A Constructive Introduction to Quantum Field Theory

Relativistic quantum fields and imaginary time

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SISSA lectures, Lecture 1 — notes: Lecture 1, §§1–2

What is constructive QFT?

- **Goal:** construct, with full rigour, *interacting* relativistic quantum fields, i.e. models satisfying axioms encoding quantum mechanics, special relativity and causality.
- **Status:**
  - $d=2$: $P(\varphi)_2$, Yukawa$_2$, sine-Gordon, ...
  - $d=3$: Φ^4_3, ...
  - $d=4$: no interacting example; critical lattice φ^4_4 has Gaussian scaling limits (Aizenman–Duminil-Copin 2021).
- **Lesson of the late 1960s–70s:** work in *imaginary time*. A QFT becomes a probability measure on $\mathscr{S}'(\mathbb{R}^d)$, which looks like the continuum limit of a classical spin system at a critical point.
- **Modern tools** (this course): stochastic quantisation, singular SPDEs, variational methods; RG and cluster expansions as complements.

Plan of the lecture

Part I. Minkowski picture

1. Wightman axioms
2. Why operator-valued distributions
3. Källén–Lehmann representation
4. The quantum wave equation

Part II. Imaginary time

1. A lattice Hamiltonian on a circle
2. Positivity of energy ⇒ Wick rotation
3. Feynman–Kac: quantum → classical
4. Statistical mechanics and criticality
5. Osterwalder–Schrader axioms

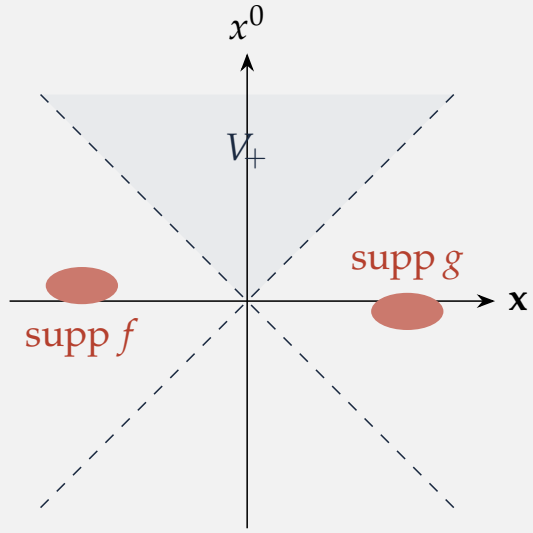
Conventions: $\hbar = c = 1$, spacetime dimension $d = 1 + s$, $\langle f, g \rangle = \int f\bar{g}$, $\hat{f}(k) = \int \mathrm{e}^{-\mathrm{i}k\cdot x} f(x)\,\mathrm{d}x$.

Relativistic quantum fields

Lecture 1

Kinematics

- Minkowski space $\mathbb{R}^{1,s}$, $x=(x^0,\mathbf{x})$, $x\cdot y = x^0y^0-\mathbf{x}\cdot\mathbf{y}$.
- x,y **spacelike separated** if $(x-y)^2<0$.
- Forward cone $V_+=\{p\colon p^0>|\mathbf{p}|\}$, closure $\bar{V}_+$.
- Restricted Lorentz group $\mathscr{L}_+^\uparrow$; Poincaré group $\mathscr{P}_+^\uparrow=\mathbb{R}^d\rtimes\mathscr{L}_+^\uparrow$ acting by $x\mapsto\Lambda x+a$.
- Wigner: a relativistic quantum system carries a (projective) unitary representation of $\mathscr{P}_+^\uparrow$.



Wightman axioms (I)

(W0) Relativistic quantum theory

Separable Hilbert space $\mathscr{H}$; strongly continuous unitary representation U of $\mathscr{P}_+^\uparrow$; $U(a,1) = \mathrm{e}^{\mathrm{i}P\cdot a}$, $H = P^0$.
Spectrum condition: $\mathrm{spec}(P) \subset \bar{V}_+$, so $H \geqslant |\mathbf{P}| \geqslant 0$.
Vacuum: unique (up to phase) unit vector Ω with $U(a,1)\,\Omega = \Omega$.

(W1) Fields

Dense $\mathscr{D} \ni \Omega$; for $f \in \mathscr{S}(\mathbb{R}^d)$ an operator $\varphi(f)$ on $\mathscr{D}$, $\varphi(f)\,\mathscr{D} \subset \mathscr{D}$, $U\,\mathscr{D} \subset \mathscr{D}$; $f \mapsto \langle \chi, \varphi(f)\,\psi\rangle$ tempered; $\langle \chi, \varphi(f)\,\psi\rangle = \langle \varphi(\bar{f})\,\chi, \psi\rangle$.

Wightman axioms (II)

(W2) Covariance, (W3) Locality, (W4) Cyclicity

$U(a,\Lambda)\,\varphi(f)\,U(a,\Lambda)^{-1} = \varphi\,(f\circ(a,\Lambda)^{-1})$; $[\varphi(f),\varphi(g)]=0$ if supports are spacelike separated; polynomials in φ applied to Ω are dense.

Formally $\varphi(f)=\int\varphi(x)\,f(x)\mathrm{d}x$, $\varphi\,(x^0,\mathbf{x}) = \mathrm{e}^{\mathrm{i}Hx^0}\varphi\,(0,\mathbf{x})\mathrm{e}^{-\mathrm{i}Hx^0}$.

Wightman functions $W_n(f_1,\ldots,f_n)=\langle\Omega,\varphi(f_1)\cdots\varphi(f_n)\,\Omega\rangle\in\mathscr{S}'(\mathbb{R}^{dn})$:

- positivity $\sum_{j,k}W_{j+k}\,(g_j^*\otimes g_k)\geqslant 0$; Poincaré invariance;
- spectral condition: Fourier support in $\bar{V}_+^{n-1}$ (difference variables);
- locality; cluster property $\Leftrightarrow$ unique vacuum.

Reconstruction

Theorem. (Wightman reconstruction; Streater–Wightman 1964) *A sequence (W_n) of tempered distributions with $W_0 = 1$ satisfying positivity, invariance, spectral condition and locality is the sequence of Wightman functions of a Wightman theory, unique up to unitary equivalence.*

- To construct a QFT it suffices to construct its Wightman functions.
- **Difficulty:** positivity involves all W_n at once and refers to real time, while what we can construct lives in imaginary time.
- The Euclidean approach replaces it by **reflection positivity**, checkable on lattice approximations and stable under limits.
- The axioms say nothing about dynamics: no Lagrangian; the interaction is hidden in the higher W_n.

The free field

$\omega(\mathbf{p}) = (|\mathbf{p}|^2 + m^2)^{1/2}$, $\mathrm{d}\Omega_m(\mathbf{p}) = \dfrac{\mathrm{d}\mathbf{p}}{(2\pi)^s\, 2\,\omega(\mathbf{p})}$ (invariant measure on $H_m^+ = \{p^2 = m^2, p^0 > 0\}$), Fock space over $L^2(\mathrm{d}\Omega_m)$:

$$\varphi(x) = \int \mathrm{d}\Omega_m(\mathbf{p})\, [a(\mathbf{p})\mathrm{e}^{-\mathrm{i}p\cdot x} + a^*(\mathbf{p})\mathrm{e}^{\mathrm{i}p\cdot x}]_{p^0=\omega(\mathbf{p})},$$

with $[a(\mathbf{p}), a^*(\mathbf{q})] = (2\pi)^s 2\,\omega(\mathbf{p})\ \delta(\mathbf{p}-\mathbf{q})$, $a(\mathbf{p})\,\Omega = 0$. Then

$$W_2(x-y) = \int \mathrm{d}\Omega_m(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}, \qquad \|\varphi(f)\,\Omega\|^2 = \int \mathrm{d}\Omega_m |\tilde{f}|^2 < \infty,$$

but

$$\|\varphi(x)\,\Omega\|^2 = W_2(0) = \int \mathrm{d}\Omega_m = +\infty \qquad (s \geqslant 1).$$

UV divergence: logarithmic for $s = 1$, power-like for $s \geqslant 2$. **Not an artefact of the free field.**

No fields at a point

Proposition. (Wightman 1964) *Assume (W0)–(W2) and* $\varphi(f)\,\Omega = \int f(x)\,U(x,1)\,\psi\,\mathrm{d}x$ *for some* $\psi \in \mathscr{H}$ *("*$\varphi(0)\,\Omega = \psi$ *exists"). Then* $\psi \in \mathbb{C}\,\Omega$*, hence* $\varphi(f)\,\Omega = c\,(\int f)\,\Omega$.

Lemma. *For* $d \geqslant 2$*, a* finite $\mathscr{L}_+^\uparrow$*-invariant measure on* $\bar{V}_+$ *is* $c\,\delta_0$.

- Combined with locality and Reeh–Schlieder: φ is a multiple of 1.
- Quantum fields are **operator-valued distributions**: nonlinear expressions in $\varphi(x)$ need a definition ⇒ renormalisation.

Proofs

Lemma. A boost of rapidity θ acts as $p^0 \pm p^1 \mapsto \mathrm{e}^{\pm\theta}(p^0 \pm p^1)$. The sets

$$S_j^{\pm} = \{p \in \bar{V}_+ : 2^j \leqslant p^0 \pm p^1 < 2^{j+1}\}, \qquad j \in \mathbb{Z},$$

are mapped to each other by the boost with $\mathrm{e}^{\theta} = 2$; they are disjoint and cover $\bar{V}_+ \setminus \{0\}$. Invariance and finiteness force $\mu(S_j^{\pm}) = 0$.

Proposition.

1. Covariance gives $\int f(x)\, U(\Lambda x)\, (U(0,\Lambda)\,\psi - \psi)\mathrm{d}x = 0$ for all f, so $U(0,\Lambda)\,\psi = \psi$.
2. The spectral measure μ of ψ for translations is finite, supported in $\bar{V}_+$ and Lorentz invariant, so $\mu = \|\psi\|^2\,\delta_0$.
3. Hence $U(a)\,\psi = \psi$ for all a, and $\psi \in \mathbb{C}\,\Omega$ by uniqueness of the vacuum.

The Källén–Lehmann representation

Let $W_2^{(M)}$ be the free two-point function of mass $M \geqslant 0$ and $\langle \Omega, \varphi(f)\,\Omega\rangle = v\int f$.

Theorem. (Källén 1952, Lehmann 1954) *Under (W0)–(W2) there is a positive, polynomially bounded measure ρ on $[0,\infty)$ (the **spectral measure** of the field) such that*

$$W_2(x-y) = v^2 + \int_{[0,\infty)} \rho(\mathrm{d}M^2)\; W_2^{(M)}(x-y).$$

($\rho(\{0\}) = 0$ if $d = 2$.)

Sketch. $\psi_f = \varphi(f)\,\Omega - v(\int f)\,\Omega$ gives $W_2 - v^2$ of positive type $\Rightarrow$ (Bochner–Schwartz) Fourier transform of a positive measure μ, supported in $\bar{V}_+$, $\mu(\{0\}) = 0$, Lorentz invariant. Orbits are the hyperboloids H_M^+ (and the cone for $M = 0$), each with a unique invariant measure $\mathrm{d}\Omega_M$. Disintegrate along M.

Reading the spectrum off ρ

- **Free field:** $\rho = \delta_{m^2}$. Conversely $\rho = \delta_{m^2}$, $m > 0$, forces $\varphi - v$ to be free, by locality (Jost–Schroer).
- **Generalised free fields** $\varphi = \sum_i c_i \varphi_{M_i}$ have $\rho = \sum_i c_i^2 \delta_{M_i^2}$ and no interaction: $\rho \neq \delta_{m^2}$ is necessary, not sufficient.
- **Particles:** an isolated atom $Z\,\delta_{m^2}$ is a particle of mass m; multi-particle continuum from $(2m)^2$ (or $(3m)^2$ with $\varphi \mapsto -\varphi$ symmetry). Weakly coupled $P(\varphi)_2$: Glimm–Jaffe–Spencer 1974.
- **No point fields again:** $W_2^T(0) = \int \rho(\mathrm{d}M^2) \int \mathrm{d}\Omega_M = \infty$.
- **Canonical fields:** CCR at sharp times $\Rightarrow \int \rho = 1$, so $Z \leqslant 1$. In $d = 4$ perturbatively $\int \rho = \infty$ (wave-function renormalisation).

The quantum wave equation

The free field solves, as an operator-valued distribution,

$$(\Box + m^2)\,\varphi = 0, \qquad \Box = \partial_t^2 - \Delta_{\mathbf{x}},$$

and its sharp-time fields $\varphi(t,g)$, $\pi(t,g)$, $\pi = \partial_t \varphi$, satisfy the CCR

$$[\varphi(t,\mathbf{x}), \pi(t,\mathbf{y})] = \mathrm{i}\,\delta(\mathbf{x}-\mathbf{y}), \qquad [\varphi,\varphi] = [\pi,\pi] = 0.$$

Picture

A *linear wave equation with quantum initial data* $(\varphi(0,\cdot), \pi(0,\cdot))$, "distributed" according to the vacuum.

Interacting version: $(\Box + m^2)\,\varphi + \lambda\,\varphi^3 = 0$ together with the CCR.

The free field on a circle

$s=1$, space $\mathbb{T}_L=\mathbb{R}/L\mathbb{Z}$, momenta $k\in\frac{2\pi}{L}\mathbb{Z}$, $\omega_k=(k^2+m^2)^{1/2}$:

$$\varphi(0,x)=\sum_k \frac{a_k e^{ikx}+a_k^* e^{-ikx}}{\sqrt{2L\omega_k}}, \qquad [a_k,a_{k'}^*]=\delta_{kk'}, \qquad H_0=\sum_k \omega_k a_k^* a_k.$$

- An infinite family of independent harmonic oscillators, one per mode.
- With a UV cutoff $|k|\leqslant R$:

$$\langle\Omega,\varphi_R(0,x)^2\Omega\rangle=\sum_{|k|\leqslant R}\frac{1}{2L\omega_k}=\frac{1}{2\pi}\log\frac{R}{m}+O(1)\to\infty.$$

- Wick ordering $:\varphi_R^2:=\varphi_R^2-\langle\Omega,\varphi_R^2\Omega\rangle$ removes exactly this divergence (Exercise 1.7).

Interactions: three problems

1. **Products of distributions.** $\varphi(x)^3$ is meaningless and must be renormalised: Wick ordering suffices in $d=2$; more counterterms in $d=3$; triviality in $d=4$.
2. **Haag's theorem.** In infinite volume the interacting field at fixed time is not unitarily equivalent to the free one: there is no interaction picture.
3. **Semiboundedness.** Even on $\mathbb{T}_L$, $H=H_0+\lambda\int{:}\varphi\,(0,x)^4{:}\,\mathrm{d}x$ is positive plus unbounded below. Nelson 1966: bounded below (hypercontractivity). Volume cutoff removed by Glimm–Jaffe; Wightman axioms for weak coupling $P(\varphi)_2$ by Glimm–Jaffe–Spencer 1974.

The Euclidean approach (Symanzik, Nelson, Osterwalder–Schrader, Guerra–Rosen–Simon) simplifies this enormously.

From real to imaginary time

Lecture 1

A lattice Hamiltonian on the circle

Lattice $\Lambda_\varepsilon = \varepsilon\,\mathbb{Z}/L\,\mathbb{Z}$, $N = L/\varepsilon$ sites, $\mathscr{H}_\varepsilon = L^2(\mathbb{R}^{\Lambda_\varepsilon}, \mathrm{d}\varphi)$, φ_x multiplication, $\pi_x = -\mathrm{i}\varepsilon^{-1}\partial_{\varphi_x}$, so $[\varphi_x, \pi_y] = \mathrm{i}\varepsilon^{-1}\mathbb{1}_{x=y}$.

$$H_\varepsilon = \sum_x \varepsilon\left[\tfrac{1}{2}\pi_x^2 + \tfrac{1}{2}(\nabla_\varepsilon\varphi)_x^2 + \tfrac{1}{2}m^2\varphi_x^2 + V(\varphi_x)\right] = -\frac{1}{2\varepsilon}\Delta_{\mathbb{R}^N} + U_\varepsilon(\varphi).$$

- Heisenberg fields solve the **lattice wave equation**

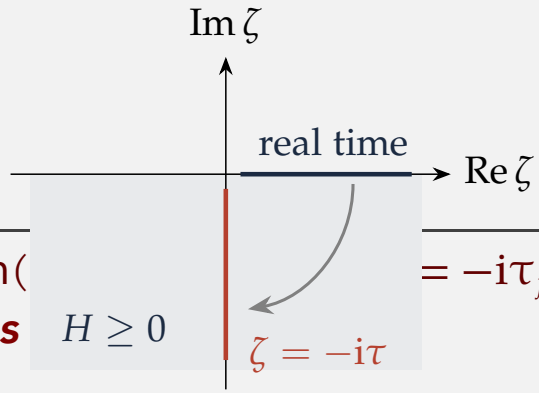
$$\partial_t^2\varphi_x = (\Delta_\varepsilon\varphi)_x - m^2\varphi_x - V'(\varphi_x)$$

  — the quantum wave equation, now rigorous.
- Schrödinger operator with confining potential: self-adjoint, compact resolvent, $\mathrm{e}^{-tH_\varepsilon}$ trace class with positive kernel.
- Perron–Frobenius: simple ground state $\Omega_\varepsilon > 0$, energy E_0. $\bar{H} := H_\varepsilon - E_0 \geqslant 0$.

Positivity of the energy ⇒ Wick rotation

For bounded F_j and $F_j(t) = e^{itH} F_j e^{-itH}$:

$$W(t_1, \dots, t_n) = \langle \Omega_\varepsilon, F_1 e^{-i(t_1 - t_2)\tilde{H}} F_2 \cdots F_n \Omega_\varepsilon \rangle.$$



Proposition. *W extends analytically to* $\mathrm{Im}(\quad) = -i\tau_j$, $\tau_1 \geqslant \dots \geqslant \tau_n$, *it gives the* ***Schwinger functions***

$$S(\tau_1, \dots, \tau_n) = \langle \Omega_\varepsilon, F_1 e^{-(\tau_1 - \tau_2)\tilde{H}} F_2 \cdots F_n \Omega_\varepsilon \rangle,$$

which determine W.

Where positivity of the energy is used

1. **Analyticity.** $\mathrm{e}^{-\mathrm{i}\zeta\bar{H}} = \int \mathrm{e}^{-\mathrm{i}\zeta\lambda}\mathrm{d}E(\lambda)$ is a contraction for $\mathrm{Im}\zeta \leqslant 0$, since $\lambda \geqslant 0$: analytic in the interior, continuous up to the boundary.
2. **Uniqueness.** An analytic function on the half plane vanishing on $-\mathrm{i}\mathbb{R}_+$ vanishes; iterate variable by variable.
3. **Semigroup.** $\mathrm{e}^{-\tau\bar{H}}$ is a contraction semigroup, which can be represented as a path integral (Feynman–Kac).
4. **Projection onto the vacuum.** $\mathrm{e}^{-T\bar{H}} \to |\Omega\rangle\langle\Omega|$ as $T \to \infty$.

Relativistic version: Wightman functions are boundary values of functions analytic in the forward tube; Lorentz covariance extends them to all non-coincident Euclidean points (Schwinger functions are *symmetric* and *Euclidean invariant*).

Discretising imaginary time

Kinetic semigroup: $e^{\frac{\delta}{2\varepsilon}\Delta}(\varphi,\varphi') = (2\pi\delta/\varepsilon)^{-N/2}\exp\left(-\frac{\varepsilon}{2\delta}|\varphi-\varphi'|^2\right)$.

Transfer matrix (symmetric Trotter step):

$$T_\delta = e^{-\delta U/2} e^{\frac{\delta}{2\varepsilon}\Delta} e^{-\delta U/2}, \qquad T_\delta(\varphi,\varphi') \propto \exp\left(-\frac{\varepsilon}{2\delta}\sum_x (\varphi_x-\varphi'_x)^2 - \frac{\delta}{2}\left(U(\varphi)+U(\varphi')\right)\right),$$

and $T_\delta^{\lfloor \tau/\delta \rfloor} \to e^{-\tau H_\varepsilon}$ (Trotter).

Space-time lattice $\Lambda = (\delta\mathbb{Z}/T\mathbb{Z}) \times (\varepsilon\mathbb{Z}/L\mathbb{Z})$, $T = J\delta$:

$$\mathrm{Tr}(T_\delta^J) \propto \int e^{-S_{\delta,\varepsilon}(\varphi)} \prod_{u\in\Lambda} \mathrm{d}\varphi_u,$$

$$S_{\delta,\varepsilon}(\varphi) = \sum_{(\tau,x)} \varepsilon\,\delta\left[\tfrac{1}{2}(\partial_\tau^\delta\varphi)^2 + \tfrac{1}{2}(\nabla_\varepsilon\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 + V(\varphi)\right].$$

The Minkowski metric has become Euclidean.

Feynman–Kac–Nelson

Lattice Gibbs measure $\mu_{\delta,\varepsilon}^{T,L}(\mathrm{d}\varphi) = Z^{-1} \mathrm{e}^{-S_{\delta,\varepsilon}(\varphi)} \prod_u \mathrm{d}\varphi_u$:

$$\frac{\mathrm{Tr}(F_1 T_\delta^{k_1} F_2 \cdots F_n T_\delta^{k_n})}{\mathrm{Tr}(T_\delta^{J})} = \int F_1(\varphi_{\tau_1}) \cdots F_n(\varphi_{\tau_n}) \; \mu_{\delta,\varepsilon}^{T,L}(\mathrm{d}\varphi).$$

Proposition. $\langle \Omega_\varepsilon, \; F_1 \mathrm{e}^{-(\tau_1-\tau_2)\bar{H}} F_2 \; \cdots \; F_n \; \Omega_\varepsilon \rangle = \lim_{T\to\infty} \lim_{\delta\to 0} \int F_1(\varphi_{\tau_1}) \cdots F_n(\varphi_{\tau_n}) \, \mathrm{d}\mu_{\delta,\varepsilon}^{T,L}$.

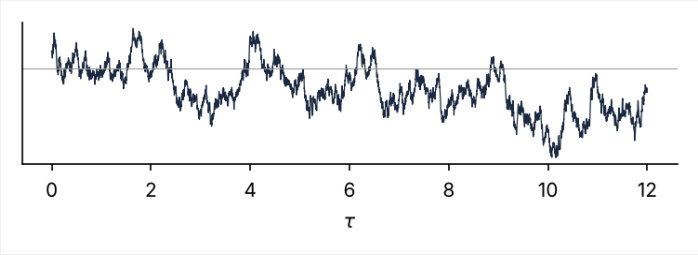
Sketch. Gaussian factor = Brownian loops with diffusion ε^{-1}; $\delta \sum_\tau U(\varphi_\tau)$ is a Riemann sum for $\int_0^T U(\omega_\tau)\mathrm{d}\tau$; then $T \to \infty$ uses the spectral gap.

Vacuum expectations in imaginary time are moments of a classical probability measure.

Example: one oscillator and the Ornstein–Uhlenbeck process

$H = -\frac{1}{2}\,\partial_q^2 + \frac{1}{2}\,m^2\,q^2$, $\Omega(q) \propto \mathrm{e}^{-mq^2/2}$, $E_0 = \frac{m}{2}$.

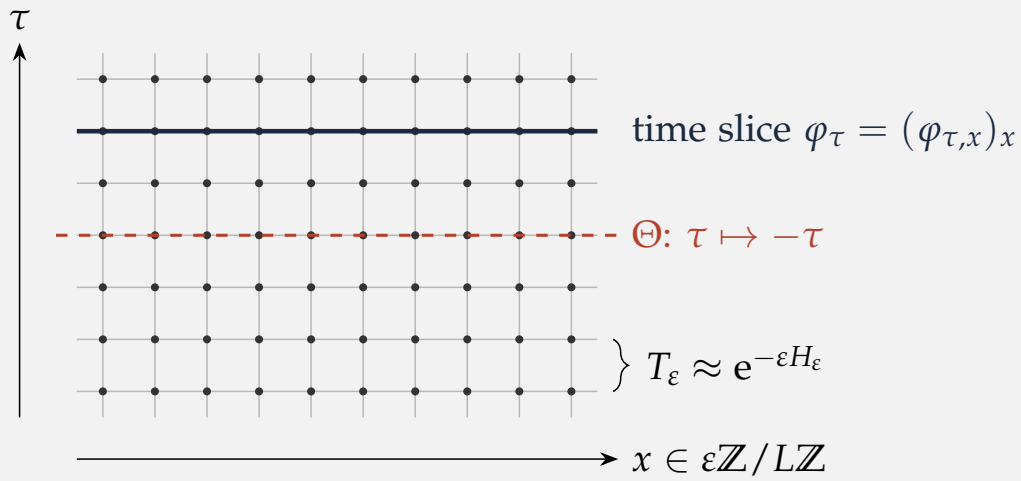
- Ground-state transform: $\Omega^{-1}\bar{H}\,\Omega = -\frac{1}{2}\partial_q^2 + m\,q\,\partial_q$, minus the generator of $\mathrm{d}X = -m\,X\mathrm{d}t + \mathrm{d}B$.
- Schwinger function $\langle \Omega,\ q\,\mathrm{e}^{-\tau\bar{H}} q\ \Omega\rangle = \dfrac{\mathrm{e}^{-m\tau}}{2\,m}$.
- This is $(-\partial_\tau^2 + m^2)^{-1}$: the **GFF in $d = 1$**.



A sample of the stationary OU process, $m = 1$.

Notes, Exercise 2.2

The space-time lattice



Rows are quantum configurations at fixed imaginary time; the transfer matrix connects consecutive rows. As $T \to \infty$, the rows form a stationary Markov process whose one-time marginal is $\Omega_\varepsilon(\varphi)^2 \mathrm{d}\varphi$.

Classical statistical mechanics

Isotropic lattice $\delta = \varepsilon$:

$$S_\varepsilon(\varphi) = \sum_{\langle u,v\rangle} \tfrac{1}{2}(\varphi_u - \varphi_v)^2 + \varepsilon^2 \sum_u \left[\tfrac{1}{2}m^2\varphi_u^2 + V(\varphi_u)\right].$$

- $\mu_\varepsilon \propto \prod_{\langle u,v\rangle} \mathrm{e}^{-\frac{1}{2}(\varphi_u-\varphi_v)^2} \prod_u \nu_\varepsilon(\mathrm{d}\varphi_u), \qquad \nu_\varepsilon(\mathrm{d}a) = \mathrm{e}^{-\varepsilon^2(\frac{1}{2}m^2a^2 + V(a))}\mathrm{d}a.$
- A **ferromagnetic nearest-neighbour spin system** with unbounded spins: short-range classical statistical mechanics.
- In dimension d: $\varepsilon^{d-2}\sum_{\langle u,v\rangle}$ and $\varepsilon^d \sum_u$.
- Nearest-neighbour ⇒ **reflection positive**: the Euclidean shadow of $\mathrm{e}^{-\varepsilon H} \geqslant 0$. Conversely, RP ⇒ Hilbert space and $H \geqslant 0$ (Exercises 2.5, 2.7).

A dictionary

Quantum (real time)	Euclidean field theory	Statistical mechanics
Hilbert space positivity, $H \geqslant 0$	reflection positivity	RP of n.n. interactions
$e^{-\varepsilon H}$	time translation by ε	transfer matrix
vacuum Ω	infinite-volume measure	infinite-volume Gibbs state
inverse temperature β	length of the time circle	—
mass gap $E_1 - E_0$	exponential decay	inverse correlation length
continuum limit	UV cutoff $\varepsilon \to 0$	critical point, scaling limit
renormalisation	counterterms	tuning to criticality

Continuum limit = critical point

- In lattice units the mass is $\varepsilon\, m \to 0$: correlation length $\xi \approx (\varepsilon\, m)^{-1} \to \infty$ lattice spacings.
- Continuum QFT = **scaling limit of short-range spin systems approaching criticality** (Wilson). Conversely, RP critical models with non-trivial scaling limits give candidate QFTs.
- Griffiths–Simon: φ^4 spins as limits of normalised sums of ferromagnetically coupled Ising spins; critical 4D Ising and φ^4_4 have Gaussian scaling limits (ADC 2021).

Renormalisation already appears

For $V = \lambda a^4$ and fixed (m, λ) the limit is trivial: $c_\varepsilon = \mathbb{E}[\varphi_u^2] = \frac{1}{2\pi} \log \frac{1}{\varepsilon\, m} + O(1) \to \infty$. Take $V_\varepsilon(a) = \lambda\,(a^4 - 6\, c_\varepsilon\, a^2)$, i.e. $\lambda{:}\, a^4$: up to a constant. Enough in $d = 2$; not in $d = 3$.

Path integrals

Formally, as $\varepsilon \to 0$ and $L, T \to \infty$:

$$\mu(\mathrm{d}\varphi) \text{``}=\text{''} Z^{-1} \exp\left(-\int_{\mathbb{R}^d} \left[\tfrac{1}{2}|\nabla\varphi|^2 + \tfrac{1}{2}m^2\varphi^2 + V(\varphi)\right]\mathrm{d}x\right) \mathscr{D}\varphi.$$

- Neither $\mathscr{D}\varphi$ nor the action make sense: typical φ are distributions.
- The Gaussian part defines the **Gaussian free field** μ_m (next lecture).
- Then $\mu \propto \mathrm{e}^{-\int V_{\mathrm{ren}}(\varphi)}\mu_m$: meaningful in finite volume for $d = 2$ (and, with more work, $d = 3$).

Question. Which measures on $\mathscr{S}'(\mathbb{R}^d)$ correspond to relativistic QFTs?

Osterwalder–Schrader axioms

$\hat{\mu}(f) = \int e^{i\varphi(f)} \mu(d\varphi), \quad \Theta(\tau, \mathbf{x}) = (-\tau, \mathbf{x}).$

OS0. *Analyticity:* $z \mapsto \hat{\mu}(\sum_j z_j f_j)$ entire.

OS1. *Regularity:* $|\hat{\mu}(f)| \leqslant \exp\left(c\left(\|f\|_{L^1} + \|f\|_{L^p}^p\right)\right)$, $p \in [1,2]$, complex f.

OS2. *Euclidean invariance.*

OS3. *Reflection positivity:* for real f_i supported in $\{\tau > 0\}$, $(\hat{\mu}(f_i - \Theta f_j))_{ij} \geqslant 0$.

OS4. *Ergodicity* under time translations.

Theorem. (Osterwalder–Schrader 1973/75) *OS0–OS3 ⇒ the moments of μ are the Schwinger functions of a unique Wightman theory; OS4 ⇔ unique vacuum.*

Reconstruction: the idea

- $\mathscr{E}_+ =$ functions of $\varphi|_{\{\tau>0\}}$, with $(F,G) = \mathbb{E}_\mu[\overline{\Theta F}\, G] \geqslant 0$ by OS3.
- Quotient by null vectors and complete: physical Hilbert space $\mathscr{H}$.
- Time translations by $t \geqslant 0$ preserve $\mathscr{E}_+$ and descend to a self-adjoint contraction semigroup e^{-tH}: **positivity of the energy is recovered from reflection positivity**.
- Vacuum $=$ class of 1; analytic continuation $\tau \to \mathrm{i}t$ gives the Wightman functions.

The Euclidean strategy

(1) regularise (lattice, torus); (2) renormalise; (3) uniform estimates, remove cutoffs; (4) verify OS axioms (RP survives limits, Euclidean invariance recovered, regularity, clustering); (5) prove non-Gaussianity.

Exercises and reading

Exercises (numbering of the notes)

- 1.3 Which restrictions of the free field make sense?
- 1.5–1.6 Canonical sum rule; generalised free fields.
- 2.2 Harmonic oscillator and Ornstein–Uhlenbeck.
- 2.4 Correlation length and criticality.
- 2.5, 2.7 Reflection positivity on the lattice; toy OS reconstruction.

Next lecture: the Gaussian free field.

Reading

- Streater–Wightman, *PCT, Spin and Statistics, and All That*.
- Glimm–Jaffe, *Quantum Physics*, Euclidean chapters.
- Simon, *Functional Integration and Quantum Physics*.
- Osterwalder–Schrader, CMP 31 (1973), 42 (1975).