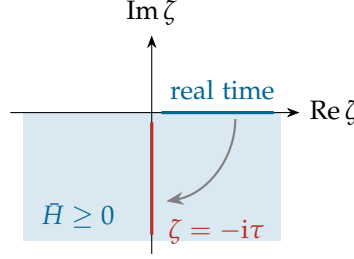


Figure 3: Wick rotation (Proposition 2.1). Since $\bar{H} \geq 0$, $e^{-i\zeta\bar{H}}$ is a contraction on the lower half-plane $\text{Im } \zeta \leq 0$, analytic in its interior. The Wightman functions on the real axis and the Schwinger functions on the negative imaginary axis are boundary values and restrictions of the same analytic function.

Proposition 2.3 (Feynman–Kac–Nelson). *For bounded continuous F_j and τ_j as above,*

$$\langle \Omega_\varepsilon, F_1 e^{-(\tau_1 - \tau_2)\bar{H}} F_2 \cdots F_n \Omega_\varepsilon \rangle = \lim_{T \rightarrow \infty} \lim_{\delta \rightarrow 0} \int F_1(\varphi_{\tau_1}) \cdots F_n(\varphi_{\tau_n}) \mu_{\delta, \varepsilon}^{T, L}(\mathrm{d}\varphi). \quad (2.11)$$

Sketch. The Gaussian factor in (2.9) gives precisely the finite-dimensional distributions, at times $\tau \in \delta\mathbb{Z}/T\mathbb{Z}$, of the (unnormalised) Brownian loop measure on $\mathbb{R}^{\Lambda_\varepsilon}$ of length T and diffusion constant ε^{-1} . The potential term $\delta \sum_\tau U_\varepsilon(\varphi_\tau)$ is a Riemann sum for $\int_0^T U_\varepsilon(\omega_\tau) \mathrm{d}\tau$. Hence, as $\delta \rightarrow 0$ (along dyadic δ , say, so that the τ_j stay on the lattice), both sides of (2.10) converge to the Feynman–Kac expression $\text{Tr}(F_1 e^{-(\tau_1 - \tau_2)H} \cdots F_n e^{-(T - \tau_1 + \tau_n)H}) / \text{Tr}(e^{-TH})$, written as an integral over Brownian loops weighted by $\exp(-\int_0^T U_\varepsilon)$ [Sim79]. The limit $T \rightarrow \infty$ uses that E_0 is simple and isolated (Exercise 2.6). $\square$

Vacuum expectations of the quantum system in imaginary time are thus *moments of a classical probability measure*. As $T \rightarrow \infty$ the time-slices $(\varphi_\tau)_{\tau \in \mathbb{R}}$ form a stationary Markov process: the ground-state transform of $e^{-\tau\bar{H}}$, whose one-time marginal is $\Omega_\varepsilon(\varphi)^2 \mathrm{d}\varphi$. For $N = 1$, $V = 0$ this is the Ornstein–Uhlenbeck process (Exercise 2.2 and Fig. 4).

2.4 Classical statistical mechanics

Take the isotropic lattice $\delta = \varepsilon$. Then (2.8) becomes

$$S_\varepsilon(\varphi) = \sum_{\langle u, v \rangle} \frac{1}{2} (\varphi_u - \varphi_v)^2 + \varepsilon^2 \sum_u \left[\frac{1}{2} m^2 \varphi_u^2 + V(\varphi_u) \right], \quad (2.12)$$

the first sum running over nearest-neighbour pairs of the two-dimensional periodic lattice Λ . So $\mu_\varepsilon \propto \prod_{\langle u, v \rangle} e^{-\frac{1}{2}(\varphi_u - \varphi_v)^2} \prod_u v_\varepsilon(\mathrm{d}\varphi_u)$, with single-site measure $v_\varepsilon(\mathrm{d}a) = e^{-\varepsilon^2(\frac{1}{2}m^2 a^2 + V(a))} \mathrm{d}a$. This is a *ferromagnetic nearest-neighbour spin system* with continuous, unbounded spins, i.e. a short-range model of classical statistical mechanics. In spacetime dimension d one gets $\varepsilon^{d-2} \sum_{\langle u, v \rangle}$ and $\varepsilon^d \sum_u$ instead (Exercise 2.3). The nearest-neighbour structure makes μ_ε *reflection positive* (Fig. 5 and Exercise 2.5), the Euclidean shadow of the positivity of $e^{-\varepsilon H_\varepsilon}$. Conversely, reflection positivity is enough to reconstruct a Hilbert space and a positive Hamiltonian (Exercise 2.7).

Remark 2.4 (Continuum limit = critical point). In lattice units the model (2.12) has mass $\varepsilon m \rightarrow 0$, so its correlation length is $\xi \approx (\varepsilon m)^{-1} \rightarrow \infty$ lattice spacings (Exercise 2.4). The continuum limit of the regularised QFT is therefore the *scaling limit of a family of short-range spin systems approaching a critical point*. Conversely, any reflection positive, short-range critical

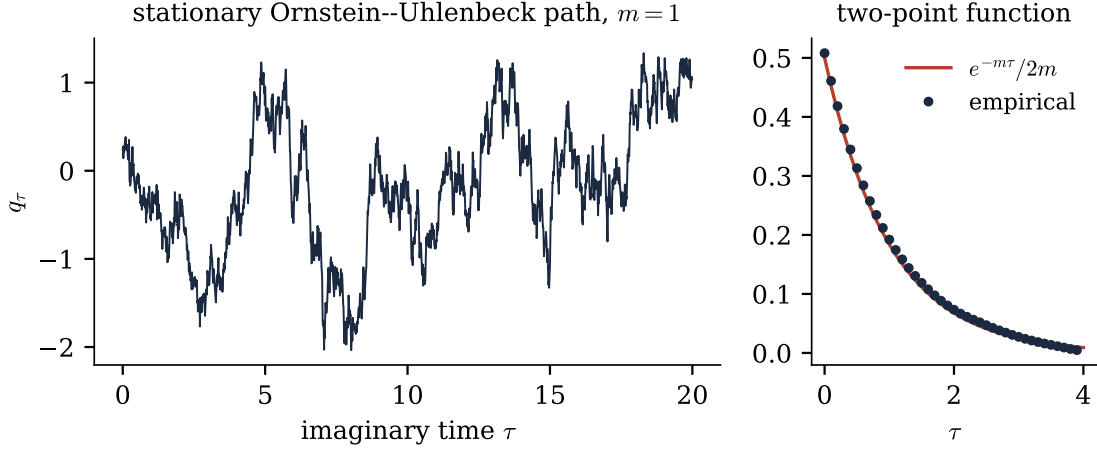


Figure 4: Left: a sample of the stationary Ornstein–Uhlenbeck process, the imaginary-time path of one harmonic oscillator, i.e. the GFF in $d = 1$ (Exercise 2.2). Right: its empirical two-point function against the Schwinger function $e^{-m\tau}/2m$.

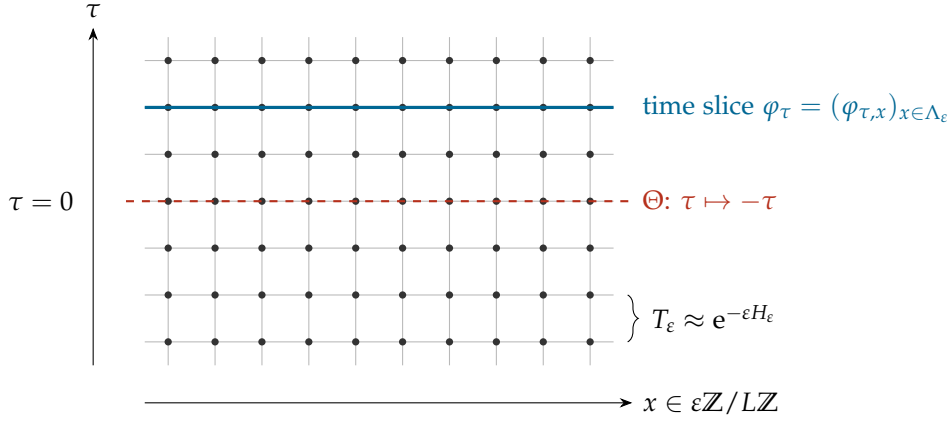


Figure 5: The Euclidean space-time lattice. Rows are configurations of the quantum system at fixed imaginary time; the transfer matrix $T_\varepsilon \approx e^{-\varepsilon H_\varepsilon}$ connects consecutive rows. The nearest-neighbour structure makes the measure reflection positive with respect to Θ .

model with a nontrivial scaling limit yields a candidate Euclidean QFT. This is Wilson’s point of view on renormalisation [WK74]. For instance, the Griffiths–Simon construction realises φ^4 single-spin measures as limits of block spins of Ising models [GS73], and the critical 4D Ising and φ^4_4 models have Gaussian scaling limits [ADC21]. See [GRS75] for the systematic use of lattice approximations for $P(\varphi)_2$.

Remark 2.5 (Renormalisation already appears). For $V(a) = \lambda a^4$, the limit $\varepsilon \rightarrow 0$ at fixed (m, λ) does not produce an interacting theory: the Gaussian part alone has $c_\varepsilon := \mathbb{E}[\varphi_u^2] = \frac{1}{2\pi} \log \frac{1}{\varepsilon m} + O(1)$ (compare (1.8)). One must take ε -dependent couplings, e.g. $V_\varepsilon(a) = \lambda(a^4 - 6c_\varepsilon a^2)$ (a divergent mass counterterm), i.e. λ times the Wick power $:a^4:$ of Section 3.3 up to a constant. In $d = 2$ this is all that is needed; in $d = 3$ more counterterms are required.

Quantum theory (real time)	Euclidean field theory	Statistical mechanics
positive Hilbert space, $H \geq 0$	reflection positivity	RP of n.n. interactions
$e^{-\varepsilon H}$	time translation by ε	transfer matrix
vacuum Ω	infinite-volume measure ($T \rightarrow \infty$)	infinite-volume Gibbs state
inverse temperature β	length T of the time circle	—
mass gap $E_1 - E_0$	exponential decay of correlations	inverse correlation length $1/\xi$
continuum limit	removal of UV cutoff $\varepsilon \rightarrow 0$	critical point, scaling limit
renormalisation	counterterms in the action	tuning parameters to criticality

Table 1: A dictionary between the three pictures.

2.5 Path integrals and the Osterwalder–Schrader axioms

Formally, as $\varepsilon \rightarrow 0$ and $L, T \rightarrow \infty$, (2.9) becomes the Euclidean path integral

$$\mu(d\varphi) \stackrel{“}{=} \stackrel{”}{=} Z^{-1} \exp\left(-\int_{\mathbb{R}^d} \left[\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi)\right] dx\right) \mathcal{D}\varphi. \quad (2.13)$$

Neither $\mathcal{D}\varphi$ nor the action make sense, since typical φ are distributions. The Gaussian part, however, defines the Gaussian free field μ_m of Section 3, and one is led to define $\mu \propto e^{-\int V(\varphi)} \mu_m$ with a renormalised V , at least in finite volume and in $d = 2, 3$. Which measures on $\mathcal{S}'(\mathbb{R}^d)$ correspond to relativistic QFTs? Let $\hat{\mu}(f) := \int e^{i\varphi(f)} \mu(d\varphi)$ be the characteristic functional and $\Theta(\tau, \mathbf{x}) = (-\tau, \mathbf{x})$, $\Theta f = f \circ \Theta$.

(OS0) Analyticity.

For real $f_1, \dots, f_n \in C_c^\infty(\mathbb{R}^d)$, the map $z \mapsto \hat{\mu}(\sum_j z_j f_j)$ is entire on $\mathbb{C}^n$.

(OS1) Regularity.

For some $p \in [1, 2]$ and $c > 0$, $|\hat{\mu}(f)| \leq \exp(c(\|f\|_{L^1} + \|f\|_{L^p}^p))$.

(OS2) Euclidean invariance.

$\hat{\mu}(f \circ g) = \hat{\mu}(f)$ for every Euclidean motion g of $\mathbb{R}^d$.

(OS3) Reflection positivity.

For real $f_1, \dots, f_n$ supported in $\{\tau > 0\}$, the matrix $(\hat{\mu}(f_i - \Theta f_j))_{i,j}$ is positive semidefinite.

(OS4) Ergodicity.

The time translations act ergodically on $(\mathcal{S}'(\mathbb{R}^d), \mu)$.

Theorem 2.6 (Osterwalder–Schrader reconstruction [OS73, OS75, GJ87]). *If μ satisfies (OS0)–(OS3), its moments are the Schwinger functions of a unique Wightman theory satisfying (W1)–(W4) and (W0) except possibly uniqueness of the vacuum. (OS4) holds iff the vacuum is unique.*

Idea. Let $\mathcal{E}_+$ be the functions of φ that depend only on its restriction to $\{\tau > 0\}$, with $(F, G) := \mathbb{E}_\mu[\Theta \bar{F} G]$. By (OS3) this is a positive semi-definite form; quotienting by null vectors and completing gives $\mathcal{H}$. Time translations by $t \geq 0$ map $\mathcal{E}_+$ to itself and descend to a self-adjoint contraction semigroup e^{-tH} , so $H \geq 0$: *positivity of the energy is recovered from reflection positivity*. The vacuum is the class of 1, and the analytic continuation $\tau \rightarrow it$ yields the Wightman functions. $\square$

The Euclidean strategy. (1) Regularise: lattice spacing ε (or another UV cutoff), finite volume. (2) Renormalise: choose ε -dependent counterterms. (3) Prove uniform estimates and

remove the cutoffs. (4) Verify the OS axioms: reflection positivity survives limits of lattice approximations; Euclidean invariance must be recovered in the limit; (OS1) needs uniform bounds; (OS4) often follows from a mass gap. (5) Show the limit is not Gaussian. In the next lectures we follow this route using stochastic analysis and PDE methods, starting from the Gaussian free field.

Exercises

Exercise 2.1 (Lattice equations of motion). Derive (2.2) from (2.1). For $V = 0$, diagonalise H_ε in Fourier modes $k \in \frac{2\pi}{L}\mathbb{Z} \cap (-\frac{\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]$ and show that the frequencies are $\omega_\varepsilon(k)^2 = m^2 + \frac{4}{\varepsilon^2} \sin^2 \frac{k\varepsilon}{2}$. Deduce that $\langle \Omega_\varepsilon, \varphi_x^2 \Omega_\varepsilon \rangle = \frac{1}{L} \sum_k \frac{1}{2\omega_\varepsilon(k)} = \frac{1}{2\pi} \log \frac{1}{\varepsilon m} + O(1)$.

Exercise 2.2 (Harmonic oscillator and Ornstein–Uhlenbeck). Let $H = -\frac{1}{2}\partial_q^2 + \frac{1}{2}m^2q^2$ on $L^2(\mathbb{R})$. (a) Show that $\Omega(q) = (m/\pi)^{1/4}e^{-mq^2/2}$ and $E_0 = m/2$. (b) Show that $\langle \Omega, q e^{-\tau \tilde{H}} q \Omega \rangle = e^{-m\tau}/(2m)$. (c) Let $(Uf)(q) = \Omega(q)f(q)$, unitary from $L^2(\Omega^2 dq)$ to $L^2(dq)$. Show that $U^{-1}\tilde{H}U = -\frac{1}{2}\partial_q^2 + mq\partial_q$, minus the generator of $dX = -mXd\tau + dB$. (d) Conclude that the Schwinger functions are the moments of the stationary Gaussian process with covariance $e^{-m|\tau-\sigma|}/(2m) = (-\partial_\tau^2 + m^2)^{-1}(\tau, \sigma)$: the GFF in $d = 1$.

Exercise 2.3 (Lattice action in general dimension). Verify (2.6)–(2.10). In spacetime dimension d (lattice $\varepsilon\mathbb{Z}^{d-1}$ in space, $\delta = \varepsilon$) show that $S_\varepsilon = \varepsilon^{d-2} \sum_{\langle u,v \rangle} \frac{1}{2}(\varphi_u - \varphi_v)^2 + \varepsilon^d \sum_u [\frac{1}{2}m^2\varphi_u^2 + V(\varphi_u)]$, so that $\varepsilon^{(d-2)/2}\varphi$ is the natural dimensionless spin variable.

Exercise 2.4 (Correlation length and criticality). On $\varepsilon\mathbb{Z}^d$ consider the lattice Green function

$$\Gamma_\varepsilon(x) = \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} \frac{e^{ik \cdot x}}{m^2 + 4\varepsilon^{-2} \sum_j \sin^2(\varepsilon k_j/2)} \frac{dk}{(2\pi)^d}.$$

(a) Show that $\Gamma_\varepsilon(x) \rightarrow G_m(x)$ for $x \neq 0$, where G_m is the Green function of $m^2 - \Delta$ on $\mathbb{R}^d$. (b) Show that in lattice units $n = x/\varepsilon$ along a coordinate axis, Γ_ε decays like $e^{-\mu_\varepsilon|n|}$ with $\cosh \mu_\varepsilon = 1 + \frac{1}{2}(\varepsilon m)^2$. Deduce that the lattice correlation length is $\xi = \mu_\varepsilon^{-1} \sim (\varepsilon m)^{-1} \rightarrow \infty$ and interpret.

Exercise 2.5 (Reflection positivity on the lattice). Let $\Lambda = (\varepsilon\mathbb{Z}/T\mathbb{Z}) \times \Lambda_\varepsilon$ with $T = 2K\varepsilon$, let μ_ε have action (2.12), and let Θ be the reflection $\tau \mapsto -\tau$, which fixes the planes $\tau = 0$ and $\tau = T/2$. Show that $\mathbb{E}[F \cdot \Theta F] \geq 0$ for every bounded real F depending only on $\{\varphi_u : \tau(u) \in [0, T/2]\}$. *Hint*: write $S_\varepsilon = S_+ + \Theta S_+$, with the terms living on the fixed planes split evenly, and integrate first over the sites off the fixed planes.

Exercise 2.6 (Vacuum, mass gap and clustering). Let H be self-adjoint with e^{-tH} trace class for $t > 0$, simple ground state Ω , and $E_1 := \min(\text{spec } H \setminus \{E_0\})$. (a) Show that $\text{Tr}(Ae^{-TH})/\text{Tr}(e^{-TH}) = \langle \Omega, A\Omega \rangle + O(e^{-T(E_1-E_0)})$ for bounded A . (b) Show that

$$|\langle \Omega, Ae^{-\tau \tilde{H}} B\Omega \rangle - \langle \Omega, A\Omega \rangle \langle \Omega, B\Omega \rangle| \leq \|A\| \|B\| e^{-\tau(E_1-E_0)} :$$

a mass gap gives exponential clustering of Schwinger functions.

Exercise 2.7 (A toy OS reconstruction). Let μ be a probability measure on $\mathbb{R}^{\varepsilon\mathbb{Z} \times \Lambda_\varepsilon}$, invariant under time translations and reflection positive with respect to Θ . On bounded functions F of $\{\varphi_u : \tau(u) \geq 0\}$ put $(F, G) = \mathbb{E}[\Theta \overline{F}G]$ and $(TF)(\varphi) = F(\varphi(\cdot + \varepsilon, \cdot))$. Show that $(F, TG) = (TF, G)$ and, by iterating Cauchy–Schwarz, $|(F, TF)| \leq (F, F)$. Deduce that T induces a self-adjoint contraction on the reconstructed Hilbert space, and that $H := -(2\varepsilon)^{-1} \log T^2 \geq 0$.

A The course

These notes cover the first of six two-hour lectures given at SISSA, Trieste, in October–November 2026, for graduate students in analysis, probability or mathematical physics. Lectures 1–3 have their own notes; the content of Lectures 4–6 is a provisional outline.

Prerequisites. Measure-theoretic probability and Itô calculus; the spectral theorem for self-adjoint operators; distributions and Fourier analysis. No previous knowledge of quantum field theory is assumed.

Approach. The course follows the Euclidean strategy of [Section 2](#). After the Gaussian free field, interacting theories are studied through the scale decomposition of the field: the Polchinski equation, its probabilistic form as a forward–backward stochastic differential equation (FBSDE) for the field at increasing resolution, and the associated variational problem. Stochastic quantisation, cluster expansions and correlation inequalities appear as complementary approaches.

Lecture	Main results
1 Relativistic quantum fields and imaginary time	No fields at a point; Källén–Lehmann; Wick rotation; Feynman–Kac; OS axioms
2 The Gaussian free field	Construction; reflection positivity; Wick powers; DS characterisation; parabolic and elliptic SQ
3 Stochastic observables, the Polchinski flow and FBSDEs	Polchinski equation and FBSDE; force coordinates and the Wetterich equation; Boué–Dupuis formula
4 Φ_2^4	Construction on $\mathbb{T}^2$ and uniform bounds; infinite volume
5 Φ_3^4	Power counting and counterterms; renormalised variational problem and FBSDE
6 Perspectives	OS axioms for limits; fermions and gauge fields; open problems

Table 2: The six lectures.

Lecture 1. Relativistic quantum fields and imaginary time ([Sections 1](#) and [2](#)). Wightman axioms; no fields at a point; Källén–Lehmann representation; quantum wave equation and its problems; lattice Hamiltonian on a circle; Wick rotation from positivity of the energy; Feynman–Kac formula; lattice spin systems, continuum limit = critical point; OS axioms and reconstruction (statement and idea). *Proofs in class:* [Proposition 1.3](#), [Theorem 1.6](#) (sketch), [Proposition 2.1](#). *Exercises:* [Exercises 1.3](#), [1.6](#), [2.2](#), [2.4](#) and [2.5](#). *Reading:* [SW64, BSZ92, GJ87, Sim79].

Lecture 2. The Gaussian free field ([Section 3](#)). Construction on $\mathbb{T}^d$ and $\mathbb{R}^d$; reflection positivity and the free Markov field; Euclidean Källén–Lehmann; Wick powers; Gaussian integration by parts; DS characterisation; parabolic and elliptic stochastic quantisation. *Proofs in class:* [Theorem 3.2](#), [Proposition 3.4](#), [Theorem 3.7](#), [Theorem 3.12](#), [Theorem 3.15](#), [Theorem 3.17](#). *Exercises:* [Exercises 3.2](#), [3.3](#), [3.6](#) to [3.8](#) and [3.13](#). *Reading:* [She07, Jan97, DPZ14, ADG20].

Lecture 3. Stochastic observables, the Polchinski flow and FBSDEs (separate notes). Scale decomposition of the GFF and stochastic observables; effective potentials and the

Polchinski equation; the Polchinski process as an FBSDE whose force is a martingale; the FBSDE in force coordinates, where the transport term is integrated exactly, and the Wetterich equation for the effective average action; the Boué–Dupuis formula, with the Polchinski equation as its Hamilton–Jacobi–Bellman equation, in finite and infinite volume. Additional material: Wick powers as observables in every dimension and the operator product expansion; perturbation theory and counterterms; Dyson–Schwinger equations at every scale. *Reading:* [Pol84, BBD24, BD98, BG20, BCG26].

Lecture 4. Φ_2^4 . Wick-ordered quartic interaction on $\mathbb{T}^2$; the effective force and its FBSDE; uniform bounds from the variational problem (coercivity of the quartic term, control of the mixed Wick terms), and Nelson’s estimate as the classical route; renormalised DS equation; infinite-volume limit with local, volume-uniform estimates; phase transitions and non-uniqueness. *Reading:* [Nel66, Sim74, GJ87, BG24, GJS75].

Lecture 5. Φ_3^4 . Power counting; the tadpole and sunset divergences and the mass counterterm; the renormalised variational problem: change of variables in the drift, paracontrolled estimates and Γ -convergence; the renormalised FBSDE; singularity of Φ_3^4 with respect to the GFF; comparison with the stochastic quantisation approach (singular SPDEs) and with the classical constructions. *Reading:* [GJ73, FO76, BG20, BG21, GH21, Hai14, GIP15, CC18].

Lecture 6. Perspectives. Reflection positivity and Euclidean invariance of the limits: the OS axioms for Φ_2^4 and Φ_3^4 ; Wilson–Itô diffusions and local observables beyond the Gaussian case; Grassmann fields and the FBSDE for Euclidean fermions; gauge fields and the two-dimensional Yang–Mills measure; open problems, from the particle structure of Φ_3^4 to the Yang–Mills mass gap. *Reading:* [BCG26, ABDG22, CCHS22, Jaf00, Sum12].

Homework and projects. Problem sets can be drawn from the exercises at the end of each section. The computer experiments of Section 3 (lattice GFF, lattice stochastic quantisation) and the zero-dimensional Polchinski process of Lecture 3 scale naturally to Φ_2^4 and are suitable as small projects. Possible reading projects: the Griffiths–Simon construction [GS73]; dimensional reduction [KLP84, ADG20]; the Euclidean approach to the particle structure of $P(\varphi)_2$ [GJS74].

References and further reading

Guide. *Axioms and general structure:* [SW64, BSZ92, Haa96, RS75]; the point-field result is from [Wig64]; the spectral representation of the two-point function from [Käl52, Leh54]; particle structure of $P(\varphi)_2$ in [GJS74]. *Hamiltonian construction of $P(\varphi)_2$ in finite volume:* [Nel66, GJ68, Seg70, BSZ92]. *Euclidean field theory and reconstruction:* [Sym69, Nel73a, Nel73b, OS73, OS75]; the textbooks [GJ87, Sim74, Sim79] are the standard references for everything in Section 2. *Lattice approximations and statistical mechanics:* [GRS75, GS73, WK74, Bry09, BBS19, ADC21]. *Surveys of the constructive programme:* [Jaf00, Sum12].

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