A Constructive Introduction to Quantum Field Theory

The Gaussian free field

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SISSA lectures, Lecture 2 — notes: Lecture 2, §3

Where we are

Last time. Positivity of the energy ⇒ imaginary time; vacuum expectations are moments of $\mu(\mathrm{d}\varphi)$ ``="$Z^{-1}\mathrm{e}^{-\int[\frac{1}{2}|\nabla\varphi|^2+\frac{1}{2}m^2\varphi^2+V(\varphi)]}\mathscr{D}\varphi$; relativistic QFTs ↔ measures on $\mathscr{S}'(\mathbb{R}^d)$ satisfying the OS axioms.

Today. The Gaussian part, rigorously.

1. Construction on $\mathbb{T}^d$ and $\mathbb{R}^d$; regularity.
2. Reflection positivity; the free field; Källén–Lehmann again.
3. Local operators: Wick powers.
4. Gaussian integration by parts: the Dyson–Schwinger (DS) equation.
5. **DS characterises the GFF.**

Definition and construction

Lecture 2

Setting and definition

$\mathscr{M} = \mathbb{T}_L^d = (\mathbb{R} / L\,\mathbb{Z})^d$ or $\mathbb{R}^d$; $\mathscr{S}(\mathscr{M})$ smooth, resp. Schwartz, functions; $\mathscr{S}'(\mathscr{M})$ with the cylinder (= Borel) σ-algebra.

$$A := m^2 - \Delta, \qquad C := A^{-1}, \qquad m > 0.$$

A, C are continuous bijections of $\mathscr{S}(\mathscr{M})$ (Fourier multipliers $(|k|^2 + m^2)^{\pm 1}$), and $\|C\|_{L^2 \to L^2} \leqslant m^{-2}$.

Definition. *The **Gaussian free field** μ_m is the centred Gaussian measure on $\mathscr{S}'(\mathscr{M})$ with*

$$\int \varphi(f)\,\varphi(g)\;\mu_m(\mathrm{d}\varphi) = \langle f, Cg\rangle, \qquad \textit{i.e.} \qquad \int \mathrm{e}^{\mathrm{i}\varphi(f)}\mu_m(\mathrm{d}\varphi) = \mathrm{e}^{-\frac{1}{2}\langle f, Cf\rangle}.$$

Formally $\mu_m(\mathrm{d}\varphi)$ ``$=$'' $Z^{-1}\mathrm{e}^{-\frac{1}{2}\langle\varphi, A\varphi\rangle}\,\mathscr{D}\,\varphi$: the Gaussian part of the Euclidean action.

The Green function

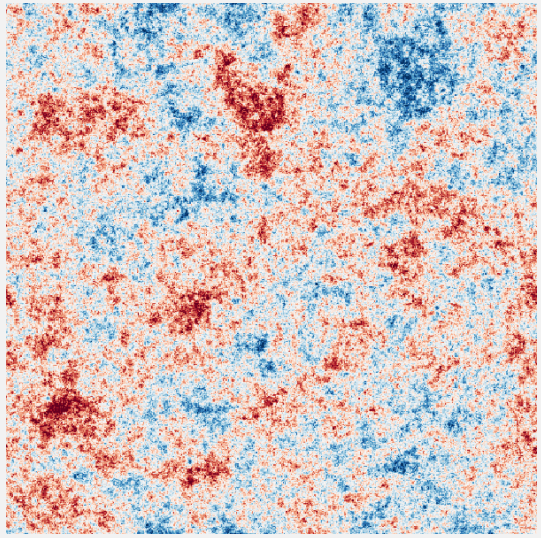
C has kernel G:

$$G_m(x) = \int \frac{e^{ik\cdot x}}{|k|^2 + m^2} \frac{dk}{(2\pi)^d}$$

on $\mathbb{R}^d$, and $\sum_{n\in\mathbb{Z}^d} G_m(x + nL)$ on $\mathbb{T}_L^d$.

- $d = 1$: $G_m(x) = e^{-m|x|}/(2m)$.
- $d = 2$: $G_m(x) \sim \frac{1}{2\pi} \log \frac{1}{|x|}$.
- $d \geqslant 3$: $G_m(x) \sim \frac{\Gamma(d/2-1)}{4\pi^{d/2}} |x|^{2-d}$.
- Decay $\approx e^{-m|x|}$ at infinity.

Singular on the diagonal for $d \geqslant 2$: the field is not a function.



A sample of the massive GFF on $\mathbb{T}^2$ ($m = 4$, 512^2 Fourier modes).

Existence on the torus

Theorem. *Let (e_n) be a real orthonormal basis of eigenfunctions of $-\Delta$ on $\mathbb{T}^d_L$, $-\Delta e_n = \kappa_n e_n$, $\lambda_n = m^2 + \kappa_n$, and (g_n) i.i.d. $N(0,1)$. For $s > \frac{d}{2} - 1$,*

$$\varphi = \sum_n \lambda_n^{-1/2} g_n \, e_n$$

converges a.s. and in $L^2(\mathbb{P}; H^{-s})$, and $\mathsf{Law}(\varphi) = \mu_m$.

Proof.

$$\mathbb{E}\|\varphi\|^2_{H^{-s}} = \sum_n \frac{1}{(1+\kappa_n)^s \lambda_n} \asymp \sum_{k \in \mathbb{Z}^d} (1+|k|^2)^{-s-1} < \infty \iff 2s+2 > d.$$

L^2-bounded H^{-s}-valued martingale $\Rightarrow$ convergence. And $\varphi(f) = \sum_n \lambda_n^{-1/2} g_n \langle e_n, f \rangle$ is Gaussian with variance $\sum_n \lambda_n^{-1} \langle e_n, f \rangle^2 = \langle f, Cf \rangle$.

Existence on $\mathbb{R}^d$ and regularity

Bochner–Minlos

$\Psi\colon \mathscr{S}(\mathbb{R}^d) \to \mathbb{C}$ continuous, positive definite, $\Psi(0) = 1 \Rightarrow$ there is a unique probability measure on $\mathscr{S}'(\mathbb{R}^d)$ with characteristic functional Ψ.

For $\Psi(f) = \mathrm{e}^{-\frac{1}{2}\langle f, Cf\rangle}$: continuity from $\langle f, Cf\rangle \leqslant m^{-2}\|f\|_{L^2}^2$; positive definiteness from $\Psi(f_j - f_k) = a_j a_k \mathrm{e}^{\langle f_j, Cf_k\rangle}$ and the Schur product theorem.

Regularity (sharp, Exercise 3.3):

$d=1$	$d=2$	$d=3$
stationary OU process	$H^{-\kappa}$, $\forall \kappa > 0$; not a function	$H^{-1/2-\kappa}$

The Cameron–Martin space H^1 (finite action) has **measure zero**.

Reflection positivity and the free field

Lecture 2

Reflection positivity

$\mathscr{M} = \mathbb{R} \times N$, $N = \mathbb{R}^{d-1}$ or $\mathbb{T}_L^{d-1}$; $\Theta(\tau, \mathbf{x}) = (-\tau, \mathbf{x})$; $A = -\partial_\tau^2 + B$, $B = m^2 - \Delta_N$, $\omega = B^{1/2}$.

Proposition. *(a)* $C(\tau, \tau') = \mathrm{e}^{-|\tau - \tau'|\omega}(2\omega)^{-1}$ *(operator-valued kernel).*
(b) If $\operatorname{supp} f \subset \{\tau > 0\}$*:*

$$\langle \Theta f, C f \rangle = \left\| (2\omega)^{-1/2} \int_0^\infty \mathrm{e}^{-\tau\omega} f(\tau, \cdot)\, \mathrm{d}\tau \right\|_{L^2(N)}^2 \geqslant 0.$$

(c) μ_m *on* $\mathbb{R}^d$ *satisfies OS0–OS4.*

Proof. (a) $\int \mathrm{e}^{\mathrm{i}kt}(k^2 + w^2)^{-1} \frac{\mathrm{d}k}{2\pi} = \frac{\mathrm{e}^{-w|t|}}{2w}$ and spectral calculus. (b) For $\tau < 0 < \tau'$, $|\tau - \tau'| = |\tau| + \tau'$: the kernel factorises. (c) OS3: $\hat{\mu}(f_i - \Theta f_j) = a_i a_j \mathrm{e}^{\langle \Theta f_i, C f_j \rangle}$; Schur.

The free field, reconstructed

- OS reconstruction of μ_m gives the **free field of mass m**: Fock space over $L^2(N)$, $H = \mathrm{d}\Gamma(\omega)$ (Nelson's free Markov field).
- The **time-zero field** $\varphi(0,\cdot)$ exists and is Gaussian with covariance $(2\omega)^{-1}$: the ground state measure $\Omega_0^2\,\mathscr{D}\varphi$ of the free Hamiltonian in the Schrödinger representation.
- The GFF is the continuum limit of the lattice spin system of Lecture 1 with $V = 0$ (Exercise 3.10).
- Markov property (Nelson): rooted in the **locality of A**.

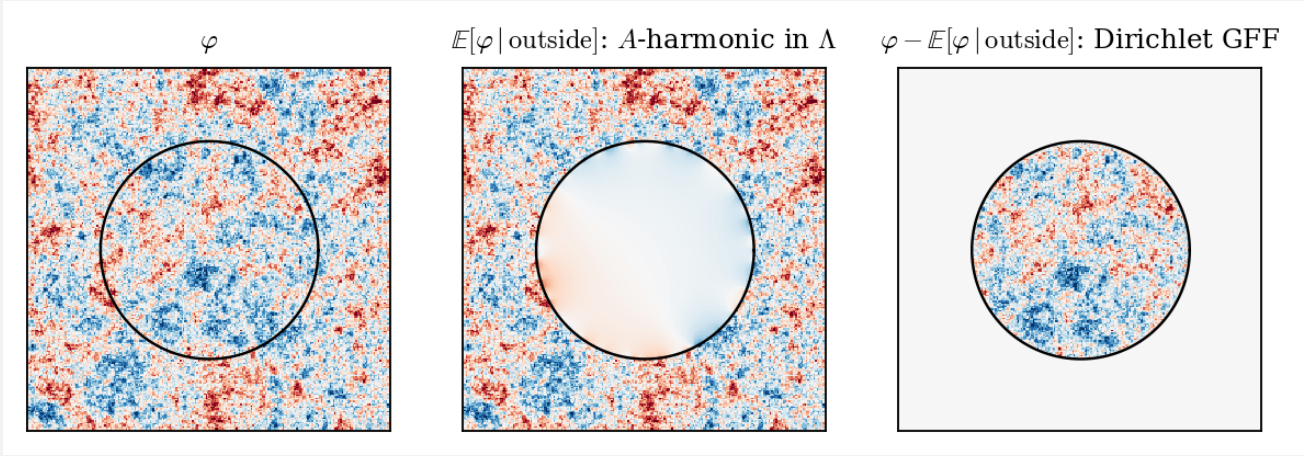
Nelson's Markov property

- **Markov** for Λ: inside and outside are conditionally independent given the field near $\partial\Lambda$.
- **The GFF is Markov for every open Λ** (Nelson 1973): $\mathbb{E}[\varphi(f)\,|\,\text{outside}] = \varphi(\Pi f)$, and $\Pi f = Au$ is supported on $\partial\Lambda$ *because* $A = m^2 - \Delta$ *is local*. Given the outside: harmonic extension + independent Dirichlet GFF (next slide).
- **Markov + sharp $\Sigma_0 \Rightarrow$ RP:** with Σ_0 the time-zero field, μ Θ-invariant, Θ trivial on Σ_0:

$$\mathbb{E}[\overline{\Theta F} F] = \mathbb{E}[|\mathbb{E}[F\,|\,\Sigma_0]|^2] \geqslant 0, \qquad \mathbb{E}[U_t \cdot |\,\Sigma_0] = \mathrm{e}^{-tH} \text{ on } L^2(\Sigma_0).$$

  Nelson's reconstruction (1973), before Osterwalder–Schrader.
- **Relevance:** preserved by local perturbations $\mathrm{e}^{-\int_\Lambda V} \Rightarrow P(\varphi)_2$ as classical statistical mechanics (Guerra–Rosen–Simon). Not stable under limits; OS use RP instead: not comparable (generalised free fields are RP, not Markov; the P^{-1} field is Markov, not RP), but closed under weak limits.

The Markov property in a picture



Lattice GFF, $m^2 \approx 10^{-3}$, Λ a disc. Inside Λ,

$$\varphi = h + \psi, \qquad (m^2 - \Delta)\, h = 0 \text{ in } \Lambda, \quad h = \varphi \text{ on } \partial\Lambda,$$

$h = \mathbb{E}[\varphi\,|\,\text{outside}]$ depends only on the field **at the boundary**; ψ is a Dirichlet GFF, **independent of the outside**.

Symanzik: fields and Brownian paths

- Symanzik (1966, 1969): QFT through Schwinger functions as moments of a classical Markov field — the Euclidean programme.
- **Paths:** $C(x,y) = \int_0^\infty \mathrm{e}^{-m^2 t} p_t(x,y)\,\mathrm{d}t$ is the Green function of a killed Brownian motion. For $\mathrm{e}^{-\int V}\mu_m$: a gas of paths and loops interacting through local times; for φ^4, intersections are penalised (lattice: Brydges–Fröhlich–Spencer 1982; triviality for $d>4$).
- **Loop soups** (Lawler–Werner 2004): Poisson cloud of Brownian loops of intensity α; on a graph $\mathbb{E}\,\mathrm{e}^{-\langle \chi, L\rangle} = \det(1+C\chi)^{-\alpha}$.
- **Le Jan's isomorphism:** for $\alpha = \frac{1}{2}$ the occupation field is $\frac{1}{2}\varphi^2$; in $d=2,3$ the renormalised one is $\frac{1}{2}{:}\varphi^2{:}$.
- Lupu 2016: loop clusters = sign clusters of the cable-graph GFF. Werner 2016: spatial Markov property of loop soups. Qian–Werner 2019. Survey: Werner–Powell.

Euclidean Källén–Lehmann

Wick rotation of $W_2^{(M)}$: $\int \mathrm{d}\Omega_M(\mathbf{p})\, \mathrm{e}^{-\omega_M(\mathbf{p})|\tau|+\mathrm{i}\mathbf{p}\cdot\mathbf{x}} = G_M(\tau, \mathbf{x})$. Hence for any theory satisfying the axioms

$$S_2(x) = v^2 + \int_{[0,\infty)} \rho(\mathrm{d}M^2)\; G_M(x), \qquad \hat{S}_2^T(k) = \int \frac{\rho(\mathrm{d}M^2)}{|k|^2 + M^2}.$$

The GFF is $\rho = \delta_{m^2}$.

- **Short distances.** G_M decreases in M and $G_M / G_m \to 1$ at 0 ($d \geqslant 2$), so $\operatorname{liminf}_{x\to 0} S_2^T(x) / G_m(x) \geqslant \rho([0,\infty)) > 0$: never less singular than free. Scale invariance $\Rightarrow$ unitarity bound $\Delta \geqslant \frac{d-2}{2}$ (Exercise 3.2).
- **Long distances.** $\operatorname{supp}\rho \subset [m_0^2, \infty) \Leftrightarrow$ decay at least like $\mathrm{e}^{-m_0|x|}$: mass gap = clustering.

Two masses: a positive but nonlocal theory

$\rho = c_1^2\,\delta_{m_1^2} + c_2^2\,\delta_{m_2^2}$, $0 < m_1 < m_2$: $C_\rho = c_1^2\,(m_1^2 - \Delta)^{-1} + c_2^2\,(m_2^2 - \Delta)^{-1}$, the law of $c_1\,\varphi_1 + c_2\,\varphi_2$ (independent GFFs). RP for each term $\Rightarrow$ OS axioms.

It reconstructs to two free species

$\mathscr{H} = \Gamma(\mathfrak{h}_{m_1}) \otimes \Gamma(\mathfrak{h}_{m_2})$, $\varphi = c_1\,\varphi_{m_1} \otimes 1 + c_2\,1 \otimes \varphi_{m_2}$, $W_2 = c_1^2\,W_2^{(m_1)} + c_2^2\,W_2^{(m_2)}$. Two kinds of free particles, created by the same field; no scattering (truncated $W_{n\geqslant 3} = 0$).

What is lost:

- $(\Box + m_1^2)\,(\Box + m_2^2)\,\varphi = 0$, but no Klein–Gordon equation; $(\varphi(0), \pi(0))$ are not complete Cauchy data.
- DS: $\mathbb{E}\,[D_h F] = \mathbb{E}\,[F\,\varphi\,(C_\rho^{-1} h)]$ with $C_\rho^{-1} = \dfrac{(m_1^2 - \Delta)\,(m_2^2 - \Delta)}{c_1^2\,(m_2^2 - \Delta) + c_2^2\,(m_1^2 - \Delta)}$ **nonlocal**; φ is not Markov.

Positivity versus locality

The **local** fourth-order action $\frac{1}{2}\langle \varphi, (m_1^2 - \Delta)\,(m_2^2 - \Delta)\,\varphi \rangle$ gives a Markov field with

$$\frac{1}{(|k|^2 + m_1^2)\,(|k|^2 + m_2^2)} = \frac{1}{m_2^2 - m_1^2}\left(\frac{1}{|k|^2 + m_1^2} - \frac{1}{|k|^2 + m_2^2}\right),$$

so $\rho = (\delta_{m_1^2} - \delta_{m_2^2}) \,/\, (m_2^2 - m_1^2)$ has a **negative weight**.

- By Euclidean Källén–Lehmann it is **not reflection positive**: the mass m_2 is a ghost of negative norm (Pais–Uhlenbeck, Lee–Wick).
- Same for any $1/\hat{C} =$ polynomial of degree $\geqslant 2$: $s\,\hat{C}(s) \to 0$, while $s \int \frac{\rho(\mathrm{d}M^2)}{s + M^2} \uparrow \rho([0,\infty))$.

Moral

For Gaussian fields, *positivity* (OS) and *locality* of the Euclidean field equation (C^{-1} a differential operator) together single out the free field: covariance $c\,(m^2 - \Delta)^{-1}$ (or white noise).

Is the nonlocal theory physical?

- **In Minkowski space, yes:** all Wightman axioms hold, including locality (W3); positive energy, unique vacuum. Two free species: trivial and redundant, not inconsistent.
- **``Nonlocal'' is Euclidean:** no Markov property $\Leftrightarrow$ φ alone is not a complete set of local degrees of freedom (no Cauchy problem, no Lagrangian in φ only). Cure: the pair (φ_1, φ_2) is local and Markov.
- **Continuous ρ** (a continuum of free species) is unphysical: no dynamical law of finite order, and too many localised states. Buchholz–Wichmann nuclearity (cf. Haag–Swieca) needs $\sum_i n_i e^{-\beta m_i} < \infty$: no sensible thermodynamics.
- **Interacting theories** also have a continuum in ρ, but produced by a *local* interaction: the Euclidean field equation of Φ^4_2, with $(m^2 - \Delta)\,\varphi + 4\,\lambda : \varphi^3 :$ in the DS equation, is local.

Local operators: Wick products

Lecture 2

Hermite polynomials

Interactions $\int V(\varphi(x))\mathrm{d}x$ need local functions of $\varphi(x)$: regularise, subtract, remove the regularisation.

$$\mathrm{e}^{ta-ct^2/2}=\sum_{n\geqslant 0}\frac{t^n}{n!}H_n(a;c):\qquad H_2=a^2-c, H_3=a^3-3ca, H_4=a^4-6ca^2+3c^2.$$

Lemma. *(X,Y) centred Gaussian, $\mathbb{E}\,X^2=a$, $\mathbb{E}\,Y^2=b$:*

$$\mathbb{E}\left[H_n(X;a)\,H_k(Y;b)\right]=\delta_{nk}\,n!\,(\mathbb{E}\left[XY\right])^n.$$

Proof. $\mathbb{E}\big[\mathrm{e}^{sX-s^2a/2}\mathrm{e}^{tY-t^2b/2}\big]=\mathrm{e}^{st\,\mathbb{E}[XY]}$; compare coefficients of $s^n t^k$.

Wick powers

Mollifier η_ε, $\varphi_\varepsilon = \eta_\varepsilon * \varphi$, $G_{\varepsilon,\delta} = \eta_\varepsilon * \eta_\delta * G$, $c_\varepsilon = G_{\varepsilon,\varepsilon}(0)$,

$$:\varphi_\varepsilon^n:(x) := H_n(\varphi_\varepsilon(x); c_\varepsilon), \qquad :\varphi_\varepsilon^n:(f) = \int f :\varphi_\varepsilon^n:.$$

Theorem. *If $n(d-2) < d$ (all n in $d=2$; $n \leqslant 2$ in $d=3$), then $:\varphi^n:(f) = \lim_{\varepsilon\to 0} :\varphi_\varepsilon^n:(f)$ exists in $L^2(\mu_m)$, is independent of η, and*

$$\mathbb{E}\left[:\varphi^n:(f):\varphi^k:(g)\right] = \delta_{nk}\, n! \iint f(x)\, G(x-y)^n g(y)\, \mathrm{d}x\, \mathrm{d}y.$$

If $n(d-2) \geqslant d$ and $0 \leqslant f \not\equiv 0$: $\mathbb{E}[:\varphi_\varepsilon^n:(f)^2] \to \infty$.

Proof

1. **Uniform bound.** $K(z) = 1 + \log_+ \frac{1}{|z|}$ $(d=2)$, $1 + |z|^{2-d}$ $(d \geqslant 3)$. With $r = \max(\varepsilon, \delta)$:
   - $|z| \geqslant 4r$: $G_{\varepsilon,\delta}(z) \leqslant \sup_{|w| \geqslant |z|/2} G(w) \leqslant C\,K(z)$;
   - $|z| < 4r$: $G_{\varepsilon,\delta}(z) \leqslant \|\eta\|_\infty\, r^{-d} \int_{B(0,6r)} G \leqslant C \frac{1}{|B(0,6r)|} \int_{B(0,6r)} K \leqslant C\,K(r) \leqslant C'\,K(z)$.
2. **Convergence.** By the lemma,

$$\mathbb{E}\big[(:\varphi^n_\varepsilon:(f) - :\varphi^n_\delta:(f))^2\big] = n! \iint f(x)\, f(y)\, (G^n_{\varepsilon,\varepsilon} - 2\,G^n_{\varepsilon,\delta} + G^n_{\delta,\delta})\,(x-y) \to 0$$

by dominated convergence: $K^n \in L^1_{\mathrm{loc}} \iff n\,(d-2) < d$.

3. **Divergence.** Otherwise $G^n \gtrsim |z|^{-n(d-2)} \notin L^1_{\mathrm{loc}}$; Fatou.

Remarks

- **Renormalisation:** $\varphi_\varepsilon(x)^n$ has no limit; subtracting $c_\varepsilon \to \infty$ (via H_n) is the simplest counterterm. A different mollifier changes $:\varphi^n:$ by lower Wick powers with *finite* coefficients (Exercise 3.4).
- **Chaos:** $:\varphi^n:(f)$ lies in the n-th Wiener chaos; $L^2(\mu_m) \simeq$ Fock space over $H^{-1}(\mathscr{M})$. Hypercontractivity $\|X\|_{L^p} \leqslant (p-1)^{n/2}\|X\|_{L^2}$ gives convergence in all L^p (and in $\mathscr{C}^{-\kappa}$ for $d=2$).
- **Towards Φ^4_2:** on $\mathbb{T}^2$, $\int :\varphi^4: \in \bigcap_p L^p$ and (Nelson) $\mathrm{e}^{-\lambda \int :\varphi^4:} \in L^1(\mu_m)$.
- **Other local operators:** $\partial^\alpha \varphi$; in $d=2$ the Wick exponentials $:\mathrm{e}^{\mathrm{i}\beta\varphi}:$, in L^2 iff $\beta^2 < 4\pi$ (sine-Gordon; Exercise 3.6).

The Dyson–Schwinger equation

Lecture 2

Beyond L^2: Wick powers do not exist

- If $n(d-2) \geqslant d$: $\mathbb{E}[:\varphi_\varepsilon^n:(f)^2] \to \infty$. Worse: there is **no random distribution** at all, in any topology or on any space. Albeverio–Liang 2007: $:\varphi^2:(f)$, $f>0$, in $d=4$ (and $d \geqslant 5$) is a Hida distribution, not a random variable.
- **Proof.** $X_\varepsilon =: \varphi_\varepsilon^n:(f) \in \mathscr{H}_n$, $\sigma_\varepsilon = \|X_\varepsilon\|_2 \to \infty$, $\mathbb{E}\, X_\varepsilon^4 \leqslant 9^n \sigma_\varepsilon^4$. Paley–Zygmund, $\mathbb{P}(Z > \theta\, \mathbb{E}\, Z) \geqslant (1-\theta)^2 (\mathbb{E}\, Z)^2 / \mathbb{E}\, Z^2$, with $Z = X_\varepsilon^2$:

$$\mathbb{P}\left(|X_\varepsilon| > \tfrac{1}{2}\sigma_\varepsilon\right) \geqslant \tfrac{9}{16} 9^{-n} \quad \Rightarrow \quad \text{not tight.}$$

- **Bounded chaos: the law is controlled by $\sigma = \|X\|_2$.** Upper tails $\|X\|_p \leqslant (p-1)^{n/2}\sigma$ (hypercontractivity); $\mathbb{P}(|X| > \frac{1}{2}\sigma) \geqslant c_n$ (Paley–Zygmund); $\mathbb{P}(|X| \leqslant t\sigma) \leqslant C n t^{1/n}$ (Carbery–Wright). So: tight $\Leftrightarrow$ bounded in L^2; in probability $\Leftrightarrow$ in L^2.
- Renormalising by σ_ε^{-1} typically gives Gaussian limits (fourth moment theorem; Albeverio–Zhou 1996): the n-th power is lost.

Gaussian integration by parts

Cylinder functions $\mathcal{F}C_b^\infty \ni F(\varphi) = g(\varphi(f_1), \dots, \varphi(f_n))$, $g \in C_b^\infty$:

$$D_h F(\varphi) = \frac{\mathrm{d}}{\mathrm{d}t} F(\varphi + t h)\Big|_{t=0} = \sum_i \partial_i g(\dots)\langle f_i, h\rangle = \langle DF, h\rangle, \qquad DF(x) = \frac{\delta F}{\delta \varphi(x)}.$$

Lemma. *$(Y, X_1, \dots, X_n)$ centred Gaussian:* $\mathbb{E}[Y\, g(X)] = \sum_i \mathbb{E}[Y X_i]\, \mathbb{E}[\partial_i g(X)]$.

Proof. For $g = \mathrm{e}^{\mathrm{i}\langle u, \cdot\rangle}$: $\mathbb{E}[Y \mathrm{e}^{\mathrm{i}\langle u, X\rangle}] = -\mathrm{i}\partial_\theta\, \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta Y + \mathrm{i}\langle u, X\rangle}]|_{\theta=0} = \mathrm{i}\sum_i u_i\, \mathbb{E}[Y X_i]\, \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle u, X\rangle}]$; then Fourier inversion and approximation.

The Dyson–Schwinger equation

Proposition. *For all* $F \in \mathcal{F}\, C_b^\infty$, $h \in \mathcal{S}(\mathcal{M})$:

$$\mathbb{E}_{\mu_m}[D_h F] = \mathbb{E}_{\mu_m}[F\, \varphi(Ah)], \qquad \textit{formally} \qquad \mathbb{E}\left[\frac{\delta F}{\delta \varphi(x)}\right] = \mathbb{E}\left[F\, (m^2 - \Delta)\, \varphi(x)\right].$$

Proof. $\mathbb{E}[\varphi(Ah)\, \varphi(f_i)] = \langle Ah, C f_i \rangle = \langle h, f_i \rangle$; apply the lemma.

With $F = \varphi(f_2) \cdots \varphi(f_n)$ one gets the hierarchy

$$(m^2 - \Delta_{x_1})\, S_n(x_1, \ldots, x_n) = \sum_{j=2}^{n} \delta(x_1 - x_j)\; S_{n-2}(\ldots \hat{x}_j \ldots),$$

which with $S_0 = 1$, $S_1 = 0$ gives Wick's theorem (Exercise 3.7).

DS = Euclidean field equation

- Minkowski: $(\Box + m^2)\,\varphi = 0$ and CCR. Euclidean: $(-\Delta + m^2)\,\varphi$ "=" $\delta\,/\,\delta\,\varphi$ inside expectations; the contact terms come from the CCR (time ordering).
- For a formal Gibbs measure $\mu_V \propto \mathrm{e}^{-\int V(\varphi)}\mu_m$:

$$\mathbb{E}_{\mu_V}[D_h F] = \mathbb{E}_{\mu_V}\left[F\left(\left(m^2 - \Delta\right)\varphi + V'(\varphi)\right)(h)\right],$$

  the Euclidean version of $(\Box + m^2)\,\varphi + V'(\varphi) = 0$.
- $V'(\varphi)$ must be a well-defined local operator: Φ^4_2: $V' = 4\lambda{:}\varphi^3{:}$; field equation for $P(\varphi)_2$: Dimock–Glimm 1974, Albeverio–Röckner 1991. For Φ^4_3 the renormalised field equation goes back to Feldman–Rączka 1977; renormalised cubic term as a random distribution: Gubinelli–Hofmanová 2021.

Defining $V'(\varphi)$: point splitting and the OPE

- $\varphi^3(x)$: product at one point. **Point splitting**: regular part of $\varphi(y_1)\,\varphi(y_2)\,\varphi(y_3)$ as $y_i \to x$.
- Under the GFF (Wick):

$$\varphi(y_1)\,\varphi(y_2)\,\varphi(y_3) =: \varphi(y_1)\,\varphi(y_2)\,\varphi(y_3) : + \sum_{i<j} G\,(y_i - y_j)\;\varphi(y_k)$$

  regular part $\to : \varphi^3 : (x)$; mollified: $: \varphi^3 : = \lim_\varepsilon (\varphi_\varepsilon^3 - 3\,c_\varepsilon\,\varphi_\varepsilon)$.
- **OPE** (Wilson 1969, Zimmermann 1973): products at nearby points = local operators × singular coefficients; composite operator = regular part. Field equation $(\Box + m^2)\,\varphi = -4\lambda N[\varphi^3]$.
- The coefficients are those of the **interacting** theory. Φ^4_2: Gaussian subtraction suffices (Schrader 1974). Φ^4_3: an extra $\delta_\varepsilon\,\varphi_\varepsilon$, $\delta_\varepsilon \sim \lambda \log(1/\varepsilon)$: $4\lambda\,\delta_\varepsilon$ is the sunset mass counterterm (Gubinelli–Hofmanová 2021).
- Lecture 3: OPE of stochastic observables, explicit at every scale.

DS characterises the GFF

Theorem. *Let ν be a probability measure on $\mathscr{S}'(\mathscr{M})$ with $\varphi(f)\in L^1(\nu)$ for all f and*

$$\mathbb{E}_\nu[D_h F] = \mathbb{E}_\nu[F\,\varphi(Ah)] \qquad \forall F\in\mathscr{F}C_b^\infty, h\in\mathscr{S}(\mathscr{M}).$$

Then $\nu=\mu_m$.

Proof. Fix f, take $h=Cf\in\mathscr{S}(\mathscr{M})$ and $F_s=\mathrm{e}^{\mathrm{i}s\varphi(f)}$, so $D_hF_s=\mathrm{i}s\langle f,Cf\rangle F_s$. With $\chi(s)=\mathbb{E}_\nu F_s$:

$$\mathrm{i}s\langle f,Cf\rangle\chi(s)=\mathbb{E}_\nu[\varphi(f)\mathrm{e}^{\mathrm{i}s\varphi(f)}]=-\mathrm{i}\chi'(s) \quad\Longrightarrow\quad \chi(s)=\mathrm{e}^{-\frac{s^2}{2}\langle f,Cf\rangle}.$$

So $\hat\nu(f)=\mathrm{e}^{-\frac12\langle f,Cf\rangle}$ for all f; linearity of $f\mapsto\varphi(f)$ and Cramér–Wold give all finite-dimensional laws.

Remarks

- The integrability assumption is exactly what is needed for DS to make sense: $\varphi(Ah)$ ranges over all $\varphi(g)$.
- DS for characteristic functions is a **first-order linear ODE**.
- $m=0$ on the torus: *no* probability measure solves DS (take $h=1$); work modulo constants (Exercise 3.9).
- Finite dimensions: DS says $\nabla(\mathrm{e}^{S}\nu)=0$, so uniqueness is easy for any S (Exercise 3.8).
- Interacting theories: DS is a singular nonlinear relation; uniqueness is hard and largely open. Results for the exponential model in $d=2$: De Vecchi–Gubinelli–Turra.

Outlook

Next lecture

- Lecture 3: stochastic quantisation; the GFF along a scale parameter; stochastic observables, products and OPE.
- Lecture 4: Polchinski equation and its FBSDE; Wetterich equation; variational structure.
- Lecture 5: observables; the field equation at every resolution and its sub-Gaussian noise.
- Lectures 6–7: a construction of Φ^4_3 on the torus via the FBSDE.

The pattern

Gaussian measure $\xrightarrow{\text{local perturbation}}$ renormalised DS equation / scale dynamics $\xrightarrow{\text{estimates}}$ Euclidean QFT $\xrightarrow{\text{OS}}$ relativistic QFT.

Exercises and reading

Exercises (numbering of the notes)

- 3.2 Unitarity bound.
- 3.3 Regularity is sharp.
- 3.6 Wick exponentials, $\beta^2 < 4\pi$.
- 3.7 Wick's theorem from DS.
- 3.8 DS in finite dimensions.
- 3.11 Computer experiments.
- 3.12 Two masses: positivity vs. locality.

Reading

- Sheffield, *GFF for mathematicians*, PTRF 2007.
- Janson, *Gaussian Hilbert Spaces*.
- Glimm–Jaffe; Simon, $P(\varphi)_2$.